Signal of Opportunity Time Difference of Arrival Estimation Using Adaptive Filtering Techniques

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Abstract—*EXACT* and *SOCRATES* are a pair of CubeSats being developed to advance the TRL of a dual-use sensor. It can function either as x-ray detector used for characterization of high energy photons or as a sensor for measuring relative ranges between users in deep space operations. The ranging function of the sensor is described in this paper. After describing the measurements generated by the sensor, the algorithms that make use of this information to generate a position solution are presented. As a validation case study, data from the Japanese spacecraft *Suzaku*'s observation of the Crab Nebula Pulsar is used to estimate the spacecraft's change in position along the line-of-sight to the pulsar, using this estimation scheme. *Suzaku*'s observations are suitable for this application because the spacecrafts Hard X-Ray Detector (HXD) is similar in effective surface area to the prototype X-Ray/gamma-ray detector currently under development for use onboard *EXACT* and *SOCRATES*. Thus, validation of this technique using data from *Suzaku* serves to verify the feasibility of this navigation technique onboard other spacecraft with detectors of similar size.

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1. INTRODUCTION

EXACT (X-ray Characterization and Timing) and *SOCRATES* (Signal of Opportunity CubeSat Ranging and Timing Experiment System) are a pair of CubeSats which will include as their main payload a dual-use hard x-ray/soft γ -ray sensor. The payload is known as the Gamma Ray Incidence Detector

(*GRID*). When flown on *EXACT*, which is sponsored by the USAF's University Nanosat Program, *GRID* will be used to measure high-energy photons from the sun to help better understand the physics behind coronal mass ejections. When flown on *SOCRATES*, sponsored by NASA's Undergraduate Student Instrumentation Program, *GRID* will serve as a novel ranging sensor that is being investigated as an enabling technology for deep-space autonomous navigation. This paper focuses on the navigation use case of the *GRID* sensor.

Navigation beyond geosynchronous orbit (GEO) presents an ongoing challenge. [1] While spacecraft in low Earth orbit (LEO) to GEO have a wide variety of navigational aides available to them, outside of GEO, navigation is much more challenging. This is because most of the current techniques used for navigation in space rely on some form of Earth-based measurements to fix the position of the spacecraft. This becomes more and more difficult to do the further the spacecraft is from Earth. In particular, while it is fairly simple to measure a spacecraft's distance from Earth in the radial direction, measuring its position in the tangential directions is more difficult because this value depends entirely on accurate measurements of the spacecraft's angular position from Earth. The effect of errors in this measurement increases linearly with distance between Earth and the spacecraft. For instance, to determine the position of a spacecraft near Pluto, the angular position of the spacecraft would have to be measured to an accuracy of 1 nanoradian in order to achieve an accuracy of 1 kilometer in the calculated position.

NASA's Deep Space Network (DSN) is capable of measuring the angular position of spacecraft to approximately 0.33 nrad. [2] However, relying on the DSN to measure position of spacecraft in deep space has multiple limitations. The most obvious limitation of relying entirely on the DSN for navigation solutions is that even with extremely high accuracy angular measurements, the error in position measurement will always linearly increase with distance away from Earth. Consequently, no matter how accurate the DSN is, it will always result in unacceptably large position errors at great enough distances from Earth. In addition to this fundamental limitation, navigation via the DSN is limited practically by time availability. The number of spacecraft that rely on the DSN for navigation has steadily increased over the last several decades, in part due to new planned missions, and in part due to old missions outliving their life expectancy (e.g. Voyager I and II). As more and more spacecraft rely

on the DSN for positioning, it will become increasingly difficult to schedule navigational updates. In order to measure a spacecraft's position to the accuracy of 0.33 nrad, the DSN uses Very Long Baseline Interferometry, which requires approximately 1.25 hours of observation time to perform the measurement. This long observation time makes navigation via the DSN difficult to schedule, and increasingly impractical as the number of space vehicles increases [2].

As an alternative to Earth-based position measurements, we present navigation via signal-of-opportunity (NAVSOP) as a means to augment or eventually replace navigation via the DSN. The signals of opportunity considered here are naturally occurring astrophysical signals, including x-ray pulsar emissions and γ -ray bursts. In order to use this navigational technique, the time-difference of arrival of signals in question between the CubeSat and a known location must be estimated. This is challenging because naturally occurring signals are often extremely weak.

In this paper, we present the framework for NAVSOP in deep space on board a CubeSat using an x-ray or γ -ray detector such as *GRID*. Accordingly, the remainder of the paper is organized as follows: In Section 2, the development of the CubeSats *EXACT* and *SOCRATES* is discussed. Section 3 briefly discusses *GRID* (Gamma Ray Incidence Detector), the hard x-ray/soft γ -ray detector being developed for NAVSOP onboard the CubeSats. Section 4 contains a brief survey of prior work on the subject of signal-of-opportunity navigation in space using pulsars. Section 5 discusses signals available for NAVSOP in deep space, as well as some of the challenges associated with using these signals. In Section 6, the algorithm implementation for NAVSOP onboard a CubeSat is discussed. Finally, in Section 7, the utility and effectiveness of the algorithm presented is demonstrated by processing data collected from the x-ray observatory satellite *Suzaku*, which had a detector of effective surface area similar to *GRID*.

2. EXACT AND SOCRATES

In order to validate concepts for NAVSOP in deep space, a pair of CubeSats, *EXACT* and *SOCRATES* are being developed which will carry *GRID* (γ -Ray Incidence Detector) as their payload. The missions of each CubeSat are outlined here.

EXACT

EXACT (Experiment for X-ray Characterization and Timing) is designed to observe x-rays from the sun. *EXACT*'s purpose is to measure the energy and arrival time of x-ray photons from the sun, particularly during solar flares and coronal mass ejections (CMEs). Data from *EXACT* will help researchers to further understand the particle acceleration mechanisms that drive flares and CMEs. This research will add to current models for predicting solar activity and the resulting effects on Earth. *EXACT* is unique among current solar x-ray observatories in that it will measure individual photon arrival times with sub- μ second timing.

Although *EXACT*'s primary mission is not navigation related, the sensor it carries will have both the energy and temporal resolution needed to demonstrate the NAVSOP techniques discussed later in this paper. *EXACT*'s secondary mission objective is to observe one or more astrophysical events and record photon arrival data from that event to be used for demonstrating the feasibility of NAVSOP techniques.

SOCRATES

SOCRATES (Signal of Opportunity CubeSat Ranging And Timing Experiment System) is designed to observe hard x-ray and soft γ -ray photons from astrophysical events for the purpose of testing NAVSOP techniques. *SOCRATES* is designed as a low-cost proof-of-concept system that will pave the way for future navigation systems that are both affordable and compact enough to fly aboard CubeSats and other missions which need auxiliary navigation outside of what is available from GPS and the DSN. *SOCRATES*'s primary mission then is to observe multiple astrophysical phenomenon, and record the arrival time of the photons from these events to sub μ second accuracy. These photon arrival times will then be used to demonstrate the feasibility of the NAVSOP techniques proposed here. Ideally, *EXACT* and *SOCRATES* will be in orbit at the same time. In this scenario, both satellites may observe the same phenomenon, in which case both relative and absolute NAVSOP techniques could be demonstrated. However, even if both satellites cannot be flown at the same time, NAVSOP techniques may still be demonstrated by using the data collected in conjunction with data from other x-ray and γ -ray observatories or known pulsar signals.

3. GRID

At the heart of both *EXACT*'s and *SOCRATES*'s mission is the dual-use sensor known as *GRID*. For completeness, a brief overview of the *GRID* instrument is presented here.

GRID is designed specifically for the high time-resolution observation of hard x-ray/soft γ -ray photons. Additionally, *GRID* is designed to be a low-cost sensor with a volume equal to one CubeSat unit, and a mass less than or equal to the mass of a single CubeSat unit. An early prototype of *GRID* is shown in Figure 1.

The *GRID* sensor detects photons through four CsI(Tl) scintillation crystals. Each crystal is $2\text{cm} \times 2\text{cm} \times 4\text{cm}$, and the four crystals are arranged in a grid pattern. When hit by a photon, the crystals emit a number of photons proportional to the energy of the impinging x-ray or γ -ray photon, typically around 54 photons per keV. This emitted light is collected using an avalanche photo-diode (APD) attached to each scintillation crystal. The outputs of the APDs are further amplified, and then the resulting waveform is measured. The peak energy of the resultant waveform is proportional to the energy of the photon measured, and the peak time is related by a fixed delay to the photon's arrival time. Thus, by recording the peak time and value of pulses from the amplifiers, a time history and energy level of the arriving photons may be measured. [3]

4. PRIOR WORK

NAVSOP generally refers to a navigation technique which uses signals native to the operating environment to estimate a position, navigation and timing (PNT) solution. A wide variety of signals have been proposed for use with NAVSOP, including signals from WiFi, cell-phones, lightning strikes, pulsars, and γ -ray bursts, to name a few. The primary advantage of NAVSOP techniques is that they do not rely on an external infrastructure designed for navigation measurements, but rather rely on signals which are already present in the environment to estimate a PNT solution. The advantage of being able to operate independently of an external

Figure 1. Early GRID prototype

infrastructure is especially attractive in the case of deep-space navigation.

The NAVSOP technique proposed here is based on measuring the time-difference of arrival, or TDOA of a naturally occurring signal at two different locations, and using this TDOA to estimate the range in that direction between those locations. From here, NAVSOP techniques can be broadly broken down into two classes, absolute navigation and relative navigation.

Absolute NAVSOP—Absolute navigation, as suggested by the title, produces an absolute navigation solution for the user. This is achievable in the case where the signal is known *a priori*. In this case, the signal values at the origin may be stored onboard the spacecraft before departure. Then, no matter how far away the spacecraft is from the origin, it will always have the value of the signal at the origin available for computing the TDOA. This scenario is illustrated in Figure 2.

In the case where the signal is not known *a priori*, as is the case with quasi-periodic signals, absolute NAVSOP may still be possible. Here, the limiting factor would be the bandwidth required to transmit a time history of the signal from the origin to the spacecraft, especially if the spacecraft is far away from the origin.

Relative NAVSOP—If the signal cannot be known *a priori*, it may still be useful for computing the relative position between two vehicles that are relatively close together. After all, the bandwidth limitation is only a real problem if the distance between the origin and the spacecraft is very large. If multiple spacecraft are close to each other, then the signals could easily be passed from user to user, allowing the cooperating spacecraft to compute their positions relative to

Figure 2. Absolute NAVSOP concept

Figure 3. Relative NAVSOP concept

each other. Then, as long as one of the spacecraft knows its absolute position, the other cooperating spacecraft could use that absolute position combined with the relative navigation solutions to compute their respective positions. One such scenario is illustrated in Figure 3.

It is apparent from examination of Figures 2 and 3 that even though the signals and applications of the two methods may be different, the basic concept is the same. In both cases, the measured TDOA is used to generate an absolute or relative PNT estimate as shown in Equation 1.

$$\mathbf{y}(\mathbf{x}) = \begin{bmatrix} \Delta t_{i \rightarrow j}^{(1)} \\ \vdots \\ \Delta t_{i \rightarrow j}^{(S)} \end{bmatrix} = \begin{bmatrix} \frac{1}{c} \left(\ell_i^{(1)} \right)^T & 1 \\ \vdots \\ \frac{1}{c} \left(\ell_i^{(S)} \right)^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{r}_{i \rightarrow j} \\ \delta t_{i \rightarrow j} \end{bmatrix} \quad (1)$$

where

- $\Delta t_{i \rightarrow j}^{(S)}$ is TDOA of the S^{th} signal between sensor i and j
- $\mathbf{x} = [\mathbf{r}_{i \rightarrow j} \quad \delta t_{i \rightarrow j}]^T$ is the state vector, consisting of the

relative or absolute position of the user and the user's clock offset

- $\mathbf{r}_{i \rightarrow j}$ is the position vector of the user
- $\delta t_{i \rightarrow j}$ is the clock offset between the two sensors
- $\mathbf{y}(\mathbf{x})$ is the measurement vector consisting of the available $\Delta t_{i \rightarrow j}^{(S)}$
- c is the speed of light
- ℓ_S is the angle-of-arrival unit vector of the S^{th} signal

Given four TDOA measurements of signals arriving from different directions, the position and clock offset of the user can be fixed.

From Equation 1, it is apparent that the TDOA is the quantity which must be determined in order to update the position and clock offset of the user. We now turn our attention to potential signals which could be used for the estimation of this TDOA.

5. NAVSOP SIGNALS

While there are many signals which occur naturally in outer space and could potentially be used as SOPs, we restrict our attention to the following three types of signals.

Pulsars

Pulsars, short for pulsating radio star, are stars which emit highly periodic beams of electromagnetic radiation with distinct timing and frequency characteristics. The precise period of pulsars make them ideal candidates for NAVSOP because a spacecraft could have *a priori* knowledge of the signal it is trying to use. However, pulsar signals have the disadvantage of being relatively weak, resulting in a relatively low signal to noise ratio (SNR). [4],[5]

Asynchronous X-ray Sources

Asynchronous X-ray sources, including quasars, blazars, and other active galactic nuclei (AGN), have the advantage that, like pulsars, they come from a constant known location. They also feature higher signal strength. However, since the signal is aperiodic, it cannot be known *a priori* by the spacecraft. To be of any use, the signal must be measured by two cooperating spacecraft and used to estimate the relative range between the sensors.

Gamma-ray Bursts

Gamma-ray bursts are thought to occur when a star in a distant galaxy implodes, and are the brightest known electromagnetic events in the universe. They are not continuous in any sense, and do not originate from a previously known location, meaning that the angle of arrival would also have to be estimated by spacecraft using these signals. However, their relatively high intensity makes them good potential candidates for some types of relative navigation. [6]

The remainder of this paper will focus on measurements of photons from x-ray pulsars. For these signals, the average flux rate cannot be directly measured. Rather, the arrival times of photons that hit the detector are measured, and from these measurements the flux rate must be estimated. The number of photons that will be observed during a given time period is described statistically by a non-homogeneous Poisson process (NHPP). The NHPP is described by Equation 2.

Figure 4. True photon flux rate and actual measured photons

$$Pr[h_k = n] = \frac{\lambda_k^n e^{-\lambda_k}}{n!} \quad (2)$$

where

- k is the time index of observation
- h_k is the number of photons observed during time step k
- $Pr[h_k = n]$ is the probability that the number of photons observed during time step k will equal n
- λ_k is the average flux rate (averaged over a very large area) of photons at the time index

The distribution at time index k has the following properties.

$$\mathbb{E}[h_k] = \lambda_k \quad (3)$$

$$var[h_k] = \lambda_k \quad (4)$$

[7]

It is important to note that for the types of signals mentioned here, and for the detector sizes that are feasible for deployment onboard a small satellite, the expected flux-rate parameter λ_k is on the order of tens of photons per second. For typical pulsar periods, this equates to only a few photons per pulse, on average. Figure 4 shows the actual photon flux rate of photons for the Crab Nebula pulsar, overlaid with the photon arrivals observed by the x-ray observatory *Suzaku* in order to illustrate the low effective signal strength.

This relatively low photon flux rate poses a problem. Recall that NAVSOP relies on being able to accurately determine the TDOA of a signal. However, if each pulse of the signal in question consists of at most a few photons, it is not immediately obvious that the time of arrival of the peak may be measured with any accuracy at all. We address the problem of estimating the TDOA from such weak signals in the next section.

Figure 5. NAVSOP Block Diagram

6. ALGORITHM IMPLEMENTATION

We now turn our attention to the specific algorithm implementation that we propose for NAVSOP using signals from astrophysical sources. We break down our algorithm into three main components: An estimator of the correlation vector between the known and measured signals, an estimator of TDOA based on the estimated correlation vector, and a navigation filter. This general structure is illustrated in Figure 5.

Correlation Estimator

The first algorithmic challenge is to estimate the correlation vector between the measured signal, which consists of individual photon arrival times, and the true signal at the origin. We address this problem using adaptive filtering techniques. We begin by defining the following terms.

- $\mathbf{h}_k$ is a vector of photon arrivals binned by time index, where $\mathbf{h}_k[p] = \mathbf{h}_{k-p}$
- $\mathbf{w}_{ok}$ is the optimal filter such that $\lambda_k - \mathbf{h}_k \mathbf{w}_{ok} = \mathbf{e}_k$, where $\mathbf{e}_k$ is minimized in the least-squares sense

As shown in the Appendix, the optimal filter $\mathbf{w}_{ok}$ is equal to a scaled version of the cross-correlation vector between the measured signal and the true signal, $c_{\lambda_k \mathbf{h}_k}$. This result is useful because, while it is not immediately obvious how to estimate the cross-correlation vector, the computation of $\mathbf{w}_{ok}$ is a problem that has been well-addressed by adaptive filtering. In particular, this filter may be estimated by using an extended Kalman Filter (EKF) where the state of the estimator consists of the taps of the optimal FIR filter. This technique is discussed in detail in [8].

TDOA Estimator

It is a well-known result from signal processing that the correlation function between a signal and a delayed version of that same signal is maximized when at the value of the delay. Using this fact, we see that the TDOA between the user and the origin is given by the delay that maximizes the correlation vector. Given an estimate of the correlation vector (consisting of sampled points from the correlation function), the task of estimating TDOA becomes simply to find the peak value of the estimated correlation vector. Figure 6 illustrates how the correlation vector is expected to shift as the spacecraft's position changes, and how the resulting correlation peak will shift.

Because the peak value could fall between two of the correla-

Figure 6. Correlation peak shifted by δ

tion vector indices, it would be naive to simply take the index that contains the maximum value and call this the TDOA. Rather, in order to estimate the TDOA, a quadratic function is fitted around the maximum point and its nearest neighbors. The resultant best-fit quadratic function is then maximized, and the result is the estimate of TDOA.

This method has the added benefit of allowing the variance of the TDOA estimate to be computed, using an unscented transform approach. The variance of each of the correlation vector taps is computed by the EKF described in the previous subsection. These variances can be used to compute a set of sigma points. These sigma points are also fitted with a quadratic function which is maximized. The result of this operation is that not only the mean TDOA, but also a set of sigma points around the TDOA are estimated. These sigma points can then be used to compute the variance of the TDOA estimate, which is key in the navigation update.

Navigation Filter

When the TDOA is finally estimated, this TDOA may simply be multiplied by the speed of light to estimate the range between the origin and the user. These range measurements may be used, as shown in Equation 1, to update the position of the user. This update is analogous to the update performed by GPS receivers after computing range measurements from satellites.

Figure 7. Earth's orbit plus satellite's motion relative to SSB

7. RESULTS TO DATE

In order to demonstrate the utility of the methods proposed here, these algorithms were used on data collected from the satellite *Suzaku's* observations of the Crab Nebula Pulsar using the Hard X-ray Detector (HXD) instrument. The photon arrivals were screened using basic criteria described in [9]. These criteria were used to screen out photons based on event grade and time interval of arrival (e.g. photons that arrived during maneuvers, or while the Earth was blocking the field of view were screened), but did not filter out background photon arrivals. From these photon arrival data, a correlation vector between the photon arrivals and the known pulsar signal at the origin was estimated, and from this correlation vector a TDOA was estimated, both using the techniques discussed in Section 6.

For the velocity input to the correlation filter, the true velocity of the satellite corrupted with additive noise was used. In practice, the true velocity would not be known, rather it would be estimated from the orbit model, which would be incorporated into the navigation filter. In order to demonstrate the effectiveness of the filter, the range estimate based *only* on the corrupted velocity measurement is also shown in the plots of estimated range, for comparison.

This TDOA was then compared to the satellite's reported trajectory plus the trajectory of the Earth relative to the solar system barycenter. For reference, an illustration of the satellite's path is shown in Figure 7.

The reported trajectory of Earth plus that of the satellite was projected onto the line-of-sight vector for the pulsar, giving the true range measurement against which to compare the estimated range measurements. Figure 8 shows the error between the range estimated using the pulsar measurements and the true position, and the error between the velocity propagation solution and the true position, shown for four different orbits of the satellite at different times of the year. As expected, the solution which uses only velocity corrupted by noise drifts significantly, while the solution that uses the pulsar measurements corrects that drift and bounds the error.

Figure 9 shows the range estimate error for the same orbits, along with 1σ and 3σ variance bounds. As shown in the figures, the error is well-bounded by the 3σ bounds, which

Figure 8. Estimated position error and velocity propagation error

indicates that the estimator is accurately estimating the solution variance as well as the range.

8. CONCLUSIONS

In this paper, we have described two CubeSats which will carry a low-cost, high resolution hard x-ray/soft γ -ray detector. These satellites will be used to demonstrate the feasibility of navigation via signal of opportunity onboard small satellites. The paper then described the NAVSOP demonstration mission of *SOCRATES* in more detail. We propose to implement NAVSOP techniques in space using signals from astrophysical sources, and in particular signals from pulsars. However, these signals are relatively weak, and require a unique estimation scheme to estimate the signal's time-difference of arrival. We have outlined the general framework for such an algorithm which takes advantage of the signal weakness to estimate the correlation vector using adaptive filtering techniques. Finally, we demonstrated the utility of these techniques by using these algorithms to process photon arrival data from the satellite *Suzaku*, and comparing our estimate of range to the true spacecraft position.

Figure 9. Estimated position error and variance bounds

APPENDIX

As mentioned in Section 6, for very weak signals, the ideal FIR filter for processing the measured signal is equal to the correlation vector between the measurement and the desired signal. We show this to be the case below. We begin with the solution to the Wiener-Hopf equations. We start by defining the following terms:

- λ is a desired signal, or the signal with noise removed.
- $\mathbf{h}$ is the time-history of the measurement of $\lambda[k]$
- $\mathbf{c}_{\lambda\mathbf{h}}$ is the correlation vector between $\mathbf{h}$ and λ
- $\mathbf{C}_{\mathbf{h}\mathbf{h}}$ is the auto-correlation matrix of the measured signal, $\mathbf{h}$

With these terms defined, the Wiener-Hopf equations give us an expression for an ideal FIR filter that, when applied to the measured signal will give us the desired signal. This solution is given in equation 5.

$$\mathbf{w}_0 = \mathbf{C}_{\mathbf{h}\mathbf{h}}^{-1} \mathbf{c}_{\lambda\mathbf{h}} \quad (5)$$

where $\mathbf{w}_0$ is the ideal filter. [10]

Now, in the context of estimating the TDOA of a pulsar, we aren't all that interested in estimating the actual signal

that is received by the user. What we care more about is estimating the correlation vector, because if we can estimate that accurately enough, then we should be able to calculate the TDOA just by looking at the argmax of the correlation vector. In other words, the quantity we want to estimate is $\mathbf{c}_{\lambda\mathbf{h}}$.

The Wiener-Hopf equations do not give us a direct way to estimate $\mathbf{c}_{\lambda\mathbf{h}}$, but they do help us. We note first of all that there are a number of adaptive filtering algorithms, (for instance LMS, nLMS, RLS, and Kalman Filtering) that give us tools for estimating $\mathbf{w}_0$. So, if we estimated $\mathbf{w}_0$ using one of the many algorithms available, and if we knew $\mathbf{C}_{\mathbf{h}\mathbf{h}}$, then we could compute $\mathbf{c}_{\lambda\mathbf{h}}$ as follows:

$$\mathbf{c}_{\lambda\mathbf{h}} = \mathbf{C}_{\mathbf{h}\mathbf{h}} \mathbf{w}_0 \quad (6)$$

In general, $\mathbf{C}_{\mathbf{h}\mathbf{h}}$ is not known beforehand. However, in the case of photon arrival times, we can actually do an excellent job of approximating it. To see how we can approximate it, we examine the matrix $\mathbf{C}_{\mathbf{h}\mathbf{h}}$ more closely. $\mathbf{C}_{\mathbf{h}\mathbf{h}}$ is defined by

$$\mathbf{C}_{\mathbf{h}\mathbf{h}} = \begin{bmatrix} c_{hh}[0] & c_{hh}[1] & \dots & c_{hh}[N-1] \\ c_{hh}[-1] & c_{hh}[0] & \dots & c_{hh}[N-2] \\ \vdots & \vdots & \ddots & \vdots \\ c_{hh}[N-1] & c_{hh}[N-2] & \dots & c_{hh}[0] \end{bmatrix} \quad (7)$$

where

$$c_{hh}[p] = \mathbb{E}[h[k]h[k-p]] \quad (8)$$

Now, we recall Section 5, where we stated that $h[k] \sim \text{Pois}[\lambda[k]]$. The statistics of Poisson process are well characterized, and we can apply that knowledge to the expectation in Equation 8. There are, generally speaking, two cases of interest; when $p = 0$ and when $p \neq 0$.

When $p = 0$, we can simply write

$$c_{hh}[0] = \mathbb{E}[h[k]h[k]] \quad (9)$$

$$= \text{var}[h[k]] \quad (10)$$

$$= \lambda[k] \quad (11)$$

On the other hand, when $p \neq 0$, we evaluate Equation 8 as follows:

$$c_{hh}[p] = \mathbb{E}[h[k]h[k-p]] \quad (12)$$

$$= \text{Cov}[h[k], h[k-p]] + \mathbb{E}[h[k]]\mathbb{E}[h[k-p]] \quad (13)$$

$$= \lambda[k]\lambda[k-p] \quad (14)$$

The last line in the 12 makes use of the fact that two measurements are independent realizations of the same Poisson

process, and so the covariance between them is zero. Substituting 9 and 12 into 7 yields the following result.

$$\mathbf{C}_{hh} = \begin{bmatrix} \lambda[k] & \lambda[k]\lambda[k+1] & \dots \\ \lambda[k]\lambda[k-1] & \lambda[k] & \dots \\ \vdots & \vdots & \ddots \end{bmatrix} \quad (15)$$

Now, we still do not know every value of $\lambda[k]$ exactly, but we note an interesting fact. Because we are assuming that $\lambda[k] \ll 1$, we can say that $\lambda[k]\lambda[k-p] \approx 0$. With this approximation, $\mathbf{C}_{hh}$ simply becomes a scaled version of the identity matrix! Under this assumption then, we can say that $\mathbf{w}_0 \approx \mathbf{a}c_{\lambda h}$. The scaling constant simply comes from the fact that $\mathbf{C}_{hh}$ will be a scaled version of the identity matrix. However, since we are only interested in finding the peak of the correlation vector, we need not be concerned with this scaling factor.

As we stated above, we have many tools at our disposal to find $\mathbf{w}_0$, so according to this analysis, we simply need to select an appropriate adaptive filtering algorithm, run it on the incoming signal $\mathbf{h}$, and the resultant filter should be the correlation vector between $\mathbf{h}$ and the true signal, λ . Furthermore, not only do we have tools to estimate $\mathbf{w}_0$, but these estimators will also estimate the *variance* of $\mathbf{w}_0$. These two points are the key takeaways from this section: *we can estimate the correlation between the true signal and the observed signal, and its variance, simply by estimating the optimal FIR filter for the observed signal, given that the observed signal is weak enough.*

It is interesting to note that the assumption of $\lambda \ll 1$ is a key component of this analysis, (as it lets us make the identity approximation), but at the same time, the fact that $\lambda \ll 1$ was what made the problem challenging. We might reasonably be skeptical of the claim that $\lambda \ll 1$ actually makes our estimation problem easier rather than harder. In fact, it does some of both.

On the one hand, $\lambda \ll 1$ does make our analysis easier, not only because it makes our approximation $c_{\lambda h} \approx \mathbf{w}_0$ reasonable, but also because it makes the matrix $\mathbf{C}_{hh}$ well-conditioned. Even though we will never actually invert $\mathbf{C}_{hh}$, the fact that it is well-conditioned actually helps the algorithms that we will end up using maintain stability.

On the other hand, as noted multiple times previously, $\lambda \ll 1$ is essentially the source of the difficulty to begin with. If λ were not much less than 1, then we would not need to estimate $c_{\lambda h}$ to begin with; we could just compute it directly. At some large enough value of λ , the effect of the individual photon arrivals would be smoothed out by the characteristics of the signal, and so the particular problem addressed in this paper would not really be a problem. So, as we would suspect intuitively, $\lambda \ll 1$ is less desirable than $\lambda \gg 1$. But, there is a “silver lining” in that $\lambda \ll 1$ helps guarantee the stability of our estimators.

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