A CAPACITABILITY THEOREM IN MEASURABLE

GAMBLING THEORY

by

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Abstract

A player in a measurable gambling house Γ defined on a Polish state space X has available, for each $x \in X$, the collection $\Sigma(x)$ of possible distributions σ for the stochastic process $x_1, x_2, \dots$ of future states. If the object is to control the process so that it will lie in an analytic subset A of $H = X \times X \times \dots$, then the player's optimal reward is

$$M(A)(x) = \sup\{\sigma(A) : \sigma \in \Sigma(x)\}.$$

The operator $M(\cdot)(x)$ is shown to be regular in the sense that

$$M(A)(x) = \inf M(\{\tau < \infty\})(x)$$

where the infimum is over Borel stopping times τ such that $A \subseteq \{\tau < \infty\}$. A consequence of this regularity property is that the value of $M(A)(x)$ is unchanged if, as in the gambling theory of Dubins and Savage, the player is allowed to use nonmeasurable strategies. This last result is seen to hold for bounded, Borel measurable payoff functions including that of Dubins and Savage.

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1. Introduction

Let X be a nonempty Borel subset of a Polish space and let $P(X)$ be the collection of countably additive probability measures defined on the Borel subsets of X . Give $P(X)$ its usual weak topology so that it too has the structure of a Borel subset of a Polish space (see Parthasarathy [16] for information about the weak topology on $P(X)$). An analytic gambling house Γ is a mapping which assigns to each $x \in X$ a nonempty subset $\Gamma(x)$ of $P(X)$ in such a way that the set

$$\Gamma = \{(x, \gamma) \in X \times P(X) : \gamma \in \Gamma(x)\}$$

is analytic. Starting at some initial state $x \in X$, a player in the house Γ chooses a measurable strategy σ available at x , which means a sequence $\sigma = (\sigma_0, \sigma_1, \dots)$ where $\sigma_0 \in \Gamma(x)$ and, for $n = 1$, σ_n is a universally measurable mapping from X^n to $P(X)$ such that $\sigma_n(x_1, x_2, \dots, x_n) \in \Gamma(x_n)$ for every $(x_1, x_2, \dots, x_n) \in X^n$. Every measurable strategy σ determines a probability measure, also denoted by σ , on the Borel subsets of

$$H = X \times X \times \dots$$

The probability measure σ can be regarded as the distribution of the coordinate process $h = (h_1, h_2, \dots)$, where h_1 has distribution σ_0 and h_{n+1} has conditional distribution $\sigma_n(x_1, x_2, \dots, x_n)$ given $h_1 = x_1, h_2 = x_2, \dots, h_n = x_n$. For each $x \in X$, let $\Sigma(x)$ be the collection of measurable strategies available at x .

The optimal reward operator M assigns to each bounded, universally measurable function $g: H \rightarrow \mathbb{R}$ the function Mg defined on X by

$$(Mg)(x) = \sup\left(\int g d\sigma : \sigma \in \Sigma(x)\right).$$

If g is the indicator function of a universally measurable set B , we write $M(B)$ for Mg . Thus, for fixed x , $M(\cdot)(x)$ is a set function. Regularity properties of this set function were studied in earlier papers [12], [13] and [19] and the major result of the present work is another such property which is analogous to the capacitability theorem of Choquet [2].

To state the result, it is convenient to introduce two topologies on H . Let T_1 be the product topology on H when X is assigned the topology under which it is a Borel subset of a Polish space and let T_2 be the product topology on H when X has the discrete topology. The words "Borel", "analytic", "coanalytic", "universally measurable", when used to qualify subsets of H , will refer to the topology T_1 , while the words "closed", "open", "clopen", " G_δ ", will refer to T_2 . For a subset E of H , define the function $M^*(E)$ on X by

$$M^*(E)(x) = \inf\{M(O)(x) : E \subseteq O \text{ and } O \text{ is Borel, open}\}.$$

Here is our main result.

Theorem 1.1. If A is an analytic subset of H , then $M(A) = M^*(A)$.

This theorem was proved in [13] for the special case when X is countable and also for the special case when $\Gamma(x)$ is finite for every x . For these cases, it was shown in [13] that $M^*(\cdot)(x)$ is a right-continuous (with respect to the topology T_1) capacity for each x . The result then follows from the capacitability theorem. It was also shown in [13] that $M^*(\cdot)(x)$ is not a capacity in general.

The proof of Theorem 1.1 will use ideas and techniques from the proof of the capacitability theorem, but also from gambling theory and effective descriptive set theory. The principal difficulty in proving Theorem 1.1 is in establishing the measurability of the function $M^*(A)(\cdot)$. It is already hard to do so when A is a Borel, closed subset of H (see [12] and remarks in [19]). We get around this problem by considering an effective surrogate of the function $M^*(A)(\cdot)$ and formulating an effective refinement of Theorem 1.1. This will involve the machinery of hyperarithmetic recursion, which will be explained in section 3. Our proof of the effective refinement of Theorem 1.1 will proceed in stages. The first step will be to establish the result for effective Borel sets which are countable intersections of open sets in the topology T_2 . This is done in Section 5. The next step, carried out in Section 6, is to extend the result to effective Borel sets which are countable unions of countable intersections of

open sets. The final step, in section 7, proves the result for effective analytic sets (Theorem 7.1). Theorem 1.1 is then deduced in section 8.

We regard Theorem 1.1 as a fundamental approximation result in gambling theory as is the usual capacitability theorem in probability and measure theory. It will be applied in section 9 to prove Theorem 1.2, which answers an old question about the adequacy of measurable strategies.

Recall that the gambling theory of Dubins and Savage [4] takes place in a very general finitely additive framework in which a player is not restricted to measurable strategies. In the theory of Dubins and Savage, as extended by Purves and Sudderth [18], the probability $\sigma(A)$ is defined for every analytically measurable (that is, measurable with respect to the σ -field generated by analytic sets) set A under every strategy σ , measurable or not. In consequence, $\int g d\sigma$ is defined for every bounded $g: H \rightarrow \mathbb{R}$ which is upper analytic in the sense that $\{h \in H: g(h) > r\}$ is analytic for every real r . For such a payoff function g , the optimal reward to a player with initial state x who is not restricted to measurable σ is

$$(\Gamma g)(x) = \sup\{\int g d\sigma: \sigma \text{ available at } x\}.$$

However, the player gains no advantage through the use of nonmeasurable σ 's.

Theorem 1.2. If g is bounded and upper analytic, then $\Gamma g = M g$.

This theorem generalizes Theorem 2.1 of Dubins et al. [5] which corresponds to the special case when $g(h) = \overline{\lim}_n u(h_n)$ for some bounded, upper analytic $u: X \rightarrow \mathbb{R}$.

2. Definitions and notation

N will denote the set of positive integers, ω the set of natural numbers and $\mathbb{R}$ the set of real numbers. If s, t are finite sequences of natural numbers, we write $|s|$ for the length of s , s_i for the i -th coordinate of s for $1 \leq i \leq |s|$, $s \supseteq t$ to denote that s extends t and st for the catenation of s followed by t . If $\alpha \in \omega^N$ and $n \in N$, the finite sequence $(\alpha(1), \alpha(2), \dots, \alpha(n))$ will be denoted by $\bar{\alpha}(n)$.

We use X exclusively to denote the state space. It will always be a Borel subset of a Polish space. For $p \in X^m$ and $(x_1, x_2, \dots, x_n) \in X^n$, $p(x_1, x_2, \dots, x_n)$ or $px_1, x_2, \dots, x_n$ will denote the element of X^{m+n} obtained by catenating p and $(x_1, x_2, \dots, x_n)$. The symbol H will be reserved for X^N , the space of histories. For $h \in H$ and $i \in N$, h_i will denote the i -th coordinate of h . If $h, h' \in H$ and $n \in \omega$, we write $h =_n h'$ if $h_i = h'_i$ for $i = 1, 2, \dots, n$. If $p \in X^m$ and $h \in H$, ph will be the element of H obtained by catenating p and h . For $k \in N$, p_k will denote the function on H to X^k defined by $p_k(h) = (h_1, h_2, \dots, h_k)$. If $B \subseteq H$ and $p \in X^m$, then Bp will denote the set of $h \in H$ such that $ph \in B$. Similarly, Bp_k will be the set-valued function on H whose value at h is the set $Bp_k(h)$.

A mapping τ from H to $NU(\omega)$ is called a stopping time if

$$\tau(h) = n \in N \text{ and } h' =_n h \text{ imply } \tau(h') = n.$$

A stop rule is an everywhere finite stopping time. A stopping time τ is Borel (universally) measurable if for each $n \in N$, the set $\{\tau \leq n\}$ is Borel (universally) measurable in H . If τ is a stopping time, then h_τ and p_τ are functions on the set $\{\tau < \infty\}$ whose values at h are $h_{\tau(h)}$ and $p_{\tau(h)}(h) = (h_1, h_2, \dots, h_{\tau(h)})$, respectively. Similarly, Bp_τ is the set-valued function on $\{\tau < \infty\}$ whose value at h is $Bp_\tau(h)$. If $x \in X$, define $\tau[x]$ on H by

$$\tau[x](h) = \tau(xh) - 1.$$

Then $\tau[x]$ is again a stopping time or identically equal to zero. We write $\tau[(x_1, x_2, \dots, x_n)]$ or just $\tau[x_1, x_2, \dots, x_n]$ for $\tau[x_1][x_2] \dots [x_n]$. If $k \in N$, define the stop rule $\tau \wedge k$ by

$$(\tau \wedge k)(h) = \text{minimum } \{\tau(h), k\}.$$

Let τ be a stop rule. We say that a function ϕ on H to a set Y is determined by time t if

$$t(h) = k \text{ \& } h =_k h' \text{ imply } \phi(h) = \phi(h').$$

A subset K of H is said to be determined by time t just in case the indicator function of K is determined by time t . It is easy to prove that a subset of H is clopen in the topology T_2 iff it is determined by time t for some stop rule t . A function ψ on $H \times N$ to Y is adapted if

$$h \equiv_k h' \text{ implies } \psi(h, k) = \psi(h', k).$$

If K is a clopen subset of H in the topology T_2 , then its structure, denoted by $\text{str}(K)$, is defined as follows. If $K = H$ or $K = 0$, we define $\text{str}(K) = 0$. Otherwise, it is defined by the formula

$$\text{str}(K) = \sup\{\text{str}(Kx) + 1 : x \in X\}.$$

Suppose σ is a measurable strategy available at x and $p \in X^m$. We define a measurable strategy $\sigma[p]$ available at $l(p)$, the last coordinate of p , as follows:

$$(\sigma[p])_0 = \sigma_m(p)$$

and, for $n \geq 1$,

$$(\sigma[p])_n(x_1, x_2, \dots, x_n) = \sigma_{m+n}(px_1x_2 \dots x_n).$$

It is easy to verify that the measures (on H induced by) $\sigma[p]$, $p \in X_m$, are a version of the conditional σ -distribution of $(h_{m+1}, h_{m+2}, \dots)$ given $(h_1, h_2, \dots, h_m) = p$. If τ is a universally measurable stopping time, $\sigma[p_\tau]$ is a mapping on $\{\tau < \infty\}$ whose value at h is $\sigma[p_\tau(h)]$.

By virtue of Lemma 2.2 in [11], we can fix a Borel measurable mapping $\nu: P(H) \times X \rightarrow P(H)$ such that for each $\mu \in P(H)$ and $x \in X$, $\nu(\mu, x)$ is a regular conditional μ -distribution of $(h_2, h_3, \dots)$ given $h_1 = x$. For $\mu \in P(H)$ and $x \in X$, we write $\mu[x]$ for $\nu(\mu, x)$; more generally, if $x_1, x_2, \dots, x_n \in X$, we shall write $\mu[(x_1; x_2, \dots, x_n)]$ or $\mu[x_1, x_2, \dots, x_n]$ for $\mu[x_1][x_2] \dots [x_n]$. If $\mu \in P(H)$, μ_0 will denote the marginal distribution of μ on the first coordinate of H . If τ is a universally measurable stopping time, $\mu[p_\tau]$ will denote the mapping from

$\{\tau < \infty\}$ to $P(H)$ whose value at h is $\mu[p_\tau(h)]$. We remark that this notation is in agreement with that of the previous paragraph in the following sense. If σ is a measurable strategy and μ the probability measure induced on H by σ , then, for almost all $(\mu)p \in X^m$, the measure induced by the strategy $\sigma[p]$ agrees with $\mu[p]$.

Suppose τ_n are universally measurable stopping times such that $\tau_n < \tau_{n+1}$ on $\{\tau_{n+1} < \infty\}$, $n \in N$. Assume that, for each $n \in N$, $\phi_n: H \times N \rightarrow P(H)$ is universally measurable and adapted. Let $\lambda \in P(H)$. Define, for each Borel subset B of X ,

$$q_0(B) = \lambda_0(B)$$

and, for $n \geq 1$,

$$\begin{aligned} q_n(B|x_1, x_2, \dots, x_n) &= (\lambda[x_1, x_2, \dots, x_n])_0(B) \\ &\quad \text{if } n < \tau_1((x_1, x_2, \dots, x_n, x^*, x^*, \dots)), \\ &= (\phi_m((x_1, x_2, \dots, x_n, x^*, x^*, \dots), i)[x_{i+1}, x_{i+2}, \dots, x_n])_0(B) \\ &\quad \text{if } i = \tau_m((x_1, x_2, \dots, x_n, x^*, x^*, \dots)) \leq n < \tau_{m+1}((x_1, x_2, \dots, x_n, x^*, x^*, \dots)), \end{aligned}$$

where x^* is an arbitrary fixed point of X and the sequence $(x_{i+1}, x_{i+2}, \dots, x_n)$ is to be regarded as the empty sequence if $n < i+1$. By [1, Proposition 7.45], there is a unique probability measure μ on the Borel subsets of H such that $\mu_0 = q_0$ and, for each $n \in N$, the μ -conditional distribution of h_{n+1} given $h_1=x_1, h_2=x_2, \dots, h_n=x_n$ is $q_n(\cdot|x_1, x_2, \dots, x_n)$ for almost all $(\mu)(x_1, x_2, \dots, x_n) \in X^n$. We call μ the sequential composition of (λ, ϕ, τ_n) .

Recall from the previous section that if Γ is an analytic gambling house then $\Sigma(x)$ is the set of measurable strategies available at x . Identifying measurable strategies with the probability measures they induce, we can consider $\Sigma(x)$ to be a set of probability measures on the Borel subsets of H . In order to establish the main result of this article, it will be convenient to work with these sets of probability measures rather than the gambling house Γ which gives rise to them. We therefore isolate the properties of Σ that will be needed and formulate the notion of a global gambling house.

A global gambling house Σ is a mapping which assigns to each $x \in X$ a nonempty subset $\Sigma(x)$ of $P(H)$ in such a way that

- (a) if $\mu \in \Sigma(x)$, then, for every $k \in \mathbb{N}$, $\mu[x_1, x_2, \dots, x_k] \in \Sigma(x_k)$ for almost all $(\mu)(x_1, x_2, \dots, x_k) \in X^k$; and
- (b) if τ_n are universally measurable stopping times such that $\tau_n < \tau_{n+1}$ on $\{\tau_{n+1} < \infty\}$ for $n \in \mathbb{N}$, if, for each $n \in \mathbb{N}$, $\phi_n: H \times \mathbb{N} \rightarrow P(H)$ is universally measurable, adapted and $\phi_n(h, k) \in \Sigma(h_k)$ for all $(h, k) \in H \times \mathbb{N}$, and if $\lambda \in \Sigma(x)$, then the sequential composition of $(\lambda, \phi_n, \tau_n)$ belongs to $\Sigma(x)$.

We remark that if Γ is an analytic gambling house and if $\Sigma(x)$ is the set of probability measures on H induced by measurable strategies available at x for each $x \in X$, then Σ is a global gambling house. Though we shall have no use for it, we remark that the converse is also true: if Σ is an analytic, global gambling house, then there is an analytic, "local" gambling house Γ such that, for each $x \in X$, $\Sigma(x)$ is precisely the set of measures induced by measurable strategies available at x in the house Γ .

3. Some effective descriptive set theory

Effective descriptive set theory takes place in Polish spaces which admit a smooth recursion theory. This is made precise in the next definition.

We say that a topological space Z is Δ_1^1 - recursively presented if Z admits a complete metric d and a dense sequence $(r_n)_{n \in \mathbb{N}}$ such that the relations

$$d(r_n, r_m) \leq p/q+1 \text{ and } d(r_n, r_m) < p/q+1$$

are Δ_1^1 in ω^4 .

Examples of such spaces are 2 , 2^ω , ω^ω , $[0,1]^\omega$, etc. (see Moschovakis [15]).

Suppose now that Z_1, Z_2 are Δ_1^1 - recursively presented compact metric spaces. Then $Z_1 \times Z_2$ and $P(Z_1)$, the set of probability measures on the Borel subsets of Z_1 , are again Δ_1^1 - recursively presented compact metric spaces (Louveau [7], [9]).

In what follows, our terminology and notation, pertaining to concepts in effective descriptive set theory, are taken from Moschovakis [15].

Following Louveau [8, p.13], we say that the pair (W, C) is a coding of Borel

subsets of a Δ_1^1 - recursively presented compact metric space Y if

- (a) W is a Π_1^1 subset of $\omega^\omega \times \omega$;
- (b) C is a Π_1^1 subset of $\omega^\omega \times \omega \times Y$ whose projection on $\omega^\omega \times \omega$ is W ;
- (c) the set $\{(\alpha, n, y) \in \omega^\omega \times \omega \times Y : (\alpha, n) \in W \text{ \& } (\alpha, n, y) \notin C\}$ is Π_1^1 ;
- (d) for fixed $(\alpha, n) \in \omega^\omega \times \omega$, the section $C_{\alpha, n} = \{y \in Y : (\alpha, n, y) \in C\}$ is $\Delta_1^1(\alpha)$;
- (e) if $P \subseteq Y$ is $\Delta_1^1(\alpha)$, then there is n such that $(\alpha, n) \in W$ and $P = C_{\alpha, n}$.

The rest of this section will be devoted to proving an effective separation theorem which will be needed in the sequel. In what follows and until further notice, X will be 2^ω , the space of sequences of 0's and 1's and H will be $X \times X \times \dots$.

Fix a coding (W, C) of Borel subsets of the Δ_1^1 - recursively presented compact metric space H . We will be interested in $(\alpha, n) \in W$ such that $C_{\alpha, n}$ is Borel and open. Define $\bar{W} \subseteq \omega^\omega \times \omega$ as follows:

$$(\alpha, n) \in \bar{W} \leftrightarrow (\alpha, n) \in W \text{ \& } (\forall h)[(\alpha, n, h) \notin C \text{ or } (\exists m)(\forall h')(h' \equiv_m h \rightarrow (\alpha, n, h') \in C)].$$

Clearly, $\bar{W}$ is Π_1^1 , and if $(\alpha, n) \in W$, then $(\alpha, n) \in \bar{W}$ if and only if $C_{\alpha, n}$ is open. Set

$$\bar{C} = C \cap (\bar{W} \times H).$$

Then

- (i) $\bar{C}$ is Π_1^1 and the projection of $\bar{C}$ to $\omega^\omega \times \omega$ is $\bar{W}$;
- (ii) the set $\{(\alpha, n, h) \in \omega^\omega \times \omega \times H : (\alpha, n) \in \bar{W} \text{ \& } (\alpha, n, h) \notin \bar{C}\}$ is Π_1^1 ;
- (iii) $P \subseteq H$ is $\Delta_1^1(\alpha)$ and open if and only if $P = \bar{C}_{\alpha, n}$ for some $(\alpha, n) \in \bar{W}$.

Lemma 3.1 If A and B are Σ_1^1 subsets of H such that A can be separated from B by an open set, then there is a Δ_1^1 - recursive stopping time τ such that

$$A \subseteq \{\tau < \infty\} \subseteq B^c.$$

Proof. Define $P \subseteq H \times N$ as follows:

$$(h,n) \in P \leftrightarrow (\forall h') (h' \equiv_n h \rightarrow h' \notin B).$$

Then P is Π_1^1 and $(\forall h \in A)(\exists n)((h,n) \in P)$. So, by the Kreisel selection theorem [15,p.203], there is a Δ_1^1 -recursive function $f: H \rightarrow N$ such that $(\forall h \in A)((h, f(h)) \in P)$. Let

$$Q = \{(h,n) \in H \times N: (\exists h' \in A)(h' \equiv_n h \text{ \& } f(h') = n)\}.$$

It follows that Q is Σ_1^1 and $Q \subseteq P$. Next, we define $S \subseteq N \times \omega$ as follows:

$$\begin{aligned} (n,m) \in S \leftrightarrow & (x^*, m) \in \bar{W} \\ & \& (\forall h)(\forall h')(h \in \bar{C}_{x^*, m} \& h \equiv_n h' \rightarrow h' \in \bar{C}_{x^*, m}) \\ & \& (\forall h)(h \in \bar{C}_{x^*, m} \rightarrow (h,n) \in P) \\ & \& (\forall h)((h,n) \in Q \rightarrow h \in \bar{C}_{x^*, m}), \end{aligned}$$

where x^* is a fixed Δ_1^1 point in ω^ω .

Then S is Π_1^1 . Use the first principle of separation for Σ_1^1 sets [15,p.204] to see that $(\forall n \in N)(\exists m)((n,m) \in S)$. Invoke the Kreisel selection theorem one more time to get a Δ_1^1 -recursive function $g: N \rightarrow \omega$ such that $(\forall n \in N)((n, g(n)) \in S)$. Finally, define τ on H as follows:

$$\begin{aligned} \tau(h) = & \text{least } n \text{ such that } h \in \bar{C}_{x^*, g(n)} \text{ if } (\exists n \in N)(h \in \bar{C}_{x^*, g(n)}), \\ & = \infty, \text{ otherwise.} \end{aligned}$$

Then it is easy to verify that τ satisfies the assertion of the lemma. $\square$

As immediate consequences of the lemma, we have the following two corollaries.

Corollary 3.2. If V is a Δ_1^1 , open subset of H , then there is a Δ_1^1 -recursive stopping time τ such that $V = \{\tau < \infty\}$.

Corollary 3.3. If K is a Δ_1^1 , clopen subset of H , then there is a Δ_1^1 - recursive stop rule τ such that K is determined by time τ .

Classical (boldface) versions of the above results were proved in [10].

4. A fundamental inequality

Let $X = 2^w$, $H = X \times X \times \dots$ and let $\Sigma \subseteq X \times P(H)$ be a global gambling house which is a Σ_1^1 subset of the Δ_1^1 - recursively presented compact metric space $X \times P(H)$.

Let $E \subseteq H$ be Σ_1^1 and let $x \in X$. We can define

$$(4.1) \quad M(E)(x) = \sup\{\mu(E) : \mu \in \Sigma(x)\}, \quad x \in X.$$

There should be no confusion with the symbol M introduced in section 1, where $\Sigma(x)$ denoted the set of measurable strategies available at x . Here we have no "local" gambling house Γ , but in case Σ above had been induced by a "local" gambling house, the M of (4.1) would have coincided with the same symbol of section 1.

Next we define the effective surrogate $\bar{M}$ of the function M^* of section 1. However, $\bar{M}$ has to be defined simultaneously for a Σ_1^1 subset of H and all its sections. So let E be a Σ_1^1 subset of H and let $x_1, x_2, \dots, x_k, x \in 2^w$. We define

$$(4.2) \quad \bar{M}(E; x_1, x_2, \dots, x_k; x) = \inf \sup\{\mu(0) : \mu \in \Sigma(x)\}$$

where the $\inf$ is taken over all $\Delta_1^1(\langle x_1, x_2, \dots, x_k, x \rangle)$, open sets O containing $E x_1 x_2 \dots x_k$. Here $\langle \alpha_1, \alpha_2, \dots, \alpha_k \rangle$ is the member of w^w which codes the finite sequence $\alpha_1, \alpha_2, \dots, \alpha_k$ of members of w^w , as described in [15, p.40]. Since $2^w \subseteq w^w$, the coding can be used for finite sequences of members of 2^w as well. We allow the finite sequence $x_1, x_2, \dots, x_k$ to be the empty sequence in (4.2). In the sequel, the left side of (4.2) will be abbreviated by $\bar{M}(E x_1 x_2 \dots x_k)(x)$. Similarly, if $\alpha \in w^w$ and F is a $\Sigma_1^1(\alpha)$ subset of H , we define the relativization $\bar{M}^\alpha$ of $\bar{M}$ as follows:

$$(4.3) \quad \tilde{M}^\alpha(F; x_1, x_2, \dots, x_k; x) = \inf \sup \{ \mu(0) : \mu \in \Sigma(x) \}$$

where now the inf is taken over all $\Delta_1^1(\langle \alpha, x_1, x_2, \dots, x_k, x \rangle)$, open sets O containing $Fx_1x_2\dots x_k$. The left side of (4.3) will be abbreviated by $\tilde{M}^\alpha(Fx_1x_2\dots x_k)(x)$.

Now suppose that $F = Ex_1x_2\dots x_k$ for some Σ_1^1 subset E of H and $x_1, x_2, \dots, x_k \in 2^\omega$. Set $\alpha = \langle x_1, x_2, \dots, x_k \rangle$. Then, as is easy to verify,

$$(4.4) \quad \tilde{M}^\alpha(F; ; x) = \tilde{M}(E; x_1, x_2, \dots, x_k; x).$$

We will use (4.4) without explicit mention in the sequel.

Lemma 4.1: If $E \subseteq H$ is Σ_1^1 , then the set

$$\{ (h, k, x, a) \in H \times N \times X \times [0, 1] : \tilde{M}(Eh_1h_2\dots h_k)(x) \geq a \}$$

is Σ_1^1 .

Proof: $\tilde{M}(Eh_1h_2\dots h_k)(x) \geq a$

$$\leftrightarrow (\forall r)(\forall n)[r < a \ \& \ (\langle h_1, h_2, \dots, h_k, x \rangle, n) \in \tilde{W}$$

$$\ \& \ Eh_1h_2\dots h_k \subseteq \tilde{C}_{\langle h_1, h_2, \dots, h_k, x \rangle, n} \rightarrow (\exists \mu \in \Sigma(x))(\mu(\tilde{C}_{\langle h_1, h_2, \dots, h_k, x \rangle, n}) > r)],$$

where r runs through the rationals in $[0, 1]$, and we think of 2^ω as being imbedded as a Π_1^0 subset of ω^ω . One sees easily by imitating the computation in the proof of Theorem 2.2.3 (a) in Kechris [6] that the relation above is Σ_1^1 . $\square$

Lemma 4.2: If $E \subseteq H$ is Σ_1^1 and τ is a Δ_1^1 - recursive stopping time on H , then the set

$$\{ (h, a) \in H \times [0, 1] : \tau(h) < \infty \ \& \ \tilde{M}(Ep_\tau)(h_\tau) \geq a \}$$

is Σ_1^1 .

Proof: $\tau(h) < \infty$ & $\bar{M}(Ep_\tau)(h_\tau) \geq a$

$\leftrightarrow (\exists k \in \mathbb{N})[\tau(h) = k \text{ \& \; } \bar{M}(Eh_1h_2\dots h_k)(h_k) \geq a]$.

The above condition is Σ_1^1 by virtue of Lemma 4.1. $\square$

Lemma 4.3: Let $E \subseteq H$ be Σ_1^1 and τ a Δ_1^1 - recursive stopping time on H . Fix $x_0 \in X$ and $\epsilon > 0$ rational. Then there is a $\Delta_1^1(x_0)$ - recursive function $g: H \rightarrow [0,1]$ such that

$$(i) \quad \tau(h) < \infty \Rightarrow g(h) \geq M(Ep_\tau)(h_\tau)$$

$$(ii) \quad \tau(h) = k < \infty \text{ \& \; } h =_k h' \Rightarrow g(h) = g(h'), \text{ and}$$

$$(iii) \quad \sup_{\mu \in \Sigma(x_0)} \int g(h) d\mu(h) \leq \sup_{\mu \in \Sigma(x_0)} \int_{\{\tau < \infty\}} \bar{M}(Ep_\tau)(h_\tau) d\mu(h) + \epsilon.$$

Proof. Let $X^* = \bigcup_{m \in \omega} X^m$. Now X^* can be endowed with a Δ_1^1 - recursive presentation so it becomes a Δ_1^1 - recursively presented Polish space. Define $\psi: H \rightarrow X^*$ as follows:

$$\begin{aligned} \psi(h) &= (h_1, h_2, \dots, h_\tau) & \text{if } \tau(h) < \infty \\ &= e & \text{if } \tau(h) = \infty, \end{aligned}$$

where e is the empty sequence. Plainly, ψ is Δ_1^1 - recursive.

Let $\tilde{\Sigma}(x_0) = \{\mu\psi^{-1}: \mu \in \Sigma(x_0)\}$, so that $\tilde{\Sigma}(x_0)$ is a $\Sigma_1^1(x_0)$ set in $P(X^*)$, and hence the set $(\nu \times \lambda: \nu \in \tilde{\Sigma}(x_0))$ is $\Sigma_1^1(x_0)$ in $P(X^* \times [0,1])$, where λ is Lebesgue measure. Finally, define $\Phi: X^* \rightarrow [0,1]$ by setting

$$\begin{aligned} \Phi(p) &= \bar{M}(Ep)(l(p)) & \text{if } p \in \bigcup_{m \geq 1} X^m \\ &= 0 & \text{if } p = e, \end{aligned}$$

where $l(p)$ is the last coordinate of p . Let

$$D = \{(p, a) \in X^* \times [0, 1] : \Phi(p) \geq a\}.$$

Note that by virtue of Lemma 4.1 D is $\Sigma_1^1(x_0)$. So, by Lemma 4.3 in [12], there is a $\Delta_1^1(x_0)$ set B in $X^* \times [0, 1]$ such that $D \subseteq B$ and

$$\begin{aligned} \sup_{\nu \in \tilde{\Sigma}(x_0)} (\nu \times \lambda)(B) &\leq \sup_{\nu \in \tilde{\Sigma}(x_0)} (\nu \times \lambda)(D) + \epsilon \\ &= \sup_{\nu \in \tilde{\Sigma}(x_0)} \int_{X^*} \Phi(p) d\nu(p) + \epsilon \\ &= \sup_{\mu \in \Sigma(x_0)} \int_{\{\tau < \infty\}} \tilde{M}(Ep_\tau)(h_\tau) d\mu(h) + \epsilon, \end{aligned}$$

where the last equality can be justified by first observing that $\tilde{M}(Ep_\tau)(h_\tau) = \Phi(\psi(h))$ if $\tau(h) < \infty$ and then applying the change of variable theorem.

We now define $g: H \rightarrow [0, 1]$ as follows:

$$\begin{aligned} g(h) &= \lambda(B_{\psi(h)}) \text{ if } \tau(h) < \infty \\ &= 0 \text{ otherwise,} \end{aligned}$$

where $B_{\psi(h)} = \{a \in [0, 1] : (\psi(h), a) \in B\}$. It is now easy to verify that g satisfies the assertions of the lemma. $\square$

The next result is the fundamental inequality of the theory and can be regarded as one-half of the optimality equation.

Theorem 4.4. If $E \subseteq H$ is Σ_1^1 and τ a Δ_1^1 - recursive stopping time on H such that $E \subseteq \{\tau < \infty\}$, then

$$\bar{M}(E)(x) \leq \sup_{\mu \in \Sigma(x)} \int_{\{\tau < \infty\}} \bar{M}(Ep_\tau)(h_\tau) d\mu(h)$$

for all $x \in X$.

Proof. Fix $x_0 \in X$ and $\epsilon > 0$ rational. Apply Lemma 4.3 with $\epsilon/3$ to get a function g satisfying the assertions of the lemma. Define $P \subseteq H \times \omega$ as follows:

$$(h, n) \in P \leftrightarrow (\exists k \in \mathbb{N})[\tau(h) = k \ \& \ \langle h_1, h_2, \dots, h_k \rangle, n \in \bar{W}$$

$$\ \& \ E h_1 h_2 \dots h_k \subseteq \bar{C}_{\langle h_1, h_2, \dots, h_k \rangle, n}$$

$$\ \& \ (\forall \mu \in \Sigma(h_k))(\mu(\bar{C}_{\langle h_1, h_2, \dots, h_k \rangle, n}) < g(h) + \epsilon/3)].$$

Then P is $\Pi_1^1(x_0)$ and one verifies easily that $(\forall h)[\tau(h) < \infty \rightarrow (\exists n)((h, n) \in P)]$. An application of Kreisel's selection theorem yields a $\Delta_1^1(x_0)$ - recursive function $f: H \rightarrow \omega$ such that $(h, f(h)) \in P$ whenever $\tau(h) < \infty$.

We now define $O \subseteq H$ by:

$$h \in O \leftrightarrow (\exists k \in \mathbb{N})[\tau(h) = k \ \& \ \langle h_1, h_2, \dots, h_k \rangle, f(\langle h_1, h_2, \dots, h_k, x^*, x^*, \dots \rangle), (h_{k+1}, h_{k+2}, \dots)) \in \bar{C}],$$

where x^* is a fixed Δ_1^1 point in X . It follows that O is a $\Delta_1^1(x_0)$, open set such that $E \subseteq O \subseteq \{\tau < \infty\}$. Hence $\bar{M}(E)(x_0) \leq M(O)(x_0)$, so there is $\nu \in \Sigma(x_0)$ such that

$$\nu(O) \geq \bar{M}(E)(x_0) - \epsilon/3.$$

Now

$$\nu(O) = \int_{\{\tau < \infty\}} \nu[p_\tau](Op_\tau)(h_\tau) d\nu(h).$$

If $\tau(h) < \infty$ and $\nu[p_\tau] \in \Sigma(h_\tau)$, then it follows from the definition of 0 that

$$\nu[p_\tau](Op_\tau) \leq g((h_1, h_2, \dots, h_\tau, x_\tau^*, x_\tau^*, \dots)) + \epsilon/3 = g(h) + \epsilon/3.$$

Consequently, since Σ is a global gambling house, we have:

$$\nu(\{h \in H: \tau(h) < \infty \ \& \ \nu[p_\tau](Op_\tau) \leq g(h) + \epsilon/3\}) = \nu(\{\tau < \infty\}).$$

Hence

$$\begin{aligned} \tilde{M}(E)(x_0) - \frac{\epsilon}{3} &\leq \int_{\{\tau < \infty\}} g(h) \, d\nu(h) + \frac{\epsilon}{3} \\ &\leq \sup_{\mu \in \Sigma(x_0)} \int_{\{\tau < \infty\}} g(h) \, d\mu(h) + \frac{\epsilon}{3} \\ &\leq \sup_{\mu \in \Sigma(x_0)} \int_{\{\tau < \infty\}} \tilde{M}(Ep_\tau)(h_\tau) \, d\mu(h) + \frac{2\epsilon}{3} \end{aligned}$$

where the last inequality involves an application of Lemma 4.3. It follows that

$$\tilde{M}(E)(x_0) \leq \sup_{\mu \in \Sigma(x_0)} \int_{\{\tau < \infty\}} \tilde{M}(Ep_\tau)(h_\tau) \, d\mu(h) + \epsilon.$$

Since ϵ is an arbitrary positive rational, this completes the proof. $\square$

Theorem 4.4 is a significant generalization of Theorem 4.6 of [12].

Corollary 4.5. If $E \subseteq H$ is Σ_1^1 and τ a Δ_1^1 - recursive stop rule, then

$$\tilde{M}(E)(x) \leq \sup_{\mu \in \Sigma(x)} \int \tilde{M}(Ep_\tau)(h_\tau) \, d\mu(h)$$

for every $x \in X$.

5. Effective G_δ 's

This section will be devoted to the proof of the effective refinement of Theorem 1.1 in the special case when A is a Δ_1^1 - intersection of Δ_1^1 , open subsets of H .

Theorem 5.1. Let $X = 2^w$, $H = 2^w \times 2^w \times \dots$ and $\Sigma \subseteq X \times P(H)$ a Σ_1^1 , global gambling house. Let V_n be open subsets of H such that $V_n \supseteq V_{n+1}$, $n \in \mathbb{N}$, and the relation $h \in V_n$ is Δ_1^1 in $N \times H$. Set $G = \bigcap_{n \in \mathbb{N}} V_n$. Then $\bar{M}(G) = M(G)$.

Proof. By Corollary 3.2, for each $n \in \mathbb{N}$, there is a Δ_1^1 - recursive stopping time τ_n such that $V_n = \{r_n < \infty\}$. We may assume without loss of generality that $\tau_n < \tau_{n+1}$ on V_{n+1} for all $n \in \mathbb{N}$.

Fix $x_0 \in X$ and $\epsilon > 0$. It will suffice to prove that $M(G)(x_0) \geq \bar{M}(G)(x_0) - \epsilon$. We will do this by defining $\mu \in \Sigma(x_0)$ such that

$$(5.1) \quad \mu(G) \geq \bar{M}(G)(x_0) - \epsilon.$$

The definition of μ depends on an inductive construction of a sequence $\{K_n\}$ of clopen subsets of H , a sequence $\{r_n\}$ of stop rules on H and a sequence $\{\phi_n\}$ of functions from $H \times \mathbb{N}$ to $P(H)$. The conditions K_n , r_n , ϕ_n must satisfy are stated in the next lemma.

Lemma 5.2. Let $x_0 \in X$, $\lambda \in \Sigma(x_0)$, $\epsilon > 0$, K_n a clopen subset of H , r_n a stop rule on H and $\phi_n: H \times \mathbb{N} \rightarrow P(H)$ such that

- (a) for every $n \in \mathbb{N}$, K_n , r_n , ϕ_n are universally measurable, K_n is determined by time r_n , $r_n < r_{n+1}$, $r_n \leq \tau_n$, $r_n = \tau_n$ on K_n , $\phi_n(h, k) \in \Sigma(h_k)$ for every $k \in \mathbb{N}$ and ϕ_n is adapted;

$$(b) \quad \int_{K_1} \bar{M}(Gq_1(h))(h_{r_1}) \, d\lambda(h) > \bar{M}(G)(x_0) - \frac{\epsilon}{2};$$

and

(c) for every $n \in \mathbb{N}$ and $h \in H$,

$$\begin{aligned} & \int_{K_{n+1}^{q_n(h)}} \bar{M}(Gq_n(h)p_{r_{n+1}}[q_n(h)])(h'_{r_{n+1}}[q_n(h)]) \phi_n(h, r_n(h)) (dh') \\ & > \bar{M}(Gq_n(h))(h_{r_n}) - \frac{\epsilon}{2^{n+1}}; \end{aligned}$$

where $q_n(h) = p_{r_n}(h)$.

Let μ be the sequential composition of (λ, ϕ_n, r_n) . Then

$$\mu(\bigcap_{n \in \mathbb{N}} K_n) \geq \bar{M}(G)(x_0) - \epsilon.$$

Proof. There is no loss of generality in assuming that $\bar{M}(G)(x_0) > \epsilon$. We first prove that $\bigcap_{n \in \mathbb{N}} K_n$ is nonempty. To see this, use (b) to get $h^1 \in K_1$ such that

$$\bar{M}(Gq_1(h^1))(h^1_{r_1}) > \frac{\epsilon}{2}. \quad \text{So, by (c), there is } h^2 \in K_2 \text{ such that } h^2 \text{ agrees with}$$

h^1 up to time $r_1(h)$ (that is, $q_1(h^1) = q_1(h^2)$) and $\bar{M}(Gq_2(h^2))(h^2_{r_2}) > \frac{\epsilon}{4}$.

Continue in this fashion to define $h^{n+1} \in K_{n+1}$ such that h^{n+1} agrees with h^n up to time $r_n(h^n)$. Then take h to be that history which agrees with h^n up to time $r_n(h^n)$ for every $n \in \mathbb{N}$. Since K_n is determined by time r_n and $h^n \in K_n$, it follows that $h \in K_n$. So $h \in \bigcap_{n \in \mathbb{N}} K_n$.

To complete the proof of the lemma, we shall establish that

$$(5.2) \quad \mu(K) \geq \tilde{M}(G)(x_0) - \epsilon$$

for every universally measurable, clopen set K containing $\bigcap_{n \in \mathbb{N}} K_n$. We do this by induction on the structure of K . If K has structure 0, then K must be H since K is nonempty. But then (5.2) holds trivially. So suppose that (5.2) holds for all universally measurable, clopen sets containing $\bigcap_{n \in \mathbb{N}} K_n$ and having structure less than ξ , where ξ is a nonzero ordinal. Suppose now K is a universally measurable, clopen set containing $\bigcap_{n \in \mathbb{N}} K_n$ and has structure ξ . Fix $h \in K_1$ and

$$\text{set } x' = h_{r_1}, \quad \epsilon' = \frac{\epsilon}{2}, \quad q = q_1(h), \quad r'_n = r_{n+1}[q], \quad G' = Gq, \quad K'_n = K_{n+1}q,$$

$$r'_n = r_{n+1}[q], \quad \lambda' = \phi_1(h, r_1(h)), \quad \phi'_n(h', k) = \phi_{n+1}(qh', r_1(h) + k). \quad \text{Then the}$$

primed objects satisfy the hypotheses of the lemma. So, if $K' = Kq$, then K' is a universally measurable, clopen set containing $\bigcap_{n \in \mathbb{N}} K'_n$ and having structure less than ξ . Hence, denoting the sequential composition of $(\lambda', \phi'_n, r'_n)$ by μ' , we obtain from the inductive hypothesis the following:

$$(5.3) \quad \mu'(K') \geq \tilde{M}(Gq_1(h))(h_{r_1}) - \frac{\epsilon}{2}.$$

This holds for all $h \in K_1$. Moreover, restoring the dependence of μ' on h , we have $\mu[q_1(h)] = \mu'(h)$ for almost all $(\mu)h$. Consequently,

$$\begin{aligned} \mu(K) &= \int \mu[q_1(h)](Kq_1(h)) d\mu(h) \\ &\geq \int_{K_1} \mu[q_1(h)](Kq_1(h)) d\mu(h) \\ &\geq \int_{K_1} \tilde{M}(Gq_1(h))(h_{r_1}) d\lambda(h) - \frac{\epsilon}{2} \end{aligned}$$

$$\geq \bar{M}(G)(x_0) - \epsilon,$$

where the second inequality is by virtue of (5.3) and the definition of μ and the last inequality uses condition (b). This concludes the proof of the lemma.

We now return to the proof of Theorem 5.1. Observe that $\mu \in \Sigma(x_0)$ since Σ is a global gambling house. Moreover, $\bigcap_{n \in \mathbb{N}} K_n \subseteq G$. So (5.1) is an immediate consequence of Lemma 5.2. It therefore remains only to construct K_n, r_n, ϕ_n as in the lemma.

By Theorem 4.4, choose $\lambda \in \Sigma(x_0)$ such that

$$\int_{(\tau_1 < \infty)} \bar{M}(Gp_{\tau_1})(h_{\tau_1}) d\lambda(h) > \bar{M}(G)(x_0) - \frac{\epsilon}{2}.$$

Find $k_1 \in \mathbb{N}$ such that

$$\int_{(\tau_1 \leq k_1)} \bar{M}(Gp_{\tau_1})(h_{\tau_1}) d\lambda(h) > \bar{M}(G)(x_0) - \frac{\epsilon}{2}.$$

Set $K_1 = \{\tau_1 \leq k_1\}$ and $r_1 = \tau_1 \wedge k_1$, so that

$$\int_{K_1} \bar{M}(Gp_{r_1})(h_{r_1}) d\lambda(h) > \bar{M}(G)(x_0) - \frac{\epsilon}{2}.$$

Assume next that $K_1, K_2, \dots, K_n, r_1, r_2, \dots, r_n, \phi_1, \phi_2, \dots, \phi_{n-1}$ have been defined. Define two functions $f: H \times N \times H \rightarrow [0, 1]$ and $g: H \times N \times H \times N \rightarrow [0, 1]$ as follows:

$$f(h, m, h') = \bar{M}(Gh_1 h_2 \dots h_m p_{\tau_{n+1}}[h_1, h_2, \dots, h_m](h')) \left(h'_{\tau_{n+1}}[h_1, h_2, \dots, h_m] \right),$$

$$\begin{aligned} & \text{if } m < \tau_{n+1}((h_1, h_2, \dots, h_m)h') < \infty \\ & = 0, \text{ otherwise;} \end{aligned}$$

and

$$g(h, m, h', k) = f(h, m, h'), \text{ if } m < \tau_{n+1}((h_1, h_2, \dots, h_m)h') \leq k, \\ = 0, \text{ otherwise.}$$

Using Lemma 4.1, one verifies that f and g are upper analytic functions. It then follows from a selection theorem [12, Lemma 2.1], (see also [1, Proposition 7.50]) that there is an analytically measurable function $\psi: H \times N \rightarrow P(H)$ such that $\psi(h, m) \in \Sigma(h_m)$ and

$$\int f(h, m, h') \psi(h, m)(dh') > \sup_{\nu \in \Sigma(h_m)} \int f(h, m, h') \nu(dh') - \frac{\epsilon}{2^{n+2}}$$

for every $h \in H$ and $m \in N$. Now, for fixed h' , f as a function in h and m is adapted. The function ψ may not be adapted, but this is easily set right by defining $\phi_n: H \times N \rightarrow P(H)$ as follows:

$$\phi_n(h, m) = \psi((h_1, h_2, \dots, h_m, x^*, x^*, \dots), m)$$

where x^* is an arbitrary fixed point in X . We then have:

$$(5.4) \quad \int f(h, m, h') \phi_n(h, m)(dh') > \sup_{\nu \in \Sigma(h_m)} \int f(h, m, h') \nu(dh') - \frac{\epsilon}{2^{n+2}}$$

for $h \in H$, $m \in N$.

We now define $k_{n+1}: H \times N \rightarrow N$ as follows: $k_{n+1}(h, m) = \text{least } k > m \text{ such that}$

$$(5.5) \quad \int g(h, m, h', k) \phi_n(h, m)(dh') > \int f(h, m, h') \phi_n(h, m)(dh') - \frac{\epsilon}{2^{n+2}}.$$

It is easily verified that k_{n+1} is universally measurable and adapted. It follows from (5.4) and (5.5) that if $\tau_{n+1}(h) > m$, then

(5.6)

$$\int_{\{\tau_{n+1}[h_1, h_2, \dots, h_m] \leq k_{n+1}(h, m) - m\}} \tilde{M}(Gh_1 h_2 \dots h_m) p_{\tau_{n+1}[h_1, h_2, \dots, h_m]}(h') \left[h'_{\tau_{n+1}[h_1, h_2, \dots, h_m]} \right] \phi_n^{(h, m)}(dh')$$

$$> \sup_{\nu \in \Sigma(h_m)} \int_{\{\tau_{n+1}[h_1, h_2, \dots, h_m] < \infty\}} \tilde{M}(Gh_1 h_2 \dots h_m) p_{\tau_{n+1}[h_1, h_2, \dots, h_m]}(h') \left[h'_{\tau_{n+1}[h_1, h_2, \dots, h_m]} \right] \nu(dh') - \frac{\epsilon}{2^{n+1}}$$

$$\geq \tilde{M}(Gh_1 h_2 \dots h_m)(h_m) - \frac{\epsilon}{2^{n+1}},$$

where the last inequality is obtained by an application of a relativized version of Theorem 4.4 to the $\Delta_1^1(<h_1, h_2, \dots, h_m>)$ set $Gh_1 h_2 \dots h_m$ and the $\Delta_1^1(h_1, h_2, \dots, h_m)$ - recursive stopping time $\tau_{n+1}[h_1, h_2, \dots, h_m]$.

Finally, define K_{n+1} and r_{n+1} as follows:

$$h \in K_{n+1} \leftrightarrow \tau_{n+1}(h) \leq k_{n+1}(h, r_n(h))$$

and

$$r_{n+1}(h) = \min(\tau_{n+1}(h), k_{n+1}(h, r_n(h))).$$

It is now a routine matter to verify that K_{n+1} , r_{n+1} and ϕ_n satisfy the conditions in (a) of Lemma 5.2. Condition (c) of Lemma 5.2 now follows by observing first that $\tau_{n+1}(h) > \tau_n(h)$ and then setting $m = r_n(h)$ in (5.6). This concludes the proof of Theorem 5.1. $\square$

6. Effective G_δ 's and the "going up" property

This section contains the second step in our program of proving the effective version of Theorem 1.1. We start with a definition. We remind the reader that X , H , Γ are as in the previous section.

A sequence $(E_n)_{n \in \mathbb{N}}$ of subsets of H is a $\Sigma_1^1(\Delta_1^1)$ sequence if the relation $h \in$

E_n is $\Sigma_1^1(\Delta_1^1)$ in $N \times H$.

Lemma 6.1. If (E_n) is a Σ_1^1 sequence of subsets of H and $\bar{M}(E_n)(x) = 0$ for each $n \in N$, then $\bar{M}(\bigcup_{n \in N} E_n)(x) = 0$.

Proof. Let $\epsilon > 0$ be rational. Define $P \subseteq N \times \omega$ by:

$$(n, m) \in P \leftrightarrow (x, m) \in \bar{W} \ \& \ E_n \subseteq \bar{C}_{x, m} \ \& \ (\forall \mu \in \Sigma(x)) \left(\mu(\bar{C}_{x, m}) < \frac{\epsilon}{2^n} \right).$$

Then P is $\Pi_1^1(x)$, and the hypothesis implies that $(\forall n \in N)(\exists m)[(n, m) \in P]$. So, by Kreisel's selection theorem, there is a $\Delta_1^1(x)$ - recursive function $f: N \rightarrow \omega$ such that $(\forall n \in N)[(n, f(n)) \in P]$. Let

$$h \in O \leftrightarrow (\exists n \in N)[h \in \bar{C}_{x, f(n)}]$$

Then O is a $\Delta_1^1(x)$, open set containing $\bigcup_{n \in N} E_n$. Hence

$$\bar{M}(\bigcup_{n \in N} E_n)(x) \leq M(O)(x) \leq \sum_n M(\bar{C}_{x, f(n)})(x) \leq \epsilon.$$

Since ϵ is arbitrary, this completes the proof of the lemma. $\square$

Lemma 6.2. Suppose that the relation $h \in B_{n, m}$ is Δ_1^1 in $H \times N \times N$ and that $B_{n, m}$ is open in H for every $n, m \in N$. Let $E_n = \bigcap_{m \in N} B_{n, m}$, $n \in N$. If $M(\bigcup_{n \in N} E_n)(x) = 0$, then $\bar{M}(\bigcup_{n \in N} E_n)(x) = 0$.

Proof. By Theorem 5.1, $\bar{M}(E_n)(x) = M(E_n)(x) = 0$ for every $n \in N$. The assertion of the lemma now follows from Lemma 6.1. $\square$

Let $E \subseteq H$ be Σ_1^1 . Fix $\epsilon > 0$ rational. Define a stopping time τ_ϵ as follows:

$$\begin{aligned} \tau_\epsilon(h) &= \text{least } k \in N \text{ such that } M(Eh_1h_2\dots h_k)(h_k) > 1 - \epsilon \\ &\quad \text{if } (\exists k \in N)[M(Eh_1h_2\dots h_k)(h_k) > 1 - \epsilon] \\ &= \infty, \quad \text{otherwise.} \end{aligned}$$

Lemma 6.3. (a) The set $\{(h,n) \in H \times N: \tau_\epsilon(h) \leq n\}$ is Σ_1^1 .
 (b) The set $\{\tau_\epsilon < \infty\}$ is Σ_1^1 in H .

Proof. The relation $M(Eh_1h_2\dots h_k)(h_k) > 1 - \epsilon$ in h,k can be rewritten as

$$(\exists \mu \in \Sigma(h_k))(\mu(Eh_1h_2\dots h_k) > 1 - \epsilon).$$

This is clearly Σ_1^1 in $H \times N$. The assertion in (a) follows immediately from this and (b) follows from (a). $\square$

The next two results were proved in [13] for measurable "local" gambling houses of the type described in section 1 of this article. However, minor modifications will make the proofs valid in the present setting. These results appeared as Lemmas 2.7 and 2.8 in [13].

Lemma 6.4. Let E and τ_ϵ be as above. Then $M(E \cap \{\tau_\epsilon = \infty\})(x) = 0$ for every $x \in X$.

Lemma 6.5. $M(\{\tau_\epsilon < \infty\})(x) \leq M(E \cap \{\tau_\epsilon < \infty\})(x) + \epsilon$ for every $x \in X$.

Here is the main result of this section.

Theorem 6.6. Let $X = 2^\omega$, $H = 2^\omega \times 2^\omega \times \dots$ and $\Sigma \subseteq X \times P(H)$ a Σ_1^1 , global gambling house. Suppose that the relation $h \in B_{n,m}$ is Δ_1^1 in $H \times N \times N$ and that $B_{n,m}$ is open in H for every $n,m \in N$. Let $E_n = \bigcap_{m \in N} B_{n,m}$, $n \in N$ and $E = \bigcup_{n \in N} E_n$. Then $\bar{M}(E) = M(E)$.

Proof. Fix $x_0 \in X$ and $\epsilon > 0$ rational. Consider the stopping time τ_ϵ associated with the set E . By Lemma 6.3, $\{\tau_\epsilon < \infty\}$ is a Σ_1^1 set. So, by a relativized version of Lemma 4.3 in [12], there is a $\Delta_1^1(x_0)$ set D in H such that $\{\tau_\epsilon < \infty\} \subseteq D$ and

$$M(D)(x_0) < M(\{\tau_\epsilon < \infty\})(x_0) + \epsilon.$$

Now the Σ_1^1 set $\{\tau_\epsilon < \infty\}$ can be separated from the $\Delta_1^1(x_0)$ set D^c by an open set. Hence, by a relativized version of Lemma 3.1, there is a $\Delta_1^1(x_0)$ - recursive stopping time t such that $\{\tau_\epsilon < \infty\} \subseteq \{t < \infty\} \subseteq D$. Set $O = \{t < \infty\}$. Then

$$\begin{aligned}
 M(O)(x_0) &< M(\{\tau_\epsilon < \infty\})(x_0) + \epsilon \\
 &\leq M(E \cap \{\tau_\epsilon < \infty\})(x_0) + 2\epsilon \\
 (6.1) \quad &\leq M(E \cap O)(x_0) + 2\epsilon,
 \end{aligned}$$

where the second inequality is by virtue of Lemma 6.5. On the other hand,

$$\begin{aligned}
 E \cap O^c &= [\bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} B_{n,m}] \cap \bigcap_{k \in \mathbb{N}} [\{t \geq k\}] \\
 &= \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} [B_{n,m} \cap \{t \geq k\}].
 \end{aligned}$$

Moreover, by virtue of Lemma 6.4,

$$M(E \cap O^c)(x_0) \leq M(E \cap \{\tau_\epsilon = \infty\})(x_0) = 0.$$

This fact together with the representation of $E \cap O^c$ above enables us to invoke a relativized version of Lemma 6.2 and conclude that

$$(6.2) \quad \tilde{M}^{x_0}(E \cap O^c)(x_0) = 0.$$

Now use (6.2) to obtain a $\Delta_1^1(x_0)$, open set O' containing $E \cap O^c$ and such that

$$(6.3) \quad M(O')(x_0) < \epsilon.$$

Then $O \cup O'$ is a $\Delta_1^1(x_0)$, open set and contains E . Hence,

$$\begin{aligned}
 \tilde{M}(E)(x_0) &\leq M(O \cup O')(x_0) \\
 &\leq M(O)(x_0) + M(O')(x_0) \\
 &\leq M(E \cap O)(x_0) + 3\epsilon \\
 &\leq M(E)(x_0) + 3\epsilon
 \end{aligned}$$

where the third inequality is by virtue of (6.1) and (6.3). Since ϵ is an arbitrary rational, this proves that $\bar{M}(E)(x_0) \leq M(E)(x_0)$. As the inequality in the opposite direction is obvious, this completes the proof. $\square$

The next result is the *raison d'être* for this intermediate step in the proof of the effective refinement of Theorem 1.1.

Corollary 6.7. Let (A_n) be a Σ_1^1 sequence of subsets of H such that $A_n \subseteq A_{n+1}$, $n \in \mathbb{N}$. Let $A = \bigcup_{n \in \mathbb{N}} A_n$. Then $\bar{M}(A) = \lim M(A_n)$.

Proof. Fix $x_0 \in X$. Plainly, $\bar{M}(A)(x_0) \geq \lim M(A_n)(x_0)$. It remains only to prove the inequality in the opposite direction. Let r be a rational in $[0, 1]$ such that $M(A)(x_0) > r$. We need to show that $\lim M(A_n)(x_0) \geq r$. Suppose not. Then, for each $n \in \mathbb{N}$, $\bar{M}(A_n)(x_0) < r$. Define $P \subseteq \mathbb{N} \times \omega$ as follows:

$$(n, m) \in P \leftrightarrow (x_0, m) \in \bar{W} \text{ \& } A_n \subseteq \bar{C}_{x_0, m} \text{ \& } (\forall \mu \in \Sigma(x_0))(\mu(\bar{C}_{x_0, m}) < r).$$

Plainly, P is $\Pi_1^1(x_0)$ and $(\forall n \in \mathbb{N})(\exists m)[(n, m) \in P]$. So, by Kreisel's selection theorem, there is a $\Delta_1^1(x_0)$ -recursive function $f: \mathbb{N} \rightarrow \omega$ such that $(\forall n \in \mathbb{N})[(n, f(n)) \in P]$. Let

$$h \in B \leftrightarrow (\exists k \in \mathbb{N})(\forall n \geq k)[h \in \bar{C}_{x_0, f(n)}].$$

Then $A \subseteq B$, and by Theorem 6.6 (relativized to x_0), $\bar{M}^{x_0}(B)(x_0) = M(B)(x_0)$.

Hence

$$\begin{aligned} \bar{M}(A)(x_0) &\leq \bar{M}^{x_0}(B)(x_0) \\ &= M(B)(x_0) \end{aligned}$$

$$\begin{aligned}
&= \sup_k M\left(\bigcap_{n \geq k} \tilde{C}_{x_0, f(n)}\right)(x_0) \\
&\leq \sup_k M\left(\tilde{C}_{x_0, f(k)}\right)(x_0) \\
&\leq r,
\end{aligned}$$

which yields the desired contradiction. $\square$

7. An effective capacitability theorem

For fixed $x \in X$, the monotonicity property and the property stated in Corollary 6.7 suggest that the set function $\tilde{M}(\cdot)(x)$ may be an "effective" capacity in the sense of Louveau [9]. However, this is not so, as the "going down" property along decreasing sequences of T_1 - compacts fails to hold in general (see [13]). In consequence, the "effective" capacitability theorem of Louveau cannot be applied directly, as Choquet's capacitability theorem was in some special cases (see [13, Theorem 3.6]), to prove Theorem 1.1. Nonetheless, we will now combine the proof of the abstract capacitability theorem - or, to more precisely identify the provenance of these ideas, Sierpinski's proof that an analytic set is measurable [21, pp.48-50] - with the proof of Theorem 5.1 to produce a proof of the effective refinement of Theorem 1.1. But first we must explain some more notation from the effective theory.

Let $U: \omega^\omega \times H \rightarrow \omega$ be a Π_1^1 - recursive partial function such that whenever $\alpha \in \omega^\omega$ and $f: H \rightarrow \omega$ is a $\Pi_1^1(\alpha)$ - recursive partial function, there is a $\Delta_1^1(\alpha)$ point $\beta \in \omega^\omega$ such that if $f(h)$ is defined, then $U(\beta, h)$ is defined and $f(h) = U(\beta, h)$. Following Kleene, we shall denote the partial function $U(\beta, \cdot)$ by $\{\beta\}$. We write $\{\beta\}(h) \downarrow$ to mean that $\{\beta\}$ is defined at h . Next, fix a Π_1^1 - recursive partial function $d: \omega^\omega \times \omega \rightarrow \omega^\omega$ such that $d(\alpha, \cdot)$ enumerates the $\Delta_1^1(\alpha)$ points of ω^ω . The existence of U is established in [15, 7A.1] and that of d is proved in [15, 4D.2].

Here at last is the effective refinement of Theorem 1.1.

Theorem 7.1. Let $X = 2^\omega$, $H = 2^\omega \times 2^\omega \times \dots$ and $\Sigma \subseteq X \times P(H)$ a Σ_1^1 , global gambling house. If $A \subseteq H$ is a Σ_1^1 set, then $\bar{M}(A) = M(A)$.

Proof. We use [15, 4A.1] to write A as follows:

$$(7.1) \quad h \in A \leftrightarrow (\exists \alpha \in {}^\omega N)(\forall n \in N)[h \in B(\bar{\alpha}(n))],$$

where $h \in B(s)$ is a Δ_1^1 relation in $\bigcup_{n \in \omega} {}^\omega n \times H$, $B(s)$ is clopen in H for each fixed $s \in \bigcup_{n \in \omega} {}^\omega n$ and $s \supseteq t$ implies $B(s) \subseteq B(t)$. Define

$$(7.2) \quad h \in A(s) \leftrightarrow (\exists m)[|s| = m \text{ and } (\exists i_1 \leq s_1)(\exists i_2 \leq s_2) \dots (\exists i_m \leq s_m)(\exists \alpha)(\forall n \in N)(h \in B((i_1, i_2, \dots, i_m) \bar{\alpha}(n)))].$$

Plainly, the relation $h \in A(s)$ is Σ_1^1 in $\bigcup_{n \in \omega} {}^\omega n \times H$.

Fix $x_0 \in X$ and $\epsilon > 0$. We will construct $\mu \in \Sigma(x_0)$ such that

$$(7.3) \quad \mu(A) \geq \bar{M}(A)(x_0) - \epsilon.$$

The definition of μ depends on an inductive construction of various objects in a manner similar to the construction in the proof of Theorem 5.1. The conditions that these objects have to satisfy are stated in the next lemma.

Lemma 7.2. Let $x_0 \in X$, $\epsilon > 0$, $\lambda \in \Sigma(x_0)$, $m_1^* \in \omega$, $m_{n+1}^*: H \rightarrow \omega$, K_n a clopen subset of H , τ_n a stop rule on H and $\phi_n: H \times N \rightarrow P(H)$ such that

(a) for every $n \in N$, m_{n+1}^* , K_n , τ_n and ϕ_n are universally measurable,
 $K_{n+1} \subseteq K_n$, $\tau_n < \tau_{n+1}$, K_n and m_{n+1}^* are determined by time τ_n , $\phi_n(h, k) \in \Sigma(h_k)$ for every $k \in N$ and ϕ_n is adapted;

(b) $h \in K_n$ iff $h \in B((i_1, i_2, \dots, i_n))$ for some $i_1 \leq m_1^*$, $i_2 \leq m_2^*(h)$, $\dots$, $i_n \leq m_n^*(h)$;

$$(c) \quad \int_{K_1} \bar{M}(A((m_1^*))q_1(h))(h_{\tau_1}) d\lambda(h) > \bar{M}(A)(x_0) - \frac{\epsilon}{2};$$

and

(d) for every $n \in \mathbb{N}$ and $h \in H$,

$$\int_{K_{n+1}^{q_n(h)}} \bar{M}\left[A((m_1^*, m_2^*(h), \dots, m_{n+1}^*(h)))q_n(h)p_{\tau_{n+1}[q_n(h)]}\right]\left[h'_{\tau_{n+1}[q_n(h)]}\right]\phi_n(h, \tau_n(h))(dh')$$

$$> \bar{M}(A((m_1^*, m_2^*(h), \dots, m_n^*(h)))q_n(h))(h_{\tau_n}) - \frac{\epsilon}{2^{n+1}},$$

where $q_n(h) = p_{\tau_n}(h)$, $n \in \mathbb{N}$.

Let μ be the sequential composition of $(\lambda, \phi_n, \tau_n)$. Then

$$(i) \quad \bigcap_{n \in \mathbb{N}} K_n \subseteq A, \quad \text{and}$$

$$(ii) \quad \mu\left(\bigcap_{n \in \mathbb{N}} K_n\right) \geq \bar{M}(A)(x_0) - \epsilon.$$

Proof. To prove (i), let $h \in \bigcap_{n \in \mathbb{N}} K_n$ and define

$$S = \{s \in \bigcup_{n \in \mathbb{N}} \omega^n : (\exists k) [|s| = k \text{ \& } s_1 \leq m_1^* \text{ \& } s_2 \leq m_2^*(h) \\ \text{\& } \dots \text{ \& } s_k \leq m_k^*(h) \text{ \& } h \in B(s)] \}.$$

Then S is closed under initial segments and contains finite sequences of every length. So, by König's Lemma [15, 4F.9], there exists $\alpha \in \omega^{\mathbb{N}}$ such that $\bar{\alpha}(n) \in S$ for every $n \in \mathbb{N}$. Hence, $h \in B(\bar{\alpha}(n))$ for every $n \in \mathbb{N}$, so that $h \in A$.

In order to prove (ii), we will assume that $\epsilon < \bar{M}(A)(x_0)$. We now prove the stronger statement:

(*) for every universally measurable, clopen set $K \supseteq \bigcap_{n \in \mathbb{N}} K_n$, $\mu(K) \geq \bar{M}(A)(x_0) - \epsilon$.

(*) will immediately imply that $\mu(K_n) \geq \bar{M}(A)(x_0) - \epsilon$ for every $n \in \mathbb{N}$. So, since $K_n \supseteq K_{n+1}$ and μ is countably additive, it will follow that $\mu(\bigcap_{n \in \mathbb{N}} K_n) \geq \bar{M}(A)(x_0) - \epsilon$.

We will prove (*) by induction on the structure of K . So suppose that structure $(K) = 0$, so that $K = H$ or $K = \emptyset$. If $K = H$, (*) is obviously true. We

now show that $K \neq \emptyset$. Use (c) to get $h^1 \in K_1$ such that $\bar{M}(A((m_1^*))_{p_{\tau_1}})(h^1_{\tau_1}) > \frac{\epsilon}{2}$.

So, by (d), there is $h^2 \in K_2$ such that h^2 agrees with h^1 up to time $\tau_1(h^1)$ and

$\bar{M}(A((m_1^*, m_2^*(h^2)))_{p_{\tau_2}})(h^2_{\tau_2}) > \frac{\epsilon}{4}$. Continue in this fashion to define

$h^{n+1} \in K_{n+1}$ such that h^{n+1} agrees with h^n up to time $\tau_n(h^n)$. Take h to be the history which agrees with h^n up to time $\tau_n(h^n)$ for every $n \in \mathbb{N}$. Then $h \in \bigcap_{n \in \mathbb{N}} K_n$, because each K_n is determined by time τ_n .

For the inductive step, suppose (*) holds for universally measurable, clopen sets of structure less than ξ , where ξ is a nonzero ordinal. Let, then, K be a universally measurable, clopen set containing $\bigcap_{n \in \mathbb{N}} K_n$ and having structure ξ .

Fix $h \in K_1$ and set $x' = h_{\tau_1}$, $\epsilon' = \frac{\epsilon}{2}$, $q = q_1(h)$, $\tau'_n = \tau_{n+1}[q]$, $K'_n = K_{n+1}q$,

$\lambda' = \phi_1(h, \tau_1(h))$, $\phi'_n(h', k) = \phi_{n+1}(qh', \tau_1(h) + k)$, $m_1^{*'} = m_2^*(h)$, $m_{n+1}^{*'}(h') = m_{n+2}^*(qh')$,

$B'(s) = \bigcup_{i \leq m_1^*} B(is)q$ and $A' = A((m_1^*))q$. Then the primed objects satisfy the

hypotheses of the lemma. So, if $K' = Kq$, then K' is a universally measurable, clopen set containing $\bigcap_{n \in \mathbb{N}} K'_n$ and having structure less than ξ . Hence, denoting the sequential composition of $(\lambda', \phi'_n, \tau'_n)$ by μ' , we obtain from the inductive hypothesis the following:

$$(7.4) \quad \mu'(K') \geq \tilde{M}(A((m_1^*))q_1(h))(h_{\tau_1}) - \frac{\epsilon}{2}.$$

This holds for every $h \in K_1$. Moreover, restoring the dependence of μ' on h , we have $\mu[q_1(h)] = \mu'(h)$ for almost all $(\mu)h$. Consequently,

$$\begin{aligned} \mu(K) &= \int \mu[q_1(h)](Kq_1(h)) \, d\mu(h) \\ &\geq \int_{K_1} \mu[q_1(h)](Kq_1(h)) \, d\mu(h) \\ &\geq \int_{K_1} \tilde{M}(A((m_1^*))q_1(h))(h_{\tau_1}) \, d\lambda(h) - \frac{\epsilon}{2} \\ &\geq \tilde{M}(A)(x_0) - \epsilon, \end{aligned}$$

where the second inequality is by virtue of (7.4) and the definition of μ and the last inequality uses condition (c) of the lemma. This concludes the proof of the lemma.

We return to the proof of Theorem 7.1. Since Σ is a global gambling house, μ , constructed in Lemma 7.2, belongs to $\Sigma(x_0)$. So (7.3) is an immediate consequence of Lemma 7.2. It therefore remains to show that the hypotheses of Lemma 7.2 are satisfied. We begin with a lemma.

Lemma 7.3. For every $n \geq 2$, there is a Δ_1^1 -recursive function $g_n: H \times \omega^n \times N \rightarrow \omega^\omega$ such that if $g_n(h, m_1, m_2, \dots, m_n, k) = \alpha$, then α is a $\Delta_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ point and (α) is a $\Delta_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ -recursive stop rule τ on H such that the $\Delta_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ -clopen set

$$\bigcup_{i_1=0}^{m_1} \bigcup_{i_2=0}^{m_2} \dots \bigcup_{i_n=0}^{m_n} B((i_1, i_2, \dots, i_n)) h_1 h_2 \dots h_k$$

is determined by time τ . Furthermore, if $h = {}_k h'$, then $g_n(h, m_1, m_2, \dots, m_n, k) = g_n(h', m_1, m_2, \dots, m_n, k)$.

Proof. Define $P \subseteq H \times \omega^n \times N \times \omega$ as follows:

$$(h, m_1, m_2, \dots, m_n, k, j) \in P \leftrightarrow d(\langle h_1, h_2, \dots, h_k \rangle, j) \downarrow$$

$$\& (\forall h') [\{d(\langle h_1, h_2, \dots, h_k \rangle, j)\}(h') \downarrow \& \{d(\langle h_1, h_2, \dots, h_k \rangle, j)\}(h') \geq 1]$$

$$\& (\forall h') (\forall h'') (\forall i) [\{d(\langle h_1, h_2, \dots, h_k \rangle, j)\}(h') = i \& h' = {}_i h''$$

$$\rightarrow \{d(\langle h_1, h_2, \dots, h_k \rangle, j)\}(h'') = i]$$

$$\& (\forall h') (\forall i) [\{d(\langle h_1, h_2, \dots, h_k \rangle, j)\}(h') = i$$

$$\rightarrow (\forall h'') (h'' = {}_i h' \rightarrow h'' \in \bigcup_{i_1=0}^{m_1} \bigcup_{i_2=0}^{m_2} \dots \bigcup_{i_n=0}^{m_n} B((i_1, i_2, \dots, i_n)) h_1 h_2 \dots h_k)$$

$$\vee (\forall h'') (h'' = {}_i h' \rightarrow h'' \notin \bigcup_{i_1=0}^{m_1} \bigcup_{i_2=0}^{m_2} \dots \bigcup_{i_n=0}^{m_n} B((i_1, i_2, \dots, i_n)) h_1 h_2 \dots h_k)].$$

Then P is Π_1^1 and an application of a relativized version of Corollary 3.3 will show that

$$(\forall h) (\forall m_1) (\forall m_2) \dots (\forall m_n) (\forall k) (\exists j) [(h, m_1, m_2, \dots, m_n, k, j) \in P].$$

So, by Kreisel's selection theorem, there is a Δ_1^1 -recursive function $f: H \times \omega^n \times N \rightarrow \omega$ such that

$$(\forall h)(\forall m_1)(\forall m_2) \dots (\forall m_n)(\forall k) [(h, m_1, m_2, \dots, m_n, k, f(h, m_1, m_2, \dots, m_n, k)) \in P].$$

Finally, set

$$g_n(h, m_1, m_2, \dots, m_n, k) = d(\langle h_1, h_2, \dots, h_k \rangle, f(\bar{h}, m_1, m_2, \dots, m_n, k)),$$

where $\bar{h} = (h_1, h_2, \dots, h_k, x^*, x^*, \dots)$ and x^* is a fixed Δ_1^1 point in X . It is now easy to verify that g_n satisfies the assertions of the lemma.

We are now ready to complete the proof of Theorem 7.1. Observe, first, that $(A(m))_{m \in \omega}$ is a Σ_1^1 sequence, $A(m) \subseteq A(m+1)$ and $\bigcup_{m \in \omega} A(m) = A$. Hence, by Corollary 6.7, there is m_1^* such that

$$(7.5) \quad \bar{M}(A((m_1^*))) (x_0) > \bar{M}(A)(x_0) - \epsilon/4.$$

Define K_1 as follows:

$$h \in K_1 \leftrightarrow (\exists i \leq m_1^*) [h \in B((i))],$$

so K_1 is a Δ_1^1 clopen set and $A((m_1^*)) \subseteq K_1$. By Corollary 3.3, there is a Δ_1^1 - recursive stop rule τ_1 such that K_1 is determined by time τ_1 . Apply Corollary 4.5 to the Σ_1^1 set $A((m_1^*))$ and Δ_1^1 - recursive stop rule τ_1 at the point x_0 to get $\lambda \in \Sigma(x_0)$ such that

$$(7.6) \quad \int_{K_1} \bar{M}(A((m_1^*))_{\tau_1})(h_{\tau_1}) d\lambda(h) > \bar{M}(A((m_1^*))(x_0) - \frac{\epsilon}{4}.$$

Since $A(m_1^*) \subseteq K_1$ and K_1 is determined by time τ_1 , combining (7.5) and (7.6), we get:

$$\int_{K_1} \bar{M}(A((m_1^*))_{\tau_1})(h_{\tau_1}) d\lambda(h) > \bar{M}(A)(x_0) - \frac{\epsilon}{2}.$$

Suppose, next, that $K_1, K_2, \dots, K_n, \tau_1, \tau_2, \dots, \tau_n, m_1^*, m_2^*, \dots, m_n^*$ and $\phi_1, \phi_2, \dots, \phi_{n-1}$ have been defined. Define $J: H \times \omega^n \times N \rightarrow \omega$ as follows:

$J(h, m_1, m_2, \dots, m_n, k) = \text{least } m \text{ such that}$

$$\tilde{M}(A(\langle m_1, m_2, \dots, m_n, m \rangle) h_1 h_2 \dots h_k)(h_k)$$

$$(7.7) \quad > \tilde{M}(A(\langle m_1, m_2, \dots, m_n \rangle) h_1 h_2 \dots h_k)(h_k) - \frac{\epsilon}{2^{n+2}}.$$

Then J is well-defined. To see this, first observe that $(A(\langle m_1, m_2, \dots, m_n, m \rangle) h_1 h_2 \dots h_k)_{m \in \omega}$ is an increasing $\Sigma_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ sequence whose union is $A(\langle m_1, m_2, \dots, m_n \rangle) h_1 h_2 \dots h_k$. Now use a relativized version of Corollary 6.7 to see that there exists m satisfying (7.7).

It follows easily from Lemma 4.1 that J is analytically measurable. Furthermore,

$$h =_k h' \text{ implies } J(h, m_1, m_2, \dots, m_n, k) = J(h', m_1, m_2, \dots, m_n, k).$$

Now set

$$(7.8) \quad m_{n+1}^*(h) = J(h, m_1^*(h), m_2^*(h), \dots, m_n^*(h), \tau_n(h)), \quad h \in H,$$

so that m_{n+1}^* is universally measurable and determined by time τ_n . Next, define K_{n+1} by the formula

$$\begin{aligned} h \in K_{n+1} \leftrightarrow & (\exists m_2)(\exists m_3) \dots (\exists m_{n+1}) [m_2^*(h) = m_2 \ \& \ m_3^*(h) = m_3 \\ & \& \dots \& \ m_{n+1}^*(h) = m_{n+1} \ \& \ (\exists i_1 \leq m_1^*)(\exists i_2 \leq m_2) \dots (\exists i_{n+1} \leq m_{n+1}) (h \in \\ & B((i_1, i_2, \dots, i_{n+1})))]. \end{aligned}$$

It is easily verified, using the facts that the functions $m_2^*, m_3^*, \dots, m_{n+1}^*$ are determined by time τ_n and that $B(s)$ is clopen in H , that K_{n+1} is clopen.

Furthermore, K_{n+1} is universally measurable, since the functions $m_2^*, m_3^*, \dots, m_{n+1}^*$

are and the sets $B(s)$ are Borel. Plainly, condition (b) is fulfilled and $K_{n+1} \subseteq K_n$.

Fix a function $g_{n+1}: H \times \omega^{n+1} \times N \rightarrow \omega^\omega$ having the properties stated in Lemma 7.3. Define $\tau_{n+1}: H \rightarrow N$ by the formula

$$(7.9) \quad \tau_{n+1}(h) = \tau_n(h) + \{g_{n+1}(h, m_1^*, m_2^*(h), \dots, m_{n+1}^*(h), \tau_n(h))\}(h'),$$

where $h' = \left(h_{\tau_n(h)+1}, h_{\tau_n(h)+2}, \dots \right)$.

Using the inductive hypothesis and Lemma 7.3, it is not hard to verify that τ_{n+1} is a universally measurable stop rule such that $\tau_n < \tau_{n+1}$ and K_{n+1} is determined by time τ_{n+1} .

In order to define ϕ_n , proceed as follows. First, define $\phi: H \times \omega^{n+1} \times N \times H \rightarrow [0,1]$ by:

$$(7.10) \quad \begin{aligned} & \phi(h, m_1, m_2, \dots, m_{n+1}, k, h') \\ &= \tilde{M} \left[A((m_1, m_2, \dots, m_{n+1})) h_1 h_2 \dots h_k p_{(g_{n+1}(h, m_1, m_2, \dots, m_{n+1}, k))}^{(h')} \right] (h'_i), \end{aligned}$$

where $i = (g_{n+1}(h, m_1, m_2, \dots, m_{n+1}, k))(h')$.

Observe first that, for fixed $m_1, m_2, \dots, m_{n+1}, h'$, ϕ as a function in h and k is adapted. Next, by doing a calculation similar to that in the proof of Lemma 4.1 and using Lemma 7.3, one verifies easily that ϕ is an upper analytic function. So, by a selection theorem [12, Lemma 2.1], there is an analytically measurable function $\psi: H \times \omega^{n+1} \times N \rightarrow P(H)$ such that for every $(h, m_1, m_2, \dots, m_{n+1}, k) \in H \times \omega^{n+1} \times N$,

$$\psi(h, m_1, m_2, \dots, m_{n+1}, k) \in \Sigma(h_k) \text{ and} \\ \int \phi(h, m_1, m_2, \dots, m_{n+1}, k, h') \psi(h, m_1, m_2, \dots, m_{n+1}, k) (dh')$$

$$(7.11) > \sup_{\nu \in \Sigma(h_k)} \int \phi(h, m_1, m_1, \dots, m_{n+1}, k, h') \nu(dh') - \frac{\epsilon}{2^{n+2}}.$$

Now an application of a relativized version of Corollary 4.5 to the $\Sigma_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ set $A((m_1, m_2, \dots, m_{n+1}))h_1 h_2 \dots h_k$ and the $\Delta_1^1(\langle h_1, h_2, \dots, h_k \rangle)$ - recursive stop rule $\{g_{n+1}(h, m_1, m_2, \dots, m_{n+1}, k)\}$ yields

$$\bar{M}(A(m_1, m_2, \dots, m_{n+1})h_1 h_2 \dots h_k)(h_k) \\ (7.12) \leq \sup_{\nu \in \Sigma(h_k)} \int \phi(h, m_1, m_2, \dots, m_{n+1}, k, h') \nu(dh')$$

for every $(h, m_1, m_2, \dots, m_{n+1}, k) \in H \times \omega^{n+1} \times N$.

Define $\phi_n: H \times N \rightarrow P(H)$ by setting

$$\phi_n(h, k) = \psi((h_1, h_2, \dots, h_k, x^*, x^*, \dots), m_1^*, m_2^*(h), \dots, m_{n+1}^*(h), k), \text{ if } k \geq \tau_n(h), \\ (7.13) \quad = \bar{\psi}(h_k), \quad \text{if } k < \tau_n(h),$$

where $\bar{\psi}: X \rightarrow P(H)$ is an analytically measurable selector for Σ regarded as a subset of $X \times P(H)$, and x^* is a fixed point in X .

Now it is easy to check that ϕ_n is universally measurable, adapted and $\phi_n(h, k) \in \Sigma(h_k)$ for every $h \in H$ and $k \in N$.

It follows from (7.9) - (7.13) and the property of ϕ mentioned immediately after (7.10) that for any $h \in H$

$$\int \tilde{M}(A((m_1^*, m_2^*(h), \dots, m_{n+1}^*(h)))q_n(h)p_{\tau_{n+1}[q_n(h)]}(h'))(h'_{\tau_{n+1}[q_n(h)]})\phi_n(h, \tau_n(h))(dh')$$

$$> \tilde{M}(A((m_1^*, m_2^*(h), \dots, m_{n+1}^*(h)))q_n(h))(h_{\tau_n}) - \frac{\epsilon}{2^{n+2}}$$

$$> \tilde{M}(A((m_1^*, m_2^*(h), \dots, m_n^*(h)))q_n(h))(h_{\tau_n}) - \frac{\epsilon}{2^{n+1}},$$

where the last inequality is by virtue of (7.7) and (7.8). Condition (d) of Lemma 7.2 now follows from the previous inequality by observing that $A(m_1^*, m_2^*(h), \dots, m_{n+1}^*(h))q_n(h)$ is contained in $K_{n+1}q_n(h)$ and that $K_{n+1}q_n(h)$ is determined by time $\tau_{n+1}[q_n(h)]$. This completes the proof of Theorem 7.1. $\square$

8. Proof of Theorem 1.1

It is now an easy matter to deduce Theorem 1.1 from Theorem 7.1. So let us return to the setting of section 1. As in Theorem 1.1, assume that Γ is an analytic gambling house on X and that A is an analytic subset of $H = X \times X \times \dots$. If X is uncountable, we can invoke the Borel isomorphism theorem and assume without loss of generality that $X = 2^\omega$. For each $x \in X$, let $\Sigma(x)$ denote the set of probability measures on the Borel sets of H induced by universally measurable strategies available at x . It was observed by Dellacherie [3] (see also Sudderth [23]) that Σ is an analytic subset of $X \times P(H)$. Furthermore, Σ is a global gambling house of the type that was considered in sections 3-7. By a result in [15, p.160], we can choose $\alpha \in \omega^\omega$ (indeed, α can be chosen in 2^ω) such that both A and Σ are $\Sigma_1^1(\alpha)$ sets. Since the global gambling house Σ is $\Sigma_1^1(\alpha)$, we modify the set function $\tilde{M}$ introduced in (4.2). Define a new set function $\bar{M}$ as follows:

$$(8.1) \quad \bar{M}(E; x_1, x_2, \dots, x_k; x) = \inf \{M(O)(x) : O \text{ is a } \Delta_1^1(\langle \alpha, x_1, x_2, \dots, x_k, x \rangle), \text{ open set containing } Ex_1x_2\dots x_k\}$$

for any $\Sigma_1^1(\alpha)$ set $E \subseteq H$ and $x_1, x_2, \dots, x_k, x \in X$. Abbreviate the left side of (8.1) by $\bar{M}(Ex_1x_2\dots x_k)(x)$.

Relativize Theorem 7.1 to α . This yields

$$\bar{M}(A)(x) = M(A)(x)$$

for every $x \in X$. But

$$M^*(A)(x) \leq \bar{M}(A)(x), \quad x \in X,$$

since a $\Delta_1^1(\langle \alpha, x \rangle)$ set is Borel. So

$$M^*(A)(x) \leq M(A)(x)$$

for every $x \in X$. The inequality in the reverse direction being obvious, we have proved Theorem 1.1 in case X is uncountable. Theorem 1.1 was proved for countable X in [13]. $\square$

We conclude this section with the observation that Theorem 1.1 implies a genuine "capacitability" result for the set functions $M^*(\cdot)(x)$, even though $M^*(\cdot)(x)$ need not be a capacity [13].

Theorem 8.1. For every $x \in X$ and analytic subset A of H ,

$$M^*(A)(x) = \sup\{M^*(K)(x) : K \subseteq A \text{ \& } K \text{ is } T_1\text{-compact}\}.$$

Proof. Since M^* is monotone, only the inequality $\leq$ needs to be established. This follows from Theorem 1.1 and the following string of inequalities:

$$\begin{aligned}
M(A)(x) &= \sup_{\sigma \in \Sigma(x)} \sigma(A) \\
&= \sup_{\sigma \in \Sigma(x)} \sup_K \{ \sigma(K) : K \subseteq A \text{ \& } K \text{ is } T_1\text{-compact} \} \\
&= \sup_K \sup_{\sigma \in \Sigma(x)} \{ \sigma(K) : K \subseteq A \text{ \& } K \text{ is } T_1\text{-compact} \} \\
&= \sup_K \{ M(K)(x) : K \subseteq A \text{ \& } K \text{ is } T_1\text{-compact} \} \\
&\leq \sup \{ M^*(K)(x) : K \subseteq A \text{ \& } K \text{ is } T_1\text{-compact} \}. \quad \square
\end{aligned}$$

9. Adequacy of measurable strategies

The gambling theory of Dubins and Savage [4] takes place in a very general finitely additive framework in which a player is permitted to use nonmeasurable strategies. An interesting question, which was posed by Dubins and Savage, is whether a player can do better if allowed to use nonmeasurable strategies or whether, to the contrary, measurable strategies are adequate. The main result of this section is that for a wide class of gambling problems nonmeasurable strategies do not give the gambler any advantage.

Suppose that Γ is a gambling house on a Borel subset X of a Polish space. Assume that every gamble γ available in Γ is defined on all subsets of X as a finitely additive probability measure, whose restriction to the Borel subsets of X is countably additive. Identifying each gamble with its restriction to the Borel sets of X , we can regard Γ as a subset of $X \times P(X)$. As in section 1, we will assume that Γ is an analytic subset of $X \times P(X)$.

Given an analytically measurable set $A \subseteq H$, we can evaluate $\sigma(A)$ for strategies σ which are not necessarily measurable by using the Dubins and Savage definition of the measure σ as extended by Purves and Sudderth [18, Theorem 5.3]. If the strategy σ is measurable, and A is as above, then $\sigma(A)$ has the same value when calculated in either setting.

We will denote the optimal reward operator when both measurable and

nonmeasurable strategies are allowed by Γ .

Theorem 9.1. If X is a Borel subset of a Polish space, Γ an analytic gambling house on X and A an analytic subset of $H = X \times X \times \dots$, then

$$\Gamma(A) = M(A).$$

Proof. It was observed in the course of the proof of Theorem 7.2 of [12] that

$$M(O) = \Gamma(O)$$

for any Borel, open set O . It therefore follows from Theorem 1.1 that, for any $x \in X$,

$$M(A)(x) = \inf\{M(O)(x) : O \supseteq A \text{ \& } O \text{ is Borel, open}\}$$

$$\geq \inf\{\Gamma(O)(x) : O \supseteq A \text{ \& } O \text{ is open}\}$$

$$\geq \Gamma(A)(x).$$

Since the inequality in the opposite direction is trivial, the theorem is proved. $\square$

The rest of this section will be devoted to the extension of Theorem 9.1 to bounded, upper analytic functions on H (Theorem 1.2). The proof uses ideas from Monticino [14] (see, in particular, Theorem 4.2 in [14]).

As in Theorem 1.2, assume that g is a bounded, upper analytic function on H . For any strategy σ , measurable or not, $\sigma g = \int g d\sigma$ is well-defined, since g is analytically measurable. For the same reason, if σ is measurable, then the value of σg , whether computed in the finitely additive or countably additive mode, is the same.

The first step in the proof of Theorem 1.2 is the uniform approximation of g by an upper analytic function with only finitely many values. Assume without

loss of generality that $0 \leq g \leq 1$ and fix $\epsilon > 0$. Let $n(\epsilon)$ be the least integer n such that $n\epsilon \geq 1$ and, for $n = 1, 2, \dots, n(\epsilon)$, define

$$A^n = \{h \in H: g(h) \geq n\epsilon\}.$$

Then set

$$\tilde{g} = \epsilon[A^1 + A^2 + \dots + A^{n(\epsilon)}].$$

where we are denoting the indicator function of the set A^i by the symbol A^i . The A^i are analytic. Thus $\tilde{g}$ is upper analytic with values in $\{0, \epsilon, 2\epsilon, \dots, n(\epsilon)\epsilon\}$. Moreover,

$$\tilde{g} \leq g \leq \tilde{g} + \epsilon.$$

Thus, it suffices to prove Theorem 1.2 for $\tilde{g}$ and, consequently, for $\epsilon^{-1}\tilde{g}$. So, without loss of generality, assume from now on that

$$(9.1) \quad g = A^1 + A^2 + \dots + A^n$$

where $A^1 \supseteq A^2 \supseteq \dots \supseteq A^n$ are analytic subsets of H .

The next step is the approximation of each analytic set A by an analytic, G_δ set G as was essentially done in [13]. Fix the analytic set $A \subseteq H$ and let $0 < \epsilon < 1$.

Define

$$B = B_\epsilon(A) = \{p \in X^*: \Gamma(Ap)(l(p)) > 1 - \epsilon\},$$

where $l(p)$ is the last coordinate of the finite sequence p and $X^* = \bigcup_{m \in \mathbb{N}} X^m$. By virtue of Theorem 9.1, the set B would be the same if we replaced Γ by M in its definition. Consequently, B is an analytic subset of X^* [13, Lemma 2.5]. Next define the analytic G_δ set

$$G = G_\epsilon(A) = \{h \in H: p_n(h) \in B \text{ infinitely often}\}.$$

Lemma 9.2. For all $x \in X$,

$$\Gamma(A \cap (H-G))(x) = 0.$$

Proof. Suppose, by way of contradiction, that, for some x and σ available at x ,

$$\sigma(A \cap (H-G)) > 0.$$

By the finitely additive Levy 0-1 law [20],

$$\sigma[p_n(h)]((A \cap (H-G)) p_n(h)) \rightarrow 1$$

for σ - almost every $h \in A \cap (H-G)$. Hence, there is an $h \in H-G$ such that $\sigma[p_n(h)](A p_n(h)) > 1 - \epsilon$ for infinitely many n . Since $\Gamma(A p_n(h))(h_n) \geq \sigma[p_n(h)](A p_n(h))$, $h \in G$, which yields the desired contradiction. $\square$

Now, for each A^i occurring in (9.1), define

$$(9.2) \quad B_i = B_\epsilon(A^i), \quad G_i = G_\epsilon(A^i).$$

Notice that

$$B_1 \supseteq B_2 \supseteq \dots \supseteq B_n \text{ and } G_1 \supseteq G_2 \supseteq \dots \supseteq G_n.$$

Now define g_ϵ on H and u_ϵ on X^* by

$$g_\epsilon = G_1 + G_2 + \dots + G_n,$$

$$u_\epsilon = B_1 + B_2 + \dots + B_n.$$

We note that both g_ϵ and u_ϵ are upper analytic functions. Finally, define, for $x \in X$,

$$(Tu_\epsilon)(x) = \sup\{\int u_\epsilon(p_t) d\sigma: \sigma \text{ available at } x, t \text{ a stop rule}\}.$$

The notation Tu_ϵ is consistent with that in [5] if we pass to the "partial history house." In particular, the result of Strauch [22], as modified in [5, Lemma 4.2], implies that $(Tu_\epsilon)(x)$ would have the same value if the supremum were taken only over measurable σ available at x and Borel stop rules t .

The final steps of the argument will show that

$$\Gamma g \leq \Gamma g_\epsilon \leq Tu_\epsilon \leq Mg + (n+1)\epsilon.$$

This will suffice to complete the proof of Theorem 1.2 since ϵ is arbitrary, n is fixed, and, obviously, $Mg \leq \Gamma g$.

Lemma 9.3. $\Gamma g \leq \Gamma g_\epsilon$

Proof. Let $x \in X$ and σ be available at x . Then

$$\begin{aligned} \sigma g &= \sigma(A^1) + \sigma(A^2) + \dots + \sigma(A^n) \\ &\leq \sigma(G_1) + \sigma(G_2) + \dots + \sigma(G_n) \\ &= \sigma g_\epsilon, \end{aligned}$$

where the inequality holds because, for each i , $\sigma(A^i \cap (H - G_1)) = 0$ by Lemma 9.2. $\square$

Lemma 9.4. $\Gamma g_\epsilon \leq Tu_\epsilon$.

Proof. First notice that, for each h ,

$$g_\epsilon(h) = G_1(h) + G_2(h) + \dots + G_n(h)$$

$$\begin{aligned}
&= \overline{\lim}_k B_1(p_k(h)) + \overline{\lim}_k B_2(p_k(h)) + \dots + \overline{\lim}_k B_n(p_k(h)) \\
&= \overline{\lim}_k (B_1 + B_2 + \dots + B_n)(p_k(h)).
\end{aligned}$$

To check the final equality, recall that $B_1 \supseteq B_2 \supseteq \dots \supseteq B_n$ so that $g_\epsilon(h) = m$ if

and only if $\overline{\lim}_k B_1(p_k(h)) = \overline{\lim}_k B_2(p_k(h)) = \dots = \overline{\lim}_k B_m(p_k(h)) = 1$ and

$$\overline{\lim}_k B_{m+1}(p_k(h)) = \dots = \overline{\lim}_k B_n(p_k(h)) = 0 \text{ for } m = 1, 2, \dots, n.$$

Now, for any σ available at x ,

$$\begin{aligned}
\sigma g_\epsilon &= \int \overline{\lim}_k (B_1 + B_2 + \dots + B_n)(p_k) d\sigma \\
&= \int \overline{\lim}_k u_\epsilon(p_k) d\sigma \\
&= \overline{\lim}_t \int u_\epsilon(p_t) d\sigma \\
&\leq \sup_t \int u_\epsilon(p_t) d\sigma.
\end{aligned}$$

Here the final equality is a trivial variation of the Fatou equation [17, Theorem 10.4] (see also [24] for a proof of a countably additive version of the Fatou equation; this proof can be adapted to the finitely additive setting by using the finitely additive Levy 0-1 law [20]). Take the supremum over σ available at x to finish the proof. $\square$

Lemma 9.5. $Tu_\epsilon \leq Mg + (n+1)\epsilon.$

Proof. Fix $x \in X$. By virtue of the remark immediately following the definition of Tu_ϵ , we can choose a measurable $\hat{\sigma}$ available at x and a Borel measurable stop rule t such that

$$\int u_\epsilon(p_t) d\hat{\sigma} > Tu_\epsilon(x) - \epsilon.$$

Next use a selection theorem [12, Lemma 2.1] to obtain analytically measurable mappings $\bar{\sigma}_k: X^* \rightarrow P(H)$ such that $\bar{\sigma}_k(p) \in \Sigma(I(p))$ and, if $p \in B_k$, then

$$\bar{\sigma}_k(p)(A^k p) > 1 - \epsilon$$

for $k = 1, 2, \dots, n$. Then define, for $p \in X^*$,

$$\begin{aligned} \bar{\sigma}(p) &= \bar{\sigma}_n(p) \quad \text{if } p \in B_n \\ &= \bar{\sigma}_{n-1}(p) \quad \text{if } p \in B_{n-1} - B_n \\ &\quad \cdot \\ &\quad \cdot \\ &= \bar{\sigma}_1(p) \quad \text{if } p \in B_1 - B_2 \\ &= \bar{\sigma}_1(p) \text{ (say),} \quad \text{elsewhere.} \end{aligned}$$

Then $\bar{\sigma}$ is analytically measurable from X^* to $P(H)$. Furthermore, since $A^1 \supseteq A^2 \supseteq \dots \supseteq A^n$, it follows that

$$\bar{\sigma}(p)(A^k p) > 1 - \epsilon \quad \text{if } p \in B_k$$

for $k = 1, 2, \dots, n$. Consequently,

$$\begin{aligned} \bar{\sigma}(p_t)(gp_t) &= \bar{\sigma}(p_t)((A^1 + A^2 + \dots + A^n)p_t) \\ &\geq (B_1 + B_2 + \dots + B_n)(p_t) - n\epsilon \\ &= u_\epsilon(p_t) - n\epsilon. \end{aligned}$$

Now let σ be the measurable strategy available at x which agrees with $\hat{\sigma}$ prior to time t and satisfies $\sigma[p_t(h)] = \hat{\sigma}(p_t(h))$ for σ - almost all h . Then

$$\begin{aligned} Mg(x) &\geq \sigma g \\ &= \int \hat{\sigma}(p_t)(gp_t) d\hat{\sigma} \\ &\geq \int u_\epsilon(p_t) d\hat{\sigma} - n\epsilon \\ &> Tu_\epsilon(x) - (n+1)\epsilon. \quad \square \end{aligned}$$

We remark that it is consistent with ZFC that Theorem 9.1 does not extend to coanalytic sets. Indeed, under the assumption that there exists a non-Lebesgue measurable PCA subset of $[0,1]$, Monticino [14] constructed a Borel gambling house Γ on a Polish space such that there is a coanalytic set $C \subseteq H$ with $\Gamma(C) \neq M(C)$. In other words, for some initial fortune x , nonmeasurable strategies provide a distinct advantage over measurable strategies. It also follows that $M^*(C)(x) > M(C)(x)$ for some $x \in X$.

10. Approximating functions

In this final section we prove an approximation result for bounded, upper analytic functions on the set H of histories, which is analogous to the result for sets proved in Theorem 1.1.

A subset of H which is a countable intersection of Borel, open subsets of H will be called a special G_δ set. Here is the approximation result for functions.

Theorem 10.1. Let Γ be an analytic gambling house on a Borel subset X of a Polish space. If g is a bounded, upper analytic function on H , then

$$(Mg)(x) = \inf(Mf)(x)$$

where the infimum is over all functions $f: H \rightarrow \mathbb{R}$ such that $f \geq g$, f takes on finitely many values and $\{f \geq c\}$ is a special G_δ for every $c \in \mathbb{R}$.

Proof. It follows by virtue of the arguments of the previous section that it suffices to prove the theorem when g is given by (9.1). Fix $\epsilon > 0$ and $x_0 \in X$. Let the sets G_i be defined by (9.2). Recall that each G_i is a countable intersection of analytic, open sets, and that, by virtue of Lemma 9.2, $M(A^i \cap (H - G_i))(x) = 0$ for all $x \in X$. We now need a lemma.

Lemma 10.2. There exist special G_δ sets $\tilde{G}_i$, $i=1,2,\dots,n$, with the following properties:

$$(a) \quad \tilde{G}_1 \supseteq \tilde{G}_2 \supseteq \dots \supseteq \tilde{G}_n,$$

$$(b) \quad G_i \subseteq \tilde{G}_i, \quad i=1,2,\dots,n, \text{ and}$$

$$(c) \quad M(\tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n)(x_0) \leq M(G_1 + G_2 + \dots + G_n)(x_0) + \epsilon.$$

Proof. Let

$$D = \{(h,a): H \times [0,n]: G_1(h) + G_2(h) + \dots + G_n(h) \geq a\}.$$

Plainly, D is analytic in $H \times [0,n]$. So, by corollary 4.4 in [12], there is a Borel subset E of $H \times [0,n]$ such that $D \subseteq E$ and

$$(10.1) \quad \sup_{\sigma \in \Sigma(x_0)} (\sigma \times \lambda)(E) \leq \sup_{\sigma \in \Sigma(x_0)} (\sigma \times \lambda)(D) + \epsilon,$$

where λ is Lebesgue measure on $[0,n]$. Set

$$\phi(h) = \lambda(E_h), \quad h \in H.$$

Clearly, then, ϕ is Borel measurable, $\phi \geq G_1 + G_2 + \dots + G_n$ and, because of (10.1),

$$(M\phi)(x_0) \leq M(G_1 + G_2 + \dots + G_n)(x_0) + \epsilon.$$

Let $C_i = \{\phi \geq i\}$, $i=1,2,\dots,n$, so C_i is Borel and $C_i \supseteq G_i$. Hence, by the proof of Corollary 6.8 in [12], there is a special G_δ set F_i such that $C_i \supseteq F_i \supseteq G_i$. Let $\tilde{G}_i = F_1 \cap F_2 \cap \dots \cap F_i$, $i=1,2,\dots,n$. Then it is straightforward to verify that the sets $\tilde{G}_i$ satisfy the conditions (a)-(c) of the lemma.

To continue with the proof of Theorem 10.1, let $\tilde{G}_1, \tilde{G}_2, \dots, \tilde{G}_n$ be as in Lemma 10.2. It follows that

$$A^i \cap (H - \tilde{G}_i) \subseteq A^i \cap (H - G_i)$$

so that

$$M(A^i \cap (H - \tilde{G}_i))(x_0) = 0.$$

Since $A^i \cap (H - \tilde{G}_i)$ is analytic, we can apply Theorem 1.1 to get a Borel, open set O'_i containing $A^i \cap (H - \tilde{G}_i)$ such that

$$M(O'_i)(x_0) < \frac{\epsilon}{n^2}, \quad i=1,2,\dots,n.$$

Set $O_i = O'_i \cup O'_{i+1} \cup \dots \cup O'_n$, so O_i is Borel, open, $O_1 \supseteq O_2 \supseteq \dots \supseteq O_n$ and, by [13, Lemma 2.1],

$$\begin{aligned} M(O_i)(x_0) &\leq M(O'_i)(x_0) + M(O'_{i+1}) + \dots + M(O'_n)(x_0) \\ &< \epsilon/n, \quad i=1,2,\dots,n. \end{aligned}$$

Define

$$f = \tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n + O_1 + O_2 + \dots + O_n.$$

Since $A^i \subseteq \tilde{G}_i \cup O_i$, it follows that $f \geq g$. Clearly, f takes on finitely many values and $\{f \geq c\}$ is a special G_δ for every $c \in \mathbb{R}$. Finally, for any $\sigma \in \Sigma(x_0)$,

$$\begin{aligned}
\sigma f &= \sigma(\tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n) + \sigma(O_1) + \sigma(O_2) + \dots + \sigma(O_n) \\
&\leq \sigma(\tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n) + M(O_1)(x_0) + M(O_2)(x_0) + \dots + M(O_n)(x_0) \\
&\leq \sigma(\tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n) + \epsilon.
\end{aligned}$$

Now take the sup over $\sigma \in \Sigma(x_0)$ to get

$$\begin{aligned}
(Mf)(x_0) &\leq M(\tilde{G}_1 + \tilde{G}_2 + \dots + \tilde{G}_n)(x_0) + \epsilon \\
&\leq M(G_1 + G_2 + \dots + G_n)(x_0) + 2\epsilon \\
&\leq (Mg)(x_0) + (n+3)\epsilon,
\end{aligned}$$

where the second inequality is by Lemma 10.2 and the final equality is by virtue of Lemmas 9.4 and 9.5. Since n is fixed and ϵ arbitrary, this completes the proof of Theorem 10.1. $\square$

According to Theorem 1.1, for any analytic subset A of H and $x \in X$, $M(A)(x)$ can be approximated by $M(O)(x)$, where O is a Borel, open set containing A . It would be reasonable to conjecture from this that if g is a bounded, upper analytic function on H , then $(Mg)(x)$ can be approximated by $(Mf)(x)$, where $f \geq g$ and $\{f \geq c\}$ is a Borel, open set for every $c \in \mathbb{R}$. We conclude with an example which shows that this conjecture is false.

Example

Let $X = \{0, 1, 2, 3\}$, $\Gamma(0) = \{\delta(1)\}$, $\Gamma(2) = \{\delta(2)\}$, $\Gamma(3) = \{\delta(3)\}$ and $\Gamma(1) = \{\delta(1), \frac{2}{3}\delta(2) + \frac{1}{3}\delta(3)\}$, where $\delta(x)$ denotes point mass at x . Let $G_1 = \{h \in H: h_n = 1 \text{ infinitely often}\}$, $i=1,2$. Let $g = G_1 + 2(G_2 - G_1)$. It is easy to see that $(Mg)(0) = 4/3$. Now let f be a function on H such that $f \geq g$ and $\{f \geq c\}$ is open for every $c \in \mathbb{R}$. Then the set $\{f \geq 1\}$ is open and contains G_1 . Let τ be a stopping time such that $\{f \geq 1\} = \{\tau < \infty\}$. Consider now the history $h = (1, 1, \dots)$. Then $h \in G_1$, so $\tau(h)$ is finite. Suppose that $\tau(h) = n$. Define a

strategy σ available at 0 as follows:

$$\sigma_0 = \delta(1),$$

and for $i \geq 1$,

$$\sigma_i(x_1, x_2, \dots, x_i) = \delta(1) \text{ if } (x_i=0) \text{ or } (x_i=1 \text{ \& } i \neq n)$$

$$= \delta(x_i) \text{ if } x_i = 2 \text{ or } 3$$

$$= \frac{2}{3} \delta(2) + \frac{1}{3} \delta(3) \text{ if } x_i=1 \text{ \& } i=n.$$

It is now easy to see that $\sigma(\{f \geq 1\}) = 1$ and $\sigma(\{f \geq 2\}) \geq \sigma(G_2 - G_1) = 2/3$.
Consequently,

$$(Mf)(0) \geq \sigma f \geq \sigma(\{f \geq 1\}) + \sigma(\{f \geq 2\}) \geq 5/3. \quad \square$$

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