PLAY-THE-WINNER SAMPLING FOR A
FIXED-SAMPLE-SIZE BINOMIAL SELECTION PROBLEM¹

by

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Introduction.

This paper deals with a fixed sample size solution for the problem of selecting the better of two binomial populations. In [4], [5], and [6] Sobel and Weiss study some procedures based on a ranking and selection formulation that was explored in detail in a monograph by Bechhofer, Kiefer, and Sobel [1]. Two binomial populations are given with unknown single-trial success probabilities p_1, p_2 with $p_1 \geq p_2$; these populations are also referred to as treatments 1 and 2, respectively. The experimenter has to decide on both i) a sampling rule and ii) a terminal decision rule for selecting the better population, i.e., the population associated with the larger p ($= p_1$).

For $p_1 > p_2$, a correct selection (CS) is the selection of treatment 1; we use $P\{CS|R\}$ to denote the probability of a correct selection under procedure R . Letting $\Delta = p_1 - p_2$, we say that a procedure R satisfies a (P^*, Δ^*) requirement if

$$(1.1) \quad P\{CS|R\} \geq P^* \quad \text{whenever} \quad \Delta \geq \Delta^*;$$

here P^* and Δ^* are preassigned constants such that $\frac{1}{2} < P^* < 1$ and $0 < \Delta^* < 1$. Our goal is to keep down to a minimum both a) the expected number of observations on the poorer treatment (i.e., treatment 2) and b) the expected total number of observations.

We are mainly concerned with a comparison of the play-the-winner (PW) and vector-at-a-time (VT) sampling rules that are used in Procedures R_{PW} and R_{VT} , respectively, both based on a fixed total sample size N . Under the PW sampling rule, we select one population at random at the outset. After each success we continue to observe the same population and after

each failure we switch to the other population. Under the VT sampling rule we take (2-component) vectors of observations where the i^{th} component is from population $i(i=1,2)$; this forces the total sample size N to be an even integer. The same terminal decision rule will be used for both procedures, namely, to select the population with the most successes and to randomize when we get equality in the number of successes. The size N for both procedures is determined so that (1.1) is satisfied for the preassigned pair (P^*, Δ^*) .

The PW rule has been studied in different connections in the past (see, e.g., [2] and [4]) but not for the fixed sample size procedure R_{PW} described above; the procedure R_{VT} was investigated in [3] and we use the results therein.

The main results of this paper are proved in sections 2 and 3. In section 2 we prove that for any even N we have exactly (i.e., for all P_1, P_2)

$$(1.2) \quad P\{CS|R_{PW}\} = P\{CS|R_{VT}\} .$$

A result similar to (1.2) was obtained in [5] for two procedures whose termination rule is based on inverse sampling. We also use (1.2) in section 2 to prove that the expected number of observations on the poorer treatment is no greater under R_{PW} than under R_{VT} . In section 3, a slight modification of both procedures R_{PW} and R_{VT} enables us to generalize the result (1.2) so that it holds for any integer N . To modify procedure R_{VT} we select a population at random at the outset and then switch after every observation; this allows N to be odd and even. The sampling rule for R_{PW} is already defined for odd N and it is not altered. The termination rule of both

procedures is the same except that we introduce a quantity θ representing the probability of selecting the population with fewer failures in case of a tie (in the number of successes) at termination. For even N such a tie also implies a tie in the number of failures and then there is no need to define θ . Let the resulting procedures be denoted by $R_{PW}(\theta)$ and $R_{VT}(\theta)$, respectively. Note that $R_{PW}(\theta)$ is not in general the same as the original R_{PW} for odd N , except for the special case of $\theta = 1/2$ when $R_{PW}(1/2) = R_{PW}$ for all N . In section 3 we prove i) that for all N we have exactly (i.e., for all pairs p_1, p_2)

$$(1.3) \quad P\{CS|R_{PW}(\theta)\} = P\{CS|R_{VT}(\theta)\}$$

if and only if $\theta = 1$. (It should be noted that for $p_1 = p_2$ we have equality in (1.3) since both sides are equal to $1/2$ and this holds for all θ . In section 3 we also prove ii) that for all N and all θ

$$(1.4) \quad P\{CS|R_{PW}(\theta)\} \geq P\{CS|R_{VT}(\theta)\}$$

and, as mentioned above, for even N equality holds independent of θ .

From an intuitive point of view, we would want to take $\theta = 1$ for a slightly better procedure (of either type) and it then follows from (1.3) that $R_{PW}(1)$ has exactly the same $P\{CS\}$ as $R_{VT}(1)$ for all values of N .

2. $P\{CS|R_{PW}\}$ vs. $P\{CS|R_{VT}\}$ for N Even.

Let $P_N\{A\}$ denote the probability of A when the total fixed sample size is N . Let S_i denote the number of successes from population i ($i = 1, 2$).

Then $P_N\{CS|R_{VT}\}$ for $N = 2n$ is easily seen to be

$$(2.1) \quad P_{2n}\{CS|R_{VT}\} = P_{2n}\{S_1 > S_2\} + \frac{1}{2} P_{2n}\{S_1 = S_2\}$$

$$= \sum_{j=1}^n \sum_{i=0}^{n-j} \binom{n}{i} \binom{n}{i+j} p_1^{n-i} q_1^i p_2^{n-i-j} q_2^{i+j} + \frac{1}{2} \sum_{i=0}^n \binom{n}{i}^2 (p_1 p_2)^{n-i} (q_1 q_2)^i.$$

We first note that for $n = 1$ in (2.1) we have

$$(2.2) \quad P_2\{CS|R_{VT}\} = p_1 q_2 + \frac{1}{2} (p_1 p_2 + q_1 q_2) = P_2\{CS|R_{PW}\}$$

by direct computation for procedure R_{PW} . The rest of the proof of (1.2) is devoted to showing that for $n = 1, 2, \dots$

$$(2.3) \quad P_{2n+2}\{CS|R_{VT}\} - P_{2n}\{CS|R_{VT}\} = P_{2n+2}\{CS|R_{PW}\} - P_{2n}\{CS|R_{PW}\}.$$

Let A_i (resp., B_i) denote the event that population 1 (resp., 2) is used on the i^{th} observation; we also use $E \cap F$ for the intersection of events E and F and ΔS for $S_1 - S_2$. For procedure R_{PW} we compare the $P\{CS\}$ values for $N+1$ and $N+2$.

In particular, we first get the increase (or decrease) in $P\{CS\}$ from N to $N+1$ when A_{N+1} occurs and then when B_{N+1} occurs. An increase occurs under A_{N+1} when we get a success from population 1 and this increase $h_N(p_1, p_2)$ is

$$(2.4) \quad h_N(p_1, p_2) = P_{N+1}\{CS \cap A_{N+1}\} - P_N\{CS \cap A_{N+1}\}$$

$$= \frac{p_1}{2} P_N\{\Delta S = 0 \cap A_{N+1}\} + \frac{p_1}{2} P_N\{\Delta S = -1 \cap A_{N+1}\}.$$

Similarly, letting IS denote an incorrect selection,

$$\begin{aligned}
(2.5) \quad & P_{N+1}\{CS \cap B_{2n+1}\} - P_N\{CS \cap B_{2n+1}\} \\
&= P_{N+1}\{B_{N+1}\} - P_{N+1}\{IS \cap B_{N+1}\} - P_N\{B_{N+1}\} + P_N\{IS \cap B_{N+1}\} \\
&= P_N\{IS \cap B_{N+1}\} - P_{N+1}\{IS \cap B_{N+1}\} = -h_N(p_2, p_1),
\end{aligned}$$

since the event B_{N+1} depends only on the first N observations and the last equality is clear by interchanging p_1 and p_2 . Adding the expressions in (2.4) and (2.5) gives for $N = 2n$

$$(2.6) \quad P_{2n+1}\{CS\} - P_{2n}\{CS\} = h_{2n}(p_1, p_2) - h_{2n}(p_2, p_1)$$

and one iteration of this easily gives

$$\begin{aligned}
(2.7) \quad & P_{2n+2}\{CS\} - P_{2n}\{CS\} = P_{2n+2}\{CS\} - P_{2n+1}\{CS\} + P_{2n+1}\{CS\} - P_{2n}\{CS\} \\
&= h_{2n+1}(p_1, p_2) + h_{2n}(p_1, p_2) - [h_{2n+1}(p_2, p_1) + h_{2n}(p_2, p_1)] \\
&= t(p_1, p_2) - t(p_2, p_1) \quad (\text{say}).
\end{aligned}$$

To obtain an explicit expression for $t(p_1, p_2)$ we let $b_x(n, q)$ denote the binomial probability $\binom{n}{x} q^x p^{n-x}$ and we define $\binom{n}{x}$ and $b_x(n, q)$ to be zero for $x < 0$ and $x > n$. Under R_{PW} for N even (resp., odd) the event $\Delta S = 0 \cap A_{N+1}$ (based on N observations) implies that sampling started with population 1 (resp., 2), i.e., that A_1 (resp., B_1) occurred. Hence for N even and odd, respectively, we obtain

$$(2.8) \quad P_{2n}\{\Delta S = 0 \cap A_{2n+1}\} = \frac{1}{2} \sum_{i=0}^n b_i(n, q_1) b_{i-1}(n-1, q_2) q_2,$$

$$(2.9) \quad P_{2n+1}\{\Delta S = 0 \cap A_{2n+2}\} = \frac{1}{2} \sum_{i=0}^n b_i(n, q_1) b_i(n, q_2) q_2,$$

since in both cases the last observation on population 2 must be a failure.

Similarly, for the event $\Delta S = -1 \cap A_{N+1}$ with N even and odd, respectively, we obtain under procedure R_{PW}

$$(2.10) \quad P_{2n}\{\Delta S = -1 \cap A_{2n+1}\} = \frac{1}{2} \sum_{i=0}^n b_i(n-1, q_1) b_i(n, q_2) q_2,$$

$$(2.11) \quad P_{2n+1}\{\Delta S = -1 \cap A_{2n+2}\} = \frac{1}{2} \sum_{i=0}^n b_{i+1}(n, q_1) b_i(n, q_2) q_2.$$

From (2.4) and (2.7) the calculation of $t(p_1, p_2)$ requires that we multiply $p_1/2$ by the sum of the (four) expressions in (2.8), (2.9), (2.10) and (2.11), obtaining

$$(2.12) \quad t(p_1, p_2) = \frac{p_1 q_2}{4} \sum_{i=0}^n b_i(n, q_1) \{b_{i-1}(n-1, q_2) + b_i(n, q_2)\} \\ + \frac{p_1 q_2}{4} \sum_{i=0}^n b_i(n, q_2) \{b_i(n-1, q_1) + b_{i+1}(n, q_1)\}.$$

For procedure R_{VT} we find by a straightforward analysis that the increase in $P\{CS\}$ between n vectors and $n+1$ vectors is

$$(2.13) \quad P_{2n+2}\{CS\} - P_{2n}\{CS\} = \frac{p_1 q_2}{2} [P_{2n}\{\Delta S = 0\} + P_{2n}\{\Delta S = -1\}] \\ - \frac{p_2 q_1}{2} [P_{2n}\{\Delta S = 0\} + P_{2n}\{\Delta S = 1\}] \\ = m(p_1, p_2) - m(p_2, p_1) \text{ say,}$$

where

$$(2.14) \quad m(p_1, p_2) = \frac{p_1 q_2}{2} \sum_{i=0}^n b_i(n, q_2) [b_i(n, q_1) + b_{i+1}(n, q_1)].$$

To prove that $t(p_1, p_2) - t(p_2, p_1) = m(p_1, p_2) - m(p_2, p_1)$ it is clearly equivalent to show that $s(p_1, p_2) = t(p_1, p_2) - m(p_1, p_2)$ is symmetric with respect to an interchange of p_1 and p_2 . From (2.12)

and (2.14) we find that

$$(2.15) \quad \frac{p_1 q_2}{4} \sum_{i=0}^n b_i(n, q_2) [b_i(n-1, q_1) - b_i(n, q_1)] \\ + \frac{p_1 q_2}{4} \sum_{i=0}^n [b_i(n, q_1) b_{i-1}(n-1, q_2) - b_{i+1}(n, q_1) b_i(n, q_2)].$$

Using the well-known identity $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$, the first summation in (2.15) can be written as

$$(2.16) \quad \frac{q_2}{4} \sum_{i=0}^n b_i(n, q_2) [\binom{n-1}{i} p_1^{n-i} q_1^i - \{\binom{n-1}{i} + \binom{n-1}{i-1}\} p_1^{n-i+1} q_1^i] \\ = \frac{p_1 q_1 q_2}{4} \sum_{i=0}^n b_i(n, q_2) [b_i(n-1, q_1) - b_{i-1}(n-1, q_1)].$$

To obtain $s(p_1, p_2)$ we add the last summation in (2.15) to the result in (2.16) and, after using the same identity as above and simplifying,

$$(2.17) \quad s(p_1, p_2) = \frac{1}{4} \sum_{i=0}^n \binom{n}{i} [\binom{n-1}{i-1} (q_1 q_2)^i (p_1 p_2)^{n-i+1} - \binom{n-1}{i+1} (q_1 q_2)^{i+1} (p_1 p_2)^{n-i}],$$

which shows the required symmetry with respect to interchanging p_1 and p_2 . This completes the proof of (2.3) and hence also that of (1.2).

One corollary of this result is that the expected number of observations on the poorer treatment, $E\{N_B | R_{PW}\}$ under R_{PW} will never be greater than the corresponding value $E\{N_B | R_{VT}\}$ under procedure R_{VT} , i.e.,

$$(2.18) \quad E\{N_B | R_{PW}\} \leq E\{N_B | R_{VT}\}.$$

To prove this result we first let $E\{N_B(M) | R\}$ denote the expected number of observations on the poorer treatment (under procedure R) on the $(M+1)^{st}$ and $(M+2)^{nd}$ trials. Let A_M^* (resp., B_M^*) denote the event that the

better (resp., poorer) treatment is used on both the $(M+1)^{\text{st}}$ and $(M+2)^{\text{nd}}$ trials under procedure R_{PW} . We now show that the inequality for all M

$$(2.19) \quad P\{A_M^* | R_{PW}\} \geq P\{B_M^* | R_{PW}\}$$

implies (2.18). We start with

$$(2.20) \quad E\{N_B(M) | R_{PW}\} = 0 \cdot P\{A_M^* | R_{PW}\} + 1 \cdot P\{C | R_{PW}\} + 2P\{B_M^* | R_{PW}\},$$

where C is the event that the $(M+1)^{\text{st}}$ and $(M+2)^{\text{nd}}$ trials contain one observation on each population. If (2.19) holds, then it follows from (2.20) that for any even M

$$(2.21) \quad E\{N_B(M) | R_{PW}\} \leq 1 = E\{N_B(M) | R_{VT}\},$$

where the last equality is obvious for procedure R_{VT} . By adding up the inequalities in (2.21), for even values of M , and using (1.2), the inequality (2.18) follows. If R_{VT} requires one more observation than R_{PW} , then (2.18) holds a fortiori. Hence we need only show (2.19) to prove our result.

We note that under procedure R_{PW}

$$(2.22) \quad P\{A_M^* | R_{PW}\} = p_1 P\{A_M | R_{PW}\}, \quad P\{B_M^* | R_{PW}\} = p_2 P\{B_M | R_{PW}\},$$

where A_M (resp., B_M) is the event that the better (resp., poorer) treatment is used on the $(M+1)^{\text{st}}$ trial. Hence, to prove (2.19), it is sufficient to prove that

$$(2.23) \quad P\{A_M | R_{PW}\} \geq P\{B_M | R_{PW}\}$$

for all even M ($M = 0, 2, \dots$). We prove this by showing for even M that $P\{A_M | R_{PW}\} \geq 1/2$. The method is by induction starting with the basic

recursion formulas (omitting R_{PW} below)

$$(2.24) \quad P\{A_{M+1}\} = p_1 P\{A_M\} + q_2 P\{B_M\}$$

$$P\{B_{M+1}\} = p_2 P\{B_M\} + q_1 P\{A_M\}.$$

An iteration on this yields for $P\{A_{M+2}\}$

$$(2.25) \quad P\{A_{M+2}\} = p_1(p_1 P\{A_M\} + q_2[1-P\{A_M\}]) + q_2(p_2[1-P\{A_M\}] + q_1 P\{A_M\}) \\ = q_2(p_1 + p_2) + P\{A_M\}(p_1 + p_2 - 1)^2.$$

By induction $P\{A_M\} \geq 1/2$ and hence we obtain from (2.25)

$$(2.26) \quad P\{A_{M+2}\} \geq \frac{1}{2}(p_1 + p_2 - 1)^2 + (1-p_2)(p_1 + p_2).$$

Straightforward algebra now shows that the right side of (2.26) is at least $1/2$ since $p_1 \geq p_2$. Since $P\{A_M\} = 1/2$ for $M = 0$ the induction is complete.

It is also useful to obtain a simple expression for $E\{N_B | R_{PW}\}$ for large N . Under procedure R_{PW} we have for asymptotically large N

$$(2.27) \quad E\{N_B | R_{PW}\} \sim E\{N_A | R_{PW}\} \frac{E\{Z(q_2)\}}{E\{Z(q_1)\}} = \frac{q_1}{q_2} E\{N_A | R_{PW}\};$$

where $Z(q)$ is the geometric random variable representing the number of trials required to get one failure when q is the single-trial probability of a failure. If we let N_{PW} denote the number of observations required by R_{PW} , then for large N

$$(2.28) \quad E\{N_B | R_{PW}\} \sim \frac{q_1}{q_1 + q_2} N_{PW}.$$

This is used to get some approximate numerical results in Table 1 of section 5 to compare procedures R_{PW} and R_{VT} .

3. $P\{CS|R_{PW}(\theta)\}$ vs. $P\{CS|R_{VT}(\theta)\}$ for all N .

As defined in the introduction we now consider the generalized procedures $R_{PW}(\theta)$ and $R_{VT}(\theta)$ with N both odd and even. For N even, if we get a tie in the number of successes, then the number of failures is also tied and we need not consider any θ ; we have shown the equality in (1.2) for any even N . Clearly for $p_1 = p_2$ we have equality in (1.3) for all θ . Hence to prove (1.3) it is sufficient to show that for any even $N = 2n$ and any θ we have for $p_1 \neq p_2$

$$(3.1) \quad P_{2n+1}\{CS|R_{PW}(\theta)\} - P_{2n}\{CS|R_{PW}(\theta)\} \geq P_{2n+1}\{CS|R_{VT}(\theta)\} - P_{2n}\{CS|R_{VT}(\theta)\},$$

with equality if and only if $\theta = 1$. The combination of (1.2) and (3.1) will also prove our inequality (1.4).

Using an analysis as in (2.4) and (2.5) the left side, say $L(\theta)$, of (3.1) can be written as

$$(3.2) \quad L(\theta) = f_{\theta}(p_1, p_2) - f_{\theta}(p_2, p_1)$$

where, for procedure $R_{PW}(\theta)$

$$(3.3) \quad \begin{aligned} f_{\theta}(p_1, p_2) &= [p_1 + (1-\theta)q_1 - \frac{1}{2}]P_{2n}\{\Delta S = 0 \cap A_{2n+1}\} + \theta p_1 P_{2n}\{\Delta S = -1 \cap A_{2n+1}\} \\ &= \frac{q_2}{2} (p_1 + (1-\theta)q_1 - \frac{1}{2}) \sum_{i=0}^n b_i(n, q_1) b_{i-1}(n-1, q_2) \\ &\quad + \frac{q_2}{2} \theta p_1 \sum_{i=0}^n b_i(n-1, q_1) b_i(n, q_2) \end{aligned}$$

and the sums above are the same as in (2.8) and (2.10) above. Similarly for $R_{VT}(\theta)$ the right hand side, say, $R(\theta)$, of (3.1) can be written as

$$(3.4) \quad R(\theta) = g_{\theta}(p_1, p_2) - g_{\theta}(p_2, p_1)$$

where for procedure $R_{VT}(\theta)$

$$\begin{aligned}
 (3.5) \quad g_{\theta}(p_1, p_2) &= (p_1 + (1-\theta)q_1 - \frac{1}{2})p_{2n} \{\Delta S = 0 \cap A_{2n+1}\} \\
 &\quad + (1-\theta)p_1 p_{2n} \{\Delta S = -1 \cap A_{2n+1}\} \\
 &= \left(\frac{p_1 + (1-\theta)q_1 - 1/2}{2} \right) \sum_{i=0}^n b_i(n, q_1) b_i(n, q_2) \\
 &\quad + \frac{(1-\theta)p_1}{2} \sum_{i=0}^n b_{i+1}(n, q_1) b_i(n, q_2).
 \end{aligned}$$

Hence the difference $D_{\theta}(p_1, p_2) = f_{\theta}(p_1, p_2) - g_{\theta}(p_1, p_2)$ becomes

$$(3.6) \quad D_{\theta}(p_1, p_2) = A(p_1, p_2) + (1-\theta)B(p_1, p_2)$$

where $A(p_1, p_2)$ is given by

$$\begin{aligned}
 (3.7) \quad A(p_1, p_2) &= \frac{1}{2} \sum_{i=0}^n \binom{n}{i} \binom{n-1}{i} (q_1 q_2)^i (p_1 p_2)^{n-i} \left[\frac{1}{2} - (p_1 + p_2) \right] \\
 &\quad + \frac{1}{2} \sum_{i=0}^n \binom{n-1}{i-1} \binom{n}{i-1} (q_1 q_2)^{i-1} (p_1 p_2)^{n-i+1}
 \end{aligned}$$

and $B(p_1, p_2)$ is given by

$$(3.8) \quad B(p_1, p_2) = -\frac{1}{2} \sum_{i=0}^n \binom{n-1}{i} b_i(n, q_1 q_2) q_1 - \frac{1}{2} \sum_{i=0}^n \binom{n-1}{i-1} b_{i-1}(n, q_1 q_2) \left(\frac{n q_1}{i} + q_2 \right).$$

It is clear from (3.7) that $A(p_1, p_2) = A(p_2, p_1)$ and from (3.8)

$$(3.9) \quad B(p_1, p_2) - B(p_2, p_1) = \frac{(q_2 - q_1)}{2} \sum_{i=0}^n \binom{n-1}{i} [b_i(n, q_1 q_2) + b_{i-1}(n, q_1 q_2)],$$

which is nonnegative since $q_2 \geq q_1$. Moreover this difference is zero only when $q_2 = q_1$. It follows that the difference of the left and right sides of (3.1) takes the form

$$(3.10) \quad L(\theta) - R(\theta) = (1-\theta)[B(p_1, p_2) - B(p_2, p_1)] \geq 0$$

and if $p_1 \neq p_2$ then equality holds if and only if $\theta = 1$. This proves (3.1) and hence completes the proof of (1.3) and (1.4).

4. Conjecture.

In this section we make a conjecture for any number of populations k if the procedures $R_{PW}(\theta)$ and $R_{VT}(\theta)$ are suitably defined as below.

For both procedures we order the k populations at the outset with a random device giving probability $1/k!$ to each ordering. If t populations are tied for first place in the number of successes at termination and $t_1 \leq t$ of these are also tied in having a common smallest number f of failures then for $t_1 < t$ the remaining $t - t_1$ populations are also tied with $f + 1$ failures. We then select one of these t populations as best using probability θ/t_1 for each population in the first set and probability $(1-\theta)/(t-t_1)$ for each population in the second set. For $t_1 = t$ we select one population at random using probability $1/t$ for each. This termination rule applies to both procedures $R_{PW}(\theta)$ and $R_{VT}(\theta)$. We conjecture that the inequality (1.4) holds for any number of populations k using the above definitions but this has not been proved.

5. Numerical Illustrations of the $E\{N_B\}$ Saving.

In this section, we illustrate by means of a table, the savings in $E\{N_B\}$ for procedure R_{PW} over procedure R_{VT} . First, for asymptotically large N , the total number of observations N_{PW} required by R_{PW} is found for $\Delta^* = .1$ and $.2$ at several values of P^* . This is done by finding the corresponding value of N_{VT} , the total number of observations required by procedure R_{VT} (given in [3]), using the fact that N_{PW} and N_{VT} differ by at most one. The quantity $E\{N_B | R_{PW}\}$ is then evaluated for the particular configuration $(p_1, p_2) = (\frac{1}{2} + \frac{\Delta^*}{2}, \frac{1}{2} - \frac{\Delta^*}{2})$, employing the asymptotic formula (2.28). The results are given in Table 1.

Table 1: Comparison of R_{PW} and R_{VT} for $(p_1, p_2) = (\frac{1}{2} + \frac{\Delta^*}{2}, \frac{1}{2} - \frac{\Delta^*}{2})$.

P^*	$\Delta^* = .1$			$\Delta^* = .2$		
	N_{PW}	$E\{N_B R_{VT}\}$	$E\{N_B R_{PW}\}$	N_{PW}	$E\{N_B R_{VT}\}$	$E\{N_B R_{PW}\}$
.990	540	270	243	134	67	53.6
.975	384	192	172.8	96	48	38.4
.950	270	135	121.5	68	34	27.2

The value of $E\{N_B | R_{VT}\}$ is one-half of N_{PW} , since we have approximated N_{PW} by the N_{VT} value from [3]. This table shows significant savings in all cases, which will be even greater if $p_1 - p_2 > \Delta^*$.

No claim is made that the procedures R_{PW} and R_{VT} are in any sense the most efficient procedures for this problem. Table 1 merely illustrates the amount of the saving and it also illustrates numerically one of the results of this paper, namely (2.18).

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