**SWELLING OF A RUBBER BALL IN
THE PRESENCE OF GOOD SOLVENT**

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SWELLING OF A RUBBER BALL IN THE PRESENCE OF GOOD SOLVENT*

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Abstract. We consider the swelling of a 3-dimensional rubber ball in the presence of a good solvent. In Lagrangian coordinates the concentration φ of rubber satisfies a diffusion equation with nonlinear nonlocal coefficients. In coordinates which are moving with the swollen rubber, φ satisfies the porous medium equation $2\varphi_t = \Delta\varphi^2$ with a moving boundary $r = s(t)$ on which $\varphi_r = -\dot{s}$ and $A\varphi_r + B\varphi = B\theta$ ($A \geq 0$, $B \geq 0$, $A^2 + B^2 > 0$, $0 < \theta < 1$); for $A = 0$, this problem also has a variational inequality formulation with a nonlinear parabolic operator. We prove existence, uniqueness and regularity of the solution and its free boundary, and determine its asymptotic behavior as $t \rightarrow \infty$.

§1. The problem. A rubber is a locally liquid polymer held together by crosslinks. When immersed in a good solvent such as gasoline it swells as much as several times its original size. The theory of equilibrium is based on minimizing the free energy function [4] [5] [13]. On the other hand the time dependent process of rubber swelling is not well understood [4]. One approach might possibly be based on the theory of mixture [2] [3], but this appears to be very complicated. A much simpler approach was recently developed by Rossi and Mazich [11] for the case of radially symmetric geometry, that is, for a swelling rubber ball: the rubber ball is immersed in a good solvent and is undergoing swelling.

It is assumed that the density of rubber ρ_R and the density of solvent ρ_S are constants (incompressibility assumption); for simplicity we take $\rho_R = \rho_S = 1$. Then the concentrations and volume fractions coincide.

Set

$$\phi_R = \text{volume fraction of rubber,}$$

$$\phi_S = \text{volume fraction of solvent.}$$

Then

$$(1.1) \quad \phi_R + \phi_S = 1 .$$

We fix the origin in the center of the ball, and denote the distance from the origin in the swelling coordinates by r , and in the original coordinates by r_0 . We may view r_0 as a Lagrange variable and r as an Euler variable. Initially (at $t = 0$) the rubber is unswollen,

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that is, $r = r_0$. As time evolves, the material point at r_0 will be displaced radially to position $r = r(r_0, t)$.

The ball initially occupies a region $r_0 \leq a$.

We may consider ϕ_R as a function of either (r_0, t) or (r, t) . The advantage of viewing it as a function of (r_0, t) is that the spatial domain is then fixed, namely, it is the ball $r_0 \leq a$.

Initially

$$(1.2) \quad \phi_R(r_0, 0) = 1 \quad \text{if} \quad r_0 < a .$$

A volume element $\Delta V(r_0, 0) = r_0^2 dr_0 d\sigma$ (where $d\sigma$ is a surface element on the unit sphere) initially located at $r = r_0$ will occupy volume $\Delta V(r_0, t) = r^2 dr d\sigma$ ($r = r(r_0, t)$) at time t , such that

$$\phi_R(r_0, t) \Delta V(r_0, t) = \phi_R(r_0, 0) \Delta V(r_0, 0) .$$

It follows that

$$(1.3) \quad \phi_R(r_0, t) r^2 dr = r_0^2 dr_0 ,$$

or

$$(1.4) \quad r^3(r_0, t) = \int_0^{r_0} \frac{3y^2 dy}{\phi_R(y, t)} .$$

Conservation of mass of rubber and solvent gives ([11]; see also [7; Chap. 5])

$$(1.5) \quad \frac{\partial \phi_S}{\partial t} = -\frac{\phi_R}{r^2} \frac{\partial}{\partial r} (\text{flux}) .$$

We assume that the flux is given by Fick's law:

$$(1.6) \quad \text{flux} = -\tilde{D}(\phi_S) r^2 \frac{\partial \phi_S}{\partial r} ,$$

where $\tilde{D}$ is a diffusion coefficient.

Substituting (1.6) into (1.5) and using (1.1), we find that ϕ_R satisfies the equation

$$(1.7) \quad \frac{\partial \phi_R}{\partial t} = \frac{\phi_R}{r^2} \frac{\partial}{\partial r} \left(D(\phi_R) r^2 \frac{\partial \phi_R}{\partial r} \right)$$

where $D(\phi_R) = \tilde{D}(1 - \phi_R)$.

For simplicity we shall make the assumption that $D(\phi_R)$ is a constant (see, however, Remark 7.2) and in fact take

$$(1.8) \quad D(\phi_R) = 1 .$$

Using (1.3) in (1.7) and setting $\varphi = \phi_R(r_0, t)$, we then get

$$(1.9) \quad \frac{\partial \varphi}{\partial t} = \frac{\varphi^2}{r_0^2} \frac{\partial}{\partial r_0} \left(\frac{r^4}{r_0^2} \varphi \frac{\partial \varphi}{\partial r_0} \right) , \quad 0 < r_0 < a, \ t > 0 .$$

We make the assumption (which clearly must hold if $\varphi \in C^1$) that

$$(1.10) \quad \frac{\partial \varphi(0, t)}{\partial r_0} = 0 , \quad t > 0 .$$

We recall the initial condition

$$(1.11) \quad \varphi(r_0, 0) = 1, \quad 0 < r_0 < a ,$$

and impose the boundary condition at $r_0 = a$

$$-A \frac{\partial \varphi}{\partial r} = B(\varphi - \theta) \quad \text{for } r = r(a, t), \ t > 0$$

where $0 < \theta < 1$ and A, B are nonnegative functions. The parameter θ arises from the Flory results [5] for the equilibrium concentration and it summarizes all polymer and network specific information of the rubbery material. In Sections 2–5 we consider the special case $A = 0, B = 1$:

$$(1.12) \quad \varphi(a, t) = \theta, \ t > 0$$

where $0 < \theta < 1$. In Section 6 we extend the results to the general boundary condition at $r_0 = a$. Finally, in Section 7 we consider the limiting case $\theta = 0$.

§2. Equivalent formulations.

DEFINITION 2.1. We refer to the system (1.9) – (1.12) as problem (P1).

Because of the relation (1.4), equation (1.9) is both nonlinear and nonlocal. Since, by the maximum principle, $\varphi \geq \theta$, the nonlinearity in φ is nondegenerate.

In this section we transform (P1) into an equivalent problem by taking the independent variable to be $r = r(r_0, t)$ instead of r_0 . For clarity we shall write

$$(2.1) \quad \psi(r, t) = \varphi(r_0, t) .$$

We can rewrite (1.3) in the form

$$(2.2) \quad \frac{\partial r}{\partial r_0} = \frac{r_0^2}{r^2} \frac{1}{\varphi(r_0, t)} = \frac{r_0^2}{r^2} \frac{1}{\psi(r, t)} .$$

Differentiating (1.4) with respect to t we get

$$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{1}{r^2} \int_0^{r_0} \left(-\frac{y^2}{\varphi^2} \right) \varphi_t dy \\ &= \frac{1}{r^2} \int_0^{r_0} \left(-\frac{y^2}{\varphi^2} \right) \frac{\varphi^2}{y^2} \frac{\partial}{\partial y} \left(\frac{r^4}{y^2} \varphi \frac{\partial \varphi}{\partial y} \right) dy \quad (\text{by (1.9)}) \\ &= -\frac{r^2}{r_0^2} \varphi \frac{\partial \varphi}{\partial r_0} \quad (\text{by (1.10)}) \\ &= -\frac{r^2}{r_0^2} \psi \frac{\partial \psi}{\partial r} \frac{r_0^2}{r^2} \frac{1}{\psi} \quad (\text{by (2.2)}). \end{aligned}$$

Hence

$$(2.3) \quad \frac{\partial r}{\partial t} = -\frac{\partial \psi}{\partial r} .$$

Using (2.1), (2.2), we can transform (1.9) into

$$\psi_t = \frac{1}{2} \left\{ (\psi^2)_{rr} + \frac{2}{r} (\psi^2)_r \right\} .$$

This equation is to be satisfied for $r < s(t)$, $t > 0$, where

$$(2.4) \quad s(t) = r(a, t) ;$$

$r = s(t)$ is the radius of the swelling rubber ball at time t . Note that

$$\dot{s} = r_t(a, t) = -\psi_r(a, t) , \quad \text{by (2.3)}.$$

We can now reformulate problem (P1) as follows:

Problem (P2). Find functions $\psi(r, t)$ and $s(t)$, with $s(0) = a$, such that

$$(2.5) \quad \psi_t = \frac{1}{2} \Delta(\psi^2) \quad \text{if} \quad 0 < r < s(t), \quad t > 0 ,$$

$$(2.6) \quad \frac{\partial \psi}{\partial r}(0, t) = 0 \quad \text{if} \quad t > 0 ,$$

$$(2.7) \quad \psi(r, 0) = 1 \quad \text{if} \quad 0 < r < a ,$$

$$(2.8) \quad \psi(s(t), t) = \theta \quad \text{if} \quad t > 0 ,$$

$$(2.9) \quad \dot{s}(t) = -\frac{\partial \psi}{\partial r}(s(t), t) \quad \text{if} \quad t > 0 ,$$

where Δ is the Laplacian in $\mathbf{R}^3$.

Problem (P2) is a free boundary problem. We expect ψ_r to be negative (this, in fact, will be proved later on). Therefore the function

$$(2.10) \quad v = \psi - \theta$$

should be a positive function for $r < s(t)$, vanishing on the free boundary, whereas the free boundary $r = s(t)$ is monotone increasing. This observation suggests possibly another formulation of the problem in terms of a variational inequality, we shall now proceed to derive.

From (2.10) we get

$$\frac{1}{2} \psi^2 = \frac{1}{2} v^2 + \theta v + \frac{1}{2} \theta^2 = w + \frac{1}{2} \theta^2$$

where

$$(2.11) \quad w = \frac{1}{2} v^2 + \theta v .$$

Introducing also the inverse function $m(r) = s^{-1}(t)$ of $r = s(t)$, we can rewrite (2.5)–(2.9) in the form

$$(2.12) \quad \begin{aligned} \frac{\partial}{\partial t} \sqrt{2w + \theta^2} - \Delta w &= 0 & \text{if} & \quad 0 < r < \infty, \quad t > m(r), \\ \frac{\partial w}{\partial r} (0, t) &= 0 & \text{if} & \quad t > 0, \\ w(r, 0) &= \frac{1 - \theta^2}{2} & \text{if} & \quad 0 < r < a, \\ w(r, m(r)) &= 0 & \text{if} & \quad a < r < \infty, \\ m_r w_r &= -\theta & \text{on} & \quad t = m(r), \quad r > 0, \\ m(r) &= 0 & \text{if} & \quad 0 < r < a. \end{aligned}$$

Observe that $w > 0$ if $t > m(r)$, since $v > 0$ if $t > m(r)$.

We now use the Duvaut transformation (cf. [6])

$$W(r, t) = \begin{cases} \int_{m(r)}^t w(r, \tau) d\tau & \text{if } t > m(r) \\ 0 & \text{if } t < m(r), \quad r > a. \end{cases}$$

We easily compute that

$$\begin{aligned}\Delta W &= \sqrt{2W_t + \theta^2} & \text{if } r > a, t > m(r), \\ \Delta W &= \sqrt{2W_t + \theta^2} - 1 & \text{if } 0 < r < a, t > 0\end{aligned}$$

and, trivially,

$$\Delta W = \sqrt{2W_t + \theta^2} - \theta \quad \text{if } r > a, t < m(r)$$

(since $W = 0$ in this set). Setting

$$(2.13) \quad f(r) = \begin{cases} 1 & \text{if } 0 < r < a \\ 0 & \text{if } r > a \end{cases}$$

we find that (2.12) reduces to the following variational inequality:

Problem (P3). Find $W(r, t)$ such that

$$(2.14) \quad \begin{aligned} \sqrt{2W_t + \theta^2} - \Delta W &\geq f, \quad W \geq 0, \\ \left(\sqrt{2W_t + \theta^2} - \Delta W - f \right) W &= 0 \quad \text{a.e. if } r > 0, t > 0, \end{aligned}$$

and

$$(2.15) \quad W_r(0, t) = 0 \quad \text{if } t > 0,$$

$$(2.16) \quad W(r, 0) = 0 \quad \text{if } r > 0.$$

We can prove the existence of a solution using either one of the three formulations (P1), (P2), (P3). However, for proving regularity we prefer to use the formulations (P1) and (P2) (the nonlinearity of the parabolic operator in (2.14) causes technical difficulties), and for proving uniqueness formulation (P3) seems the most suitable.

§3. Existence and regularity. In this section we prove:

THEOREM 3.1. *There exists a solution φ of problem (P1) and a corresponding solution (ψ, s) of problem (P2) such that*

$$\varphi \in C^\infty\{[0, r_0] \times (0, \infty)\}, \quad s \in C^\infty(0, \infty);$$

furthermore, $\dot{s}(t) > 0$ and $\partial\varphi/\partial r_0 < 0$ if $0 < r_0 \leq a$, $t > 0$.

The function φ is continuous at $(r_0, 0)$, $0 \leq r_0 < a$ and at (a, t) , $t > 0$, but it is discontinuous at $(a, 0)$.

It will be convenient to work with the function $u = \varphi^2$ rather than with φ . Setting $x = r_0$ in problem (P1), we begin by approximating this problem by more regular ε -problems

$$\begin{aligned}
 (3.1) \quad & u_t = \frac{u^{3/2}}{x^2} \frac{\partial}{\partial x} \left(\frac{\sigma^4}{x^2} \frac{\partial u}{\partial x} \right), \quad \varepsilon < x < a, \quad t > 0, \\
 & u(a, t) = \theta^2, \quad t > 0, \\
 & u_x(\varepsilon, t) = 0, \quad t > 0, \\
 & u(x, 0) = h_\varepsilon(x), \quad \varepsilon < x < a
 \end{aligned}$$

where ε is a small positive number,

$$\sigma^3 = \sigma^3(u)(x, t) = \int_{\varepsilon}^x \frac{3y^2 dy}{\sqrt{u(y, t)}} + \frac{\varepsilon^3}{\theta}$$

and $h_\varepsilon(x)$ is a function in $C^{2+\alpha}[\varepsilon, a]$ such that

$$h_\varepsilon(x) = 1 \text{ if } \varepsilon \leq x \leq a - 2\varepsilon, \quad h_\varepsilon(x) = \theta^2 \text{ if } a - \varepsilon \leq x \leq a, \text{ and } h'_\varepsilon(x) \leq 0.$$

Set

$$Q_T^\varepsilon = (\varepsilon, a) \times (0, T).$$

We shall use the notation $u \in C^{2+\alpha, 1+\alpha/2}$ to mean that the spatial derivatives ∇u , $\nabla^2 u$ and the time derivative u_t are uniformly in C^α in the space variable and in $C^{\alpha/2}$ in t . By $u \in C^{\beta, \beta/2}$ we mean that u is uniformly in C^β in the space variable and in $C^{\beta/2}$ in t ; these and similar spaces are defined in [10].

LEMMA 3.2. *the system (3.1) has a solution u_ε in $C^{2+\alpha, 1+\alpha/2}(Q_T^\varepsilon)$ for some $\alpha \in (0, 1)$.*

Proof. For any $\lambda > 0$, $0 < \beta < 1$, set

$$B_\lambda = \left\{ v \in C(\overline{Q_T^\varepsilon}), \quad \theta^2 \leq v \leq 1, \quad \|v\|_{C^{\beta, \beta/2}(Q_T^\varepsilon)} \leq \lambda \right\}.$$

We define an operator $\mathcal{M}$ on B_λ such that, for any $v \in B_\lambda$, $u = \mathcal{M}v$ is the unique solution to

$$\begin{aligned}
(3.2) \quad & u_t = \frac{v^{3/2}}{x^2} \frac{\partial}{\partial x} \left(\frac{\sigma^4(v)}{x^2} \frac{\partial u}{\partial x} \right), \quad \varepsilon < x < a, \quad 0 < t < T, \\
& u(a, t) = \theta^2, \quad 0 < t < T, \\
& u_x(\varepsilon, t) = 0, \quad 0 < t < T, \\
& u(x, 0) = h_\varepsilon(x), \quad \varepsilon < x < a.
\end{aligned}$$

By the maximum principle

$$(3.3) \quad \theta^2 \leq u \leq 1.$$

Since

$$1 \leq \frac{\sigma^3(v)}{x^3} \leq \frac{1}{\theta}, \quad \left| \frac{\partial \sigma^3(v)}{\partial x} \right| \leq \frac{3a^2}{\theta},$$

we can apply the well known Hölder estimate for the uniformly parabolic equation in (3.1) (see [9]) to get

$$(3.4) \quad \|u\|_{C^{\alpha, \alpha/2}(Q_T^\varepsilon)} \leq C_* \quad \text{for some } \alpha \in (0, 1),$$

where C_* and α are constants depending on ε , but not on β and λ . Choosing $\beta = \alpha$ and using (3.4), we can then apply the Schauder estimate to the equation in (3.1) to get

$$(3.5) \quad \|u\|_{C^{2+\alpha, 1+\alpha/2}(Q_T^\varepsilon)} \leq C$$

where C depends on C_* and ε . Hence, if we choose $\lambda = C_*$ then $\mathcal{M}$ maps B_λ into itself.

Using the estimate (3.5) one can easily deduce that $\mathcal{M}(B_\lambda)$ lies in a compact subset of B_λ and $\mathcal{M}$ is a continuous mapping on B_λ , where B_λ is endowed with the $C^{\alpha, \alpha/2}$ topology. Hence, by the Schauder fixed point theorem, there exists a solution u_ε of (3.1). Fix a small δ , $0 < \delta \leq \frac{a}{4}$, and set

$$\Omega_T^\delta = Q_T^\delta \setminus \{[a - \delta, a] \times [0, \delta]\}.$$

By (3.3) and a priori Hölder estimates [9],

$$(3.6) \quad \|u_\varepsilon\|_{C^{\alpha, \alpha/2}(\Omega_T^\delta)} \leq C(\delta)$$

for any $0 < \varepsilon < \delta/4$ where $\alpha \in (0, 1)$ and $C(\delta)$ is a constant depending on δ but independent of ε . Using (3.6) we deduce, by the Schauder estimates, that

$$(3.7) \quad \|u_\varepsilon\|_{C^{2+\alpha, 1+\alpha/2}(\Omega_T^\delta)} \leq C(\delta)$$

with another constant $C(\delta)$.

We next wish to estimate derivatives of u_ε in $\varepsilon \leq x \leq \delta$. We begin by establishing bounds on the derivative $u_{\varepsilon, x}$.

LEMMA 3.3. *The solution u_ε satisfies*

$$(3.8) \quad \frac{\partial}{\partial x} u_\varepsilon < 0 \quad \text{in} \quad Q_T^\varepsilon .$$

Proof. Set $v = \partial u_\varepsilon / \partial x$. Differentiating the equation in (3.1) with respect to x and dropping ε in u_ε , we get

$$(3.9) \quad \begin{aligned} v_t = & \frac{\sigma^4}{x^4} u^{3/2} v_{xx} + \left[\frac{\partial}{\partial x} \left(\frac{\sigma^4}{x^4} u^{3/2} \right) + \frac{1}{x^2} u^{3/2} \frac{\partial}{\partial x} \left(\frac{\sigma^4}{x^2} \right) \right] v_x \\ & + \left[\frac{\partial}{\partial x} \left(\frac{u^{3/2}}{x^2} \frac{\partial}{\partial x} \left(\frac{\sigma^4}{x^2} \right) \right) \right] v \end{aligned}$$

where $\sigma = \sigma(u)$. Clearly also

$$\begin{aligned} v(\varepsilon, t) &= 0, \quad 0 < t < T, \\ v(x, 0) &= h'_\varepsilon(x) \leq 0, \quad \varepsilon < x < a, \end{aligned}$$

and

$$v(a, t) \leq 0, \quad 0 < t < T$$

since u takes its minimum at $x = a$. The assertion (3.8) now follows by applying the maximum principle to v .

LEMMA 3.4. *The solution u_ε satisfies*

$$(3.10) \quad \frac{\partial}{\partial x} u_\varepsilon \geq -Ax \quad \text{in} \quad [\varepsilon, \delta] \times [0, T]$$

where A is a positive constant independent of ε and δ .

Proof. Setting $\varphi_\varepsilon = \sqrt{u_\varepsilon}$ and $\psi(r, t) = \varphi_\varepsilon(r_0, t)$ where $r_0 = x$ and $r = r(r_0, t)$ is defined slightly differently than in (1.4), namely, $r = \sigma(u_\varepsilon)(r_0, t)$, one can easily find that ψ still satisfies (2.5). Differentiating this equation with respect to r we find that $\zeta = \psi_r$ satisfies

$$(3.11) \quad \mathcal{L}\zeta \equiv \zeta_t - \psi \zeta_{rr} - \left(3\psi_r + \frac{2}{r} \psi \right) \zeta_r - \left(\frac{2}{r} \psi_r - \frac{2}{r^2} \psi \right) \zeta = 0 .$$

Since $\partial r / \partial r_0 > 0$ (by (2.2)) and $\partial \varphi_\varepsilon / \partial r_0 \leq 0$ (by Lemma 3.3), we have that $\psi_r < 0$. Hence the coefficient of ζ in (3.11) is ≥ 0 , so that the maximum principle can be applied.

The function $\bar{\zeta} \equiv -Ar$ is a subsolution to (3.11) since

$$\mathcal{L}\bar{\zeta} = 5A\psi_r < 0 .$$

Using (3.7) with $\delta = a/4$ we find that $\zeta = \psi_r$ majorizes $\bar{\zeta}$ on the parabolic boundary in the (r, t) -domain corresponding to $r_0 = a/4$, $r_0 = \varepsilon$ and $t = 0$, provided A is sufficiently large. Hence $\psi_r \geq -Ar$, from which (3.10) follows.

Remark 3.1. It seems difficult to prove (3.10) by working directly with (3.9); we can show that the coefficient of v in (3.9) is ≤ 0 (so that the maximum principle can be applied) only when $\theta \geq \frac{2}{3}$.

LEMMA 3.5. *There holds:*

$$(3.12) \quad \|u_\varepsilon\|_{C^{\alpha, \alpha/2} \{[\varepsilon, 2\delta] \times [0, T]\}} \leq \tilde{C}$$

where $\tilde{C}$ and α are positive constants independent of ε and δ .

Proof. Rewrite the differential equation in (3.1), for $u = u_\varepsilon$, in the form

$$(3.14) \quad u_t = \frac{\sigma^4}{x^4} u^{3/2} u_{xx} + f$$

where

$$(3.15) \quad f = \frac{4\sigma}{x} u^{3/2} \left(\frac{1}{\sqrt{u}} - \frac{1}{2} \frac{\sigma^3}{x^3} \right) \frac{u_x}{x} .$$

Recall that

$$(3.16) \quad \theta^2 \leq u \leq 1, \quad 1 \leq \frac{\sigma^3}{x^3} \leq \frac{1}{\theta} .$$

From (3.8), (3.10) we see that

$$\left| \frac{u_x}{x} \right| \leq A$$

Using also (3.16) we conclude that f is bounded in $[\varepsilon, \delta] \times [0, T]$ uniformly with respect to ε and δ . Hence (3.14) is uniformly parabolic equation with uniformly bounded coefficients. By a priori Hölder estimates [9], the assertion (3.12) follows.

Proof of Theorem 3.1. Using (2.7) and (3.12), we can choose a sequence u_{ε_n} which converges uniformly to a function u in any region $\overline{\Omega_T^\delta}$, with $\delta \rightarrow 0$. It follows that $\psi = \sqrt{u}$ is a solution of (1.9), (1.11), (1.12). From the estimate

$$\left\| \frac{\partial}{\partial x} u_{\varepsilon_n}(x, t) \right\|_{L^\infty([\varepsilon_n, \delta] \times [0, T])} \leq Ax \quad \text{for} \quad \delta = \frac{a}{4}$$

derived above we conclude that

$$(3.17) \quad \left| \frac{\partial u}{\partial x}(x, t) \right| \leq Ax ,$$

so that (1.10) is also satisfied.

If we take a sequence of times $T = T_n \rightarrow \infty$ then, by the estimates established above for the corresponding solutions $u = u_n$ we can deduce that, for a subsequence, u_n is convergent to a solution of (1.9) – (1.12) for all $t > 0$.

By standard parabolic estimates, the corresponding solution ψ is in C^∞ for $0 < r < s(t)$, $t > 0$. Take $\delta \in (0, a)$ and denote by $\Psi(X, t)$ ($X \in \mathbb{R}^3$) the solution of (2.5) in $\{|X| < \delta, t > 0\}$ with $\Psi(X, t) = \psi(|X|, t)$ for $|X| = \delta$ and for $t = 0$. Then, by standard regularity result for the nondegenerate parabolic equation (2.5), $\Psi(X, t)$ is C^∞ in a neighborhood of $\{(O, t), t > 0\}$ where O is the origin in $\mathbb{R}^3$. By uniqueness, it is radially symmetric and therefore $\nabla_X \Psi(O, t) = 0$. Then, again by uniqueness, $\Psi(X, t) = \psi(|X|, t)$. It follows that ψ is in C^∞ in a neighborhood of $\{(O, t); t > 0\}$ where O is the origin in $\mathbb{R}^3$. The $C^{2+\alpha, 1+\alpha/2}$ regularity established for u (by (3.7)) up to the free boundary, for $t \geq \delta$, implies that $\dot{s} \in C^{\alpha/2}$ in $\{t \geq \delta\}$. From Lemma 3.3 we deduce that $\psi_r < 0$ if $0 < r < s(t)$ and, by the strong maximum principle for ψ , $\psi_r < 0$ also on $r = s(t)$. Consequently $\dot{s}(t) > 0$ for $t > 0$.

To prove higher regularity for $s(t)$ and for ψ at the free boundary, we can proceed similarly to the corresponding situation for the Stefan problem [12].

We have thus completed the proof of Theorem 3.1. Note that the solution φ is not continuous at $(a, 0)$.

§4. Uniqueness. By a classical solution of (P1) we mean a solution φ which is continuous in $\{[0, a] \times [0, \infty)\} \setminus (a, 0)$, with φ_{r_0} continuous in $[0, a] \times (0, \infty)$. By the maximum principle it then follows that $\theta^2 \leq \varphi \leq 1$ and $\partial\varphi/\partial r_0 < 0$ at $r_0 = a$; therefore also $\dot{s}(t) > 0$ if $t > 0$.

THEOREM 4.1. *There exists at most one classical solution of (P1).*

Proof. Suppose there exist two such solutions φ and $\widehat{\varphi}$ with their corresponding (ψ, s) and $(\widehat{\psi}, \widehat{s})$. Denote the corresponding solutions of (P3) by W and $\widehat{W}$. It suffices to show that $W \equiv \widehat{W}$. If this is not true then we may assume that there exists a point (r_*, t_*) with $t_* > 0$ such that $W(r_*, t_*) > \widehat{W}(r_*, t_*)$. Denote by G the largest connected open set in $[0, a] \times [0, \infty]$ which contains (r_*, t_*) and on which $W > \widehat{W}$. Then

$$\sqrt{2W_t + \theta^2} - \Delta W = f, \quad \sqrt{2\widehat{W}_t + \theta^2} - \Delta \widehat{W} \geq f$$

in G . It follows that

$$\frac{2(W - \widehat{W})_t}{\sqrt{2W_t + \theta^2} + \sqrt{2\widehat{W}_t + \theta^2}} - \Delta(W - \widehat{W}) \leq 0 \quad \text{in } G.$$

Observe that G must lie in $\{r < s(t)\}$ and thus $G_T \equiv G \cap \{t < T\}$ is a bounded set for any $T > 0$. At each point on the parabolic boundary of G either $W - \widehat{W} = 0$ or (if $r = 0$

at such a point) $(W - \widehat{W})_r = 0$. Hence, by the maximum principle, $W - \widehat{W} \leq 0$ in G_T . If we take $T > t_*$ then we conclude that $W(r_*, t_*) \leq \widehat{W}(r_*, t_*)$, which is a contradiction.

Remark 4.1. One can prove the existence of a solution to problem (P3) by the penalty method. Introduce ε -family of C^∞ functions $\beta_\varepsilon(\lambda)$ satisfying

$$\begin{aligned}\beta_\varepsilon(\lambda) &= 0 \quad \text{if } \lambda \geq \varepsilon, \quad \beta_\lambda(0) = -1, \\ \beta'_\varepsilon(\lambda) &\geq 0, \quad \beta''_\varepsilon(\lambda) \leq 0, \\ \beta_\varepsilon(\lambda) &\rightarrow -\infty \quad \text{if } \lambda < 0 \quad \text{and } \varepsilon \downarrow 0.\end{aligned}$$

Consider the parabolic problems:

$$(4.1) \quad \sqrt{2W_t + \theta^2} - \Delta W + \theta\beta_\varepsilon(W) = f_\varepsilon, \quad 0 < r < R, \quad t > 0,$$

$$(4.2) \quad W_r(0, t) = 0, \quad t > 0,$$

$$(4.3) \quad W(R, t) = 0, \quad t > 0,$$

$$(4.4) \quad W(r, 0) = 0, \quad 0 < r < R$$

where f_ε is a smooth approximation to f with $f'_\varepsilon \leq 0$. Setting $v = 2W_t + \theta^2$ and differentiating (4.1) with respect to t , we get

$$(4.5) \quad \frac{v_t}{\sqrt{v}} - \Delta v + \theta\beta'_\varepsilon \left(\int_0^t \frac{1}{2} (v(r, \tau) - \theta^2) d\tau \right) (v - \theta^2) = 0.$$

Clearly also

$$\begin{aligned}(4.6) \quad v_r(0, t) &= 0, \quad t > 0, \\ v(R, t) &= \theta^2, \quad t > 0, \\ v(r, 0) &= (f_\varepsilon + \theta)^2, \quad 0 < r < R.\end{aligned}$$

One can now prove, by a fixed point argument, that (4.5), (4.6) has a solution and this yields a solution W_ε of (4.1) – (4.4). (It seems harder to prove directly that the latter system has a solution.) Next one can apply the maximum principle to deduce that

$$\theta^2 \leq v \leq (1 + \theta)^2.$$

Going back to (4.1) – (4.4) we can proceed to derive a lower bound on $\beta_\varepsilon(W_\varepsilon)$ which is independent as ε and R . It is then easy to show that for some subsequences $\varepsilon = \varepsilon_n \rightarrow 0, R = R_n \rightarrow \infty, W_\varepsilon$ converges to a solution W of (P3) (cf. [8] where this is done for the Stefan problem). However, it seems difficult to establish regularity of the free boundary by working directly with the variational inequality formulation.

§5. Asymptotic behavior as $t \rightarrow \infty$. If the solution W of (P3) converges to a limit W_0 as $t \rightarrow \infty$, then formally W_0 is the solution to the elliptic variational inequality

$$(5.1) \quad \begin{aligned} -\Delta W_0 &\geq f - \theta, \quad W_0 \geq 0, \\ (-\Delta W_0 - f + \theta)W_0 &= 0 \quad \text{a.e. in } \{r > 0\}, \end{aligned}$$

$$(5.2) \quad \frac{\partial}{\partial r} W_0(0) = 0.$$

Since $f - \theta = -\theta$ if $r > a$, W_0 must have compact support (see [6; p. 9]). In fact, we can compute W_0 explicitly:

$$(5.3) \quad W_0(r) = \begin{cases} -\frac{1-\theta}{6} r^2 + \frac{a^2}{2} (1 - \theta^{1/3}), & 0 < r < a \\ \frac{\theta}{6} r^2 + \frac{a^3}{3r} - \frac{a^2 \theta^{1/3}}{2}, & a < r < \rho \\ 0, & r > \rho \end{cases}$$

where

$$(5.4) \quad \rho = \frac{a}{\theta^{1/3}}.$$

In order to compare W with W_0 , let us introduce the principal eigenvalue λ and a corresponding eigenfunction $\zeta(r)$ of

$$(5.5) \quad \begin{aligned} \Delta \zeta + \lambda \zeta &= 0 \quad \text{in } 0 < r < \rho, \\ \zeta_r(0) &= 0, \quad \zeta(\rho) = 0, \\ \zeta(r) &> 0 \quad \text{if } 0 < r < \rho. \end{aligned}$$

THEOREM 5.1. (i) For all $r > 0$, $t > 0$,

$$(5.6) \quad W(r, t) \leq W_0(r);$$

(ii) There exists a positive constant K such that

$$(5.7) \quad W_0(r) - W(r, t) \leq K \zeta(r) e^{-\lambda \theta t}$$

for all $0 < r < \rho$, $t > 0$.

Proof. The function $W(r, t)$, for each fixed t , satisfies the same variational inequality as W_0 but with $f - \theta$ replaced by

$$\widehat{f} \equiv f - \sqrt{2W_t + \theta^2} .$$

Since $W_t \geq 0$, $\widehat{f} \leq f - \theta$ and therefore, by a comparison theorem for variational inequalities (see [6; Chap. 1]), $W(r, t) \leq W_0(r)$.

To prove (5.7) set

$$Lv \equiv \frac{1}{\theta} v_t - \Delta v .$$

The function $v = W_0 - W$, in $\{0 < r < \rho, t > 0\}$, satisfies

$$\begin{aligned} Lv &= -\frac{1}{\theta} W_t - \Delta W_0 + \Delta W = -\frac{1}{\theta} W_t + f - \theta + \Delta W \\ &\leq -\frac{1}{\theta} W_t + f - \theta + \left(-f + \sqrt{2W_t + \theta^2}\right) \quad (\text{by (2.14)}) \\ &= -\left(\frac{1}{\theta} W_t + \theta\right) + \sqrt{2W_t + \theta^2} \leq 0 \end{aligned}$$

since $W_t \geq 0$.

On the boundary:

$$\zeta_r = v_r = 0 \quad , \quad r = 0, t > 0 ,$$

$$\zeta = v = 0 \quad , \quad r = \rho, t > 0$$

and, if K is large enough,

$$v(r, 0) = W_0(r) - W(r, 0) \leq K\zeta(r), \quad 0 < \rho < a .$$

Since

$$L[K\zeta(r)e^{-\lambda\theta t}] = 0 ,$$

we can apply the maximum principle to $v - K\zeta e^{-\lambda\theta t}$ to deduce the assertion (5.7).

COROLLARY 5.2. *The free boundary satisfies:*

$$(5.8) \quad s(t) \uparrow \frac{a}{\theta^{1/3}} \quad \text{as} \quad t \uparrow \infty .$$

§6. More general boundary conditions. In this section we replace the boundary condition (1.12) by the flux condition

$$-A \frac{\partial \varphi}{\partial r} = B(\varphi - \theta) \quad \text{for } r = r(a, t), \quad t > 0,$$

where θ is a constant, $0 < \theta < 1$, and A, B are smooth functions of (r, t) , with $A > 0, B \geq 0$. For simplicity we shall take A and B to be constants; see however Remark 6.3.

In view of (2.2), we can rewrite this flux condition in the form

$$(6.1) \quad \frac{Ar^2 \varphi}{a^2} \frac{\partial \varphi}{\partial r_0} + B\varphi = B\theta \quad \text{for } r_0 = a, \quad t > 0.$$

DEFINITION. If in problem (P1) we replace (1.12) by (6.1), then we refer to the new problem as problem (P1'). Similarly we define problem (P2') whereby (2.8) is replaced by

$$(6.2) \quad A \frac{\partial \psi}{\partial r} + B\psi = B\theta \quad \text{for } r = s(t), \quad t > 0.$$

THEOREM 6.1. *There exists a solution to problem (P1') having the same regularity as the solution established for problem (P1) in Theorem 3.1; furthermore, $\partial \varphi / \partial r_0 < 0$ if $0 < r_0 \leq a, t > 0$ and*

$$(6.3) \quad \varphi_{r_0} \in L^\infty((0, a) \times (0, \infty)), \quad \varphi \in C^\alpha([0, a] \times [0, \infty))$$

for some $\alpha \in (0, 1)$.

Proof. As in §3 we first consider the ε -approximate problem (with (1.12) replaced by (6.1)) for which we can prove existence of a solution φ_ε as well as the estimate

$$(6.4) \quad \theta \leq \varphi_\varepsilon(r_0, t) \leq 1.$$

We can choose the initial data for the ε -approximate problem such that $\zeta = \partial \varphi_\varepsilon / \partial r_0$ is continuous at $(a, 0)$ and, at the same time, it is nonpositive and bounded on $\{t = 0\}$ independently of ε . Applying the maximum principle to ζ (cf. (3.11)) we conclude that

$$(6.5) \quad -C \leq \zeta \leq 0$$

where C is a constant independent of ε .

From (2.9) and (6.4), (6.5) we deduce that $0 \leq \dot{s} \leq C$. Using this and (6.5) we can then derive (as in [10]; Chap. 10) Hölder estimate for ψ_ε , and hence also for φ_ε , in a neighborhood of $(a, 0)$, with Hölder exponent and coefficient independent of ε . finally, by following the arguments in the proof of Theorem 3.1 we can complete the proof of Theorem 6.1 as we let $\varepsilon \rightarrow 0$.

COROLLARY 6.2. *There exists a solution (ψ, s) to problem $(P2')$ with $\dot{s}(t) > 0$ if $t > 0$; furthermore,*

$$(6.6) \quad s \in C^{1+\alpha}[0, \infty) ,$$

$$(6.7) \quad s \in C^\infty(0, \infty) .$$

Indeed, (6.6) follows from (6.2), (2.9) and the Hölder continuity of ψ , whereas the proof of (6.7) is similar to the corresponding proof in the case of (1.12).

By a classical solution to problem $(P1')$ we mean a solution φ which satisfies (6.3). The next theorem establishes uniqueness for such solutions.

THEOREM 6.3. *There exists at most one classical solution to problem $(P1')$.*

Remark 6.1. Problem $(P1')$ cannot be recast as a variational inequality; consequently the uniqueness proof for problem $(P1)$ cannot be extended to the present case. Note however that the classical solution to problem $(P1')$ is more regular at $(a, 0)$ than the solution to problem $(P1)$ (the solution to $(P1)$ is discontinuous at $(a, 0)$). This additional regularity will enable us to work directly with the formulation $(P1)$ (something we are unable to do in the case of problem $(P1)$).

Proof. Suppose $\hat{\varphi}$ and φ are two solutions and set

$$\hat{u}(x, t) = \hat{\varphi}^2(r_0, t), \quad u(x, t) = \varphi^2(r_0, t), \quad x = r_0 .$$

Then both $\hat{u}$ and u satisfy

$$(6.8) \quad v_t = \frac{v^{3/2}}{x^2} \frac{\partial}{\partial x} \left(\frac{\sigma^4}{x^2} v_x \right) , \quad 0 < x < a, \quad t > 0 ,$$

$$(6.9) \quad v_x(0, t) = 0 , \quad t > 0$$

$$(6.10) \quad \frac{A}{2a^2} \sigma^2 v_x + Bv^{1/2} = B\theta , \quad x = a, \quad t > 0 ,$$

$$(6.11) \quad v(x, 0) = 1 , \quad 0 \leq x < a$$

with

$$\sigma^3 = \hat{r}^3 = \int_0^x \frac{3y^2}{\sqrt{\hat{u}(y, t)}} dy \quad \text{for } \hat{u} ,$$

$$\sigma^3 = r^3 = \int_0^x \frac{3y^2}{\sqrt{u(y, t)}} dy \quad \text{for } u .$$

Set $w = \hat{u} - u$.

LEMMA 6.4. *There holds:*

$$(6.12) \quad |\widehat{r}^4 - r^4| \leq Cx \int_0^x |w| 3y^2 dy .$$

Proof. Using the estimates

$$(6.13) \quad \begin{aligned} \theta^2 &\leq \widehat{u}, \quad u \leq 1 , \\ x^3 &\leq \widehat{r}^3, \quad r^3 \leq \frac{x^3}{\theta} , \end{aligned}$$

we have

$$\begin{aligned} |\widehat{r}^4 - r^4| &= |(\widehat{r}^3)^{4/3} - (r^3)^{4/3}| \\ &= \left| \int_0^1 \frac{4}{3} [r^3 + \eta(\widehat{r}^3 - r^3)]^{1/3} d\eta \right| \cdot |\widehat{r}^3 - r^3| \\ &\leq Cx \left| \int_0^x \left(\frac{1}{\sqrt{\widehat{u}}} - \frac{1}{\sqrt{u}} \right) 3y^2 dy \right| \leq Cx \int_0^x |\widehat{u} - u| 3y^2 dy . \end{aligned}$$

LEMMA 6.5. *For any $\varepsilon > 0$ there is a positive constant $C(\varepsilon)$ such that*

$$(6.14) \quad \int_0^T \int_0^a \frac{1}{x^4} \left(\int_0^x |w| 3y^2 dy \right)^2 dx dt \leq \varepsilon \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon) \int_0^T \int_0^a x^2 w^2 .$$

for any $0 < T < \infty$.

Proof. By Schwarz's inequality, the left-hand side of (6.14) is bounded by

$$\begin{aligned} &\int_0^T \int_0^a \frac{1}{x} \left(\int_0^x w^2 3y^2 dy \right) dx dt \\ &= \int_0^T \log x \cdot \left[\int_0^x w^2 3y^2 dy \right]_{x=0}^{x=a} dt \\ &\quad - \int_0^T \int_0^a \log x \cdot w^2 3x^2 dx dt \quad (\text{by integration by parts}) \end{aligned}$$

$$\begin{aligned}
&\leq 3|\log a| \int_0^T \int_0^a x^2 w^2 dx dt + \int_0^T \left[\left(\frac{x^3}{3} - x^3 \log x \right) w^2 \right]_{x=0}^{x=a} dt \\
&\quad + \int_0^T \int_0^a 2w_x w \left(x^3 \log x - \frac{x^3}{3} \right) dx dt \quad (\text{by integration by parts}).
\end{aligned}$$

Since $|x^3 \log x - x^3/3| \leq Cx^2$,

$$\int_0^T |w_x w| x^2 \leq \varepsilon \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon) \int_0^T \int_0^a w^2,$$

and

$$(6.15) \quad \int_0^T w^2(a, t) dt \leq \varepsilon \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon) \int_0^T \int_0^a w^2,$$

the assertion (6.14) follows.

Subtracting the equation (6.8) for u from the one for $\hat{u}$, then multiplying by w and integrating, we get

$$\begin{aligned}
(6.16) \quad &\int_0^T \int_0^a x^2 w_t w = \int_0^T \int_0^a \left[\hat{u}^{3/2} \frac{\partial}{\partial x} \left(\frac{\hat{r}^4}{x^2} \hat{u}_x \right) - u^{3/2} \frac{\partial}{\partial x} \left(\frac{r^4}{x^2} u_x \right) \right] w \\
&= \int_0^T \int_0^a \frac{\partial}{\partial x} \left[\frac{\hat{r}^4}{x^2} \hat{u}^{3/2} \hat{u}_x - \frac{r^4}{x^2} u^{3/2} u_x \right] w \\
&\quad - \int_0^T \int_0^a \left[\frac{\hat{r}^4}{x^2} \hat{u}_x \frac{3}{2} \sqrt{\hat{u}} \hat{u}_x - \frac{r^4}{x^2} u_x \frac{3}{2} \sqrt{u} u_x \right] w \\
&= \int_0^T \left[\left(\frac{\hat{r}^4}{x^2} \hat{u}^{3/2} \hat{u}_x - \frac{r^4}{x^2} u^{3/2} u_x \right) w \right]_{x=0}^{x=a} - \int_0^T \int_0^a \left(\frac{\hat{r}^4}{x^2} \hat{u}^{3/2} \hat{u}_x - \frac{r^4}{x^2} u^{3/2} u_x \right) w_x \\
&\quad - \frac{3}{2} \int_0^T \int_0^a \left(\frac{\hat{r}^4}{x^2} \sqrt{\hat{u}} \hat{u}_x^2 - \frac{r^4}{x^2} \sqrt{u} u_x^2 \right) w \equiv I_1 - I_2 - \frac{3}{2} I_3.
\end{aligned}$$

From (6.10) we deduce that

$$(6.17) \quad (\widehat{r}^2 \widehat{u}_x - r^2 u_x)(\widehat{u} - u) \leq 0 \quad \text{at} \quad x = a .$$

We now proceed to estimate the terms I_j . First

$$\begin{aligned} I_1 &= \frac{1}{a^2} \int_0^T \left[\left(\widehat{r}^4 \widehat{u}^{3/2} \widehat{u}_x - r^4 u^{3/2} u_x \right) w \right] (a, t) dt \\ &= \frac{1}{a^2} \int_0^T \left[\widehat{r}^2 \widehat{u}^{3/2} (\widehat{r}^2 \widehat{u}_x - r^2 u_x) w \right] (a, t) dt + \frac{1}{a^2} \int_0^T [\widehat{r}^2 (\widehat{u}^{3/2} - u^{3/2}) r^2 u_x w] (a, t) dt \\ &\quad + \frac{1}{a^2} \int_0^T \left[(\widehat{r}^2 - r^2) u^{3/2} r^2 u_x w \right] (a, t) . \end{aligned}$$

The first term on the right-hand side is ≤ 0 by (6.17). The second term is bounded by $C \int_0^T w^2(a, t) dt$. The third term can be estimated (using Lemma 6.4) by

$$C \int_0^T \int_0^a |w| 3y^2 |w(a, t)| dy dt ,$$

which is bounded by

$$C \int_0^T \int_0^a x^2 w^2 + C \int_0^T w^2(a, t) .$$

Recalling (6.15) we conclude that

$$I_1 \leq \varepsilon \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon) \int_0^T \int_0^a x^2 w^2 .$$

Next, we can write

$$\begin{aligned} -I_2 &= - \int_0^T \int_0^a \frac{\widehat{r}^4}{x^2} \widehat{u}^{3/2} w_x^2 - \int_0^T \int_0^a \left(\frac{\widehat{r}^4}{x^2} \widehat{u}^{3/2} - \frac{r^4}{x^2} u^{3/2} \right) u_x w_x \\ &\leq -\theta^3 \int_0^T \int_0^a x^2 w_x^2 - \int_0^T \int_0^a \frac{\widehat{r}^4}{x^2} (\widehat{u}^{3/2} - u^{3/2}) u_x w_x \\ &\quad - \int_0^T \int_0^a \left(\frac{\widehat{r}^4}{x^2} - \frac{r^4}{x^2} \right) u^{3/2} u_x w_x \end{aligned}$$

where (6.13) was used. The second integral on the right-hand side is estimated, by the mean value theorem, by

$$C \int_0^T \int_0^a x^2 |w w_x| ,$$

and the third integral is estimated, using Lemma 6.4, by

$$\int_0^T \int_0^a \frac{1}{x^2} \cdot C x \left(\int_0^x |w| 3y^2 dy \right) \cdot |u_x| |w_x| .$$

Hence

$$\begin{aligned} -I_2 &\leq -\theta^3 \int_0^T \int_0^a x^2 w_x^2 + \varepsilon' \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon') \int_0^T \int_0^a x^2 w^2 \\ &\quad + C(\varepsilon') \int_0^T \int_0^a \frac{1}{x^4} \left(\int_0^x |w| 3y^2 dy \right)^2 , \end{aligned}$$

for any small $\varepsilon' > 0$.

Applying Lemma 6.5 with $\varepsilon \ll 1/C(\varepsilon')$, we conclude that

$$-I_2 \leq -\frac{1}{2}\theta^3 \int_0^T \int_0^a x^2 w_x^2 + C \int_0^T \int_0^a x^2 w^2$$

where C is a large constant (depending on θ).

Finally,

$$\begin{aligned} -I_3 &= \int_0^T \int_0^a \frac{\widehat{r}^4}{x^2} \sqrt{\widehat{u}} (\widehat{u}_x^2 - u_x^2) w + \int_0^T \int_0^a \frac{\widehat{r}^4}{x^2} (\sqrt{\widehat{u}} - \sqrt{u}) u_x^2 w \\ &\quad + \int_0^T \int_0^a \left(\frac{\widehat{r}^4}{x^2} - \frac{r^4}{x^2} \right) \sqrt{u} u_x^2 w . \end{aligned}$$

Using the relations

$$\widehat{u}_x^2 - u_x^2 = (\widehat{u}_x + u_x) w_x , \quad \sqrt{\widehat{u}} - \sqrt{u} = \frac{w}{\sqrt{\widehat{u}} + \sqrt{u}}$$

and Lemma 6.4, we get

$$-I_3 \leq \varepsilon \int_0^T \int_0^a x^2 w_x^2 + C(\varepsilon) \int_0^T \int_0^a x^2 w^2 + C \int_0^T \int_0^a \frac{1}{x^4} \left(\int_0^x |w| 3y^2 dy \right)^2 .$$

Applying Lemma 6.5 and then combining the resulting estimate for $-I_3$ with those for I_1 and $-I_2$, we deduce from (6.16) that

$$\frac{d}{dt} \int_0^T \int_0^a x^2 w^2 = \int_0^T \int_0^a x^2 w_t w \leq C(\varepsilon) \int_0^T \int_0^a x^2 w^2$$

provided ε is small enough. It follows that $w \equiv 0$, i.e., $\hat{u} \equiv u$.

We finally consider the asymptotic behavior of the solution, assuming that $B > 0$ (if $B = 0$ then $\varphi \equiv 1$).

Let λ and ζ be the principal eigenvalue and eigenfunction for the eigenvalue problem

$$(6.18) \quad \begin{aligned} \theta^{5/3} \left(\zeta_{xx} + \frac{2}{x} \zeta_x \right) + \lambda \zeta &= 0, \quad 0 < x < a, \\ \zeta'(0) &= 0, \quad \frac{A}{2\theta^{2/3}} \zeta'(a) + \frac{B}{1+\theta} \zeta(a) = 0, \end{aligned}$$

with $\zeta(x) > 0$ if $0 < x < a$ and $\zeta(a) = 1$; recall that $\lambda > 0$. By the maximum principle ($\zeta_{xx} + \frac{2}{x} \zeta_x < 0$), the minimum of ζ is positive and it is attained at $x = a$, so that

$$(6.19) \quad \zeta(x) \geq 1 .$$

THEOREM 6.6. *The solution φ of problem (P1') satisfies*

$$(6.20) \quad 0 \leq \varphi - \theta \leq \frac{\zeta(x)}{2\theta} e^{-\lambda\theta^{4/3}t}$$

and the free boundary $x = s(t)$ satisfies (5.8).

Proof. Set $w = \varphi^2 - \theta^2$ (w is $\frac{1}{2}$ times the function w defined in (2.11)). We wish to compare w with

$$w_0 = \zeta(x) e^{-\lambda\theta^{4/3}t} .$$

For this purpose we introduce the operator

$$Lv = v_t - \frac{r^4}{x^4} (w + \theta^2)^{3/2} \left(v_{xx} + \frac{2}{x} v_x \right) .$$

As seen from (3.14), (3.15),

$$Lw = \frac{4r}{x^2} (w + \theta^2)^{3/2} \left(\frac{1}{\sqrt{w + \theta^2}} - \frac{r^3}{x^3} \right) w_x .$$

From (1.4) and the inequality $\varphi_x \leq 0$ we get

$$r^3 \leq \int_0^x \frac{3y^2 dy}{\varphi(x, t)} = \frac{x^3}{\sqrt{w + \theta^2}} .$$

Since also $w_x = 2\varphi\varphi_x \leq 0$, it follows that $Lw \leq 0$

On the other hand

$$\begin{aligned} Lw_0 &= \lambda w_0 \left[-\theta^{4/3} + \frac{r^4}{x^4} (w + \theta^2)^{3/2} \frac{1}{\theta^{5/3}} \right] \\ &\geq \lambda w_0 \left[-\theta^{4/3} + \theta^3 \theta^{-5/3} \right] = 0 . \end{aligned}$$

On $\{x = a\}$, (6.10) implies

$$\frac{A}{2a^2} r^2 w_x + B \sqrt{w + \theta^2} = B\theta .$$

Since $r^3 \leq a^3/\theta$ (by (6.13)) and $w > 0, w_x < 0$, we get

$$\frac{A}{2\theta^{2/3}} w_x + \frac{B}{1 + \theta} w < 0 \quad \text{at } x = a .$$

Also

$$\frac{A}{2\theta^{2/3}} w_{0,x} + \frac{B}{1 + \theta} w_0 = 0 \quad \text{at } x = a .$$

Next, on the boundary $\{x = 0\}$, $w_x = w_{0,x} = 0$.

Finally, at $t = 0$, $w = 1 - \theta^2 < 1 \leq \zeta(x) = w_0$. Applying the maximum principle to $w_0 - w$ we get $w_0 \geq w$, which implies the right-hand inequality in (6.20).

Finally, (5.8) follows from (1.4) and (6.20).

Remark 6.2. One can establish Corollary 5.2 by the method of proof of Theorem 6.6.

Remark 6.3. The results of this section can be extended with minor modifications to the case where A and B are functions of (r, t) .

§7. **The case $\theta = 0$.** For problem (P2) the case $\theta = 0$ is simply the Cauchy problem for the porous medium equation $2\psi_t = \Delta\psi^2$. This problem has been studied extensively in the literature (see [6] and the references therein).

Consider next problems (P1') and (P2') for small $\theta > 0$ and denote the solutions by φ_θ and (ψ_θ, s_θ) . By (6.2), (2.9),

$$A\dot{s}_\theta = B(\psi_\theta - \theta) \leq B(1 - \theta) .$$

It follows that

$$(7.1) \quad 0 \leq \dot{s}_\theta(t) \leq C , \quad a \leq s_\theta(t) \leq a + Ct \quad \text{for all } t > 0 ,$$

where $C = B(1 - \theta)/A$.

LEMMA 7.1. *For any $T > 0$ there exists an $\varepsilon > 0$ such that*

$$(7.2) \quad \psi_\theta(s_\theta(t), t) \geq \varepsilon^2 \quad \text{if } 0 < t < T , \quad 0 < \theta < 1 .$$

Proof. Let $x_0 = s_\theta(t_0)$, $0 < t_0 < T$, and introduce the function

$$W = \varepsilon(x_0 - r)^m + \varepsilon^2 \quad \text{for } \frac{a}{2} < r < s_\theta(t), \quad 0 < t < t_0 .$$

where m is a positive integer. Then

$$W_t - \frac{1}{2} \Delta W^2 = -\frac{\varepsilon^2}{2} \Delta(x_0 - r)^{2m} - \frac{2\varepsilon^3}{2} \Delta(x_0 - r)^m .$$

Clearly

$$\Delta(x_0 - r)^{2m} = 2m(x_0 - r)^{2m-2} \left[(2m-1) - \frac{2}{r}(x_0 - r) \right] > 0$$

if m is large enough, independently of φ (by (7.1)), and the same holds for $\Delta(x_0 - x)^m$.

Hence W is a subsolution: $W_t - \frac{1}{2} \Delta W^2 \leq 0$.

On $r = s_\theta(t)$, $0 < t < t_0$,

$$AW_r + BW = (x_0 - r)^{m-1} \varepsilon [-Am + B(x_0 - r)] + B\varepsilon^2 < 0$$

if m is large (depending on T) and ε is small, depending on m and T (both m and ε are independent of θ).

On $t = 0$ we have $W < 1 = \psi_\theta(r, 0)$ if $\frac{a}{2} < r < a$ provided ε is small enough depending on m .

Finally, in order to compare ψ_θ with W on $r = \frac{a}{2}$ we recall [1] that there exist solutions of the porous medium equation of the form $f(r)g(t)$ for $0 < r < a$, $t > 0$ with $f(r) > 0$, $g(t) > 0$ if $0 \leq r < a$, $t \geq 0$ such that $f(a) = 0$. By comparison, $\psi_\theta(r, t) \geq f(r)g(t)$. Consequently

$$\psi_\theta\left(\frac{a}{2}, t\right) > W\left(\frac{a}{2}\right)$$

if ε is sufficiently small (depending on m and T).

We now apply comparison to deduce that $\psi_\theta > W$ for all $\frac{a}{2} \leq r \leq s_\theta(t)$, $0 \leq t \leq t_0$ and, in particular, (7.2) follows at $t = t_0$.

From Lemma 7.1 we see that equations (2.5) and (1.9) are uniformly parabolic for $0 < \theta < 1$, $0 \leq t \leq T$, for any $T > 0$. Hence the estimates established for $\varphi_\theta, s_\theta, \psi_\theta$ hold uniformly with respect to θ . We can therefore take sequences of solutions which converge, as $\theta \rightarrow 0$, to solutions φ and (ψ, s) of (P1') and (P2') with $\theta = 0$. The proof of Theorem 6.3 extends to the present case. We summarize:

THEOREM 7.1. *For $\theta = 0$, there exists a solution φ to problem (P1') and a solution (ψ, s) to problem (P2'). The solutions have the same regularity as the solutions in the case $\theta > 0$, and φ, ψ are uniformly positive for $0 \leq r \leq s(t)$, $0 \leq t \leq T$, for any $T > 0$. Finally, the classical solutions to problems (P1') and (P2') are unique*

Remark 7.1. It is clear that $s(t) \rightarrow \infty$ if $t \rightarrow \infty$. Furthermore, from the conservation law

$$(7.3) \quad \int_0^{s(t)} r^2 \psi(r, t) dr = \frac{a^3}{3}$$

(which follows by integrating (2.5)) and the relations $\psi_r < 0$, $\psi(s(t), t) = \frac{A}{B} \dot{s}$, it follows that

$$s(t) \leq \left(\frac{Ba^3}{A} t + a^4 \right)^{1/4}.$$

One can formally deduce a more precise estimate

$$(7.3) \quad s(t) \sim \gamma t^{1/5} \quad \text{as } t \rightarrow \infty$$

where γ is a positive constant which can be expressed in terms of a . Indeed, the functions $f = f_\lambda$, $\sigma = \sigma_\lambda$ with

$$f_\lambda(r, t) = \lambda^3 \psi(\lambda r, \lambda^5 t), \quad \sigma_\lambda(t) = \frac{\sigma(\lambda^5 t)}{\lambda} \quad (\lambda \gg 1)$$

solve the system

$$f_t = \frac{1}{2} \Delta f^2 \quad \text{if } 0 < r < \sigma(t), \quad t > 0 ,$$

$$f_r(0, t) = 0 , \quad t > 0 ,$$

$$f(\sigma(t), t) = \frac{A}{B} \frac{\dot{\sigma}(t)}{\lambda} , \quad f_r(\sigma(t), t) = -\dot{\sigma}(t), \quad t > 0 ,$$

$$f(r, 0) = \lambda^3 \quad \text{if } 0 < r < \frac{a}{\lambda} ,$$

and

$$\int_0^{\sigma(t)} r^2 f(r, t) dr = \frac{a^3}{3} .$$

If $\lambda \rightarrow \infty$ then the initial data converge to $\frac{a^3}{3}$ times the Dirac function whereas $f(\sigma(t), t) \rightarrow 0$. The solution of the limit problem is (see [6; p. 603])

$$\frac{b}{t^{3/5}} \left(1 - \frac{r^2}{\gamma^2 t^{2/5}} \right)^+$$

where b and γ are determined by

$$b = 10\gamma^2 , \quad 2\gamma^3 b = 5a^3$$

Hence

$$\sigma_\lambda(t) = \frac{\sigma(\lambda^5 t)}{\lambda} \rightarrow \gamma t^{1/5} \quad \text{as } \lambda \rightarrow \infty ,$$

and (7.3) follows.

Remark 7.2. The results of this paper can be extended without difficulty to the case where the diffusion coefficient $D(\phi_R)$ in (1.7) is a function of ϕ_R ; for example, $D(\phi_R) = \phi_R^m$, $m > 0$.

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