**HIGHER DIMENSIONAL BLOW UP FOR
SEMILINEAR PARABOLIC EQUATIONS**

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HIGHER DIMENSIONAL BLOW UP FOR SEMILINEAR PARABOLIC EQUATIONS*

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Abstract. We consider the Cauchy problem

$$\begin{aligned} (1) \quad & u_t = \Delta u + u^p \quad \text{if } x \in \mathbb{R}^N, \quad t > 0, \\ (2) \quad & u(x, 0) = u_o(x) \quad \text{if } x \in \mathbb{R}^N, \end{aligned}$$

where $N \geq 1$, $p > 1$ and $u_o(x)$ is a continuous, nonnegative and bounded function. It is well known that the solution $u(x, t)$ of (1), (2) may blow up in a finite time $T < +\infty$. We shall be concerned here with the asymptotic behaviour of $u(x, t)$ as blow up is approached. In particular the final profile of $u(x, T)$ near blow up points is studied, and the fact that the blow up set has zero Lebesgue measure is proved.

Key words. Blow up, asymptotic behaviour, semilinear parabolic equations

AMS(MOS) subject classifications. 35B40, 35K55, 35K57

1. Introduction. This paper deals with the following problem:

$$\begin{aligned} (1.1) \quad & u_t = \Delta u + u^p \quad \text{when } x \in \mathbb{R}^N (N \geq 1), t > 0, p > 1, \\ (1.2) \quad & u(x, 0) = u_o(x) \quad \text{when } x \in \mathbb{R}^N \end{aligned}$$

where $u_o(x)$ is continuous, nonnegative and bounded. It is well known that solutions of (1.1), (1.2) may develop singularities in finite time. In particular, we shall say that $u(x, t)$ blows up in a finite time T if $u(x, t)$ satisfies (1.1), (1.2) in $S_T = \mathbb{R}^N \times (0, T)$, and

$$\limsup_{t \uparrow T} \left(\sup_{x \in \mathbb{R}^N} u(x, t) \right) = +\infty$$

A point x_o is then called a blow up point if there exist sequences $\{x_n\}$, $\{t_n\}$ such that $\lim_{n \rightarrow \infty} x_n = x_o$, $\lim_{n \rightarrow \infty} t_n = T$ and $\lim_{n \rightarrow \infty} u(x_n, t_n) = \infty$. The set S consisting of blow up points is called the blow up set. We refer to [BBE], [B], [FM], [L], [W] for recent results in blow up problems.

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Throughout this paper, we shall assume that

$$\begin{aligned}
(1.3a) \quad & u(x, t) \text{ blows up at } x = 0, t = T < +\infty, \\
(1.3b) \quad & u(x, t) \leq M(T - t)^{-\frac{1}{p-1}} \text{ for any } x \in \mathbf{R}^N, t < T, \text{ and some} \\
& \text{constant } M > 0, \\
(1.3c) \quad & \lim_{t \uparrow T} (T - t)^{\frac{1}{p-1}} u(x(T - t)^{1/2}, t) = (p - 1)^{-\frac{1}{p-1}} \text{ uniformly on} \\
& \text{bounded sets } |x| \leq R \text{ with } R > 0.
\end{aligned}$$

We remark that conditions (1.3) hold under the present assumptions on the initial value $u_o(x)$ if $p < \frac{N+2}{N-2}$ (cf. [GP], [GK1], [GK2], [GK3]). Following [GP], [GK1], we now introduce similarity variables as follows

$$\begin{aligned}
(1.4a) \quad & u(x, t) = (T - t)^{-\frac{1}{p-1}} \phi \left(\frac{x}{(T - t)^{1/2}}, -\log(T - t) \right) \\
(1.4b) \quad & y = \frac{x}{(T - t)^{1/2}}, \quad \tau = -\log(T - t)
\end{aligned}$$

We define $\psi(y, \tau)$ by

$$(1.5a) \quad \phi(y, \tau) = (p - 1)^{-\frac{1}{p-1}} + \psi(y, \tau)$$

so that $\psi(y, \tau)$ satisfies

$$(1.5b) \quad \frac{d\psi}{d\tau} = \Delta\psi - \frac{1}{2}y \cdot \nabla\psi + \psi + f(\psi) \equiv A\psi + f(\psi)$$

where $f(\psi) = \left((p - 1)^{-\frac{1}{p-1}} + \psi \right)^p - \frac{p}{p-1}\psi - (p - 1)^{-\frac{p}{p-1}}$. Here and henceforth, in the symbol $y \cdot \nabla$, differentiation is understood to be performed with respect to the explicit space variable appearing to the left of operator ∇ . As in [HV1] (where problem (1.1), (1.2) with $N = 1$ was considered), the particular form of operator $A\psi$ above suggests our choice of function spaces: For $k \geq 1$, $1 < q < +\infty$, we shall consider the weighted spaces

$$\begin{aligned}
L_w^q(\mathbf{R}^N) &= \left\{ f \in L_{\text{loc}}^q(\mathbf{R}^N) : \int_{\mathbf{R}^N} |f(s)|^q e^{-s^2/4} ds < +\infty \right\} \\
H_w^k(\mathbf{R}^N) &= \{ f \in L_w^2(\mathbf{R}^N) : D^\alpha f \in L_w^2(\mathbf{R}^N) \text{ for } \alpha = (\alpha_1, \dots, \alpha_N) \text{ such that } |\alpha| = \alpha_1 + \dots + \alpha_N \leq k \}
\end{aligned}$$

where we are using the standard notation $D^\alpha f = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}} f$. Norms in $L_w^q(\mathbf{R}^N)$, $H_w^k(\mathbf{R}^N)$ are defined in a natural way. In particular,

$$\|f\|_{q,w}^q = \int_{\mathbf{R}^N} |f(s)|^q e^{-s^2/4} ds$$

Since the L_w^2 -norm will be repeatedly used in the sequel, we shall write it in the abridged form $\| \cdot \|_{2,w} \equiv \| \cdot \|$. It is well known that operator $A\psi$ in (1.5) is self-adjoint in $L_w^2(\mathbb{R}^N)$ with domain $H_w^2(\mathbb{R}^N)$ and eigenfunctions $H_\alpha(y)$ given by $H_\alpha(y) = H_{\alpha_1}(y_1) \dots H_{\alpha_N}(y_N)$, where $\alpha = (\alpha_1, \dots, \alpha_N)$, α_i is a nonnegative integer for $1 \leq i \leq N$, $y = (y_1, \dots, y_N)$, $H_m(y_i) = c_m \tilde{H}_m(\frac{y_i}{2})$, $c_m = (2^{m/2}(4\pi)^{1/4}(n!)^{1/2})^{-1}$ and $\tilde{H}_m(s)$ is the standard m^{th} -Hermite polynomial. The corresponding eigenvalues are given by $\lambda_\alpha = 1 - \frac{|\alpha|}{2}$, where $|\alpha| = \alpha_1 + \dots + \alpha_N$.

We are concerned here with the asymptotic behaviour near blow up of solutions of (1.1), (1.2) which satisfy (1.3). Before proceeding to describe our results, it is convenient to recall the following theorem which was proved in [V].

THEOREM A. *Let $u(x, t)$ be the solution of (1.1), (1.2) and assume that (1.3) holds. Let $\psi(y, \tau)$ be the function given in (1.5). If $\psi(y, \tau) \not\equiv 0$ for some $\tau > 0$, then the following alternative holds: Either there exists an orthogonal transformation of the coordinate axes such that, denoting still by y the new coordinates,*

$$(1.6a) \quad \psi(y, \tau) = -\frac{C_p}{\tau} \sum_{k=1}^{\ell} H_2(y_k) + o\left(\frac{1}{\tau}\right) \text{ as } \tau \rightarrow \infty.$$

$$\text{where } 1 \leq \ell \leq N, C_p = \frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}p} \text{ and } y = (y_1, \dots, y_N),$$

or there exist an even number m , $m \geq 4$, and constants C_α not all zero, such that

(1.6b)

$$\psi(y, \tau) = -e^{(1-\frac{m}{2})\tau} \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha H_\alpha(y) + o\left(e^{(1-\frac{m}{2})\tau}\right) \text{ as } \tau \rightarrow \infty,$$

$$\text{where the homogeneous multilinear form } B(x) = \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha x^\alpha \text{ is}$$

$$\text{nonnegative. Here } \bar{c}_\alpha = c_{\alpha_1} \dots c_{\alpha_N}, \text{ and } x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}.$$

In (1.6a) and (1.6b), convergence takes place in $H_w^1(\mathbb{R}^N)$ as well as in $C_{\text{loc}}^{k,\gamma}(\mathbb{R}^N)$ for any $k \geq 0$ and $\gamma \in (0, 1)$.

Theorem A was previously proved in [HV1], [HV2] for the case $N = 1$ (cf. also [FK]). On the other hand, related results were independently and simultaneously obtained in [FL], where the authors proved that, under our present assumptions, (1.6a) must hold unless there is some exponential decay of $\psi(\cdot, \tau)$ as $\tau \rightarrow \infty$.

For $N = 1$ it was proved in [HV2], [HV3] that the behaviours corresponding to (1.6a), (1.6b) determine the asymptotic profile of the singularities appearing at $t = T$. Furthermore, the blow-up set was shown to consist only of isolated points (cf. [CM] for earlier results in this direction). In deriving these facts, use was made of a technique consisting of

extending the asymptotics obtained in the one-dimensional version of Theorem A to larger regions which, for instance, contain curves of the form $|x| \approx ((T-t)|\log(T-t)|)^{1/2}$. This particular scaling was suggested by the analysis previously performed in [GHV].

The main goal of this paper is to develop a new technique for extending the results in [HV1], [HV2], [HV3] to the higher dimensional case. As a first step in such direction, we shall prove:

THEOREM 1. *Assume that the hypotheses of Theorem A hold. Then:*

(a) *If (1.6a) holds, then in the same coordinate system*

$$(1.7a) \quad \lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} u(\xi(T-t)^{1/2} |\log(T-t)|^{1/2}, t) = \left((p-1) + \bar{C}_p \sum_{k=1}^{\ell} \xi_k^2 \right)^{-\frac{1}{p-1}}$$

uniformly on sets $|\xi| \leq R$ where $\bar{C}_p = \frac{(p-1)^2}{4p}$.

(b) *If (1.6b) holds, then*

$$(1.7b) \quad \lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} u(\xi(T-t)^{1/m}, t) = \left((p-1) + K_p \sum_{|\alpha|=m} C_\alpha \bar{c}_\alpha \xi^\alpha \right)^{-\frac{1}{p-1}}$$

uniformly on sets $|\xi| \leq R$, where $K_p = (p-1)^{\frac{p}{p-1}}$ and $C_\alpha, \bar{c}_\alpha$ are as in Theorem A.

We next describe the asymptotic profile at the blow up time $t = T$, when blow up points are approached from outside of some cones

THEOREM 2. *Assume that the hypotheses of Theorem 1 are satisfied. Then*

i) *If (1.6a) is satisfied, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$(1.8a) \quad S \cap B_\delta(0) \subset \Omega_\varepsilon \equiv \left\{ x \in \mathbb{R}^N : \sum_{k=1}^{\ell} x_k^2 \leq \varepsilon \sum_{k=1}^N x_k^2 \right\}.$$

Moreover,

$$(1.8b) \quad \lim_{\substack{|x| \rightarrow 0 \\ x \in \mathbb{R}^N - \Omega_\varepsilon}} \left(\frac{|x|^2}{|\log|x||} \right)^{\frac{1}{p-1}} u(|x|\vec{n}, T) = \left(C_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}}$$

uniformly for any fixed $\varepsilon > 0$, where C_p is given in (1.7a), and $\vec{n} = (n_1, \dots, n_N) \in S^{N-1} \cap (\mathbb{R}^N - \Omega_\varepsilon)$, S^{N-1} denoting the unit sphere in $\mathbb{R}^N$.

ii) *If (1.6b) is satisfied, for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$(1.9a) \quad S \cap B_\delta(0) \subset \tilde{\Omega}_\varepsilon \equiv \left\{ x \in \mathbb{R}^N : \sum_{|\alpha|=m} C_\alpha x^\alpha \leq \varepsilon |x|^\alpha \right\}.$$

Moreover there holds

$$\lim_{\substack{|x| \rightarrow 0 \\ x \in \mathbf{R}^N - \tilde{\Omega}_\epsilon}} |x|^{\frac{m}{p-1}} u(|x|\vec{n}, T) = \left(K_p \sum_{|\alpha|=m} C_\alpha(\vec{n})^\alpha \right)^{-\frac{1}{p-1}}$$

uniformly for any fixed $\epsilon > 0$, where K_p is given in (1.7b).

Notice that the sets $\Omega_\epsilon, \tilde{\Omega}_\epsilon$ may be empty, in which case (1.8), (1.9) yield single point blow up.

We shall use (1.8a), (1.9a) to prove:

THEOREM 3. *Assume that the hypotheses of Theorem 1 are satisfied. Then*

$$L^N(S) = 0$$

where L^N denotes the standard N -dimensional Lebesgue measure.

Theorem 1 will be proved in Section 2 below. The proofs of Theorems 2 and 3 are given in Section 3.

2. Analysis of the intermediate region. In this section we shall prove Theorem 1. To this end, we will make use of some regularizing effects in suitable weighted spaces, which are of independent interest. For $r \geq 0$, $1 < q < \infty$ and $\psi \in L^q_{\text{loc}}(\mathbf{R}^N)$, we define

$$(2.1) \quad L^q_r(\psi) \equiv \sup_{|\xi| \leq r} \left(\int_{\mathbf{R}^N} |\psi(y)|^q \exp\left(-\frac{(y-\xi)^2}{4}\right) dy \right)^{1/q}$$

It is readily seen that, for fixed r and q , L^q_r satisfies the norm properties. We next consider the linear operator

$$(2.2) \quad A\psi \equiv \Delta\psi - \frac{1}{2}y \cdot \nabla\psi + \psi$$

defined in $L^2_w(\mathbf{R}^N)$ with domain $H^2_w(\mathbf{R}^N)$, which was introduced in the previous Section. We shall denote the corresponding semigroup by $S(\tau) \equiv S_A(\tau)$.

A key point in our approach is the following linear regularizing effect, that may be considered as a generalization of Proposition 2.1 in [HV1].

PROPOSITION 2.1. *Assume that $1 < q, \beta < +\infty$ and $r, \tilde{r} \geq 0$. Set $\beta' = \frac{\beta}{\beta-1}$. Then, for any $\tau > 0$ and any ψ such that $L^q_r(\psi) < +\infty$.*

$$(2.3) \quad L^q_r(S(\tau)\psi) \leq \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{N/2}} \cdot \left(\frac{4\pi\beta(1-e^{-\tau})}{\beta'(\beta-(1-e^{-\tau}))} \right)^{\frac{N}{2\beta'}} \cdot \left(\frac{4\pi(\beta-(1-e^{-\tau}))}{(\beta-1)-(q-1)e^{-\tau}} \right)^{\frac{N}{2q}} \exp\left(\frac{e^{-\tau}(r-\tilde{r}e^{\tau/2})_+^2}{4((\beta-1)-(q-1)e^{-\tau})} \right) L^\beta_{\tilde{r}}(\psi)$$

where $s_+ = \max\{s, 0\}$.

Informally stated, (2.3) estimates in a rather precise form the effect of the convective term $-\frac{y \cdot \nabla}{2}$ on (2.2).

Proof. As has been observed in [HV1] (and can be readily checked), the action of the semigroup $S(\tau) \equiv S_A(\tau)$ is described by

$$S(\tau)\psi(y) = \frac{e^\tau}{(4\pi(1 - e^{-\tau}))^{N/2}} \int_{\mathbf{R}^N} \exp\left(-\frac{ye^{-\frac{\tau}{2}} - \lambda)^2}{4(1 - e^{-\tau})}\right) \psi(\lambda) d\lambda$$

Fix now $\xi \in \mathbf{R}^N$ such that $|\xi| \leq r$ and proceed to estimate the following integral:

$$\begin{aligned} I &\equiv \int_{\mathbf{R}^N} |S(\tau)\psi(y)|^q \exp\left(-\frac{(y - \xi)^2}{4}\right) dy \\ &\leq \frac{e^{q\tau}}{(4\pi(1 - e^{-\tau}))^{Nq/2}} \int_{\mathbf{R}^N} \exp\left(-\frac{(y - \xi)^2}{4}\right) \cdot \\ &\quad \left(\int_{\mathbf{R}^N} \exp\left(-\frac{(ye^{-\frac{\tau}{2}} - \lambda)^2}{4(1 - e^{-\tau})}\right) e^{\frac{(\lambda - \mu)^2}{4\beta}} |\psi(y)| e^{-\frac{(\lambda - \mu)^2}{4\beta}} d\lambda \right)^q dy \end{aligned}$$

where $\mu \in \mathbf{R}^N$, $|\mu| \leq \tilde{r}$, but is otherwise arbitrary. Using Hölder's inequality as well as the identity

$$\begin{aligned} &\frac{\beta'}{4} \left(\frac{(\lambda - \mu)^2}{\beta} - \frac{(ye^{-\frac{\tau}{2}} - \lambda)^2}{1 - e^{-\tau}} \right) = \\ &-\frac{\beta'(\beta - (1 - e^{-\tau}))}{4\beta(1 - e^{-\tau})} \left((\lambda - \mu) - \frac{\beta(ye^{-\tau} - \mu)}{(\beta - (1 - e^{-\tau}))} \right)^2 + \frac{\beta'e^{-\tau}(y - \mu e^{\frac{\tau}{2}})^2}{4(\beta - (1 - e^{-\tau}))} \end{aligned}$$

we obtain that

$$\begin{aligned}
I &\leq \frac{e^{q\tau}}{(4\pi(1-e^{-\tau}))^{Nq/2}} \int_{\mathbf{R}^N} \exp\left(-\frac{(y-\xi)^2}{4}\right) \cdot \left(\int_{\mathbf{R}^N} |\psi(\lambda)|^\beta \exp\left(-\frac{(\lambda-\mu)^2}{4}\right) d\lambda \right)^{q/\beta} \\
&\quad \left(\int_{\mathbf{R}^N} \exp\left(-\frac{\beta'(ye^{-\frac{\tau}{2}}-\lambda)^2}{4(1-e^{-\tau})} + \frac{\beta'(\lambda-\mu)^2}{\beta \cdot 4}\right) d\lambda \right)^{q/\beta'} \\
&\leq \frac{e^{q\tau}}{(4\pi(1-e^{-\tau}))^{Nq/2}} (L_{\tilde{r}}^\beta(\psi))^q \int_{\mathbf{R}^N} \exp\left(-\frac{(y-\xi)^2}{4}\right) \\
&\quad \left(\int_{\mathbf{R}^N} \exp\left(-\frac{\beta'(\beta-(1-e^{-\tau}))}{4\beta(1-e^{-\tau})} \left((\lambda-\mu) - \frac{\beta(ye^{-\tau}-\mu)}{(\beta-(1-e^{-\tau}))}\right)^2 + \frac{\beta'e^{-\tau}(y-\mu e^{\tau/2})^2}{4(\beta-(1-e^{-\tau}))}\right) d\lambda \right)^{1/\beta'} dy \\
&= \frac{e^{q\tau}}{(4\pi(1-e^{-\tau}))^{Nq/2}} (L_{\tilde{r}}^\beta(\psi))^q \left(\frac{4\pi\beta(1-e^{-\tau})}{\beta'(\beta-(1-e^{-\tau}))} \right)^{\frac{Nq}{2\beta'}} \\
&\quad \int_{\mathbf{R}^N} \exp\left(-\frac{(y-\xi)^2}{4} + \frac{qe^{-\tau}(y-\mu e^{\tau/2})^2}{4(\beta-(1-e^{-\tau}))}\right) dy.
\end{aligned}$$

Taking now the infimum in $\mu \in \mathbf{R}^N$ with $|\mu| \leq \tilde{r}$, and then the supremum in $\xi \in \mathbf{R}^N$ with $|\xi| \leq r$, we arrive at

$$L_r^q(S(\tau)\psi) \leq \frac{e^\tau}{(4\pi(1-e^{-\tau}))^{N/2}} \left(\frac{4\pi\beta(1-e^{-\tau})}{\beta'(\beta-(1-e^{-\tau}))} \right)^{\frac{N}{2\beta'}} A(q, \beta, r, \tilde{r})(L_{\tilde{r}}^\beta(\psi))$$

where

$$A(q, \beta, r, \tilde{r}) \equiv \sup_{|\xi| \leq r} \inf_{|\mu| \leq \tilde{r}} \left(\int_{\mathbf{R}^N} \exp\left(-\frac{(y-\xi)^2}{4} + \frac{qe^{-\tau}(y-\mu e^{\tau/2})^2}{4(\beta-(1-e^{-\tau}))}\right) dy \right)^{1/q}.$$

Standard (but tedious) computations yield

$$A(q, \beta, r, \tilde{r}) = \left(\frac{4\pi(\beta-(1-e^{-\tau}))}{(\beta-1)-(q-1)e^{-\tau}} \right)^{\frac{N}{2q}} \exp\left(\frac{e^{-\tau}}{4((\beta-1)-(q-1)e^{-\tau})} (r - \tilde{r}e^{\tau/2})_+^2 \right)$$

(cf. Appendix A), whence the result. $\square$

Notice that for $r = 0$, $L_r^q(\psi) = \|\psi\|_{q,w}$, so that (2.3) reduces to Proposition 2.1 in [HV1] when $r = \tilde{r} = 0$. We shall later make use of a Gronwall – type lemma which reads as follows:

LEMMA 2.2. Let ϵ , C , R and δ be positive constants, $0 < \delta < 1$, and assume that $H(\tau)$ is a family of continuous functions satisfying

$$(2.4) \quad H(\tau) \leq \epsilon e^\tau + C \int_0^{(\tau-R)_+} \frac{e^{\tau-s} (H(s))^2}{(1 - e^{-(\tau-s-R)})^\delta} ds \text{ for } \tau > 0.$$

Then there exist $\theta = \theta(R, C, \delta)$ and $\epsilon_o = \epsilon_o(R, C, \delta)$ such that, for any $\epsilon \in (0, \epsilon_o)$ and any τ for which $\epsilon e^\tau \leq \theta$, we have

$$(2.5) \quad H(\tau) \leq 2\epsilon e^\tau.$$

Proof. Set $\bar{\tau} = \sup\{\tau > 0 : H(\tau) \leq 2\epsilon e^\tau\}$. By (2.4), we clearly have that $\bar{\tau} > 0$. Then, if $0 < \tau < \bar{\tau}$,

$$H(\tau) \leq \epsilon e^\tau + 4\epsilon^2 C \int_0^{(\tau-R)_+} \frac{e^{\tau+s}}{(1 - e^{-(\tau-s-R)})^\delta} ds.$$

Setting $\tau - s - R = \lambda$ and $\tau(R) = (\tau - R) - (\tau - R)_+$ (so that $\tau(R) = \tau - R$ if $\tau < R$), we deduce that, for $0 < \tau < \bar{\tau}$

$$\begin{aligned} H(\tau) &\leq \epsilon e^\tau + 4\epsilon^2 C e^{2(\tau-R)} \int_0^{(\tau-R)_+} \frac{e^{R-\lambda}}{(1 - e^{-\lambda})^\delta} d\lambda \\ &\leq \epsilon e^\tau + 4\epsilon^2 C K e^{-R} e^{2\tau} \equiv \epsilon e^\tau + \gamma \epsilon^2 e^{2\tau}, \end{aligned}$$

where $K = \int_0^\infty e^{-\lambda} (1 - e^{-\lambda})^{-\delta} d\lambda$. Suppose now that $\epsilon e^{\bar{\tau}} \leq \frac{1}{2\gamma}$. Then $H(\bar{\tau}) \leq \frac{3\epsilon e^{\bar{\tau}}}{2}$, which contradicts the definition of $\bar{\tau}$. Therefore, $\epsilon e^{\bar{\tau}} > \frac{1}{2\gamma}$, and the result follows if we take $\theta = \frac{1}{2\gamma}$ and $\epsilon_o e^{\bar{\tau}} = \theta$. $\square$

We next establish (1.7a).

PROPOSITION 2.3. Assume that (1.6a) holds. Then, in the same coordinate system used there,

$$(2.6) \quad \lim_{t \uparrow T} (T - t)^{\frac{1}{p-1}} u(\xi((T - t)|\log(T - t)|)^{1/2}, t) = \left((p - 1) + \overline{C}_p \sum_{k=1}^{\ell} \xi_k^2 \right)^{-\frac{1}{p-1}},$$

uniformly on sets $|\xi| \leq R$ with $R > 0$, where $\xi = (\xi_1, \xi_2, \dots, \xi_N)$ and $\overline{C}_p$ is given in (1.7a).

Proof. We define

$$(2.7) \quad G(\xi) = \left((p - 1) + \overline{C}_p \sum_{k=1}^{\ell} \xi_k^2 \right)^{-\frac{1}{p-1}},$$

$$(2.8) \quad \overline{\Phi} = G\left(\frac{y}{\sqrt{\tau}}\right) + \beta \frac{H_o}{\tau}, \text{ where } \beta = \ell \frac{(p-1)^{-\frac{1}{p-1}}}{2p}.$$

Standard calculations yield then

$$(2.9) \quad -\frac{1}{2}\xi\nabla G + G^p - \frac{G}{p-1} = 0 \text{ where } \xi = \frac{y}{\sqrt{\tau}},$$

$$(2.10) \quad \|\Phi - \bar{\Phi}\| = o\left(\frac{1}{\tau}\right) \text{ as } \tau \rightarrow \infty.$$

The formal idea of the proof of (2.6) is that $\bar{\Phi}$ solves in an approximate way the same equation as Φ for $\tau \rightarrow \infty$. By (2.10), Φ and $\bar{\Phi}$ are very close to the region $|y| \sim 1$ if $\tau \rightarrow \infty$. The difference $(\Phi - \bar{\Phi})$ satisfies an equation essentially linear with a strong convective term $-\frac{y \cdot \nabla}{2}$. By means of Proposition 2.1 it is possible to estimate $(\Phi - \bar{\Phi})$ in the region $|y| \sim \sqrt{\tau}$.

We readily check that $W = \Phi - \bar{\Phi}$ satisfies

$$\begin{aligned} W_\tau = \Delta W - \frac{1}{2}y \cdot \nabla W + (\bar{\Phi} + W)^p - \bar{\Phi}^p - \frac{W}{p-1} + \frac{\xi \nabla G}{2\tau} + \beta \frac{H_o}{\tau^2} + \\ \frac{\Delta G}{\tau} + \left(\left(G(\xi) + \beta \frac{H_o}{\tau} \right)^p - G(\xi)^p \right) - \beta \frac{H_o}{(p-1)\tau}. \end{aligned}$$

Therefore, using Kato's inequality,

$$\Delta g \cdot \text{sgn}(g) \leq \Delta(|g|) \quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

we obtain for $Z = |W|$

$$(2.11a) \quad \begin{aligned} Z_\tau - \Delta Z + \frac{1}{2}y \cdot \nabla Z - Z \leq & \left(\frac{(\bar{\Phi} + W)^p - \bar{\Phi}^p}{W} - \frac{p}{p-1} \right) Z \\ & + \frac{1}{2\tau} |\xi \nabla G(\xi)| + \beta \frac{|H_o|}{\tau^2} + L(y, \tau), \end{aligned}$$

where

$$(2.11b) \quad L(y, \tau) = \left| \frac{\Delta G}{\tau} + \left(G(\xi) + \beta \frac{H_o}{\tau} \right) - G(\xi)^p - \frac{\beta}{p-1} \cdot \frac{H_o}{\tau} \right|.$$

Fix now any $R > 0$, and write $\chi_1(y, \tau) = 1$ if $\frac{y}{\sqrt{\tau}} \geq R$, and zero otherwise. Using Taylor's formula in (2.11b), we readily obtain that

$$(2.12) \quad |L(y, \tau)| \leq C_1 \left(\frac{|\xi|^2}{\tau} + \frac{1}{\tau^2} + \chi_1(y, \tau) \right)$$

for some $C_1 = C_1(N, p, R) > 0$. On the other hand, recalling (1.3b), we see that

$$(2.13) \quad \frac{(\bar{\Phi} + W)^p - \bar{\Phi}^p}{W} - \frac{p}{p-1} \leq C_2(Z + \chi_1(y, \tau) + \frac{1}{\tau}),$$

where $C_2 = C_2(n, p, R, M) > 0$, M being the constant in (1.3b). We shall denote henceforth by C a generic positive constant, possibly changing from line to line, which depends on the same quantities as C_2 . Taking into account (2.12) and (2.13), (2.11a) yields

$$(2.14) \quad Z_\tau - \Delta Z + \frac{1}{2}y \cdot \nabla Z - Z - \frac{c}{\tau}Z \leq C \left(Z^2 + \frac{|y|^2}{\tau^2} + \frac{1}{\tau^2} + \chi_1(y, \tau) \right).$$

We now arrive at a crucial point in the proof. We take $\tau_o > 0$ large enough and $\tau > \tau_o$ such that

$$(2.15a) \quad e^{\frac{\tau - \tau_o}{2}} \leq \sqrt{\tau}.$$

We choose τ as in (2.15a) because, informally speaking, with this choice the convective term $\frac{1}{2}y \cdot \nabla(\cdot)$ has enough time to propagate the asymptotic behavior previously obtained for $|y| \sim 1$. (cf. (1.6a)) to a region of size $|y| \sim \sqrt{\tau}$. A precise estimate of this effect will be obtained with the help of Proposition 2.1.

We use variation of constants formula in (2.14)

$$(2.15b) \quad \begin{aligned} Z(y - \tau) &\leq \exp \left(\int_{\tau_o}^{\tau} \frac{C}{s} ds \right) S(\tau - \tau_o) Z(\cdot, \tau_o) + \\ &+ \int_{\tau_o}^{\tau} \exp \left(\int_{\tau_o}^{\tau} \frac{C}{s} ds \right) S(\tau - s) \left\{ CZ^2 + \frac{|y|^2}{s} + \frac{1}{s^2} + X_1(\cdot, s) \right\} ds. \end{aligned}$$

Setting

$$(2.15c) \quad \xi_o = \frac{R}{2}; r \equiv r(\tau; \tau_o) = \xi_o e^{\frac{\tau - \tau_o}{2}},$$

and taking L_r^2 -norms in (2.15b) we obtain

$$(2.16) \quad \begin{aligned} L_r^2(Z(\cdot, \tau)) &\leq c L_r^2(S(\tau - \tau_o) Z(\cdot, \tau_o)) + C \int_{\tau_o}^{\tau} L_r^2(S(\tau - s) Z(\cdot, \tau_o)^2) ds \\ &+ C \int_{\tau_o}^{\tau} L_r^2 \left(S(\tau - s) \left(\frac{|y|^2}{s^2} + \frac{1}{s^2} \right) \right) ds + C \int_{\tau_o}^{\tau} L_r^2(S(\tau - s) \chi_1(y, \tau)) ds \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

We next proceed to estimate the terms $J_1, \dots, J_4$ in (2.16). To begin with, we make use of Proposition 2.1 with $q = 2$, β large (say, $\beta = 20N$), $r \equiv r(\tau, \tau_o)$ (cf. (2.15)) and $\tilde{r} = r(s, \tau_o)$, to obtain that

$$\begin{aligned} L_r^2 \left(S(\tau - s) \left(\frac{|y|^2}{s^2} + \frac{1}{s^2} \right) \right) &\leq \frac{ce^{\tau-s}}{(1 - e^{-(\tau-s)})^{1/40}} \left(L_{\tilde{r}}^{20N} \left(\frac{|y|^2}{s^2} + \frac{1}{s^2} \right) \right) \\ &\leq \frac{Ce^{\tau-s}}{(1 - e^{-(\tau-s)})^{1/40}} \cdot \frac{1}{s^2} (1 + e^{s-\tau_o}). \end{aligned}$$

Therefore, if τ_o is large enough so that $s \leq 2\tau_o$, we arrive at

$$\begin{aligned} (2.17) \quad |J_3| &\leq \frac{C}{\tau_o^2} \left(\int_{\tau_o}^{\tau} e^{\tau-\tau_o} (1 - e^{-(\tau-s)})^{-1/40} ds \right. \\ &\quad \left. + \int_{\tau_o}^{\tau} e^{\tau-s} (1 - e^{-(\tau-s)})^{-1/40} ds \right) \leq C \frac{e^{\tau-\tau_o}}{\tau_o^2} (1 + (\tau - \tau_o)). \end{aligned}$$

In a similar way, we obtain that for large enough τ_o

$$\begin{aligned} (2.18) \quad |J_4| &\leq C \int_{\tau_o}^{\tau} \frac{e^{(\tau-s)}}{(1 - e^{-(\tau-s)})^{1/40}} L_{\tilde{r}}^{20N}(\chi_1(\cdot, s)) ds \\ &\leq C \int_{\tau_o}^{\tau} \frac{e^{(\tau-s)}}{(1 - e^{-(\tau-s)})^{1/40}} \left(\int_{|\lambda| \geq \frac{R\sqrt{s}}{4}} e^{-\lambda^2/4} d\lambda \right)^{1/20N} ds \\ &\leq Ce^{-\theta\tau_o}, \quad \text{for some } \theta = \theta(R) \end{aligned}$$

To bound $|J_1|$, we may assume without loss of generality that $\tau - \tau_o \geq 2$, which certainly holds if τ_o is large. The factor $(r - \tilde{r}e^{\tilde{r}/2})_+$ in the exponential factor of (2.3) makes it convenient estimate $|J_1|$ by applying Proposition 2.1 twice. We first take advantage of (2.3) with $r = r(\tau; \tau_o)$ (cf. (2.15)), $\tilde{r} = r(\tau_o + 1, \tau_o)$, $\beta = q = 2$, and then we use Proposition 2.1 again with $\tilde{r} = 0$, $\beta = q = 2$. One is thus led to

$$\begin{aligned} (2.19) \quad |J_1| &= |L_r^2(S(\tau - \tau_o - 1)S(1)Z(\cdot, \tau_o))| \\ &\leq Ce^{\tau-\tau_o-1} L_{r(\tau_o+1, \tau_o)}^2(S(1)Z(\cdot, \tau_o)) \\ &\leq Ce^{\tau-\tau_o} \|Z(\cdot, \tau_o)\| \leq e^{\tau-\tau_o} o \left(\frac{1}{\tau_o} \right) \quad \text{as } \tau_o \rightarrow \infty, \end{aligned}$$

where we have used (2.10) to derive the last inequality. It remains to deal with the quadratic term J_2 in (2.16). To this end, some auxiliary results are needed. In particular, we shall need the inequality

$$(2.20) \quad L_{r(s; \tau_o)}^{20N}(Z(\cdot, s)) \leq C \left(L_{r(s-R; \tau_o)}^2(Z(\cdot, s-R)) + \frac{1}{\tau_o} \right) \quad \text{for } s \geq \tau_o > R.$$

We obtain (2.20) by recalling that Z is bounded and using (2.15b). This yields

$$\begin{aligned} L_{r(s;\tau_o)}^{20N}(Z(\cdot, s)) &\leq e^{CR} L_{r(s;\tau_o)}^{20N}(S(R)Z(\cdot, s-R)) + \\ &C \int_{s-R}^s e^{C(s-\lambda)} L_{r(s;\tau_o)}^{20N} \left(S(s-\lambda) \left(\frac{|y|^2+1}{\lambda^2} \right) + X_1(y, \tau) \right) d\lambda \\ &\equiv L_1 + L_2. \end{aligned}$$

For $R = R(N) \geq 2$, we readily check that

$$(2.21a) \quad L_{r(s;\tau_o)}^{20N}(S(R)Z(\cdot, s-R)) \leq CL_{r(s-R;\tau_o)}^2(Z(\cdot, s-R)).$$

where $r = r(s; \tau_o)$ is defined as in (2.15b) even for $s < \tau_o$. On the other hand, arguing as in the previous bounds for J_2 , J_4 , we easily obtain that

$$(2.21b) \quad L_2 = o\left(\frac{1}{\tau_o}\right) \quad \text{as } \tau_o \rightarrow \infty.$$

Putting together (2.21a) and (2.21b), (2.20) follows.

We now use Proposition 2.1 and (2.20) to estimate J_2 :

$$\begin{aligned} (2.22a) \quad |J_2| &\leq \int_{\tau_o}^{\tau} L_{r(\tau;\tau_o)}^2(S(\tau-s)Z(\cdot, s)^2) ds \\ &\leq C \int_{\tau_o}^{\tau} \frac{e^{\tau-s}}{(1-e^{-(\tau-s)})^{1/20}} (L_{r(s;\tau_o)}^{20N}(Z(\cdot, s)))^2 ds \\ &\leq C \int_{\tau_o}^{\tau} \frac{e^{\tau-s}}{(1-e^{-(\tau-s)})^{1/20}} \left(L_{r(s-R;\tau_o)}^2(Z(\cdot, s-R))^2 + \frac{1}{\tau_o^2} \right) ds \\ &= C \int_{\tau_o-R}^{\tau-R} \frac{e^{\tau-s-R}}{(1-e^{-(\tau-s-R)})^{1/20}} (L_{r(s;\tau_o)}^2 Z(\cdot, s))^2 ds + \frac{C}{\tau_o^2} \int_0^{\tau-\tau_o} \frac{e^{\lambda}}{(1-e^{-\lambda})^{1/20}} d\lambda \\ &\leq C \int_{\tau_o-R}^{\tau} \frac{e^{\tau-s-R}}{(1-e^{-(\tau-s-R)})^{1/20}} (L_{r(s;\tau_o)}^2 Z(\cdot, s))^2 ds + \\ &\quad C \int_{\tau_o}^{(\tau-R)_+} \frac{e^{\tau-s-R}}{(1-e^{-(\tau-s-R)})^{1/20}} (L_{r(s;\tau_o)}^2 Z(\cdot, s))^2 ds + \frac{C}{\tau_o^2} e^{\tau-\tau_o} \equiv S_1 + S_2 + S_3. \end{aligned}$$

To proceed further, we next remark that Proposition 2.1, together with (2.15b) and the boundedness of Z , yield that for any $\sigma > 0$

$$(2.22b) \quad L_o^2(Z(\cdot, \tau_o)) = o\left(\frac{1}{\tau_o}\right) \quad \text{as } \tau_o \rightarrow \infty,$$

whence

$$|S_1| \leq e^{\tau - \tau_o} o\left(\frac{1}{\tau_o^2}\right) \quad \text{as } \tau_o \rightarrow \infty,$$

so that

$$(2.23) \quad |J_2| \leq C \int_{\tau_o}^{(\tau-R)_+} \frac{e^{\tau-s-R}}{(1 - e^{-(\tau-s-R)})^{1/20}} (L_{r(s; \tau_o)}^2 Z(\cdot, s))^2 ds \\ + \frac{C}{\tau_o^2} e^{\tau - \tau_o},$$

Summing up (2.17) – (2.23), and setting $g(s; \tau_o) = L_{r(s; \tau_o)}^2(Z(\cdot, s))$ we arrive at

$$g(\tau; \tau_o) \leq e^{\tau - \tau_o} o\left(\frac{1}{\tau_o}\right) + C \int_{\tau_o}^{(\tau-R)_+} \frac{e^{\tau-s-R}}{(1 - e^{-(\tau-s-R)})^{1/20}} (g(s; \tau_o))^2 ds.$$

It then follows from Lemma 2.2 that

$$g(\tau; \tau_o) = e^{\tau - \tau_o} o\left(\frac{1}{\tau_o}\right) \quad \text{as } \tau_o \rightarrow \infty \text{ if } e^{\tau - \tau_o} \leq \tau \text{ and } \tau_o \text{ is large enough.}$$

We now fix $\tau = e^{\tau - \tau_o}$ to obtain

$$(2.24) \quad L_{\xi_o \sqrt{\tau}}^2(Z(\cdot, \tau)) = o(1) \quad \text{as } \tau \rightarrow \infty.$$

To conclude the proof, we have yet to improve the convergence obtained in (2.24). To this end, notice that

$$Z(\cdot, \tau) \leq e^{CR} S(R) Z(\cdot, \tau - R) + C \int_{\tau-R}^{\tau} e^{C(\tau-\lambda)} S(\tau - \lambda) \left(\frac{(|y|^2 + 1)}{\lambda^2} + \chi_1(\cdot, s) \right) d\lambda \\ \equiv M_1 + M_2.$$

Assume now that $|y| \leq \frac{R}{4} \sqrt{\tau}$. Then, arguing as in [HV1], Lemma 6.5, we deduce that

$$(2.25) \quad \sup_{|y| \leq \frac{4\sqrt{\tau}}{4}} |M_2(\cdot, \tau)| \leq \frac{C}{\tau}.$$

On the other hand, for $|y| \leq \frac{R\sqrt{\tau}}{4}$, we have that

$$\begin{aligned} |M_1(y, \tau)| &\leq e^{cR} \int_{\mathbf{R}^N} \frac{\exp\left(-\frac{(ye^{-\tau/2}\lambda)^2}{4(1-e^{-\tau})}\right)}{(4\pi(1-e^{-R})^{N/2})} \cdot Z(\lambda, \tau-R) d\lambda \\ &\leq CL_{\frac{R}{2}\sqrt{\tau}}^2 (Z(\cdot, \tau-R)) I(R) \end{aligned}$$

where

$$I(R) = \sup_{|y| \leq \frac{R}{4}\sqrt{\tau}} \inf_{|\mu| \leq \frac{R}{2}\sqrt{\tau}} \left(\int_{\mathbf{R}^N} \exp\left(-\frac{((ye^{-R/2}-\mu)-\lambda)^2}{2(1-e^{-R})} + \frac{\lambda^2}{4}\right) d\lambda \right)^{1/2}.$$

It is readily seen (cf. for instance a similar computation in Appendix A below) that $I(R) \leq C$ for some $C = C(R) > 0$, hence by (2.24)

$$(2.26) \quad \sup_{|y| \leq \frac{R}{4}\sqrt{\tau}} |M_1(\cdot, \tau)| = o(1) \text{ as } \tau \rightarrow \infty$$

and putting together (2.25) and (2.26), the proof of Proposition 2.3 is complete. $\square$

To conclude the proof of Theorem 1, we need yet to examine the case of the flatter profiles in (1.6b). This is done in the next Proposition.

PROPOSITION 2.4. *Assume that (1.6b) holds. Then*

$$(2.27) \quad \lim_{t \uparrow T} (T-\tau)^{\frac{1}{p-1}} u(\xi(T-t)^{1/m}, t) = \left((p-1) + K_p \sum_{|\alpha|=m} C_\alpha \bar{c}_\alpha \xi^\alpha \right)^{-\frac{1}{p-1}}$$

where $K_p, \bar{c}_\alpha, C_\alpha$ are as in (1.7b).

Proof. The proof is very similar to that of Proposition 2.3. We shall therefore just sketch it, emphasizing such points where differences appear. Define

$$G(\xi) \equiv (p-1)^{-\frac{1}{p-1}} \left(1 + (p-1)^{\frac{p}{p-1}} \sum_{|\alpha|=m} C_\alpha \bar{c}_\alpha \xi^\alpha \right)^{-\frac{1}{p-1}},$$

Then

$$\frac{\xi \nabla G}{m} + \frac{G}{p-1} = G^p.$$

Set

$$(2.28) \quad \bar{\Phi}(y, \tau) = G(ye^{-(\frac{1}{2}-\frac{1}{m})\tau}) + e^{(1-\frac{m}{2})\tau} \left(\sum_{|\alpha|=m} C_\alpha H_\alpha(y) - \sum_{|\alpha|=m} C_\alpha \bar{c}_\alpha y^\alpha \right).$$

Then it is readily seen that

$$(2.29) \quad \|\Phi - \bar{\Phi}\| = o(e^{(1-\frac{m}{2})\tau}) \quad \text{as } \tau \rightarrow \infty.$$

From now on, we set $\xi = ye^{-(\frac{1}{2}-\frac{1}{m})\tau}$ and $\chi_2(y, \tau) = 1$ for $|y| \geq Re^{(\frac{1}{2}-\frac{1}{m})\tau}$, $\chi_2(y, \tau) = 0$ otherwise, where $R > 0$ is any fixed number. Define $W = \Phi - \bar{\Phi}$, $Z = |W|$. Arguing as in Proposition 2.3, we then have

$$(2.30) \quad \begin{aligned} & Z_t - \Delta Z + \frac{y}{2} \nabla Z - Z \\ & \leq CZ^2 + \left(G(\xi)^p - (G(\xi) + e^{(1-\frac{m}{2})\tau} \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha))^p \right) \\ & + \left(\frac{p}{p-1} - \frac{m}{2} \right) \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} - e^{-(1-\frac{2}{m})\tau} \Delta G(\xi) \\ & + \sum_{|\alpha|=m} C_\alpha \Delta (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} + C\chi_2(y, \tau). \end{aligned}$$

From now on C will denote a generic constant depending only on p, N, R, m, M and the coefficients $\{C_\alpha\}$. By using Taylor's expansion, we get

$$(2.31) \quad \begin{aligned} & \left| \left(G(\xi) + e^{(1-\frac{m}{2})\tau} \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha) \right)^p - G(\xi)^p \right. \\ & \quad \left. - \frac{p}{p-1} \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} \right| \\ & \leq Ce^{2(1-\frac{m}{2})\tau} (1 + |y|^{2m-2}) + C\chi_1(y, \tau). \end{aligned}$$

On the other hand

$$(2.32) \quad \begin{aligned} & \left| \frac{m}{2} \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} + e^{-(1-\frac{2}{m})\tau} \Delta G(\xi) \right. \\ & \quad \left. - \sum_{|\alpha|=m} C_\alpha \Delta (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} \right| \leq \\ & \left| \frac{m}{2} \sum_{|\alpha|=m} C_\alpha (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau} + e^{-(1-\frac{2}{m})\tau} \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha \Delta \xi(\xi^\alpha) \right. \\ & \quad \left. - \sum_{|\alpha|=m} C_\alpha \Delta (H_\alpha \Delta (H_\alpha(y) - \bar{c}_\alpha y^\alpha) e^{(1-\frac{m}{2})\tau}) \right| + Ce^{-(1-\frac{m}{2})\tau} |\xi|^{2m-2} + C\chi_2(y, \tau) \\ & = Ce^{2(1-\frac{m}{2})\tau} |y|^{2m-2} + C\chi_2(y, \tau). \end{aligned}$$

By using (2.31), (2.32) in (2.39), we obtain

$$(2.33) \quad Z_\tau - \Delta Z + \frac{y}{2} \nabla Z - Z - C \frac{1}{\tau} Z \leq CZ^2 + Ce^{2(1-\frac{m}{2})\tau}(1 + |y|^{2m-2}) + C\chi_2(y, \tau).$$

Let us apply now variation of constants formula and take the norm $L^2_{r(\tau; \tau_o)}$ where $r(\tau; \tau_o)$ is as in Proposition 2.3. Assume that $\tau_o \leq \tau \leq \frac{m}{2}\tau_o$. This yields

$$(2.34) \quad \begin{aligned} L^2_{r(\tau; \tau_o)}(Z(\cdot, \tau)) &\leq CL^2_{r(\tau; \tau_o)}((S(\tau - \tau_o)Z(\cdot, \tau_o)) \\ &\quad + C \int_{\tau_o}^{\tau} L^2_{r(\tau; \tau_o)}(S(\tau - s)Z(\cdot, s)^2)ds + \\ &\quad C \int_{\tau_o}^{\tau} L^2_{r(\tau; \tau_o)}(S(\tau - s)(e^{2(1-\frac{m}{2})s}(1 + |y|^{2m-2})))ds + \\ &\quad C \int_{\tau_o}^{\tau} L^2_{r(\tau; \tau_o)}(S(\tau - s)\chi_2(\cdot, 2))ds \equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Arguing as in Proposition 2.3 we arrive at the estimate

$$(2.35) \quad |J_3| \leq Ce^{\tau-(m-1)\tau_o}(1 + (\tau - \tau_o)) \quad \text{for } \tau \geq \tau_o,$$

and

$$(2.36) \quad |J_4| \leq Ce^{-\theta e^{(1-2/m)\tau_o}} \quad \text{for some } \theta > 0.$$

A slight modification of the argument in Proposition 2.3 also gives

$$(2.37) \quad |J_1| = e^{(\tau-\tau_o)} o(e^{(1-m/2)\tau_o}) \quad \text{as } \tau_o \rightarrow \infty$$

for $\tau - \tau_o \geq 2$.

Using now (2.33), the global boundedness of Z and the variation of constants formula as in Proposition 2.3, we obtain

$$(2.38) \quad L^{20N}_{r(s; \tau_o)}(Z(\cdot, s)) \leq CL^2_{r(s-R, \tau_o)}(Z(\cdot, s-R)) + Ce^{-(m-1)\tau_o}e^s,$$

whence, arguing as in Proposition 2.3

$$(2.39) \quad \begin{aligned} |J_2| &\leq C \int_{\tau_o}^{\tau} \frac{e^{(\tau-s)}}{(1 - e^{-(\tau-s)})^{1/20}} (L^2_{r(s-R, \tau_o)}(Z(\cdot, s-R)))^2 ds \\ &\quad + Ce^{2\tau-2(m-1)\tau_o}. \end{aligned}$$

Making use of (2.35), (2.36), (2.37) and (2.39) in (2.34) and defining $g(\tau; \tau_o) \equiv L_{r(\tau; \tau_o)}^2(Z(\cdot, \tau))$, we find that

$$(2.40) \quad g(\tau; \tau_o) \leq e^{\tau - \tau_o} o(e^{(1-m/2)\tau_o}) + Ce^{\tau - (m-1)\tau_o} (1 + (\tau - \tau_o)) + \\ Ce^{-\theta e^{(1-2/m)\tau_o}} + C \int_{\tau_o}^{\tau} \frac{e^{(\tau-s)}}{(1 - e^{-(\tau-s)})^{1/20}} (g(s - R; \tau_o))^2 + \\ Ce^{2\tau - 2(m-1)\tau_o} \quad \text{as } \tau_o \rightarrow \infty,$$

for $\tau \leq \frac{m}{2}\tau_o$, $\tau - \tau_o \geq 2$. Arguing as in (2.22b) we see that (2.40) holds for $\tau_o \leq \tau \leq \frac{m}{2}\tau_o$.

It is readily seen that (2.40) implies

$$(2.41) \quad g(\tau; \tau_o) \leq e^{\tau - \tau_o} o(e^{(1-m/2)\tau_o}) + C \int_{\tau_o}^{(\tau-R)_+} \frac{e^{\tau-s-R}}{(1 - e^{-(\tau-s-R)})^{1/20}} (g(s; R))^2 ds,$$

whence, by Lemma 2.2

$$g(\tau; \tau_o) = o(1) \quad \text{as } \tau_o \rightarrow \infty \quad \text{for } \tau_o \leq \tau \leq \frac{m}{2}\tau_o.$$

By using the regularizing effects of the semigroup $S(\tau)$ as in Proposition 2.3 we then obtain

$$(2.42) \quad \sup \left\{ Z(y, \tau) : |y| \leq \frac{\xi_o}{4} e^{(1/2-m/2)\tau} \right\} = o(1) \quad \text{as } \tau \rightarrow \infty$$

for any fixed $\xi_o > 0$, and this concludes the proof. $\square$

3. Singularities and profiles. In this section we shall prove of Theorems 2 and 3. We shall need some auxiliary results which provide technical conditions to prevent blow up. Suppose for simplicity that $T = 1$, and consider the equation

$$(3.1) \quad \phi_\tau = \Delta \phi - \frac{1}{2} y \nabla \phi + \phi^p - \frac{\phi}{p-1}, \quad \text{where } y \in R, \quad \tau > 0.$$

PROPOSITION 3.1. *Suppose that $0 \leq \phi \leq M < +\infty$, and assume also that*

$$(3.2) \quad \|\phi(\cdot, 0) - C\| \leq \epsilon, \quad \text{where } C \leq (p-1)^{-\frac{1}{p-1}} - \delta, \quad \text{for some} \\ \delta \in (0, (p-1)^{-\frac{1}{p-1}}) \text{ and } \epsilon > 0.$$

Then there exist $K > 0$ and $\epsilon_o > 0$ depending on N, p, δ and M (cf. (1.3a)) such that, for any $\tau > 0$ and $\epsilon \in (0, \epsilon_o)$,

$$(3.3) \quad \|\phi(\cdot, \tau) - \tilde{\phi}(\cdot, \tau)\| \leq \epsilon K e^{-\frac{\tau}{p-1}}, \quad \text{where} \\ \tilde{\phi}(y, \tau) = \left((p-1) + (C^{-(p-1)} - (p-1))e^\tau \right)^{-\frac{1}{p-1}},$$

C being as in (3.2). Moreover, for any $R > 0$ and $\tau \geq 1$,

$$(3.4) \quad \sup_{y \in B_R(0)} |\phi(\cdot, \tau) - \tilde{\phi}(\cdot, \tau)| \leq \epsilon K_1 e^{-\frac{\tau}{p-1}},$$

where $K_1 > 0$ depends on N, p, δ, M and R .

Proof. Define $W(y, \tau) = \phi(y, \tau) - \tilde{\phi}(y, \tau)$, $Z(y, \tau) = |W(y, \tau)|$. We readily see that

$$(3.5) \quad \begin{aligned} W_\tau &= \Delta W - \frac{1}{2} y \nabla W - \frac{W}{p-1} + (\phi^p - \tilde{\phi}^p), \\ Z_\tau &\leq \Delta Z - \frac{1}{2} y \nabla Z - \frac{Z}{p-1} + (\tilde{\phi} + Z)^p - \tilde{\phi}^p. \end{aligned}$$

Let $L > 0$ be a constant to be determined later. By the choice of $\tilde{\phi}$ in (3.3), there exists an $\eta = \eta(L) > 0$ such that

$$\tilde{\phi}(y, \tau) \geq \tilde{\phi}(y, L) \equiv \eta \text{ if } 0 \leq \tau \leq L, \quad \tilde{\phi}(y, \tau) \leq \tilde{\phi}(y, L) \text{ if } \tau \geq L.$$

Using the elementary inequality $(a + b)^p - a^p \leq C_1(a^{p-1}b + b^p)$ which is valid for $a \geq 0$, $b \geq 0$, $p > 1$ and some constant $C_1 > 0$, we deduce from (3.5) that

$$(3.6) \quad Z_\tau \leq \Delta Z - \frac{1}{2} y \nabla Z - \frac{Z}{p-1} + C_1(\tilde{\phi}^{p-1}Z + Z^p).$$

Notice that for $\tau \geq L$ we have

$$(3.7) \quad Z_\tau \leq \Delta Z - \frac{1}{2} y \nabla Z - \frac{Z}{p-1} + C_1 \eta^{p-1} Z + C_1 Z^p,$$

whereas the global boundedness of Z yields

$$(3.8) \quad Z_\tau \leq \Delta Z - \frac{1}{2} y \nabla Z + CZ \quad \text{for any } \tau > 0,$$

with $C = C(M, p) > 0$. We shall denote by C a positive generic constant depending only on N, M, p and δ . By Proposition 2.1 and (3.8),

$$(3.9) \quad \|Z(\cdot, \tau)\|_{2p, w} \leq C \|Z(\cdot, \tau - \tilde{R})\|,$$

where $\tilde{R} = \tilde{R}(p) > 0$. We now select $L > 0$ (depending only on p) such that $L > \tilde{R}$, and

$$(3.10) \quad -\left(\frac{p}{p-1} - pC_1\eta^{p-1}\right) + \frac{1}{p-1} < 0.$$

Set $\mu = \left(\frac{1}{p-1} - C_1\eta^{p-1}\right)$. Since $S(\tau)$ is a bounded linear operator in $L_w^2(\mathbf{R}^N)$, with norm less or equal to one, using (3.9) and variation of constants formula in (3.1) we obtain

$$\begin{aligned}
(3.11) \quad \|Z(\cdot, \tau)\| &\leq e^{-\mu(\tau-L)} \|S(\tau-L)Z(\cdot, L)\| \\
&+ C \int_L^\tau e^{-\mu(\tau-s)} \|S(\tau-s)Z(\cdot, s)^p\| ds \leq e^{-\mu(\tau-L)} \|Z(\cdot, L)\| \\
&+ C \int_L^\tau e^{-\mu(\tau-s)} \|Z(\cdot, s)\|_{2p,w}^p ds \leq e^{-\mu(\tau-L)} \|Z(\cdot, L)\| \\
&+ C e^{-\mu\tau} \int_L^\tau e^s \|Z(\cdot, s - \tilde{R})\|^p ds.
\end{aligned}$$

On the other hand, by (3.8) and (3.2),

$$(3.12) \quad \|Z(\cdot, \tau)\| \leq \epsilon e^{CL} \quad \text{for } 0 \leq \tau \leq L.$$

Using (3.12) with $\tau = L$ in (3.11) yields

$$(3.13) \quad \|Z(\cdot, \tau)\| \leq \epsilon C e^{-\mu(\tau-L)} + C e^{-\mu\tau} \int_L^\tau e^{\mu s} \|Z(\cdot, s - \tilde{R})\|^p ds.$$

Recalling the fact that C is a generic constant, we see that (3.12) can be recast as

$$(3.14) \quad \|Z(\cdot, \tau)\| \leq \epsilon C e^{-\mu(\tau-L)} \quad \text{for } 0 \leq \tau \leq L.$$

Set now

$$\bar{\tau} = \inf \left\{ \tau > 0 : \|Z(\cdot, \tau)\| \leq 2\epsilon C e^{-\mu(\tau-L)} \right\}.$$

Clearly, $\bar{\tau} > 0$ and it may depend on L . We want to show that for $\epsilon > 0$ small enough, $\bar{\tau} = \infty$. To this end suppose $\bar{\tau} < +\infty$. Then by (3.14) and (3.10), if $\tau \leq \bar{\tau}$

$$\|Z(\cdot, \tau)\| \leq \epsilon C e^{-\mu(\tau-L)} + \epsilon^p C e^{-\mu(\tau-L)},$$

Choosing $\epsilon > 0$ small enough (depending on M, N, L and p), we arrive at the inequality

$$\|Z(\cdot, \tau)\| \leq \frac{3}{2} C e^{-\mu(\tau-L)},$$

which contradicts the choice of $\bar{\tau}$, thereby proving that $\bar{\tau} = +\infty$, i. e.

$$\|Z(\cdot, \tau)\| \leq 2\epsilon C e^{-\mu(\tau-L)}, \quad \text{for any } \tau \geq 0.$$

We now use again variation of constants formula in (3.6), (3.9) and (3.14) to obtain (since η can be selected arbitrarily small) that

$$\|Z(\cdot, \tau)\| \leq C\epsilon e^{-\frac{\tau}{p-1}} + \int_0^\tau e^{-\frac{1}{p-1}(\tau-s)} C_1 (e^{-s} 2\epsilon C e^{-\mu(s-L)} + (2\epsilon C)^p e^{-p\mu(s-L)}) ds,$$

with C_1 and C are as before. Taking into account (3.10), we readily arrive at (3.3) for $\tau \geq L$. The proof of (3.3) for $\tau \leq L$ is obtained from (3.14). Finally (3.4) follows at once from (3.8), (3.3) and standard regularizing effects of $S(\tau)$ that may be easily derived from the explicit formula for the associated semigroup (cf. the proof of Proposition 2.1). $\square$

Proposition 3.1 is close in spirit to results in Section 3 in [GK3] (cf. for instance Corollary 3.6 there), as well as to Proposition 3.1 in [HV2]. Related arguments, controlling how fast perturbations propagate in parabolic equations, have been also used in different contexts, cf. for instance Lemma 2.1 in [CF].

For ℓ, ϵ such that $1 \leq \ell \leq N$ and $\epsilon > 0$, set

$$\Omega_\epsilon = \left\{ x \in \mathbb{R}^N : \sum_{k=1}^\ell x_k^2 \leq \epsilon(x_1^2 + \cdots + x_N^2) \right\}.$$

Notice that Ω_ϵ is a cone that for ϵ small is concentrated near of the zero set of $\sum_{k=1}^\ell x_k^2$.

Let S be the blow up set of the solution under consideration. We then have

PROPOSITION 3.2. *Assume that (1.6a) holds. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$(3.15) \quad S \cap B_\delta(0) \subset \Omega_\epsilon.$$

Moreover,

$$(3.16) \quad \lim_{\substack{|x| \rightarrow 0 \\ x \in \mathbb{R}^N - \Omega_\epsilon}} \left(\frac{|x|^2}{|\log |x||} \right)^{\frac{1}{p-1}} u(|x|\vec{n}, T) = \left(\overline{C}_p \sum_{k=1}^\ell n_k^2 \right)^{-\frac{1}{p-1}},$$

where $\vec{n} = (n_1, \dots, n_N) \in S^{N-1} \cap (\mathbb{R}^N - \Omega_\epsilon)$, and the convergence is uniform for any fixed $\epsilon > 0$.

Proof. Following [HV3], we define, for $0 < s < T$

$$v_s(x, t) = (T - s)^{\frac{1}{p-1}} u(\lambda(s) + x(T - s)^{1/2}, s + t(T - s)),$$

where $\lambda(s) = ((T-s)|\log(T-s)|)^{1/2}\vec{n}$ and $\vec{n} = (n_1, \dots, n_N) \in S^{N-1} \cap (\mathbb{R}^N - \Omega_\epsilon)$.

Since $v_s(x, t)$ solves (1.1) for any fixed s , it follows from Proposition 2.3 that

$$v_s(x, 0) = \left((p-1) + \overline{C}_p \sum_{k=1}^{\ell} \xi_k^2 \right)^{-\frac{1}{p-1}} + o(1) \text{ as } s \uparrow T.$$

uniformly on sets $|x| \leq |\log(T-s)|^{1/2}$ where $\overline{C}_p$ is as in (1.7a).

We then have that

$$\left\| v_s(x, 0) - \left((p-1) + \overline{C}_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}} \right\| \rightarrow 0 \text{ as } s \uparrow T,$$

uniformly on $\vec{n} \in S^{N-1}$. Let now $\phi_s(\cdot, \tau)$ be the function given by

$$\phi_s(x(1-t)^{-1/2}, -\log(1-t)) = (1-t)^{\frac{1}{p-1}} v_s(x, t).$$

Clearly, ϕ_s satisfies (3.1) with initial condition

$$\phi_s(y, 0) = \left((p-1) + \overline{C}_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}} + g_s(y),$$

where $\|g_s(\cdot)\| = o(1)$ as $s \uparrow T$, uniformly for $\vec{n} \in S^{N-1}$.

Assume now that $\vec{n} \in \mathbb{R}^N - \Omega_\epsilon$. Then there exists a $\delta > 0$ (depending on ϵ) such that

$$\left((p-1) + \overline{C}_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}} \leq (p-1)^{-\frac{1}{p-1}} - \delta.$$

Recalling (1.3b), we may use Lemma 3.1 to deduce that

$$(3.17) \quad |\phi_s(0, \tau) - \phi^*(0, \tau)| \leq \epsilon K \|g_s(\cdot)\| e^{-\frac{\tau}{p-1}}$$

where

$$\phi^*(y, \tau) = e^{-\frac{\tau}{p-1}} \left(\overline{C}_p \sum_{k=1}^{\ell} n_k^2 + (p-1)(1-t) \right)^{-\frac{1}{p-1}}$$

and $K = K(N, p, \delta, M)$. By the definition of ϕ_s , we see that (3.17) can be rewritten in the form

$$\left| v_s(0, t) - \left((p-1)(1-t) + \overline{C}_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}} \right| \leq \epsilon K \|g_s(\cdot)\| = o(1)$$

as $s \uparrow T$, uniformly for $\vec{n} \in S^{N-1} \cap (\mathbb{R}^N - \Omega_\epsilon)$,

whence

$$\begin{aligned}
& u((T-s)|\log(T-s)|)^{1/2} \vec{n}, s + t(T-s)) = \\
& \left((p-1)(1-t) + \bar{C}_p \sum_{k=1}^{\ell} n_k^2 \right)^{-\frac{1}{p-1}} (T-s)^{-\frac{1}{p-1}} + o((T-s)^{-\frac{1}{p-1}}) \\
& \text{as } s \uparrow T, \text{ uniformly for } \vec{n} \in S^{N-1} \cap (\mathbf{R}^N - \Omega_\epsilon), 0 \leq t \leq 1.
\end{aligned}$$

In particular, setting in (3.18) $t = 1$ and $|x| = ((T-s)|\log(T-s)|)^{1/2}$, (3.16) follows. $\square$

To conclude the proof of Theorem 2, we prove:

PROPOSITION 3.3. *Assume that (1.6b) holds. Then, for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$(3.19) \quad S \cap B_\delta(0) \subset \tilde{\Omega}_\epsilon = \left\{ x \in \mathbf{R}^N : \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha x^\alpha \leq \epsilon |x|^\alpha \right\}.$$

Moreover,

$$(3.20) \quad \lim_{\substack{|x| \rightarrow 0 \\ x \in \mathbf{R}^N - \tilde{\Omega}_\epsilon}} |x|^{\frac{m}{p-1}} u(|x| \vec{n}, T) = \left(K_p \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha \vec{n}^\alpha \right)^{-\frac{1}{p-1}}$$

uniformly for $\vec{n} \in S^{N-1} \cap (\mathbf{R}^N - \tilde{\Omega}_\epsilon)$.

Proof. The proof is similar to the proof of Proposition 3.2, and we shall therefore only sketch it briefly. For $\epsilon > 0$ given, and $0 < s < T$, we define $v_s(x, t)$ as follows:

$$v_s(x, t) = (T-s)^{\frac{1}{p-1}} u(\lambda(s) + x(T-s)^{1/2}, s + t(T-s)),$$

where $\lambda(s) = (T-s)^{1/m} \vec{n}$ and $\vec{n} \in S^{N-1} \cap (\mathbf{R}^N - \tilde{\Omega}_\epsilon)$. Defining ϕ_s as in the previous case, we readily check that ϕ_s satisfies (3.1) and

$$\phi_s(y, 0) = \left((p-1) + K_p \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha (\vec{n})^\alpha \right)^{-\frac{1}{p-1}} + g_s(y),$$

where $\|g_s(\cdot)\| = o(1)$ as $s \uparrow T$, uniformly for $\vec{n} \in S^{N-1}$.

Again, we see that there exists $\delta > 0$ such that

$$\left((p-1) + K_p \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha (\vec{n})^\alpha \right)^{-\frac{1}{p-1}} \leq (p-1)^{-\frac{1}{p-1}} - \delta$$

whereupon we obtain, as in the previous case,

$$u((T-s)^{1/m}\vec{n}, s+t(T-s)) = \left((p-1)(1-t) + K_p \sum_{|\alpha|=m} C_\alpha(\vec{n})^\alpha \right)^{-\frac{1}{p-1}} (T-s)^{-\frac{1}{p-1}} + o((T-s)^{-\frac{1}{p-1}}) \text{ as } s \uparrow T,$$

uniformly for $0 \leq t \leq 1$ and $\vec{n} \in S^{N-1} \cap (\mathbf{R}^N - \tilde{\Omega}_\epsilon)$, which implies (3.20) if we just set $t = 1$, $|x| = (T-s)^{1/m}$ above. $\square$

We next proceed to show our last result

Proof of Theorem 3.

It clearly suffices to prove that, for any fixed $R > 0$.

$$L^N(S \cap B_R(0)) = 0,$$

where L^N is the standard N -dimensional Lebesgue measure. Consider now an arbitrary multilinear form $B(x) = \sum_{|\alpha|=m} \bar{c}_\alpha C_\alpha x^\alpha$, where $m \geq 2$ and $B \neq 0$. For any given $\epsilon > 0$, let $A_\epsilon = A_\epsilon(B)$ be the set defined by

$$(3.21) \quad A_\epsilon = A_\epsilon(B) = \{\vec{n} \in S^{N-1} : |B(\vec{n})| \leq \epsilon\}$$

Let k be any positive integer. The k -dimensional Hausdorff measure of a set A is defined by (cf. for instance [F]).

$$H^k(A) = \sup_{\delta > 0} \inf \left\{ \sum_{S \in G} \zeta(S) : G \text{ is any countable covering of } A \text{ by sets of diameter less than } \delta \right\}$$

where

$$\zeta(S) = \frac{\Gamma(1/2)^k}{\gamma(k/2 + 1)} 2^{-k} (\text{diam } S)^k,$$

and $\Gamma(s)$ is the standard gamma function. We claim that

$$(3.22) \quad H^{N-1}(A_\epsilon) \rightarrow 0 \text{ as } \epsilon \rightarrow 0.$$

Indeed, if (3.22) does not hold, then there is a sequence (say, $\epsilon_n = \frac{1}{n}$) such that $\lim_{n \rightarrow \infty} H^{N-1}(A_{1/n}) \neq 0$. By classical results (cf. for instance [F], p. 55), we then have

$$H^{N-1}(A_o) = H^{N-1} \left(\bigcap_{n=1}^{\infty} A_{1/n} \right) = \lim_{n \rightarrow \infty} H^{N-1}(A_{1/n}) \neq 0,$$

and a measure theoretic argument gives that $B \equiv 0$ (see Appendix B below), a contradiction.

We next claim that

$$(3.23) \quad \begin{aligned} &\text{For any } x_o \in S \cap B_R(0), \text{ and for any } \eta > 0, \text{ there} \\ &\text{exists } B_\delta(x_o) \subset B_{2R}(0) \text{ such that} \\ &L^N(S \cap B_\delta(x_o)) \leq \frac{1}{N} \eta \delta^N. \end{aligned}$$

Indeed, by the results of [V], we know that, if $u(x, t) \neq ((p-1)(T-t))^{-\frac{1}{p-1}}$, then either (1.6a) or (1.6b) must be satisfied. Then, by (3.15) for any $x_o \in S \cap B_R(0)$ there exists $B_\delta(x_o) \subset B_{2R}(0)$ such that

$$S \cap B_\delta(x_o) \subset \left\{ x \in B_\delta(x_o) : \frac{x}{|x|} \in A_\epsilon, A_\epsilon \text{ as in (3.21)} \right\}$$

where $B = B(x)$ is the corresponding multilinear form given by the right hand sides in (1.8b), (1.9b). By (3.22), we can select $\delta > 0$ small enough so that $H^{N-1}(A_\epsilon) \leq \eta$. We then notice that

$$L^N(B_\delta(x_o) \cap S) \leq \int_{A_\epsilon} d\omega \int_0^\delta r^{N-1} dr = \frac{\delta^N}{N} H^{N-1}(A_\epsilon) \leq \frac{\delta^N}{N} \eta$$

and (3.23) follows.

To conclude, we make use of Besicovitch Covering Theorem (cf. for instance [F]) that may be stated as follows:

BESICOVITCH COVERING THEOREM. *Assume that $A \subset \mathbb{R}^N$. Let F be a family of closed balls such that each point of A is a center of some member of F . Then, there exists an integer θ_N depending only on N such that A is contained in the union of θ_N disjoint subfamilies of F .*

By this Theorem there exists at most θ_N families of disjoint balls $\mathcal{F}_i = \{B_{\delta_i}(x_i)\}$, $1 \leq i \leq \theta_N$, such that

$$S \cap B_R(0) \subset \bigcup_{i=1}^{\theta_N} \bigcup_{\mathcal{F}_i} (B_{\delta_i}(x_i) \cap S)$$

Therefore

$$\begin{aligned} L^N(S \cap B_R(0)) &\leq \sum_{i=1}^{\theta_N} \sum_{\mathcal{F}_i} L_N(B_{\delta_i}(x_i) \cap S) \leq \sum_{i=1}^{\theta_N} \sum_{\mathcal{F}_i} \frac{\eta \delta_i^N}{N} = \\ \frac{\eta}{\omega_N N} &\leq \sum_{i=1}^{\theta_N} \sum_{\mathcal{F}_i} L^N(B_{\delta_i}(x_i)) = \frac{\eta}{\omega_N N} \sum_{i=1}^{\theta_N} L^N\left(\bigcup_{\mathcal{F}_i} B_{\delta_i}(x_i)\right) \leq \\ \frac{\theta_N \eta}{\omega_N N} &L^N(B_{2R}(0)) \end{aligned}$$

and letting $\eta \rightarrow 0$, the proof is concluded. $\square$

APPENDIX A. We shall briefly sketch here the argument leading to the formula for $A(q, \beta, r, \tilde{r})$ in Proposition 2.1. To this end, we notice that

$$\begin{aligned} \int_{\mathbf{R}^N} \exp \left(-\frac{(y - \xi)^2}{4} + \frac{qe^{-\tau}(y - \xi + (\xi - \mu e^{\tau/2}))^2}{4(\beta - (1 - e^{-\tau}))} \right) dy = \\ \int_{\mathbf{R}^N} \exp \left(-\frac{y^2}{4} + \frac{qe^{-\tau}(y - \theta)^2}{4(\beta - (1 - e^{-\tau}))} \right) dy \equiv I, \end{aligned}$$

where $\theta = \mu e^{\tau/2} - \xi$. On the other hand, by completing squares, we readily check that

$$\begin{aligned} \frac{y^2}{4} - \frac{qe^{-\tau}(y - \theta)^2}{4(\beta - (1 - e^{-\tau}))} = \frac{((\beta - 1) - (q - 1)e^{\tau})}{4(\beta - (1 - e^{-\tau}))} \left(y + \frac{qe^{-\tau}\theta}{((\beta - 1) - (q - 1)e^{-\tau})} \right)^2 \\ - \frac{qe^{-\tau}\theta^2}{4((\beta - 1) - (q - 1)e^{-\tau})}. \end{aligned}$$

Therefore

$$\begin{aligned} I \equiv \exp \left(\frac{qe^{-\tau}\theta^2}{4((\beta - 1) - (q - 1)e^{-\tau})} \right) \int_{\mathbf{R}^N} \exp \left(-\frac{((\beta - 1) - (q - 1)e^{\tau})}{4(\beta - (1 - e^{-\tau}))} y^2 \right) dy = \\ \left(\frac{4\pi(\beta - (1 - e^{-\tau}))}{((\beta - 1) - (q - 1)e^{-\tau})} \right)^{N/2} \exp \left(\frac{qe^{-\tau}\theta^2}{4((\beta - 1) - (q - 1)e^{-\tau})} \right), \end{aligned}$$

We then have

$$A(q, \beta, r, \tilde{r}) = \left(\frac{4\pi(\beta - (1 - e^{-\tau}))}{((\beta - 1) - (q - 1)e^{-\tau})} \right)^{N/2q} \exp \left(\frac{e^{-\tau}}{4((\beta - 1) - (q - 1)e^{-\tau})} \left(\sup_{|\xi| \leq r} \inf_{|\mu| \leq \tilde{r}} \theta^2 \right) \right),$$

and it only remains to compute

$$\sup_{|\xi| \leq r} \inf_{|\mu| \leq \tilde{r}} \theta^2, \quad \text{with } \theta = \mu e^{\tau/2} - \xi.$$

Now, if $r \leq \tilde{r}e^{\tau/2}$, $\inf_{|\mu| \leq \tilde{r}} (\mu e^{\tau/2} - \xi)^2 = 0$. If on the other hand $r > \tilde{r}e^{\tau/2}$, then $\alpha \equiv \inf_{|\mu| \leq \tilde{r}} (\mu e^{\tau/2} - \xi)^2 = (|\xi| - \tilde{r}e^{\tau/2})^2$ if $|\xi| \geq \tilde{r}e^{\tau/2}$, and $\alpha = 0$ otherwise. In both cases, we have

$$\sup_{|\xi| \leq r} \inf_{|\mu| \leq \tilde{r}} \theta^2 = (r - \tilde{r}e^{\tau/2})_+^2,$$

and the result follows. $\square$

APPENDIX B: On the size of the zero set of Homogeneous Polynomials. For the reader convenience we shall recall here some known properties of homogeneous polynomials (cf. [F]) which have been used in the proof of Theorem 3. In particular, we shall prove

PROPOSITION B. *Let $P(x)$ be an homogeneous polynomial in $\mathbf{R}^N$, and define $A = \{x \in S^{N-1} : P(x) = 0\}$. Then, if $H^{N-1}(A) > 0$, $P(x)$ is identically zero.*

Proof. We argue by induction on N . The result is obvious for $N = 1$. Suppose now that it holds when N is replaced by $N - 1$. Since $H^{N-1}(A) > 0$, it follows (cf. [F], Thm. 2.9.11) that, there exists $e \in A$ such that, if $\tilde{\beta}_r(e)$ is the ball in S^{N-1} centered at e ,

$$(B1) \quad \lim_{r \rightarrow 0} \frac{H^{N-1}(A \cap \tilde{\beta}_r(e))}{H^{N-1}(\tilde{\beta}_r(e))} = 1.$$

Without loss of generality, we may assume that $N > 2$ and $e = (0, 0, \dots, 1)$. On the other hand, we certainly can write

$$P(x) = P(e) + \sum_{k=m}^{\ell} \sum_{|\beta|=k}^N a_{\beta,k} (x - e)^{\beta},$$

where $a_{\beta,m} \neq 0$, for some β . For $x \in \mathbf{R}^N$, we write $x = (x_1, \dots, x_{N-1}, x_N) \equiv (\tilde{x}, x_N)$. Let us then define the transformation $T : e + (\mathbf{R}^{N-1}x\{0\}) \rightarrow S^{N-1}$ given by

$$T(x) = T(\tilde{x}, 1) = \frac{x}{|x|}.$$

Notice that T is one-to-one between $e + (\mathbf{R}^{N-1}x\{0\})$ and $S^{N-1} \cap \{x_N > 0\}$. Write now $\Sigma = T^{-1}(A) \cap T^{-1}(\tilde{\beta}_r(e))$. Recalling (B1) and making use of Thm. 3.25 in [F], we see that

$$(B2) \quad 1 = \lim_{r \rightarrow 0} \frac{H^{N-1}(A \cap \tilde{\beta}_r(e))}{H^{N-1}(\tilde{\beta}_r(e))} = \lim_{r \rightarrow 0} \frac{\int_{\Sigma} J_{N-1}(T)(x) dL^{N-1}(x)}{\int_{T^{-1}(\tilde{\beta}_r(e))} J_{N-1}(T)(x) dL^{N-1}(x)} =$$

$$\lim_{r \rightarrow 0} \frac{L^{N-1}(T^{-1}(A) \cap B_r(e))}{L^{N-1}(B_r(e))},$$

where $J_{N-1}(T)$ denotes the Jacobian corresponding to the first $(N - 1)$ components of T . Since $P(x)$ is homogeneous, it follows from (B2) that $P(x) = 0$ for any $x \in T^{-1}(A)$. Take now $y \in S^{N-2} = \{y = (y_1, \dots, y_{N-1}, 0) : |y| = 1\}$. Then by (B2) and Fubini's Theorem, there exists a sequence $r_n \rightarrow 0$ as $n \rightarrow \infty$, a family of sets $\{B_n\}$ and a $\delta > 0$, such that $P(e + r_n y) = 0$ for $y \in B_n \subset S^{N-2}$ and $H^{N-2}(B_n) \geq \delta > 0$. Set

$$M(y) = \lim_{r \rightarrow 0} \frac{P(e + ry)}{r^n}.$$

Clearly, M is an homogeneous polynomial in y . Define

$$L_r = \{y \in S^{N-2} : P(e + ry) \neq 0\}, \quad C = \{y \in S^{N-2} : M(y) = 0\}.$$

Denoting by χ_A the characteristic function of a set A , we readily see that $y \in C$ if $\liminf_{r \rightarrow 0} \chi_{L_r}(y) = 0$. We then have that

$$\begin{aligned} H^{N-2}(C) &\geq \int_{S^{N-2}} (1 - \liminf_{r \rightarrow 0} \chi_{L_r}(y)) dH^{N-2}(y) \geq \\ &\int_{S^{N-2}} dH^{N-2}(y) - \liminf_{r \rightarrow 0} \int_{S^{N-2}} \chi_{L_r}(y) dH^{N-2}(y) \geq \\ &\int_{S^{N-2}} dH^{N-2}(y) - \liminf_{n \rightarrow \infty} \int_{S^{N-2}} \chi_{L_{r_n}}(y) dH^{N-2}(y) \geq \\ &\int_{S^{N-2}} dH^{N-2}(y) - \int_{S^{N-2}} dH^{N-2}(y) + \delta = \delta > 0, \end{aligned}$$

so that, by the induction hypothesis, $M \equiv 0$. Iterating this argument, we conclude that $P(x) = (x_N - 1)Q(x)$ for some $\gamma > 0$ and some polynomial Q . By the homogeneity assumption, this implies that $P \equiv 0$, whence the result. $\square$

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