SPIKE-LAYERS IN SEMILINEAR ELLIPTIC SINGULAR PERTURBATION PROBLEMS

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SPIKE-LAYERS IN SEMILINEAR ELLIPTIC SINGULAR PERTURBATION PROBLEMS†

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The purpose of this expository paper is to describe a new method, introduced in a series of papers [LNT], [NT1,2], [NPT] and [J], in handling "spikes" (or "point-condensation" phenomena) for singularly perturbed semilinear elliptic equations of the form

$$\varepsilon^2 \Delta u + f(u) = 0 \tag{1}$$

where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in $\mathbb{R}^n$ and ε is a small positive number. To illustrate the flexibility of our method, we shall treat *both Dirichlet and Neumann boundary value problems using the same method.*

We begin with the following Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, ν is the unit outer normal to $\partial\Omega$, and $1 < p < \frac{n+2}{n-2}$ ($1 < p < \infty$ when $n = 2$). Problem (2) arises in various models in pattern formation in mathematical biology, e.g. the Keller-Segel model in chemotaxis, and the shadow system for the non-saturated case of an activator-inhibitor system in morphogenesis proposed by Gierer and Meinhardt. (The idea of "diffusion-driven instability", however, goes back to A. Turing. See [LNT] and the references therein for the background of the problem (2).)

In [LNT], [NT1, 2] it was established that (2) *has a solution u_ε (in fact, a "least-energy" solution, see §1 below) which possesses a single spike-layer at the boundary. Moreover, the* "amplitude" *and the* "location" *of the spike can be determined. Mathematically, the following statements hold.*

(i) *For every ε sufficiently small, u_ε has a unique local (thus, global) maximum point P_ε in $\overline{\Omega}$, and, P_ε must lie on the boundary $\partial\Omega$.*

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(ii) *$u_\varepsilon \to 0$ everywhere in $\overline{\Omega}$ outside a small neighborhood of P_ε and $u_\varepsilon(P_\varepsilon) \to w(0)$, as $\varepsilon \to 0$, where w is the unique solution of the problem*

$$(3) \qquad \begin{cases} \Delta w - w + w^p = 0 & \text{in } \mathbf{R}^n, \\ w > 0 \text{ in } \mathbf{R}^n, w \to 0 & \text{at } \infty, \\ w(0) = \max_{\mathbf{R}^n} w. \end{cases}$$

(iii) *P_ε must be situated at the "most curved" part of $\partial\Omega$ when ε is sufficiently small. More precisely, $H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$ where $H(P)$ denotes the mean curvature of $\partial\Omega$ at P.*

Our method of proof will be described in §1 below. It turns out that a suitable modification of the techniques can be used to handle the critical case of (2); i.e. if we replace p by $\tau = \frac{n+2}{n-2}$ in (2), $n \geq 3$. Note that the corresponding result must be somewhat different since (3) does not admit any solution if p is replaced by τ. Nonetheless, our approach applies. This will be discussed in §2.

Our method also carries over automatically (with minimal changes) to the Dirichlet problem which we obtain from (2) by replacing the original Neumann boundary condition in (2) by the homogeneous Dirichlet condition "$u = 0$ on $\partial\Omega$". Thus, to further illustrate the power of our method, we shall include, in §3, a modification of our approach ([J]) to the following Dirichlet problem

$$(4) \qquad \begin{cases} \varepsilon^2 \Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where a typical example of the nonlinearity is the well-known "cubic" or "loop" case $f(u) = u(1-u)(u-a)$, $0 < a < \frac{1}{2}$, which has attracted lots of attention in recent years. Finally, we conclude this article with some remarks in §4.

1. Neumann problem: Sub-critical case. In this section we shall concentrate on the problem

$$(1.1) \qquad \begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ is the Laplacian, Ω is a bounded smooth domain in $\mathbf{R}^n$, ν is the unit outer normal to $\partial\Omega$, and $\varepsilon > 0$, $1 < p < \frac{n+2}{n-2}(1 < p < \infty$ if $n = 2)$ are constants. We wish to give an outline of our method and refer the interested readers to [NT1,2] for the full details. We first define an "energy" functional J_ε on $H^1(\Omega)$ by

$$(1.2) \qquad J_\varepsilon(u) = \frac{1}{2}\int_\Omega (\varepsilon^2 |Du|^2 + u^2) - \frac{1}{p+1}\int_\Omega (u_+)^{p+1}$$

where $u_+ = \max\{u, 0\}$. It turns out that the Mountain-Pass Lemma applies and gives that

$$c_\varepsilon = \inf_{h\in\Gamma} \max_{0\le t\le 1} J_\varepsilon(h(t)) \tag{1.3}$$

where $\{h(t)|0 \le t \le 1\}$ denotes a continuous path connecting 0 and e (an arbitrary but fixed positive function in $H^1(\Omega)$ with $J_\varepsilon(e) = 0$) and Γ is the set of all such paths, is a *positive critical value* of J_ε. Moreover,

$$c_\varepsilon = \min\{J_\varepsilon(u)|u \in \mathcal{M}\} \tag{1.4}$$

where

$$\mathcal{M} = \left\{ 0 < u \in H^1(\Omega)| \int_\Omega (\varepsilon^2|Du|^2 + u^2) = \int_\Omega u_+^{p+1} \right\}. \tag{1.5}$$

(Thus c_ε is independent of the choice of e.) Since $\mathcal{M}$ clearly contains all possible solutions of (1.1), we call the corresponding critical point u_ε a *least-energy* solution. Then, standard elliptic regularity results show that u_ε is smooth on $\overline{\Omega}$.

The main results in [LNT], [NT1,2] may be summarized as follows.

THEOREM 1.6. *For every $\varepsilon > 0$, (1.1) possesses a least-energy solution u_ε* (i.e. $J_\varepsilon(u_\varepsilon) = c_\varepsilon$) *with the following properties:*

(i) *$u_\varepsilon \equiv 1$ for every ε sufficiently large, and, u_ε has a unique local (thus, global) maximum point P_ε in $\overline{\Omega}$ for every ε sufficiently small. Furthermore, P_ε must lie on the boundary $\partial\Omega$.*

(ii) *$u_\varepsilon \to 0$ everywhere in $\overline{\Omega}$ outside an arbitrarily small neighborhood of P_ε and $u_\varepsilon(P_\varepsilon) \to w(0)$, as $\varepsilon \to 0$, where w is the unique solution of (3).*

(iii) *$H(P_\varepsilon) \to \max_{P\in\partial\Omega} H(P)$ as $\varepsilon \to 0$ where $H(P)$ denotes the mean curvature of $\partial\Omega$ at P.*

A few remarks are in order. First, it was shown in [LNT] that for ε large, (1.1), in fact, can only have one trivial solution $u \equiv 1$. As for the uniqueness of w in (3), we include a brief discussion about it in §4. Finally, we remark that the "asymptotic profile" (suitably rescaled as $\varepsilon \to 0$) is given by w. (See the outline of the proof below, or [NT1,2] for full details.)

The proof of Theorem 1.6 is lengthy and technical, and, is carried out in several steps. First, for ε small we obtain a good upper bound for c_ε; namely,

$$c_\varepsilon \le \varepsilon^n \left\{ \frac{1}{2} I(w) - (n-1)H\gamma\varepsilon + 0(\varepsilon) \right\} \tag{1.7}$$

where H is the maximum of $H(P)$ on $\partial\Omega$,

(1.8)
$$I(w) = \frac{1}{2}\int_{\mathbf{R}^n}(|Dw|^2 + w^2) - \frac{1}{p+1}\int_{\mathbf{R}^n} w^{p+1},$$
$$\gamma = \frac{1}{n+1}\int_{\mathbf{R}^n_+}[w'(|x|)]^2 x_n dx,$$

and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. This is done by choosing suitable paths in (1.3).

Now, let P_ε be a local maximum point of u_ε in $\overline{\Omega}$. Our second step is to show that $\mathrm{dist}(P_\varepsilon, \partial\Omega) \leq C\varepsilon$ for ε sufficiently small. Suppose for contradiction that this does not hold. Then, using a scaling argument and the fact that u_ε is a least-energy solution we deduce that there exists a sequence $\varepsilon_j \to 0$ such that a scaled u_{ε_j} tends to w and $c_{\varepsilon_j} \geq \varepsilon_j^n(I(w) - \delta)$ where $\delta > 0$ is an arbitrarily small constant. This contradicts (1.7).

Next, we prove that P_ε actually lies on $\partial\Omega$ for sufficiently small ε. The proof here is more complicated than that of the previous step but the idea is somewhat similar. (The proof above deals with an "interior" situation while here we need to handle the boundary.) Again, suppose that there exists a sequence $\varepsilon_j \to 0$ such that $P_{\varepsilon_j} \to P \in \partial\Omega$ (by step 2 above). We then extend u_{ε_j} by reflection with respect to $\partial\Omega$ near P. (This can be done since u_{ε_j} satisfies the homogeneous Neumann boundary condition.) Although the extended u_{ε_j}, which we denote by $\bar{u}_{\varepsilon_j}$, satisfies a different equation near P, this equation does converge to the equation in (3) and a scaled $\bar{u}_{\varepsilon_j}$ converges to w as $\varepsilon_j \to 0$. Thus, if $P_{\varepsilon_j} \notin \partial\Omega$, then $\bar{u}_{\varepsilon_j}$ would have two peaks near P, and, as the scaled $\bar{u}_{\varepsilon_j}$ converges to w, these two peaks converge to P. Since w is radial and $w'' < 0$ at P, this is not possible.

From the arguments above we observe that local maximum points are isolated. (In fact, $\frac{1}{\varepsilon}\,\mathrm{dist}(P_\varepsilon, P'_\varepsilon) \to \infty$ as $\varepsilon \to 0$ for any two local maximum points P_ε and P'_ε.) Then, from the "energy" consideration it follows that $J_\varepsilon(u_\varepsilon) \geq \varepsilon^n(I(w) - \delta)$ for ε sufficiently small if u_ε has more than one local maximum points (where δ again denotes a small constant). This again contradicts (1.7) and essentially finishes the proof of parts (i) and (ii) of Theorem 1.6.

It remains to prove part (iii). This relies on a second order approximation of c_ε; namely,

(1.9)
$$c_\varepsilon = \varepsilon^n\left\{\frac{1}{2}I(w) - (n-1)H(P_\varepsilon)\gamma\varepsilon + o(\varepsilon)\right\}$$

as $\varepsilon \to 0$, where the quantities $I(w)$ and γ are defined in (1.8). Once (1.9) is established, (iii) follows immediately from (1.7).

To prove (1.9), it requires first to obtain a second order approximation (in ε) to u_ε near P_ε. That is, if Φ is a suitable diffeomorphism which straightens the boundary $\partial\Omega$ near P_ε, then u_ε admits the following approximation

(1.10)
$$u_\varepsilon(\Phi^{-1}(\varepsilon y)) = w(y) + \varepsilon\phi(y) + o(\varepsilon)$$

as $\varepsilon \to 0$, where ϕ is the unique solution of the linearized problem

$$(1.11)\qquad \begin{cases} \Delta\phi - \phi + pw^{p-1}\phi = h & \text{in } \mathbf{R}^n, \\ \phi \to 0 & \text{at } \infty \end{cases}$$

with the condition that for $j = 1, \ldots, n$

$$\int_{\mathbf{R}^n} \phi \frac{\partial w}{\partial y_j} = 0,$$

where the inhomogeneous term h depends on w and Φ (which, in turn, involves the mean curvature of $\partial\Omega$ at P_ε). Substituting (1.10) into (1.2), we derive (1.9) from the fact that $c_\varepsilon = J_\varepsilon(u_\varepsilon)$.

2. Neumann problem: Critical case. In this section, we take up the problem

$$(2.1)\qquad \begin{cases} \varepsilon^2\Delta u - u + u^\tau = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\tau = \frac{n+2}{n-2}$ and $n \geq 3$. As was pointed out before that the problem

$$(2.2)\qquad \begin{cases} \Delta w - w + w^\tau = 0 & \text{in } \mathbf{R}^n, \\ w > 0 \text{ in } \mathbf{R}^n \text{ and } w \to 0 & \text{at } \infty \end{cases}$$

does not admit any solutions. Nonetheless, to a certain extent our method described in §1 does carry over to this critical case with some modifications. Thus we will be brief in this section.

The first general existence result for least-energy solutions of (2.1) is due to X.-J. Wang [W] who also used the variational approach described in §1. Then, a different kind of spike-layers (than those given by Theorem 1.6) is established in [NPT].

THEOREM 2.3. *Let u_ε be a least-energy solution of (2.1). Then, for ε sufficiently small, the maximum of u_ε in $\overline{\Omega}$ is attained at exactly one point P_ε, and P_ε must lie on the boundary $\partial\Omega$. Furthermore, the following statements hold.*

(i) *$u_\varepsilon(P_\varepsilon) \to \infty$ and $u_\varepsilon \to 0$ in $\overline{\Omega}$ outside an arbitrarily small neighborhood of P_ε as $\varepsilon \to 0$.*

(ii) *$\varepsilon^{-n} \int_\Omega u_\varepsilon^{\tau+1} \to \frac{1}{2} S^{n/2}$ as $\varepsilon \to 0$ where $S = n(n-2)\pi \left[\Gamma\left(\frac{n}{2}\right)/\Gamma(n)\right]^{2/n}$ is the best Sobolev constant.*

(iii) *For any $\delta > 0$, there exist constants ε_0 and R such that for $0 < \varepsilon < \varepsilon_0$ it holds that for all $x \in \Omega \cap B_{\beta_\varepsilon \varepsilon R}(P_\varepsilon)$*

$$(2.4)\qquad \left| \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{L^\infty(\Omega)}} - U\left(\frac{\Phi^{-1}(x)}{\beta_\varepsilon \varepsilon}\right) \right| < \delta$$

where $\beta_\varepsilon = \|u_\varepsilon\|_{L^\infty(\Omega)}^{-2/(n-2)}$, Φ *is a suitable diffeomorphism straightening a boundary portion of* $\partial\Omega$ *around* P_ε, *and*

$$U(x) = \left[1 + \frac{|x|^2}{n(n-2)}\right]^{-(n-2)/2}$$

satisfies

$$\Delta U + U^\tau = 0 \tag{2.5}$$

in $\mathbf{R}^n$ *with* $U(0) = 1$.

In fact, part (i) is a consequence of the fact that (2.2) does not have any solutions. Note also that in Theorem 2.3 we only prove that u_ε has one global maximum point in $\overline{\Omega}$, which is a weaker result than its counterpart (Theorem 1.6 (i)) for the sub-critical case. On the other hand, in contrast to the finite-amplitude spikes in the sub-critical case, we now have the peaks $u_\varepsilon(P_\varepsilon)$ (in Theorem 2.3) tending to ∞ as $\varepsilon \to 0$. It is also interesting to note the different scaling appeared in (2.4).

We omit the proof here but refer the interested readers to [NPT] for complete details.

3. Dirichlet problem. As we remarked earlier that if we simply replace the homogeneous Neumann boundary condition in (2) or (1.1) (*but not* (2.1)) by the homogeneous Dirichlet condition "$u = 0$ on $\partial\Omega$", the method described in §1 still applies with minimal modification. (Of course, in the Dirichlet case, locating the peak of a least-energy solution requires new ideas and still remains open.)

To illustrate the power and the flexibility of our method, we shall, however, include a recent result of J. Jang in this section. In his Ph.D. thesis [J], Jang considered the following problem

$$\begin{cases} \varepsilon^2\Delta u + f(u) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where, for simplicity, we assume that $f(u) = u(1-u)(u-a), 0 < a < \frac{1}{2}$, and that Ω is convex.

As in the Neumann case, we define an "energy" functional in $H_0^1(\Omega)$ (i.e. the completion of $C_0^\infty(\Omega)$ under the H^1-norm) by

$$\tilde{J}_\varepsilon(u) = \int_\Omega \left[\frac{\varepsilon^2}{2}|Du|^2 - F(u)\right] \tag{3.2}$$

where $F(u) = \int_0^u f(t)dt$. It is well-known that (3.1) possesses a "boundary-layer" solution $\tilde{u}_\varepsilon$ for each ε sufficiently small. This solution $\tilde{u}_\varepsilon$ may be obtained by using the standard monotone iteration scheme with the iteration started at the super-solution $u \equiv 1$ of (3.1). It is not hard to show *that $\tilde{u}_\varepsilon$ is the maximal solution of* (3.1) *and that* $J_\varepsilon(\tilde{u}_\varepsilon) < 0$ *for ε small.* (See our discussion below.) Thus, using $u \equiv 0$ and $\tilde{u}_\varepsilon$ we can again apply the Mountain-Pass Lemma to ensure the existence of a second solution u_ε – a "mountain-pass" solution.

Jang's result may be stated as follows.

THEOREM 3.3. *For each sufficiently small ε, (3.1) possesses a mountain-pass solution u_ε which has only one local maximum point P_ε in Ω. Moreover, the scaled function $v_\varepsilon(y) \equiv u_\varepsilon(P_\varepsilon + \varepsilon y)$, for $P_\varepsilon + \varepsilon y \in \Omega$, tends to v in $C^2_{\text{loc}}(\mathbb{R}^n)$ where v is the unique solution of*

$$(3.4) \qquad \begin{cases} \Delta v + f(v) = 0 & \text{in } \mathbb{R}^n \\ v > 0 \text{ in } \mathbb{R}^n, \ v \to 0 & \text{at } \infty, \\ v(0) = \max\limits_{\mathbb{R}^n} v. \end{cases}$$

We remark that the convexity assumption of Ω can be relaxed. (See [J].) For the uniqueness of (3.4), we also include a brief discussion in §4.

In order to apply the method described in §1, it is crucial to obtain that $\tilde{J}_\varepsilon(\tilde{u}_\varepsilon) < 0$ for ε small and that

$$(3.5) \qquad \tilde{J}_\varepsilon(u_\varepsilon) \le C\varepsilon^n$$

for ε small. To establish these estimates it is first shown that the maximal solution $\tilde{u}_\varepsilon$ is actually a global minimizer of the functional $\tilde{J}_\varepsilon$ in $H^1_0(\Omega)$. A key ingredient in the proof of this fact is the following "local" uniqueness result for $\tilde{u}_\varepsilon$, which follows from a theorem of Clément and Sweers [CS].

LEMMA 3.6. *There exists a constant $\delta > 0$, independent of ε, such that if u^*_ε is a solution of (3.1) with $\max\limits_\Omega u^*_\varepsilon \ge 1 - \delta$, then $u^*_\varepsilon \equiv \tilde{u}_\varepsilon$.*

Once (3.5) is obtained, one can then show that a mountain-pass solution u_ε has at most finitely many local maximum points (i.e. peaks). To prove that u_ε has exactly one peak, we need a more accurate estimate of $J_\varepsilon(u_\varepsilon)$; namely, for ε small,

$$(3.7) \qquad \tilde{J}_\varepsilon(u_\varepsilon) = \varepsilon^n(J^* + o(1))$$

where

$$J^* = \int_{\mathbb{R}^n} \left[\frac{1}{2}|Dv|^2 - F(v)\right] > 0.$$

The initial step in proving (3.7) is furnished by a result of [GP] which asserts that *for sufficiently large $R > 0$, the following boundary value problem has exactly two positive solutions*

$$\begin{cases} v'' + \frac{n-1}{r} v' + f(v) = 0, & r \in (0, R), \\ v'(0) = v(R) = 0. \end{cases}$$

Now, Theorem 3.3 may be handled in a similar fashion by our method as in §1 since one can derive from (3.7) that

$$\|u_\varepsilon\|^p_{L^p(\Omega)} = \varepsilon^n \left(\|v\|^p_{L^p(\mathbf{R}^n)} + o(1) \right)$$

for ε sufficiently small.

4. Remarks.

4.1. Other types of layer solutions, such as boundary layers or interior transition layers, also arise in various branches of applied mathematics (e.g. in mathematical biology, and in phase transition) and have received lots of attention. There have been many papers appeared in the literature dealing with them and we shall not touch them here.

As for spike layers, it seems that little was known prior to our work, even in the formal level. We ought to mention the recent work of Kelley and Ko [KeK] which contains some interesting examples (or counterexamples). We should also mention an earlier survey [NT3] on (1.1) and (2.1), which contains, among other things, results from the point of view of bifurcation.

4.2. In case Ω is a ball, it is well-known that solutions of (4) (for any Lipschitz continuous nonlinearity f) must be radially symmetric. However, the story changes for Neumann problems. Both Theorems 1.6 and 2.3 show that u_ε cannot be radially symmetric for ε small. Nonetheless, solutions u_ε do possess certain symmetries, e.g. Steiner symmetry. (See [NT1, §5].)

Radial solutions of (1.1) and (2.1) have also been studied by various authors, e.g. [BKP] and [N2]. Other work related to (2.1) include e.g. [CK1,2].

4.3. Multiple single-peak spike-layer solutions have been constructed by [NO1] and [Wa]. In [NO1], it is proved that for any given point $P \in \partial\Omega$ with the second fundamental form of $\partial\Omega$ at P being nondegenerate, there is a spike-layer solution of (1.1) with its only peak located near P, for ε sufficiently small. In [Wa], a topological argument is used to show that for ε small, the number of single-peak spike-layer solutions of (1.1) is bounded below by the Ljusternik-Schnirelman category of $\partial\Omega$, which is a much weaker result.

Using the approach in [NO1], multiple-peak spike-layer solutions are also constructed in [NO2].

4.4. The Mountain-Pass Lemma is due to [AR] originally. The fact that the mountain-pass critical value c_ε is actually independent of the particular choice e and is given by (1.4) for a large class of nonlinearities (formulated in §1, (1.3)–(1.5)) is first observed in [DN] and [N1]. In this connection, we should also mention the work of Nehari [Ne] which gives a different variational approach and eventually leads to the observation (1.4) explicitly appeared in [N1].

4.5. For the uniqueness of (3) and (3.4), we first apply a result of [GNN; Theorem 2] to show that a (any) solution of (3) or (3.4) must be radially symmetric, and thereby reduces (3) and (3.4) to ordinary differential equations.

Extending earlier work of [C], [MS] and [Z], Kwong eventually establishes the uniqueness of (3) in [K]. Further extensions of [K] include [CL], [KZ] and [M].

The uniqueness of the ODE (3.4) is proved by [PS1], which also has been further extended by [PS2] and [KaK].

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