**THE TORSION PROBLEM FOR A NONCONVEX
STORED ENERGY FUNCTION**

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1. INTRODUCTION

Motivation for this work is contained in the experimental observations of Bader and Nadai [1], Takenaka [19], and Nakanishi [14] concerning the torsional deformation of steel bars. Some examples of these observations, taken from Nadai [13], are shown in Figures 4-7 at the end of Section 5 of this paper. These observations show that as the angle of twist is increased, the internal deformation pattern in a steel bar eventually undergoes a transition in which many small sliver-like zones of large shear occur. These zones are generally sector shaped with a periodic circumferential distribution when viewed at in a right cross section, as indicated in Figures 4-7. Sometimes the zones are seen to be wedge-shaped and distributed in the axial direction, in a longitudinal cross section of the bar [14, 15, 19]. The experimental observations show that with increasing angle of twist the distribution of zones of large shear becomes more dense and that eventually the cross section of the bar contains essentially three regions of shear behavior: an inner core of classical torsion with small shear, an outer annular-like region of relatively large shear, and a mixture region in between, in which thin sector shaped, or wedge-shaped zones of large shear are distributed periodically among zones of small shear.

Plasticity theory has been applied by some to study this phenomenon, but it is not yet well understood. In 1928, Nakanishi [14] suggested that to study this problem one should develop the idea that in torsion, mild steel bars of circular cross section will yield when an appropriate weighted mean of the stress in the cross section reaches the lower yield point. Later, Nadai [13] attributed this phenomenon to the “instability of the uniform elastic-plastic mode of yielding under torsion in a round bar as contrasted with a nonuniform yielding in distinct plasticized wedge-shaped layers surrounded by elastic regions.” The basis of his conjecture was Reuss’s [16] theoretical work on a twisted circular bar with a single, thin, radial flow-layer. In addition to these works, which attempt to give a theoretical description of this phenomenon, more recently Shioya and Shioiri [17] have given a numerical calculation which is based upon a known empirical constitutive relation for plasticity and a proposal of theirs concerning the production of initial, mobile dislocations. They modelled the elastic-plastic field as a continuous distribution of dislocations. In the case of a bar of non-circular cross section, we are not aware of any progress on the theoretical modelling of this phenomenon. In 1980, however, Lee and Swedlow [11] gave a numerical computation

based upon classical plasticity theory for the case of a bar with a rectangular cross section. They expressed some concern about the possibility of a mesh dependence in their solution, and were cautious to remark that their results were not definitive in capturing the shear banding process.

We shall consider this phenomenon within the framework of minimization in non-linear elasticity theory for a non-convex stored energy function. This approach will offer a new perspective on the origin of the sector shaped, or the wedge-shaped zones of large shear. From an energetic point of view, the optimal deformed configuration of an elastic body is one which associates with the lowest total potential energy among all those that are admissible. If one considers the experimentally observed sector shaped, or the wedge-shaped zones of large shear as being of prime importance in determining the optimal deformed configuration of highly twisted cylinders, then our goal is to provide a theoretical basis for why there exists a minimizing configuration of the total potential energy with such complicated structures.

The existence of zones of large shear distributed among the zones of small shear is evidence that there are two phases – large shear and small shear phases – in equilibrium coincidently. Questions concerning the coexistence of equilibrium phases in solid, fluid, and fluid-like materials have been discussed by many authors. Some of these works, which have a bearing on the present study, are: Ball and James [2], Bauman and Phillips [3], Dunn and Fosdick [4], Ericksen [5], Fosdick [6], Fosdick and MacSithigh [7], Gurtin [9], James [10], and Silling [18]. These investigations propose that the coexistence of distinct equilibrium phases is related to the loss of local convexity of the stored energy function for the material.

The idea that the stored energy function has a loss of local convexity in parts of its domain of definition is adopted in our analysis. We shall consider the torsion of a solid incompressible, isotropic, and homogeneous elastic bar of right circular cross section. This problem is special in that it has a universal solution in nonlinear elasticity theory. Of course, this universal solution may not be stable, and to a large extent the present work contributes to this question. We show, for those stored energy functions that satisfy the mild constitutive assumptions given in Section 2, that when the angle of torsion is sufficiently large there is a convergent coarse-mixture sequence of admissible deformations for which the corresponding total potential energy converges to its infimum. We restrict our considerations to a class of plane deformations which preserve the height of the cylinder.

The remainder of this introduction contains a brief outline of the structure and results of this work. In Section 2 we give some preliminary considerations, including boundary conditions and the main Constitutive Hypotheses for the stored energy function. The boundary conditions (4) and (5) state that the solid cylinder is traction free on its lateral surface, and that its ends are held planar and twisted through a given relative angle such that the length of the cylinder remains unchanged. An incompressible, isotropic, and homogeneous elastic solid is considered, and the corresponding stored energy function is assumed to depend only on the first principal invariant of the left Cauchy-Green strain tensor. We replace this dependence with a more convenient, but equivalent, strain invariant κ , and state our main Constitutive Hypotheses on the stored energy function so that it will lose its local convexity only when κ lies in a certain finite interval. The main minimization problem (10) is formulated, and the admissible class of functions $\mathcal{A}$ for this problem is defined in (9) as the set of all isochoric plane deformations which satisfy the displacement boundary conditions, and which belong to the Sobolev space $W^{1,p}$, where $p \geq 1$.

As a first step in our analysis of problem (10), we examine, in Section 3, certain kinematic requirements on the deformation gradient of a possible minimizing field. Lemma 2 develops two important elementary properties of the strain invariant κ associated with any deformation $\chi \in \mathcal{A}$. This suggests the introduction of a subset $\mathring{\mathcal{A}} \subset \mathcal{A}$, in which a solution of (10) must be sought, and in Lemma 3 and Theorem 1 we give the technical details which support this claim. Finally, we show, in Theorem 2, that in order for a deformation field $\chi \in \mathring{\mathcal{A}}$ to minimize the total potential energy per unit cross section area of the cylinder, the associated strain invariant κ must either be a point of convexity of the stored energy function, or be a convex combination of two common support points of the stored energy function. This requirement on the strain invariant κ is very important since it provides a basis upon which a minimizing sequence for the main problem (10) is constructed in Section 4.

We devote Section 4 to the detailed construction of a sequence of continuous, isochoric, admissible deformation fields in $\mathcal{A}$ for moderate angles of twists. Every member of the sequence resembles a coarse-mixture of small and large shears smoothly connected to a region of classical torsion by a small transition zone, and the limit of the associated deformation gradients satisfies the hypotheses of Theorems 1 and 2. We then briefly discuss the similarities between this construction for moderate twists and the analogous construction for any other possible angle of twist, without introducing

details that would be repetitive. Let it suffice to say, here, that there are three ranges of the angle of twist which distinguish characteristic changes in the construction of this sequence and the corresponding structure of a solution of (10). For small angles of twist the problem is now classical, albeit nonlinear, and the well-known universal and elementary torsion field applies. For moderate angles of twist, according to our construction, the axial core of the cylinder is supposed to support a universal (single-phase) deformation field and in the remaining annular region we introduce, roughly, the (two-phase) limit of a sequence of deformation fields, each member of which is a coarse-mixture structure with small transition zone for interpolation. For large angles of twist, both the axial core of the cylinder and an annular region adjacent to the lateral boundary are supposed to support a universal (single-phase) deformation field, while the deformation of the remaining annular domain between these two regions is considered to be the (two-phase) limit of a sequence of coarse-mixture fields with appropriate interpolation zones.

In Section 5, we make use of the above construction and show the existence of a planar-like solution to the minimization problem (10) which preserves the height of the cylinder for any prescribed angle of twist. This solution is either the classical smooth torsion field for small angles of twist, or for moderate and large angles of twist it corresponds to the limit of a minimizing sequence. To pursue this goal, we first prove in Theorem 5 that for moderate angles of twist the sequence of admissible deformation fields given in (50) induces a corresponding sequence of total potential energies which converges to its infimum in $\mathcal{A}$. Although the limit of the sequence (50) itself is not a minimizer of problem (10), and, in fact, is not in $\mathcal{A}$, nevertheless, Theorem 5 delivers (50) as a minimizing sequence. Considering the similarities, we simply state, with brief justification, analogous results for all other possible angles of twist in Theorem 6.

In the closing remarks of Section 5, we show some photographs from Nadai [13] which illustrate the cross sectional morphology that is induced in soft iron bars subject to a large torsion. We briefly discuss the experimental observations and note some similarities between the observed deformation-induced microstructures and the minimizing sequence that we constructed for this problem.

2. PRELIMINARIES

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ denote the usual orthonormal set of base vectors for a fixed rectangular cartesian coordinate frame, and let $\mathcal{B}$ denote a natural reference configuration for an elastic, incompressible, homogeneous, and isotropic cylinder of unit radius and length H . We choose the axis of $\mathcal{B}$ to coincide with $\mathbf{e}_3$, and locate the particles $\mathbf{X} \in \mathcal{B}$ either by their rectangular cartesian coordinates (X_1, X_2, X_3) or their related cylindrical coordinates (R, Θ, X_3) . In particular,

$$\mathcal{B} \equiv \{(R, \Theta, X_3) | 0 \leq R \leq 1, 0 \leq \Theta < 2\pi, 0 \leq X_3 \leq H\}.$$

The cylinder is said to undergo a plane deformation $\mathbf{X} \rightarrow \mathbf{x} = \chi(\mathbf{X})$ if

$$\begin{aligned} x_1 &= x_1(\mathbf{X}), \\ x_2 &= x_2(\mathbf{X}), \\ x_3 &= X_3, \end{aligned} \tag{1a}$$

where (x_1, x_2, x_3) denotes the rectangular coordinates of $\mathbf{x} \in \chi(\mathcal{B})$. Alternatively, we may express this plane deformation in terms of the related cylindrical coordinates (r, θ, x_3) and then write

$$\begin{aligned} r &= r(R, \Theta, X_3), \\ \theta &= \theta(R, \Theta, X_3), \\ x_3 &= X_3. \end{aligned} \tag{1b}$$

The deformation gradient $\mathbf{F} = \nabla \chi$ has the form

$$\mathbf{F} = \mathbf{e}_1 \otimes \nabla x_1 + \mathbf{e}_2 \otimes \nabla x_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{2}$$

where here and throughout this work, ∇ denotes the gradient operator with respect to $\mathbf{X} \in \mathcal{B}$. Clearly, the deformation is isochoric if and only if

$$\det \mathbf{F} = (\nabla x_1 \times \nabla x_2) \cdot \mathbf{e}_3 = 1. \tag{3}$$

We shall decompose the boundary of the cylinder $\mathcal{B}$ into two complementary parts: the lateral surface (traction part) $\partial_2 \mathcal{B}$ and the ends (place part) $\partial_1 \mathcal{B}$. Thus,

$$\partial_1 \mathcal{B} \equiv \{(R, \Theta, X_3) \in \mathcal{B} | 0 \leq R < 1, 0 \leq \Theta < 2\pi, X_3 = 0 \text{ or } H\},$$

and

$$\partial_2 \mathcal{B} \equiv \{(R, \Theta, X_3) \in \mathcal{B} | R = 1, 0 \leq \Theta < 2\pi, 0 < X_3 < H\}.$$

For torsion, while $\partial_2 \mathcal{B}$ is supposed to be traction-free, the end $X_3 = 0$ is fixed, so that

$$\begin{aligned} r(R, \Theta, 0) &= R, \\ \theta(R, \Theta, 0) &= \Theta, \\ x_3 &= 0, \end{aligned} \tag{4}$$

and the other end, $X_3 = H$, is rigidly rotated through a given angle of twist $\Psi > 0$, i.e.,

$$\begin{aligned} r(R, \Theta, H) &= R, \\ \theta(R, \Theta, H) &= \Theta + \Psi, \\ x_3 &= H. \end{aligned} \tag{5}$$

Let W denote the stored energy function for $\mathcal{B}$ per unit reference volume, and let I and II denote the first and second principal invariant of $\mathbf{F}\mathbf{F}^T = \mathbf{B}$, respectively. Then, $I = \text{tr}\mathbf{B} = \mathbf{F} \cdot \mathbf{F} = |\mathbf{F}|^2$, and $II = \frac{1}{2}((\text{tr}\mathbf{B})^2 - \text{tr}(\mathbf{B}^2))$, and from isotropy, objectivity, homogeneity, and incompressibility, it follows that W may be expressed as a function of I and II . In this work, W is assumed to depend on the first invariant I only, and so we write $W = W(I)$. Alternatively, the following lemma allows us to introduce a somewhat more convenient invariant than I .

Lemma 1. *For any isochoric deformation, the first invariant satisfies $I \geq 3$, where equality holds if and only if the three principal stretches of $\mathbf{B}$ are equal to one; i.e., the deformation is rigid.*

Proof. Let λ_i , $i = 1, 2, 3$, be the three principal stretches of $\mathbf{B}$, respectively. The first invariant of $\mathbf{B}$ then can be written as

$$I(\lambda_i) = \lambda_1 + \lambda_2 + \lambda_3.$$

Since the deformation field is isochoric, we have $\lambda_1 \lambda_2 \lambda_3 = 1$, and so

$$I(\lambda_i) = \hat{I}(\lambda_2, \lambda_3) = \frac{1}{\lambda_2 \lambda_3} + \lambda_2 + \lambda_3.$$

Since λ_i must be real and positive it then follows that,

$$\frac{\partial^2 \hat{I}(\lambda_2, \lambda_3)}{\partial \lambda_2^2} \frac{\partial^2 \hat{I}(\lambda_2, \lambda_3)}{\partial \lambda_3^2} - \left(\frac{\partial^2 \hat{I}(\lambda_2, \lambda_3)}{\partial \lambda_2 \partial \lambda_3} \right)^2 = 3(\lambda_2 \lambda_3)^{-4} > 0,$$

and so $\frac{\partial \hat{I}(\lambda_2, \lambda_3)}{\partial \lambda_2} = 0$ and $\frac{\partial \hat{I}(\lambda_2, \lambda_3)}{\partial \lambda_3} = 0$ hold if and only if $\lambda_2 = \lambda_3 = 1$. Thus, $\hat{I}(1, 1) = 3$ is the unique minimum of $\hat{I}(\lambda_2, \lambda_3)$, i.e., $I(\lambda_i) \geq 3$. Since $\lambda_2 = \lambda_3 = \lambda_1 = 1$ is the unique minimizer of $I(\lambda_i)$, then $I(\lambda_i) = 3$ holds if and only if $\lambda_2 = \lambda_3 = \lambda_1 = 1$. Δ

We now introduce the strain invariant

$$\kappa = \kappa(R, \Theta, X_3) \equiv \sqrt{I(R, \Theta, X_3) - 3} \geq 0, \tag{6}$$

and write

$$W = W(I) = \overline{W}(\kappa) \Big|_{\kappa=\sqrt{I-3}}. \tag{7}$$

Figure 1

In the following discussion we shall require $\bar{W}(\cdot)$ to have the properties shown in Figures 1 and 2. More specifically, if $\bar{W}'(\cdot)$ and $\bar{W}''(\cdot)$ denote the first and second derivatives of $\bar{W}$ with respect to κ , we require the following

Constitutive Hypotheses:

- (i) $\bar{W}(\cdot)$ is class C^2 ;
- (ii) there exist κ_3 and κ_4 with $0 < \kappa_3 < \kappa_4 < \infty$ such that
 - (a) $\bar{W}''(\kappa_3) = \bar{W}''(\kappa_4) = 0$,
 - (b) $\bar{W}''(\kappa) > 0 \quad \forall \kappa \in [0, \kappa_3) \cup (\kappa_4, \infty)$,
 - $\bar{W}''(\kappa) < 0 \quad \forall \kappa \in (\kappa_3, \kappa_4)$;
- (iii) there exist κ_1 and κ_2 with $0 < \kappa_1 < \kappa_3 < \kappa_4 < \kappa_2 < \infty$ such that
 - (a) $\bar{W}'(\kappa_1) = \bar{W}'(\kappa_2)$,
 - (b) $\bar{W}(\kappa_2) - \bar{W}(\kappa_1) = (\kappa_2 - \kappa_1)\bar{W}'(\kappa_1)$,
 - (c) $\bar{W}'(0) \geq 0$.

These conditions [7] require that $\bar{W}(\cdot)$ be sufficiently smooth, convex in the domain $[0, \kappa_1) \cup (\kappa_2, \infty)$, and strictly non-convex in the interval (κ_1, κ_2) . The values of κ_3 and κ_4 correspond to points in the interval (κ_1, κ_2) where local convexity breaks down. It also is noteworthy that the Constitutive Hypotheses imply that $\bar{W}'(\kappa) > 0$ for $\kappa \in (0, \kappa_3) \cup (\kappa_2, \infty)$, and that there is a unique point $\kappa = \kappa_5 \in (\kappa_3, \kappa_4)$, such that $\bar{W}'(\kappa_5) = \bar{W}'(\kappa_1)$, $\bar{W}'(\kappa) > \bar{W}'(\kappa_1)$ for $\kappa \in (\kappa_3, \kappa_5)$, and $\bar{W}'(\kappa) < \bar{W}'(\kappa_1)$ whenever $\kappa \in (\kappa_5, \kappa_4)$.

Since we shall not consider body forces in this work and since $\partial\mathcal{B}$ is traction-free, we may write the total potential energy of $\mathcal{B}$ as the volume integral

$$E[\chi] = \int_{\mathcal{B}} \bar{W}(\kappa(R, \Theta, X_3)) dV. \quad (8)$$

Figure 2

Then, if $\mathcal{A}$ denotes the class of functions

$$\mathcal{A} \equiv \{\chi \in W^{1,p}(\mathcal{B}) \mid \chi \text{ is of the planar form (1), } \det \nabla \chi = 1, \text{ a.e., } \chi = \chi^* \text{ on } \partial_1 \mathcal{B}\}, \quad (9)$$

where χ^* denotes the boundary data given by (4) and (5) and $p > 2$ is a positive integer,¹ the minimization problem which we study here is to determine $\tilde{\chi} \in \mathcal{A}$ so that

$$E[\tilde{\chi}] = \min_{\chi \in \mathcal{A}} E[\chi]. \quad (10)$$

For any $\chi \in \mathcal{A}$, $F \equiv \nabla \chi$ must exist almost everywhere, and be measurable. Also note that because of the boundary data (4) and (5), we must have

$$\int_0^H F e_3 \cdot e_\Theta dX_3 = R\Psi \quad (11)$$

¹The choice of $p > 2$ is for the purpose of a technical requirement later in Theorem 4.

for any $R \in [0, 1]$, where $\mathbf{e}_\Theta = -\sin \Theta \mathbf{e}_1 + \cos \Theta \mathbf{e}_2$. In the remainder of this work we shall use $\{\mathbf{e}_R, \mathbf{e}_\Theta, \mathbf{e}_3\}$ to denote the usual cylindrical orthonormal basis at $\mathbf{X} \in \mathcal{B}$ and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$ to denote the analogous basis at $\mathbf{x} \in \chi(\mathcal{B})$.

3. SETUP OF THE VARIATIONAL PROBLEM

In this section we investigate certain technical requirements that the deformation gradient field for any minimizer of (10) must satisfy. These requirements will be used in Section 4 to construct an admissible deformation field that in general leads to a minimizing sequence for the minimization problem.

Lemma 2. *Let $\chi \in \mathcal{A}$, and according to (6), define $\kappa = \sqrt{I - 3}$, where $I = \text{tr} \mathbf{F} \mathbf{F}^T$ and $\mathbf{F} = \nabla \chi$. Then,*

$$\kappa \geq \mathbf{F} \mathbf{e}_3 \cdot \mathbf{e}_\Theta \quad (12)$$

almost everywhere in $\mathcal{B}$. In addition, if Ψ is the prescribed angle of twist as given in the boundary condition (5), then

$$\int_0^H \kappa(R, \Theta, X_3) dX_3 = R\Psi \quad (13)$$

for any $R \in [0, 1]$, $\Theta \in [0, 2\pi)$, if and only if

$$\mathbf{F} = 1 + \kappa \mathbf{e}_\Theta \otimes \mathbf{e}_3 \text{ a.e.} \quad (14)$$

Proof. Note that for any $\chi \in \mathcal{A}$ we have

$$\begin{aligned} \mathbf{F} = & F_{RR} \mathbf{e}_R \otimes \mathbf{e}_R + F_{\Theta\Theta} \mathbf{e}_\Theta \otimes \mathbf{e}_\Theta + F_{R\Theta} \mathbf{e}_R \otimes \mathbf{e}_\Theta + F_{\Theta R} \mathbf{e}_\Theta \otimes \mathbf{e}_R \\ & + F_{R3} \mathbf{e}_R \otimes \mathbf{e}_3 + F_{\Theta 3} \mathbf{e}_\Theta \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_3, \end{aligned}$$

so that

$$\kappa^2 = F_{RR}^2 + F_{\Theta\Theta}^2 + F_{R\Theta}^2 + F_{\Theta R}^2 + F_{R3}^2 + F_{\Theta 3}^2 - 2.$$

Now, since χ is isochoric we must satisfy

$$F_{RR} F_{\Theta\Theta} - F_{R\Theta} F_{\Theta R} = 1,$$

and so

$$\kappa = \sqrt{(F_{RR} - F_{\Theta\Theta})^2 + (F_{R\Theta} + F_{\Theta R})^2 + F_{R3}^2 + F_{\Theta 3}^2}. \quad (15)$$

This clearly implies

$$\kappa \geq F_{\Theta 3}$$

to complete the proof of (12).

Now, suppose that (14) holds. Then,

$$\mathbf{F} \mathbf{e}_3 \cdot \mathbf{e}_\Theta = \kappa,$$

and because of (11) we see that (13) also must hold. On the other hand, suppose (13) holds. Then, since for any $\chi \in \mathcal{A}$ (23) holds, we conclude, using (11), that $\kappa(R, \Theta, X_3) = \mathbf{F} \mathbf{e}_3 \cdot \mathbf{e}_\Theta$ a.e. in $\mathcal{B}$. Therefore, to reach (14), we must show that all components of $\mathbf{F} - \mathbf{1}$, except $F_{\Theta 3}$, are zero. First, from (15) we see that for such a deformation field, the following relations must hold:

$$F_{RR} - F_{\Theta\Theta} = 0, \quad (16)$$

$$F_{R\Theta} + F_{\Theta R} = 0, \quad (17)$$

and

$$F_{R3} = 0. \quad (18)$$

Since (18) implies $\frac{\partial r}{\partial X_3} = 0$, i.e., $r(R, \Theta, X_3) = r(R, \Theta, 0) = r(R, \Theta, H)$ for all $R \in [0, 1]$, $\Theta \in [0, 2\pi]$, and $X_3 \in [0, H]$, we see, using (4) and (5) that

$$r(R, \Theta, X_3) = R \quad (19)$$

for all $R \in [0, 1]$, $\Theta \in [0, 2\pi]$. Consequently, $F_{RR} \equiv \frac{\partial r}{\partial R} = 1$, and then (16) yields $F_{\Theta\Theta} = 1$. In addition, since $F_{R\Theta} \equiv \frac{1}{R} \frac{\partial r}{\partial \Theta} = 0$, we see from (17) that $F_{\Theta R} = 0$, to complete this lemma. Δ

Let $\mathring{\mathcal{A}}$ be a subspace of $\mathcal{A}$ such that

$$\mathring{\mathcal{A}} \equiv \left\{ \chi \in \mathcal{A} \left| \int_0^H \kappa dX_3 = R\Psi \text{ for any } R \in [0, 1], \Theta \in [0, 2\pi), \text{ and } \kappa \text{ is defined by (6)} \right. \right\}.$$

Clearly, only those deformations in $\mathcal{A}$ which also satisfy (14) can belong to $\mathring{\mathcal{A}}$. For example, any deformation such that

$$\begin{aligned} r &= R, \\ \theta &= \Theta + g(X_3), \\ x_3 &= X_3 \end{aligned} \quad (20)$$

where $g'(X_3)$ is continuous for all $X_3 \in [0, H]$, $g(0)$, and $g(H) = \Psi$, is a member of $\mathring{\mathcal{A}}$, though (20) is not a complete representation of $\mathring{\mathcal{A}}$. We shall discuss this further in a remark after Theorem 1, later in the present section.

Lemma 3. Let $\tilde{\chi} \in \mathring{\mathcal{A}}$ and define $\tilde{\kappa} \equiv \sqrt{|\nabla \tilde{\chi}|^2 - 3}$. Suppose that for such $\tilde{\chi}$, $\overline{W}'(\tilde{\kappa}) > 0$ is independent of X_3 . Then,

$$\int_{\mathcal{B}} (\kappa - \tilde{\kappa}) \overline{W}'(\tilde{\kappa}) dV \geq 0 \quad (21)$$

for all $\chi \in \mathcal{A}$, where $\kappa \equiv \sqrt{|\nabla \chi|^2 - 3}$. The equality holds if and only if $\chi \in \mathring{\mathcal{A}}$.

Proof. Note that

$$\int_{\mathcal{B}} (\kappa - \tilde{\kappa}) \overline{W}'(\tilde{\kappa}) dV = \int_0^1 \int_0^{2\pi} \int_0^H (\kappa - \tilde{\kappa}) \overline{W}'(\tilde{\kappa}) R dR d\Theta dX_3,$$

and observe that for any $\chi \in \mathcal{A}$ we have

$$\int_0^H \kappa dX_3 \geq R\Psi, \quad (22)$$

since $\kappa \geq \mathbf{F} \mathbf{e}_3 \cdot \mathbf{e}_\Theta$ pointwise by Lemma 2 and $\mathbf{F} \mathbf{e}_3 \cdot \mathbf{e}_\Theta$ satisfies (11). Note also that for any $\tilde{\chi} \in \mathring{\mathcal{A}}$, we have $\int_0^H \kappa dX_3 = R\Psi$. Therefore, if $\overline{W}'(\tilde{\kappa}) > 0$ and is independent of X_3 , we may write,

$$\int_0^H (\kappa - \tilde{\kappa}) \overline{W}'(\tilde{\kappa}) dX_3 = \overline{W}'(\tilde{\kappa}) \int_0^H (\kappa - \tilde{\kappa}) dX_3 \geq 0,$$

and (21) follows immediately.

Since equality in (22) holds if and only if $\chi \in \mathring{\mathcal{A}}$ (because of the second part of Lemma 2), the equality in (21) then holds if and only if $\chi \in \mathring{\mathcal{A}}$. $\triangle$

Note that since

$$\int_{\mathcal{B}} \overline{W}(\kappa) dV = \int_0^1 \int_0^{2\pi} \int_0^H \overline{W}(\kappa) R dR d\Theta dX_3,$$

then

$$\bar{E}[\chi](R, \Theta) \equiv \int_0^H \overline{W}(\kappa(R, \Theta, X_3)) dX_3$$

is the total potential energy per unit cross section area of the cylinder. If $\inf \bar{E}(R, \Theta)$ denotes the infimum of $\bar{E}[\chi](R, \Theta)$ at (R, Θ) among all $\chi \in \mathcal{A}$, then, of course, we have

$$\inf \bar{E}(R, \Theta) \leq \bar{E}[\chi](R, \Theta), \quad (23)$$

and the equality in (23) can be achieved for some fixed $\tilde{\chi} \in \mathcal{A}$, for any $R \in [0, 1]$ and $\Theta \in [0, 2\pi)$, if and only if

$$\bar{E}[\tilde{\chi}](R, \Theta) \leq \bar{E}[\chi](R, \Theta)$$

for all $\chi \in \mathcal{A}$, i.e., if and only if $\tilde{\chi}$ is minimizer of $\bar{E}[\cdot](R, \Theta)$ in $\mathcal{A}$.

Since it is not necessary for $\inf \bar{E}(R, \Theta)$ to be achieved by any one member of $\mathcal{A}$ for all $R \in [0, 1]$, $\Theta \in [0, 2\pi)$, then, we must have

$$E[\chi] \geq \int_0^1 \int_0^{2\pi} \inf \bar{E}(R, \Theta) R dR d\Theta, \quad (24)$$

for all $\chi \in \mathcal{A}$, and it is easy to show that the equality of (24) holds if and only if there is a $\chi \in \mathcal{A}$ such that $\bar{E}[\chi](R, \Theta) = \inf \bar{E}(R, \Theta)$ for all $R \in [0, 1]$, $\Theta \in [0, 2\pi)$.

Next we shall discuss the conditions under which $\bar{E}[\chi](R, \Theta)$ reaches its infimum $\inf \bar{E}(R, \Theta)$ for any fixed $R \in [0, 1]$, and $\Theta \in [0, 2\pi)$.

Theorem 1. Suppose $\tilde{\chi} \in \mathring{\mathcal{A}}$ is a minimizer of $\bar{E}[\cdot](R, \Theta)$ at fixed (R, Θ) such that $\tilde{\kappa} \equiv \sqrt{|\nabla \tilde{\chi}|^2 - 3}$ is a point of convexity of $\bar{W}(\cdot)$, and $\bar{W}'(\tilde{\kappa}) > 0$ is independent of X_3 at (R, Θ) . Then,

$$\int_0^H \bar{W}(\kappa) dX_3 \geq \int_0^H \bar{W}(\tilde{\kappa}) dX_3 \quad (25)$$

for all $\chi \in \mathcal{A}$, where $\kappa \equiv \sqrt{|\nabla \chi|^2 - 3}$. Moreover, the equality is possible only if $\chi \in \mathring{\mathcal{A}}$.

Proof. Let $\tilde{\chi} \in \mathring{\mathcal{A}}$ and let $\tilde{\kappa}(R, \Theta, X_3)$ be a point of convexity of $\bar{W}(\cdot)$ for any given $(R, \Theta, X_3) \in \mathcal{B}$. Then

$$\bar{W}(\lambda) \geq \bar{W}(\tilde{\kappa}(R, \Theta, X_3)) + (\lambda - \tilde{\kappa})\bar{W}'(\tilde{\kappa}(R, \Theta, X_3)), \quad \forall \lambda \in (0, \infty),$$

and we see that, for any $\chi \in \mathcal{A}$,

$$\int_0^H \bar{W}(\kappa) dX_3 \geq \int_0^H \bar{W}'(\tilde{\kappa}) dX_3 + \int_0^H (\kappa - \tilde{\kappa})\bar{W}'(\tilde{\kappa}) dX_3, \quad \forall R \in [0, 1], \Theta \in [0, 2\pi).$$

Thus, by Lemma 3 and the hypothesis that $\bar{W}'(\tilde{\kappa}) > 0$ is independent of X_3 , we see that (25) holds.

If $\chi \in \mathring{\mathcal{A}}$, then

$$\int_0^H (\kappa - \tilde{\kappa}) dX_3 = 0,$$

and the equality in (25) is now possible. Δ

The key assumptions for Lemma 3 and Theorem 1 are that $\tilde{\chi} \in \mathring{\mathcal{A}}$, and that $\overline{W}'(\tilde{\kappa})$ is independent of X_3 . If $\tilde{\kappa}$ does not depend on X_3 then neither does $\overline{W}'(\tilde{\kappa})$. However, for the given material, this assumption may still hold even if $\tilde{\kappa}$ does depend on X_3 . To see this, note that if for a given $R \in [0, 1]$ we have $R\Psi/H \notin (\kappa_1, \kappa_2)$, it is natural to try $\tilde{\kappa} = R\Psi/H$ which is point of convexity of $\overline{W}(\cdot)$ and such that $\overline{W}'(\tilde{\kappa})$ is independent of X_3 . However, if for such an R , we have $R\Psi/H \in (\kappa_1, \kappa_2)$, then it would be more natural to try

$$\kappa = \begin{cases} \kappa_1 & \text{for } X_3 \in \mathcal{P}, \\ \kappa_2 & \text{for } X_3 \in [0, H] \setminus \mathcal{P} \end{cases} \quad (26)$$

where $\mathcal{P}$ is a collection of subintervals of $[0, H]$ of total length λH with

$$\lambda = \frac{\kappa_2 - R\Psi/H}{\kappa_2 - \kappa_1}. \quad (27)$$

In this situation, it follows (see Figure 1) that $\overline{W}'(\tilde{\kappa})$ is independent of X_3 and

$$\begin{aligned} \int_0^H \tilde{\kappa} dX_3 &= \int_{\mathcal{P}} \kappa_1 dX_3 + \int_{\mathcal{Q}} \kappa_2 dX_3 \\ &= (\lambda \kappa_1 + (1 - \lambda) \kappa_2) H \\ &= R\Psi, \end{aligned}$$

where $\mathcal{Q} \equiv [0, H] \setminus \mathcal{P}$. Clearly, $\tilde{\kappa}$ is a function of X_3 in this case. Now, it is straightforward to show that if $\tilde{\kappa} = R\Psi/H$ everywhere in $\mathcal{B}$, then the corresponding deformation field $\tilde{\chi}$ is given by (20) with $g(X_3) = \frac{\Psi}{H} X_3$. However, if $\tilde{\kappa}$ is given by (26) and (27), there is a question about the existence of such a corresponding deformation field in $\mathcal{A}$. In fact, it has been noted by MacSithigh [12] that in this case it is impossible to construct a continuous deformation field which is piecewise C^1 such that the associated deformation gradient satisfies (26) and (27). Thus, since the Sobolev space $W^{1,p}(\cdot)$ is a completion of the space $C^0 \cap \text{piecewise } C^1$, it may still be true that there is no such deformation in $\mathcal{A}$. Later, in Section 4, we shall show that even though for a certain range of the angle of twist Ψ there is no such deformation field in $\mathcal{A}$, nevertheless, it is possible to construct a coarse sequence of deformation fields in $\mathcal{A}$ for which the limit of the associated deformation gradients converge to (26) and (27) not in $\mathcal{A}$, but in a measure theoretic sense.

In Theorem 1, we assumed that there is a deformation field $\tilde{\chi} \in \mathring{\mathcal{A}}$ such that for each $R \in [0, 1]$ and $\Theta \in [0, 2\pi)$ $\bar{E}[\chi](R, \Theta)$ reaches its minimum value $\inf \bar{E}(R, \Theta)$ in the class $\mathring{\mathcal{A}}$. For determining $\bar{E}[\chi](R, \Theta)$, we have the following elementary

Theorem 2. *Suppose $\tilde{\chi} \in \mathring{\mathcal{A}}$ is a minimizer of $\bar{E}[\cdot](R, \Theta)$ at (R, Θ) , and $\tilde{\kappa} \equiv \sqrt{|\nabla \tilde{\chi}|^2 - 3}$. Let the stored energy function $\bar{W}(\kappa)$ be as depicted in Figure 1 and satisfy the associated Constitutive Hypotheses of Section 2. Note that κ_1 and κ_2 are two common support points of convexity of $\bar{W}(\kappa)$ as shown in Figure 1. (a) If $R\Psi/H$ is a point of convexity of $\bar{W}(\kappa)$, then $\inf \bar{E}(R, \Theta) = \bar{W}(R\Psi/H)H$, $\tilde{\kappa} = R\Psi/H$, and $\tilde{\kappa}$ is unique. (b) If $R\Psi/H \in (\kappa_1, \kappa_2)$, then, $\inf \bar{E}(R, \Theta) = (\lambda \bar{W}(\kappa_1) + (1 - \lambda) \bar{W}(\kappa_2))H$, and $\tilde{\kappa}$ is given by (26) and (27) with the corresponding collection of subintervals $\mathcal{P}$ unique up to a “rearrangement”.*

Proof. (a) Let $\tilde{\kappa} = R\Psi/H$. Since $R\Psi/H$ is a point of convexity of $\bar{W}(\cdot)$, then for all $\chi \in \mathring{\mathcal{A}}$ we have

$$\bar{W}(\kappa) > \bar{W}(\tilde{\kappa}) + (\kappa - \tilde{\kappa})\bar{W}'(\tilde{\kappa}) \quad \forall \kappa \neq \tilde{\kappa},$$

and the conclusion follows from Theorem 1 due to the fact that $\tilde{\kappa}$, as well as $\bar{W}'(\tilde{\kappa})$, is independent of X_3 . (b) Let $\tilde{\kappa}$ be given by (26) with λ given by (27). Then, for all $\chi \in \mathring{\mathcal{A}}$,

$$\bar{W}(\kappa) \geq \bar{W}(\tilde{\kappa}) + (\kappa - \tilde{\kappa})\bar{W}'(\tilde{\kappa}),$$

and the equality holds if and only if $\kappa = \kappa_1$, or κ_2 . Using Theorem 1 as above, we have

$$\int_0^H \bar{W}(\kappa) dX_3 \geq \int_0^H \bar{W}(\tilde{\kappa}) dX_3$$

and the equality holds if and only if $\kappa = \kappa_1$, or κ_2 in agreement with the relations (26) and (27). Finally, by using (26) and (27) on the right hand side above, we reach the stated conclusion. $\triangle$

4. AN ADMISSIBLE DEFORMATION FIELD

In the last section, we introduced certain kinematic requirements on $\nabla \chi$ such that the infimum of the total potential energy might be reached by a minimizing sequence in $\mathcal{A}$. There, we noted, in the latter half of a remark following Theorem 1, that there may not be such a deformation field in $\mathcal{A}$, itself. In the present section, we shall construct a minimizing sequence of admissible deformation fields χ_N in $\mathcal{A}$, $N = 1, 2, \dots$, each of which satisfies the displacement boundary conditions (4) and

(5) on $\partial_1 \mathcal{B}$. The limit, $N \rightarrow \infty$, of $\nabla \chi_N$ will have the properties stated in Theorem 1, i.e., the limit of κ_N is associated only with the convex points of the stored energy function $\overline{W}(\cdot)$, and the limit of $\overline{W}'(\kappa_N)$ (which is positive) is independent of X_3 . These properties will allow us to prove, in Section 5, that the sequence of potential energies associated with χ_N converges to the infimum of the total potential energy.

If, for a given Ψ , $R\Psi/H$ is a point of convexity of $\overline{W}(\cdot)$ for all $R \in [0, 1]$, as is the case when $\Psi/H \leq \kappa_1$, then, by Theorems 1 and 2, there is a unique minimizer of (10) in $\mathcal{A}$. This minimizer is, in fact, the classical torsion field $\chi_0(\mathbf{X}) = \mathbf{Q}(\frac{\Psi}{H}X_3)\mathbf{X}$, where here and throughout the remainder of this work the orthogonal transformation $\mathbf{Q}(\lambda)$ denotes a rotation about the $\mathbf{e}_3$ -axis of angle λ . In cylindrical coordinates χ_0 has the specific form

$$\begin{aligned} r &= R, \\ \theta &= \Theta + \frac{\Psi}{H} X_3, \\ x_3 &= X_3. \end{aligned} \tag{28}$$

However, if $R\Psi/H \in (\kappa_1, \kappa_2)$ for some $R \in [0, 1]$, then (28) is not a solution of (10) since this field is not stable at those values of R . In this case, to construct an admissible field in $\mathcal{A}$, let us consider, for illustrative purposes, the situation $\kappa_1 < \Psi/H \leq \kappa_2$. Then, defining $R_1 \equiv \frac{\kappa_1}{\Psi/H} < 1$ and $R_2 \equiv \frac{\kappa_2}{\Psi/H} \geq 1$, we see that for $R \in (R_1, 1)$, we have $R\Psi/H \in (\kappa_1, \kappa_2)$.

To begin our construction, we first divide the cylinder $\mathcal{B}$ into N disks of equal height L , so that $L = H/N$. A generic point (R, Θ, X_3) of $\mathcal{B}$ is said to be in the i -th disk if $(i-1)L < X_3 < iL$. In a conventional torsion problem (i.e., one with a convex stored energy function), since (28) is the solution of (10), it follows that every cross section of the cylinder deforms identical to one another so that the final configuration of a generic cross section is described by a X_3 -dependent rotation about the axis of the cylinder. Keeping this in the mind, we now concentrate on the construction of a suitable deformation field for a single disk instead of the whole cylinder. The deformation field for the whole cylinder then will be obtained by stacking N such disks together with an appropriate rigid body rotation of each, and then we let $N \rightarrow \infty$.

Following the above idea, we let $\mathcal{D}$ denote the reference configuration of a single representative disk, so that

$$\mathcal{D} \equiv \{\mathbf{X} = (R, \Theta, \zeta) \mid 0 \leq R \leq 1, 0 \leq \Theta < 2\pi, 0 \leq \zeta \leq L\},$$

and assign the following displacement boundary data:

$$\begin{aligned} r(R, \Theta, 0) &= R, \\ \theta(R, \Theta, 0) &= \Theta, \\ x_3(R, \Theta, 0) &= 0, \end{aligned} \tag{29}$$

and

$$\begin{aligned} r(R, \Theta, L) &= R, \\ \theta(R, \Theta, L) &= \Theta + \alpha L, \\ x_3(R, \Theta, L) &= L, \end{aligned} \tag{30}$$

where here and in the remainder of this work we let $\alpha = \Psi/H$.

Now, for $R \in [0, R_1)$, we take $\kappa = \alpha R < \kappa_1$ so that κ is a point of convexity of $\overline{W}(\cdot)$ for each such R . Therefore, the deformation field for this sub-region of $\mathcal{D}$ is of the form (28). We call this sub-region at the core of the disk the *single-phase* domain $\mathcal{D}_c \subset \mathcal{D}$ and denote the corresponding deformation field as χ^s . Thus, we write

$$\chi^s(\mathbf{X}) = \mathbf{Q}(\alpha\zeta)\mathbf{X}, \tag{31}$$

for $\mathbf{X} = (R, \Theta, \zeta) \in \mathcal{D}_c$. Recall that $\mathbf{Q}(\lambda)$ is a rotation tensor about the $\mathbf{e}_3$ -axis of angle $= \lambda$, so $\mathbf{X}$ on the right hand side of (31) should be interpreted as the position vector of the point (R, Θ, ζ) , and $\mathbf{Q}(\alpha\zeta)$ operates on this vector.

For the annular region $R \in (R_1, 1)$, it is clear that $\alpha R \in (\kappa_1, \kappa_2)$ and so αR is not a point of convexity of $\overline{W}(\cdot)$. Here, we shall construct an admissible deformation field χ^m which consists of an “appropriate” mixture of two phases. We call this annular sub-region of $\mathcal{D}$ the *mixture-phase* domain, and denote it as $\mathcal{D}_a$ so that $\mathcal{D} = \mathcal{D}_c \cup \mathcal{D}_a$. To begin, notice that the boundary conditions (29) and (30) map circular top and bottom surfaces of the disk $\mathcal{D}$ to circular surfaces that are axi-symmetric; also the field (31) is axi-symmetric. We shall construct an axi-symmetric deformation field for the annular region $\mathcal{D}_a$ by first considering a deformation of the generic plane $\Theta = \text{constant}$, and then allow Θ to vary over the interval $[0, 2\pi)$.

Consider a generic fixed plane $\Theta = \text{constant}$, and denote the four corners where it intersects with the annular region $\mathcal{D}_a^+$ by $\{P_1, P_2, P_3, P_4\}$ (see Figure 3). Here, $\mathcal{D}_a^+ \supset \mathcal{D}_a$ and is given by

$$\mathcal{D}_a^+ = \{\mathbf{X} = (R, \Theta, \zeta) \mid R_1 \leq R \leq R_2, 0 \leq \Theta < 2\pi, 0 \leq \zeta \leq L\}.$$

Let p_1, p_2, p_3 and p_4 be the corresponding images of the boundary points P_1, P_2, P_3 , and P_4 of $\mathcal{D}_a^+$ after they are deformed according to (29) and (30), assuming these apply

for $R > 1$. Then, for any *admissible* deformation χ^m of $\mathcal{D}_a$, the points p_1, p_2, p_3 , and p_4 must be arranged as shown in Figure 3.

Figure 3: Deformation of a Disk

Now, divide the rectangle whose corners are at P_1, P_2, P_3 and P_4 in the plane $\Theta = \text{const.}$ into two triangular parts: Part 1 having vertices at P_1, P_2 , and P_3 , and Part 2 having vertices at P_2, P_3 , and P_4 . Let the point $\mathbf{X} \in \mathcal{D}_a$ in Part 1 and Part 2 undergo *affine* transformations χ_1^m and χ_2^m , respectively, where χ_1^m is characterized by

$$\chi_1^m(\mathbf{X}) = \mathbf{X} + 2R_1(\sin \frac{\alpha}{2} L) \frac{\zeta}{L} \mathbf{Q}(\frac{\alpha}{2} L) \mathbf{e}_\Theta,$$

for $\mathbf{X} \in \mathcal{D}_a^1$, where

$$\mathcal{D}_a^1 \equiv \{ \mathbf{X} = (R, \Theta, \zeta) \mid R \in (R_1, 1), \Theta \in [0, 2\pi), \zeta \in (0, \frac{R_2 - R}{R_2 - R_1} L) \},$$

and where χ_2^m is characterized by

$$\chi_2^m(\mathbf{X}) = \mathbf{Q}(\alpha L) \mathbf{X} - 2R_2(\sin \frac{\alpha}{2} L) (\frac{L - \zeta}{L}) \mathbf{Q}(\frac{\alpha}{2} L) \mathbf{e}_\Theta, \quad (33)$$

for $\mathbf{X} \in \mathcal{D}_a^2$, where

$$\mathcal{D}_a^2 \equiv \{ \mathbf{X} = (R, \Theta, \zeta) \mid R \in (R_1, 1), \Theta \in [0, 2\pi), \zeta \in (\frac{R_2 - R}{R_2 - R_1} L, L) \}.$$

In Figure 3, we show the configurations of the plane $\Theta = \text{constant}$ in the undeformed and deformed states. Clearly, the deformation χ^m , which we define as the union of χ_1^m and χ_2^m , maps the undeformed plane into two planes with different slopes. According to (32), a point $\mathbf{X}$ in the lower triangle of the undeformed plane with unit normal $\mathbf{e}_\Theta$ in Figure 3 is sheared in the direction $\mathbf{Q}(\frac{\alpha}{2}L)\mathbf{e}_\Theta$ of the chord from P_3 to p_3 by the amount $2R_1(\sin \frac{\alpha}{2} L)\frac{\zeta}{L}$. Notice that here $2R_1 \sin \frac{\alpha}{2} L$ is the chord length between P_3 and p_3 and ζ denotes the height to each point in the lower triangle. According to (33), all points $\mathbf{X}$ in the upper triangle of the undeformed plane with unit normal $\mathbf{e}_\Theta$ in Figure 3 are first rigidly rotated to the places $\mathbf{Q}(\alpha L)\mathbf{e}_\Theta$. Then, each of the points is sheared (backward) in the direction $-\mathbf{Q}(\frac{\alpha}{2}L)\mathbf{e}_\Theta$ of the chord from p_4 to P_4 by the amount $2R_2(\sin \frac{\alpha}{2} L)(\frac{L-\zeta}{L})$. Here, $2R_2 \sin \frac{\alpha}{2} L$ is the chord length between p_4 and P_4 , and ζ is the height to each point in the upper triangle.

Clearly, from (32) and (33) the deformation field $\chi^m \equiv \chi_1^m \cup \chi_2^m$ for the mixture-phase sub-domain $\mathcal{D}_a = \mathcal{D}_a^1 \cup \mathcal{D}_a^2$ satisfies the boundary conditions (29) and (30). Also, it is straightforward to show that χ^m is continuous in $\mathcal{D}_a$. While it is clear from (31), (32) and (33) that $\chi^s = \chi^m \equiv \chi_1^m \cup \chi_2^m$ at $R = R_1$, and all $\Theta \in [0, 2\pi)$ for $\zeta = 0$ and $\zeta = L$, in general $\chi^s \neq \chi^m$ for $\zeta \in (0, 1)$, i.e., on the whole cylindrical surface $R = R_1$ which separates $\mathcal{D}_c$ and $\mathcal{D}_a$. Therefore, $\chi^s \cup \chi^m$ does not represent a continuous field in the whole disk $\mathcal{D}$. In order to rectify this deficiency, consider the function

$$\chi \equiv \chi^m + g(R; \varepsilon)(\chi^s - \chi^m) \quad \text{for } R \in [0, 1], \quad (34)$$

where, for each small $\varepsilon > 0$ with $R_1 + \varepsilon < R_2$, $g(\cdot; \varepsilon)$ is an auxiliary *match* function which is class C^2 in $[0, 1]$ and characterized by

$$g(R; \varepsilon) = \begin{cases} 1, & R < R_1, \\ 6\frac{(R_1+\varepsilon-R)^5}{\varepsilon^5} - 15\frac{(R_1+\varepsilon-R)^4}{\varepsilon^4} + 10\frac{(R_1+\varepsilon-R)^3}{\varepsilon^3}, & R_1 \leq R \leq R_1 + \varepsilon, \\ 0, & R > R_1 + \varepsilon. \end{cases} \quad (35)$$

Clearly,

$$\nabla g(R; \varepsilon) = \begin{cases} 0, & R < R_1, \\ \text{bounded}, & R_1 \leq R \leq R_1 + \varepsilon, \\ 0, & R > R_1 + \varepsilon, \end{cases} \quad (36)$$

and $\nabla g = O(\varepsilon^{-1})$, $\nabla(\nabla g) = O(\varepsilon^{-2})$ for all $R \in (R_1, R_1 + \varepsilon)$ as $\varepsilon \rightarrow 0$. The interval $(R_1, R_1 + \varepsilon)$ is called a *transition* interval, and, for convenience, we let $\mathcal{D}_\varepsilon$ denote the *transition* domain, i.e.,

$$\mathcal{D}_\varepsilon \equiv \{\mathbf{X} = (R, \Theta, \zeta) \mid R \in (R_1, R_1 + \varepsilon), \Theta \in [0, 2\pi), \zeta \in [0, L]\}.$$

It is easy to show that χ in (34) is a continuous and piecewise infinitely differentiable function in $\mathcal{D}$, and satisfies the boundary conditions (29), (30) at the bottom and top of $\mathcal{D}$.

We now have the following

Lemma 4. *Let $g(\cdot; \varepsilon) \in C^2$ be the function as given in (35), and set $\varepsilon = L$. Then, for any given $L > 0$, χ defined in (34) belongs to the Sobolev space $W^{1,p}(\mathcal{D})$, for any $p \in (1, \infty)$. In addition, $\nabla \chi$ remains bounded in $\mathcal{D}$ in the limit $L \rightarrow 0$.*

Proof. Fix $L > 0$ and $\varepsilon = L$. To show that $\chi \in W^{1,p}(\mathcal{D})$, we must show that

$$\int_{\mathcal{D}} (|\chi|^p + |\nabla \chi|^p) dV < \infty.$$

Since $|\chi| < \infty$ and $\mathcal{D}$ is a finite domain for any given L , it suffices to show that $|\nabla \chi| < \infty$ in $\mathcal{D}$. Clearly, from (34) we have

$$\nabla \chi = \nabla \chi^m + g(R; \varepsilon)(\nabla \chi^s - \nabla \chi^m) + (\chi^s - \chi^m) \otimes \nabla g(R; \varepsilon), \quad (37)$$

and so by (31), (32), (33), (35) and (36), it follows that $|\nabla \chi| < \infty$ for all $L > 0$.

Now, to see that $\nabla \chi$ is bounded in $\mathcal{D}$ as $L \rightarrow 0$ we must be sure that $\nabla \chi^s$ and $\nabla \chi^m$ remain bounded, and that $\chi^s - \chi^m$ vanishes sufficiently fast to cancel the $O(\varepsilon^{-1}) = O(L^{-1})$ growth in ∇g in $\mathcal{D}_\varepsilon$. Recall that $\nabla g \equiv 0$ in $\mathcal{D} \setminus \mathcal{D}_\varepsilon$ to (36). First, by using the identities

$$\begin{aligned} Q(\lambda)e_\Theta &= -(\sin \lambda)e_R + (\cos \lambda)e_\Theta, \\ Q(\lambda)e_R &= (\cos \lambda)e_R + (\sin \lambda)e_\Theta \end{aligned}$$

for any rotation tensor $Q(\lambda)$ of angle λ and axis e_3 , it is straight-forward to show from (31), (32), and (33) that

$$\chi^s - \chi_1^m = (R_1 \frac{\zeta}{L}(1 - \cos \alpha L) - R(1 - \cos \alpha \zeta))e_R + (R \sin \alpha \zeta - R_1 \frac{\zeta}{L} \sin \alpha L)e_\Theta$$

in $\mathcal{D}_a^1$, and

$$\begin{aligned} \chi^s - \chi_2^m &= ((R - \frac{L - \zeta}{L}R_2)(1 - \cos \alpha L) - R(1 - \cos \alpha \zeta))e_R \\ &\quad + (R \sin \alpha \zeta - (R - \frac{L - \zeta}{L}R_2) \sin \alpha L)e_\Theta \end{aligned}$$

in $\mathcal{D}_a^2$. Then, for that part of the transition domain $\mathcal{D}_a^1 \cap \mathcal{D}_\varepsilon$ where $R = R_1 + \beta \varepsilon$ for $\beta \in (0, 1)$, and $\zeta \in [0, \frac{R_2 - R}{R_2 - R_1}L]$, we find, using $\varepsilon = L$, that

$$\chi^s - \chi_1^m = O(L^2)e_R + O(L^2)e_\Theta.$$

In addition, in $\mathcal{D}_a^2 \cap \mathcal{D}_\varepsilon$ where $\zeta \in [\frac{R_2-R}{R_2-R_1}L, L]$ it also follows that

$$\chi^s - \chi_2^m = O(L^2)e_R + O(L^2)e_\Theta.$$

Thus, using (35) and (36) we see that

$$|(\chi^s - \chi^m) \otimes \nabla g(R; \varepsilon)| = O(L)$$

in $\mathcal{D}_\varepsilon$, in the limit $L \rightarrow 0$.

Finally, to finish this lemma we observe from (31) that

$$\nabla \chi^s = Q(\alpha\zeta)(1 + \alpha R e_\Theta \otimes e_3)$$

in $\mathcal{D}_c$, and, by using the identity $\nabla e_\Theta = -(1/R)e_R \otimes e_\Theta$ we readily find from (32) and (33) that

$$\nabla \chi_1^m = 1 + \bar{\kappa}_1 Q\left(\frac{\alpha}{2}L\right)(e_\Theta \otimes e_3 - \frac{\zeta}{R}e_R \otimes e_\Theta) \quad (38)$$

in $\mathcal{D}_a^1$, and

$$\nabla \chi_2^m = Q(\alpha L)\{1 + \bar{\kappa}_2 Q\left(-\frac{\alpha}{2}L\right)(e_\Theta \otimes e_3 + \frac{L-\zeta}{R}e_R \otimes e_\Theta)\} \quad (39)$$

in $\mathcal{D}_a^2$, where

$$\bar{\kappa}_i \equiv 2R_i(\sin \frac{\alpha}{2}L)/L, \quad (i = 1, 2). \quad (40)$$

Clearly, as $L \rightarrow 0$ we have the following limiting behavior:

$$\begin{aligned} \nabla \chi^s &\rightarrow 1 + \alpha R e_\Theta \otimes e_3 \quad \text{in } \mathcal{D}_c \\ \nabla \chi_1^m &\rightarrow 1 + \alpha R_i e_\Theta \otimes e_3 \quad \text{in } \mathcal{D}_a^i, \quad (i = 1, 2). \end{aligned} \quad (41)$$

Thus, the gradients remain bounded and we may finally conclude that $\nabla \chi$ is bounded in $\mathcal{D}$ as $L \rightarrow 0$. Δ

While the introduction of the transition domain and related match function takes care of the smoothness and regularity requirements of a possible minimizing deformation field, nevertheless the field given in (34) is not isochoric and we must now address this problem. In fact, in Theorem 3 we shall provide a method to transform (34) into a closely related and useful deformation that is isochoric. First, we record some elementary, but relevant, information concerning the deformation as it stands. In particular, observe that $\det \nabla \chi^s = 1$ in $\mathcal{D}_c$, and, from (38) and (39), that

$$\det \nabla \chi_1^m = 1 - \bar{\kappa}_1 \frac{\zeta}{R} \sin \frac{\alpha}{2}L \quad (42)$$

in $\mathcal{D}_a^1$, and

$$\det \nabla \chi_2^m = 1 - \bar{\kappa}_2 \left(\frac{L - \zeta}{R} \right) \sin \frac{\alpha}{2} L \quad (43)$$

in $\mathcal{D}_a^2$. Thus, in $\mathcal{D}_a \setminus \mathcal{D}_\epsilon$ where, from (37) $\nabla \chi$ is either $\nabla \chi_1^m$, or $\nabla \chi_2^m$, it follows that $\det \nabla \chi \neq 1$, but nevertheless *at the top and bottom of $\mathcal{D}_a \setminus \mathcal{D}_\epsilon$ we have*

$$\det \nabla \chi = 1.$$

Further, from (37) and the fact that at $\zeta = 0, L$ we have $\chi^s = \chi^m$, we see that

$$\nabla \chi = \nabla \chi^m + g(R; \epsilon)(\nabla \chi^s - \nabla \chi^m)$$

at $\zeta = 0, L$ in $\mathcal{D}_\epsilon$. More specifically, for $\zeta = 0$ we use (38) to obtain

$$\begin{aligned} \nabla \chi &= 1 + \bar{\kappa}_1 Q\left(\frac{\alpha}{2} L\right) e_\Theta \otimes e_3 \\ &\quad + g(R; \epsilon)(\alpha R e_\Theta \otimes e_3 - \bar{\kappa}_1 Q\left(\frac{\alpha}{2} L\right) e_\Theta \otimes e_3), \end{aligned}$$

from which it follows that $\det \nabla \chi = 1$ at $\zeta = 0$ in $\mathcal{D}_\epsilon$. Similarly, by using (39) we find that $\det \nabla \chi = 1$ at $\zeta = L$ in $\mathcal{D}_\epsilon$. In summary, then, *at the top and bottom of $\mathcal{D}_a$ the deformation (34) satisfies*

$$\det \nabla \chi = 1, \quad (44)$$

and with the aid of (41) we may further conclude that

$$\lim_{L \rightarrow 0} \det \nabla \chi = 1$$

almost everywhere in the limiting mixture-phase sub-domain $\mathcal{D}_a$ as $L \rightarrow 0$. Of course, as noted earlier, $\nabla \chi = \nabla \chi^s$ in the core region, so in $\mathcal{D}_c$ $\det \nabla \chi = 1$ is satisfied identically.

We now follow a technique of Fosdick and MacSithigh [8] in order to transform the deformation field (34) into one that is isochoric everywhere in $\mathcal{D}$. In the present circumstances, this technique depends upon the result of the following

Theorem 3. *Let $\chi(X)$ be continuous and piecewise twice differentiable for $X \in \mathcal{D}$, and suppose that $\det \nabla \chi > 0$ everywhere in $\mathcal{D}$. Let $\mathbf{x}$ be a point in $\chi(\mathcal{D})$ with corresponding cylindrical coordinates (r, θ, x_3) and orthonormal cylindrical basis $\{\mathbf{e}_r(\mathbf{x}), \mathbf{e}_\theta(\mathbf{x}), \mathbf{e}_3\}$. Define*

$$\phi(\mathbf{x}) = \sqrt{\int_0^r 2r(1 + f(\chi^{-1}(\mathbf{x}))) dr} - 1, \quad (45)$$

where

$$f(\mathbf{X}) = \frac{1 - \det \nabla \chi(\mathbf{X})}{\det \nabla \chi(\mathbf{X})}. \quad (46)$$

Then, the deformation

$$\mathbf{y}(\mathbf{X}) = \chi(\mathbf{X}) + \phi(\chi(\mathbf{X}))\mathbf{e}_r(\chi(\mathbf{X})) \quad (47)$$

is continuous and piecewise differentiable for $\mathbf{X} \in \mathcal{D}$, and is isochoric, i.e., $\det \nabla \mathbf{y} \equiv 1$.

Proof. Consider the function $\mathbf{y}(\mathbf{X})$ defined in (47). Clearly,

$$\nabla \mathbf{y} = (\mathbf{1} + \mathbf{e}_r \otimes \text{grad} \phi + \phi \text{grad} \mathbf{e}_r) \nabla \chi, \quad (48)$$

and since $\text{grad} \mathbf{e}_r = (\mathbf{e}_\theta \otimes \mathbf{e}_\theta)/r$ it is straightforward to obtain

$$\det \nabla \mathbf{y} = (1 + (1 + \frac{\phi}{r})\frac{\partial \phi}{\partial r} + \frac{\phi}{r}) \det \nabla \chi.$$

If we now set $\det \nabla \mathbf{y} = 1$ and define $f(\mathbf{X})$ through (46) it follows that

$$(1 + \frac{\phi}{r})\frac{\partial \phi}{\partial r} + \frac{\phi}{r} = f(\chi^{-1}(\mathbf{x})).$$

Then, to integrate this partial differential equation, introduce the auxiliary function

$$u(\mathbf{x}) \equiv (r + \phi(\mathbf{x}))^2$$

and observe that

$$\frac{\partial u}{\partial r} = 2r(1 + f(\chi^{-1}(\mathbf{x}))).$$

The right hand side of this equation is positive for $r > 0$, since $f(\mathbf{X}) > -1$, and so the solution which satisfies $u(\mathbf{x}) = 0$ at $r = 0$ is

$$u(\mathbf{x}) = \int_0^r 2r(1 + f(\chi^{-1}(\mathbf{x})))dr \geq 0.$$

Finally, by restoring $\phi(\mathbf{x})$ we arrive at (45) to complete the major part of this proof. The fact that $\phi(\mathbf{x})$ is piecewise differentiable follows from inspection of (44) and the structure of the function $f(\mathbf{X})$ in (46). Δ

Corollary. Let $\chi(\mathbf{X})$ of Theorem 3 be given in $\mathcal{D}$ by (34) with $\varepsilon = L$. Then, for any given $L > 0$, $\mathbf{y}(\mathbf{X})$ in (47) is isochoric and belongs to the Sobolev space $W^{1,p}(\mathcal{D})$

for any $p \in (1, \infty)$, $\mathbf{y}(\mathbf{X})$ satisfies the boundary conditions (29), (30) on the bottom and top faces of $\mathcal{D}$, and $\nabla \mathbf{y}$ remains bounded in $\mathcal{D}$ in the limit $L \rightarrow 0$.

Proof. The fact that $\mathbf{y}(\mathbf{X})$ in (47) is isochoric, is part of the Theorem 3; that it belongs to $W^{1,p}(\mathcal{D})$ follows from the observation that $\chi \in W^{1,p}(\mathcal{D})$ from Lemma 4, and the boundedness of $\text{grad} \phi$ in $\mathcal{D}$ for $L > 0$ as follows from (45), (46), and the smoothness properties of χ in (34).

To see that $\mathbf{y}(\mathbf{X})$ satisfies the boundary conditions (29) and (30), first recall from (34) that in the core $\mathcal{D}_c$ we have $\chi = \chi^s$ so that $\det \nabla \chi = 1$ in $\mathcal{D}_c$. Then, (45), (46) and (47) show that $g(\chi(\mathbf{X})) = 0$, and $\mathbf{y}(\mathbf{X}) = \chi(\mathbf{X})$ for $\mathbf{X} \in \mathcal{D}_c$. Thus, $\mathbf{y}(\mathbf{X})$ satisfies (29), (30) on the top and bottom faces of $\mathcal{D}_c$. Now, we argue in (44) that on the top and bottom faces of the mixture-phase region $\mathcal{D}_a$, $\det \nabla \chi = 1$. But, recall the deformation χ in (34) is such that $x_3 = \zeta$, for $\zeta \in [0, L]$. Thus, in the integral in (45) we may set $f(\chi^{-1}(\mathbf{x})) = 0$ for $\mathbf{x} = (r, \theta, 0)$ and $\mathbf{x} = (r, \theta, L)$, and conclude that $\phi(\chi(\mathbf{X})) = 0$ for $\mathbf{X}$ on the top and bottom faces of $\mathcal{D}_a$ (i.e., $\zeta = 0, L$). Then, according to (47), $\mathbf{y}(\mathbf{X}) = \chi(\mathbf{X})$ on the top and bottom faces of $\mathcal{D}_a$ and we know from (34) that $\chi(\mathbf{X})$ satisfies the boundary condition (29), (30).

We now turn to the boundedness of $\nabla \mathbf{y}$ in $\mathcal{D}$ in the limit $L \rightarrow 0$. In Lemma 4 we argued that $\nabla \chi$ remains bounded in $\mathcal{D}$ in the limit $L \rightarrow 0$. Thus, from (48) it suffices to show that ϕ and $\text{grad} \phi$ have this regularity property. For ϕ , we need recall the definition (45), (46) and note that since $\nabla \chi$ is bounded and $\det \nabla \chi > 0$ then both f and ϕ are bounded in $\mathcal{D}$ as $L \rightarrow 0$. Actually, we observed earlier that $\det \nabla \chi \rightarrow 1$ in $\mathcal{D}$ as $L \rightarrow 0$ and in this case we have $f \rightarrow 0$ and $\phi \rightarrow 0$ as $L \rightarrow 0$.

Finally, for $\text{grad} \phi$, note from (45) that

$$\text{grad} \phi = \frac{1}{r + \phi(\mathbf{x})} \left\{ -\phi(\mathbf{x}) \mathbf{e}_r + \text{grad} \int_0^r r f(\chi^{-1}(\mathbf{x})) dr \right\}.$$

Since the intergrad vanishes at $r = 0$ the grad operator can be moved, equivalently, inside the integral and we then have

$$\text{grad} \phi = \frac{1}{r + \phi(\mathbf{x})} \left\{ -\phi(\mathbf{x}) \mathbf{e}_r + \mathbf{e}_r \int_0^r f(\chi^{-1}(\mathbf{x})) dr + \int_0^r r \text{grad} f(\chi^{-1}(\mathbf{x})) dr \right\}.$$

The first two terms on the right hand side are bounded and, in fact, vanish in the limit $L \rightarrow 0$, as we saw above. The third term requires some further consideration, and we start by observing that the chain rule gives

$$\text{grad} f = (\nabla \chi)^{-1r} \nabla f,$$

where, from (46),

$$\nabla f = -\frac{1}{(\det \nabla \chi)^2} \nabla (\det \nabla \chi).$$

Thus, since $\nabla \chi$ is bounded and $\det \nabla \chi > 0$, we must be assured that $\nabla(\det \nabla \chi)$ is bounded in $\mathcal{D}$ as $L \rightarrow 0$. Then we may conclude that both $\text{grad} f$ and $\text{grad} \phi$ are bounded in $\chi(\mathcal{D})$ in this limit. In $\mathcal{D}_c$ we know from earlier calculations, based upon (34) and given in the proof of Lemma 4, that $\det \nabla \chi = 1$. Similarly, we found the simple formulae (42) and (43) for determining $\det \nabla \chi$ in the respective regions $\mathcal{D}_a^1 \setminus \mathcal{D}_\epsilon$ and $\mathcal{D}_a^2 \setminus \mathcal{D}_\epsilon$. From these it is clear that $\nabla(\det \nabla \chi)$ is bounded (and in fact vanishes) in $\mathcal{D}_a \setminus \mathcal{D}_\epsilon$ in the limit $L \rightarrow 0$. Finally, in $\mathcal{D}_\epsilon$ we must return to the complicated expression for $\nabla \chi$ given in (37) and first compute $\det \nabla \chi$, and then the resulting gradient $\nabla(\det \nabla \chi)$. In so doing, it is clear that the second gradient of the match function $\nabla(\nabla g)$ will appear, and this term represents a most singular contribution to the final result (i.e., $O(\epsilon^{-2})$, where $\epsilon = L$) in the limit $L \rightarrow 0$. Nevertheless, from this rather lengthy calculation it follows that $\nabla(\det \nabla \chi)$ is bounded in $\mathcal{D}_\epsilon$ in the limit $L \rightarrow 0$. Δ

In the above corollary we showed that the isochoric deformation field $\mathbf{y}(\mathbf{X})$ in (47) satisfies the boundary conditions (29) and (30), and that $\nabla \mathbf{y}$ is bounded everywhere in $\mathcal{D}$ in the limit $L \rightarrow 0$. These properties are important because they will enable us to appropriately stack N disks together in order to build an admissible isochoric deformation field $\chi_N(\mathbf{X})$, $\mathbf{X} \in \mathcal{B}$, in $\mathcal{A}$ for the original cylinder $\mathcal{B}$, and at the same time eliminate undesirable contributions to the total potential energy of $\mathcal{B}$ for members of the sequence $\chi_N(\mathbf{X})$ as $N \rightarrow \infty$. In Section 5, we will show that the sequence so constructed is a minimizing sequence for the minimization problem (10).

At this stage, recall that we have assumed that the angle of twist per unit length $\alpha = \Psi/H$ satisfies $\kappa_1 < \alpha \leq \kappa_2$. In this case we have constructed in the above corollary a deformation field $\mathbf{y}(\mathbf{X})$ that has a single transition zone $\mathcal{D}_\epsilon$ which connects a single-phase core region $\mathcal{D}_c$ to an annular mixture-phase region that extends completely out to the lateral boundary of $\mathcal{D}$ at $R = 1$. When $\alpha > \kappa_2$, we have $R_2 = \frac{\kappa_2}{\Psi/H} < 1$ and our construction is no longer complete in the sense that the annular mixture-phase region stops at the radius $R = R_2$. For this situation, we must define the deformation of $\mathcal{D}$ for $R \in (R_2, 1)$. Following the same ideas as presented above, we may successfully do this by introducing a second single-phase sub-domain of $\mathcal{D}$ for $R \in (R_2, 1)$ in addition to the one for $R \in [0, R_1)$. In this new single-phase

sub-domain, the deformation field is the same (c.f., (28) or (31)) as that for the inner core single-phase sub-domain. Then, we introduce a corresponding transition interval with a sufficiently smooth *match* function $h(\cdot; \varepsilon)$ for each small $\varepsilon > 0$ with $R_2 - \varepsilon > R_1$, similar to (35) and such that

$$h(R; \varepsilon) = \begin{cases} 1, & R > R_2 \\ 6\frac{(R-R_2+\varepsilon)^5}{\varepsilon^5} - 15\frac{(R-R_2+\varepsilon)^4}{\varepsilon^4} + 10\frac{(R-R_2+\varepsilon)^3}{\varepsilon^3}, & R_2 - \varepsilon \leq R \leq R_2 \\ 0, & R < R_2 - \varepsilon. \end{cases} \quad (49)$$

We shall not carry out the details of this construction here as they are similar in all respects to what we have done above for the case $\kappa_1 < \alpha \leq \kappa_2$.

We turn now to the description of an admissible deformation field for the whole cylinder $\mathcal{B}$, assuming $\kappa_1 < \alpha \leq \kappa_2$, which is composed of a stacking of N disks, each one being similar to $\mathcal{D}$ and of height $L = H/N$. For this purpose, let (R, Θ, X_3) denote a generic point in the i -th disk, so that $(i-1)L < X_3 < iL$ for $i = 1, 2, \dots$, or N , and set

$$\zeta = X_3 - (i-1)L$$

in the deformation field $\mathbf{y}(\mathbf{X}) = \mathbf{y}(R, \Theta, \zeta)$ of (47). In addition, take the primitive field $\chi(\mathbf{X}) = \chi(R, \Theta, \zeta)$ in (47) to be given by (34). Then, with a minor change of notation so that $\mathbf{X}$ denotes the triple (R, Θ, X_3) , define

$$\chi_N(\mathbf{X}) \equiv \mathcal{Q}((i-1)\alpha L)\mathbf{y}(R, \Theta, X_3 - (i-1)L) + (i-1)L\mathbf{e}_3 \quad (50)$$

for each integer $i = 1, 2, \dots, N$ and corresponding $\mathbf{X} \in \mathcal{B}$ with $X_3 \in ((i-1)L, iL)$. The deformation field (50), including the limiting interfacial points between disks on the cross sections at $X_3 = (i-1)L$ for $i = 2, \dots, N$, is continuous and isochoric in $\mathcal{B}$. According to (50) it is composed of a stack of rotated and translated images of the field (47) for the single disk $\mathcal{D}$. Thus, the boundary conditions (4) and (5) on $\partial_1 \mathcal{B}$ are satisfied, and from the Corollary of Theorem 3 we infer that $\chi_N \in W^{1,p}(\mathcal{B})$ for each fixed N and for any $p \in (1, \infty)$; i.e., $\chi_N \in \mathcal{A}$.

While the Corollary of Theorem 3 shows that $\nabla \chi_N$ remains bounded everywhere in $\mathcal{B}$ as $N \rightarrow \infty$, some additional limiting properties of χ_N and $\nabla \chi_N$ as $N \rightarrow \infty$ are contained in the following.

Theorem 4. *Recall that χ_0 represent the classical torsion field (28) in $\mathcal{B}$ and suppose that $\alpha = \Psi/H$ satisfies $\kappa_1 < \alpha \leq \kappa_2$. Then, for $p > 2$ the deformation field χ_N given by (50) satisfies $\chi_N \rightharpoonup \chi_0$ in $W^{1,p}(\mathcal{B})$ as $N \rightarrow \infty$ ². Moreover, for any fixed N ,*

²The symbol $\rightharpoonup$ denotes weak convergence in the usual sense.

$\nabla \chi_N = \nabla \chi_0 = 1 + \alpha R \mathbf{e}_\Theta \otimes \mathbf{e}_3$ in the single-phase domain of $\mathcal{B}$ (i.e., $R < R_1$), and $\nabla \chi_N \rightarrow \lambda(R)\mathbf{F}_1 + (1 - \lambda(R))\mathbf{F}_2$ in L^p in the mixture-phase domain of $\mathcal{B}$ (i.e., $R_1 \leq R < 1$) as $N \rightarrow \infty$, where

$$\mathbf{F}_i = \mathbf{1} + \kappa_i \mathbf{e}_\Theta \otimes \mathbf{e}_3, \quad \kappa_i = \alpha R_i, \quad (i = 1, 2),$$

and where $\lambda(R) \equiv \frac{R_2 - R}{R_2 - R_1}$, $R \in (R_1, 1)$, is a measure of the volume fraction of the “total” deformation gradient that is equal to $\mathbf{F}_1$ at the radius R in the mixture-phase domain.

Proof. We wish to show that both $\chi_N \rightarrow \chi_0$ and $\nabla \chi_N \rightarrow \nabla \chi_0$ in $L^p(\mathcal{B})$ as $N \rightarrow \infty$. To do this first observe that for any given point $\mathbf{X} \in \mathcal{B}$, the fixed X_3 coordinate is in a disk bounded by the interval $[(i-1)L, iL]$ for some index $i \in \{1, 2, \dots, N\}$. Thus as $N \rightarrow \infty$ we have $L = H/N \rightarrow 0$ and consequently the index $i \rightarrow \infty$ so that $(i-1)L \rightarrow iL \rightarrow X_3$. Then, from (50) we see that

$$\chi_N(\mathbf{X}) \rightarrow \mathcal{Q}(\alpha X_3) \mathbf{y}(R, \Theta, 0) + X_3 \mathbf{e}_3,$$

and, from an argument like that given in the Corollary of Theorem 3, we find $\mathbf{y}(R, \Theta, 0) = \mathbf{X} - X_3 \mathbf{e}_3$. Since $\mathcal{Q}(\alpha X_3)$ is simply a rotation with $\mathbf{e}_3$ as its axis we may equivalently write

$$\chi_N(\mathbf{X}) \rightarrow \mathcal{Q}(\alpha X_3) \mathbf{X} = \chi_0(\mathbf{X}),$$

where the last equality follows from (28). The convergence here is pointwise so, in particular, $\chi_N \rightarrow \chi_0$ in $L^p(\mathcal{B})$.

Now, we note that

$$\int_{\mathcal{B}} \nabla \chi_N \cdot \mathbf{M} dV = - \int_{\mathcal{B}_0} \text{Div} \mathbf{M} \cdot \chi_N dV$$

for any $\mathbf{M} \in C_0^\infty(\mathcal{B}, \text{Lin})$.³ Thus, since $\nabla \chi_N \cdot \mathbf{M}$ and $\text{Div} \mathbf{M} \cdot \chi_N$ are bounded, $\mathcal{B}$ is a finite domain, and $\chi_N \rightarrow \chi_0$ pointwise, the order of the limit and integration operations can be interchanged to obtain

$$\lim_{N \rightarrow \infty} \int_{\mathcal{B}} \nabla \chi_N \cdot \mathbf{M} dV = \int_{\mathcal{B}} \nabla \chi_0 \cdot \mathbf{M} dV$$

³Lin represents the set of all linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$.

for any $\mathbf{M} \in C_0^\infty(\mathcal{B}, \text{Lin})$. Thus, $\nabla \chi_N \rightharpoonup \nabla \chi_0$ in $L^p(\mathcal{B})$. In addition to this, it is of interest to observe from (50), (47), and the fact that $\chi = \chi^s$ in the core region of each disk $\mathcal{D}_c$, as follows from (34), that

$$\chi_N(\mathbf{X}) = \mathcal{Q}((i-1)\alpha L)\chi^s(R, \Theta, X_3 - (i-1)L) + (i-1)L\mathbf{e}_3$$

in the single-phase domain for $X_3 \in ((i-1)L, iL)$. Then, with the aid of (31) it readily follows that $\chi_N(\mathbf{X}) = \mathcal{Q}(\alpha X_3)\mathbf{X}$, and with (28) that $\chi_N(\mathbf{X}) = \chi_0(\mathbf{X})$ for any fixed N . Thus, $\nabla \chi_N = \nabla \chi_0$ in the single-phase (core) domain of $\mathcal{B}$ (i.e., $R \in [0, R_1)$) for any fixed N . To show that $\nabla \chi_N \rightharpoonup \lambda(R)\mathbf{F}_1 + (1 - \lambda(R))\mathbf{F}_2$ in L^p in the mixture-phase domain of $\mathcal{B}$, we first recall, from Proposition 2 of Ball and James [2], that for any bounded, open, and connected domain $\mathcal{M} \subset \mathbb{R}^3$, if $\mathbf{y}_N \rightharpoonup \mathbf{y}_0$ in $W^{1,p}(\mathcal{M})$, $p > 2$, and if, for every $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \text{meas}\{\mathbf{X} \in \mathcal{M} : |\nabla \mathbf{y}_N(\mathbf{X}) - \mathbf{A}| > \varepsilon \text{ and } |\nabla \mathbf{y}_N(\mathbf{X}) - \mathbf{B}| > \varepsilon\} = 0,$$

where $\mathbf{A}$ and $\mathbf{B} \in \text{Lin}$ are distinct, then,

$$\nabla \mathbf{y}_0 = \lambda(\mathbf{X})\mathbf{A} + (1 - \lambda(\mathbf{X}))\mathbf{B} \quad \text{a.e. } \mathbf{X} \in \mathcal{M}$$

for some measurable function λ satisfying $0 \leq \lambda(\mathbf{X}) \leq 1$ a.e. To apply this proposition to our analysis, we identify $\mathbf{y}_N$ with χ_N , $\mathbf{y}_0$ with χ_0 , $\mathbf{A}$ with $\mathbf{F}_1$, $\mathbf{B}$ with $\mathbf{F}_2$, and $\mathcal{M}$ with the mixture-phase domain of $\mathcal{B}$, i.e.,

$$\mathcal{M} \equiv \{(R, \Theta, X_3) \mid R \in (R_1, 1), \Theta \in [0, 2\pi), X_3 \in [0, H]\}.$$

Clearly, $\mathcal{M}$ is bounded, open, and connected. Also, it follows from (50), (48), (34), (41), and the conclusions drawn in the proof of the Corollary of Theorem 3 that $\phi \rightarrow 0$ in $\mathcal{D}$ and $\text{grad } \phi \rightarrow 0$ in $\mathcal{D}_a \setminus \mathcal{D}_\varepsilon$ as $L \rightarrow 0$, that for every $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty} \text{meas}\{\mathbf{X} \in \mathcal{M} : |\nabla \chi_N(\mathbf{X}) - \mathbf{F}_1| > \varepsilon \text{ and } |\nabla \chi_N(\mathbf{X}) - \mathbf{F}_2| > \varepsilon\} = 0.$$

Then, since we showed above that $\chi_N \rightarrow \chi_0$ pointwise in $\mathcal{B}$ and also we have $\nabla \chi_N \rightharpoonup \nabla \chi_0$ in $L^p(\mathcal{M})$, it follows that

$$\nabla \chi_N \rightharpoonup \nabla \chi_0 = \lambda(\mathbf{X})\mathbf{F}_1 + (1 - \lambda(\mathbf{X}))\mathbf{F}_2 \quad \text{a.e. } \mathbf{X} \in \mathcal{M}.$$

Finally, to determine the value of $\lambda(\mathbf{X})$ in the mixture-phase domain of $\mathcal{B}$ we note from the above convex combination relationship and the known forms of $\nabla \chi_0$, $\mathbf{F}_1$ and $\mathbf{F}_2$ that

$$\alpha R = \lambda(\mathbf{X})\alpha R_1 + (1 - \lambda(\mathbf{X}))\alpha R_2 \quad \text{a.e. } \mathbf{X} \in \mathcal{M},$$

and so

$$\lambda(\mathbf{X}) = \lambda(R) = \frac{R_2 - R}{R_2 - R_1} \in [0, 1] \quad a.e. \ R \in (R_1, 1),$$

and for each $\alpha = \Psi/H \in (\kappa_1, \kappa_2]$. Δ

5. MINIMIZING SEQUENCE

In this final section, we shall first show that, for any prescribed angle of twist Ψ such that $\alpha = \Psi/H \in (\kappa_1, \kappa_2]$, the deformation fields χ_N given by (50) for each $N = 1, 2, \dots$, form a minimizing sequence as $N \rightarrow \infty$ in the class $\mathcal{A}$ of (9), for the total potential energy of the problem (10). We show in Theorem 5 that the sequence of total potential energies associated with this minimizing sequence converges to the infimum of (8) as $N \rightarrow \infty$. Then, we summarize in Theorem 6, without details, our final conclusions that for any other prescribed angle of twist Ψ there is a solution to the minimization problem (10) which is either the unique (classical) minimizer in $\mathcal{A}$ when $\frac{\Psi}{H} \leq \kappa_1$, or the weak limit of a minimizing sequence in $\mathcal{A}$ when $\frac{\Psi}{H} > \kappa_2$.

Theorem 5. *Let the stored energy function $\overline{W}(\cdot)$ be as depicted in Figure 1 and suppose that it satisfies the associated Constitutive Hypotheses of Section 2. Let κ_1 and κ_2 be two common support points of convexity of the stored energy function $\overline{W}(\cdot)$ as shown in Figure 1. Suppose Ψ is given so that $\alpha = \Psi/H \in (\kappa_1, \kappa_2]$, and let χ_N be given by (50). Then, $E[\chi]$ of (8) satisfies*

$$\lim_{N \rightarrow \infty} E[\chi_N] = \inf_{\chi \in \mathcal{A}} E[\chi]. \quad (51)$$

Note that for a fixed finite N , and $\mathbf{X} \in \mathcal{B}$, $\kappa_N \equiv \sqrt{|\nabla \chi_N|^2 - 3}$ generally does not correspond to a point of convexity of $\overline{W}(\cdot)$. Therefore, in anticipation of the proof of this theorem, we explicitly draw attention to elementary and useful geometrical properties of the graph of $\overline{W}(\cdot)$ as depicted in Figures 1 and 2. For this purpose, it is first convenient to introduce the auxiliary function

$$T_{\tilde{\kappa}}(\kappa) = \overline{W}(\tilde{\kappa}) + (\kappa - \tilde{\kappa})\overline{W}'(\tilde{\kappa}) \quad \forall \kappa > 0, \quad (52)$$

where $\tilde{\kappa}$ is any positive number. This, of course, corresponds to the straight line which is tangent to the graph of $\overline{W}(\cdot)$ at $\tilde{\kappa}$. Then, we have the following: (i) If $\tilde{\kappa} \in (\kappa_3, \kappa_5)$ and if there is a $\kappa = \kappa'$ with $\kappa' \in (0, \kappa_5)$ such that $T_{\tilde{\kappa}}(\kappa') = \overline{W}(\kappa')$, then κ' is unique and satisfies $\kappa' \in (0, \kappa_3)$. (ii) If $\tilde{\kappa} \in (\kappa_5, \kappa_4)$, then there is a unique

$\kappa = \kappa''$ with $\kappa'' \in (\kappa_4, \infty)$ such that $T_{\tilde{\kappa}}(\kappa'') = \overline{W}(\kappa'')$. These are most easily observable geometrically using Figure 1 and appropriate tangent line constructions.

We now turn to the

Proof of Theorem 5. For this, we define the following pseudo stored energy function for all $\kappa \geq 0$:

$$f(\kappa, \tilde{\kappa}) = \overline{W}(\kappa) + \begin{cases} 0 & \text{if } \tilde{\kappa} \leq \kappa_1 \text{ or } \tilde{\kappa} \geq \kappa_2, \\ \zeta(\kappa, \tilde{\kappa}) & \text{if } \tilde{\kappa} \in (\kappa_1, \kappa_2), \end{cases} \quad (53)$$

where the function $\zeta(\kappa, \tilde{\kappa})$ is characterized by the three alternatives;

$$(a) \text{ if } \tilde{\kappa} \in (\kappa_1, \kappa_3] \text{ then } \zeta(\kappa, \tilde{\kappa}) = \begin{cases} T_{\tilde{\kappa}}(\kappa) - T_{\kappa_1}(\kappa) & \text{for } \kappa \geq \tilde{\kappa} \\ 0 & \text{for } \kappa < \tilde{\kappa}, \end{cases} \quad (54a)$$

$$(b) \text{ if } \tilde{\kappa} \in [\kappa_4, \kappa_2) \text{ then } \zeta(\kappa, \tilde{\kappa}) = \begin{cases} T_{\tilde{\kappa}}(\kappa) - T_{\kappa_1}(\kappa) & \text{for } \kappa \leq \tilde{\kappa} \\ 0 & \text{for } \kappa > \tilde{\kappa}, \end{cases} \quad (54b)$$

$$(c) \text{ if } \kappa \in (\kappa_3, \kappa_4) \text{ then } \zeta(\kappa, \tilde{\kappa}) = \begin{cases} T_{\tilde{\kappa}}(\kappa) - T_{\kappa_1}(\kappa) & \text{for } \kappa \in [\kappa', \kappa''] \\ 0 & \text{for } \kappa < \kappa' \text{ or } \kappa > \kappa''. \end{cases} \quad (54c)$$

In the third alternative, here, the numbers κ' and κ'' are determined from the geometric properties (i) and (ii) above. If $\kappa' \in (0, \kappa_3)$ in property (i) does not exist then set $\kappa' = 0$ in alternative (c).

Since $T_{\tilde{\kappa}}(\kappa) - T_{\kappa_1}(\kappa)$ represents the difference between the two straight lines which are tangent to the graph of $\overline{W}(\cdot)$ at $\tilde{\kappa}$ and κ_1 , and since $T_{\kappa_1}(\kappa) = T_{\kappa_2}(\kappa)$, it is clear that $\zeta(\kappa, \tilde{\kappa}) \rightarrow 0$ as $\tilde{\kappa} \rightarrow \kappa_1$ or κ_2 . In addition, notice that $\zeta(\kappa, \tilde{\kappa}) \geq 0$ and is bounded for any $\tilde{\kappa} \in (\kappa_1, \kappa_2)$ and for all $\kappa \in [0, \infty)$. Finally, observe that

$$f(\kappa, \tilde{\kappa}) \geq T_{\tilde{\kappa}}(\kappa) \quad \forall \tilde{\kappa} \in [0, \infty) \text{ and } \kappa \in [0, \infty). \quad (55)$$

We now let χ be an arbitrary deformation field in $\mathcal{A}$ with $\kappa(\mathbf{X})$ given by $\kappa = \sqrt{|\nabla \chi|^2 - 3}$, and use (55) and (52) to generate the key inequality

$$\int_{\mathcal{B}} f(\kappa(\mathbf{X}), \kappa_N(\mathbf{X})) dV \geq \int_{\mathcal{B}} (\overline{W}(\kappa_N(\mathbf{X})) + (\kappa(\mathbf{X}) - \kappa_N(\mathbf{X})) \overline{W}'(\kappa_N(\mathbf{X}))) dV, \quad (56)$$

where $\kappa_N(\mathbf{X})$ is given by $\kappa_N = \sqrt{|\nabla \chi_N|^2 - 3}$, and χ_N is defined in (50) for every $N = 1, 2, \dots$. In order to analyze this inequality, it is convenient to decompose the region $\mathcal{B}$ into the following core, transition, and annular sub-regions:

$$\begin{aligned} \mathcal{B}_c &= \{\mathbf{X} \in \mathcal{B} | R \leq R_1\}, \\ \mathcal{B}_\epsilon &= \{\mathbf{X} \in \mathcal{B} | R \in (R_1, R_1 + \epsilon)\}, \end{aligned}$$

$$\begin{aligned}\mathcal{B}_a^1 &= \bigcup_{i=1}^N \{ \mathbf{X} \in \mathcal{B} | R \in (R_1, 1), X_3 \in ((i-1)L, (i-1)L + \frac{R_2 - R}{R_2 - R_1}L) \}, \\ \mathcal{B}_a^2 &= \bigcup_{i=1}^N \{ \mathbf{X} \in \mathcal{B} | R \in (R_1, 1), X_3 \in ((i-1)L + \frac{R_2 - R}{R_2 - R_1}L, iL) \}.\end{aligned}$$

We are interested in the situation when N is large for which, according to Lemma 4 and subsequent usage, $\varepsilon = L = H/N$. In this case, if we recall (see the Corollary of Theorem 3) that $\nabla \chi_N$ is bounded everywhere in $\mathcal{B}$ as $N \rightarrow \infty$, that $\nabla \chi_N = \nabla \chi_0$ (so that $\kappa_N = \alpha R$) in $\mathcal{B}_c$ (see Theorem 4), and note that the volume measure of $\mathcal{B}_\varepsilon$ is $O(\varepsilon) = O(L)$, then with (53) we see that the left hand side of (56) may be written as

$$\int_{\mathcal{B}} f(\kappa(\mathbf{X}), \kappa_N(\mathbf{X})) dV = \int_{\mathcal{B}} \overline{W}(\kappa(\mathbf{X})) dV + \int_{\mathcal{B}_a \setminus \mathcal{B}_\varepsilon} \zeta(\kappa(\mathbf{X}), \kappa_N(\mathbf{X})) dV + O(1/N),$$

where $\mathcal{B}_a = \mathcal{B}_a^1 \cup \mathcal{B}_a^2$. Moreover, in the proof of Theorem 4 we saw that κ_N is close to either κ_1 or κ_2 in $\mathcal{M}$ as $N \rightarrow \infty$ so it follows from (54) that

$$\lim_{N \rightarrow \infty} \int_{\mathcal{B}_a \setminus \mathcal{B}_\varepsilon} \zeta(\kappa(\mathbf{X}), \kappa_N(\mathbf{X})) dV = 0.$$

Thus, from (56) and (8) we have

$$E[\chi] \geq \lim_{N \rightarrow \infty} E[\chi_N] + \lim_{N \rightarrow \infty} \int_{\mathcal{B}} (\kappa(\mathbf{X}) - \kappa_N(\mathbf{X})) \overline{W}'(\kappa_N(\mathbf{X})) dV. \quad (57)$$

Finally, by interchanging the limit and integration operators, the second term on the right hand side becomes

$$\int_{\mathcal{B}} (\kappa(\mathbf{X}) - \lim_{N \rightarrow \infty} \kappa_N(\mathbf{X})) \lim_{N \rightarrow \infty} \overline{W}'(\kappa_N(\mathbf{X})) dV, \quad (58)$$

which we now show is non-negative. To see this, recall from Lemma 2 that

$$\int_0^H \kappa(\mathbf{X}) dX_3 \geq R\Psi$$

in $\mathcal{B}$. In addition, note from Theorem 4 that in the single-phase domain of $\mathcal{B}$ we have $\kappa_N(\mathbf{X}) = \alpha R$ for all N , and that in the mixture-phase domain of $\mathcal{B}$ $\kappa_N(\mathbf{X})$ converges weakly to $\alpha R = \lambda(R)\kappa_1 + (1 - \lambda(R))\kappa_2$. Thus

$$\int_0^H \lim_{N \rightarrow \infty} \kappa_N(\mathbf{X}) dX_3 = R\Psi$$

in $\mathcal{B}$. Finally note that since $\lim_{N \rightarrow \infty} \overline{W}'(\kappa_N(\mathbf{X}))$ is independent of X_3 and can only correspond to the slope of the graph of $\overline{W}(\cdot)$ at its convex points, we have from the Constitutive Hypothesis of Section 2 that

$$\lim_{N \rightarrow \infty} \overline{W}'(\kappa_N(\mathbf{X})) \geq 0.$$

Thus, like Lemma 3 we see that the integral of (58) is non-negative and conclude from (57) that

$$E[\chi] \geq \lim_{N \rightarrow \infty} E[\chi_N]$$

for all $\chi \in \mathcal{A}$. Then, since for any $\varepsilon > 0$ there is a deformation $\chi_M \in \mathcal{A}$, for M a sufficiently large integer possibly dependent on ε , such that

$$\lim_{N \rightarrow \infty} E[\chi_N] < E[\chi_M] < \lim_{N \rightarrow \infty} E[\chi_N] + \varepsilon,$$

it follows that

$$\inf_{\chi \in \mathcal{A}} E[\chi] = \lim_{N \rightarrow \infty} E[\chi_N]. \quad \Delta$$

The status of the two additional cases $\Psi/H \leq \kappa_1$ and $\Psi/H > \kappa_2$ is summarized in the following

Theorem 6. *Suppose $\overline{W}(\cdot)$ satisfies the hypotheses of Theorem 5. If $\Psi/H \leq \kappa_1$, then the unique minimizer for the problem (10) is given in $\mathcal{A}$ by the classical torsion field χ_0 of (28). If $\Psi/H > \kappa_2$, then a minimizing sequence in $\mathcal{A}$ for the problem (10) is given by χ_N of (50) and $\mathbf{y}$ of (47) as $N \rightarrow \infty$, with χ defined according to*

$$\chi = \chi^m + g(R; \varepsilon)(\chi^s - \chi^m) + h(R; \varepsilon)(\chi^s - \chi^m), \quad (59)$$

and $g(R; \varepsilon)$ and $h(R; \varepsilon)$ ($\varepsilon = L = H/N$) given by (35) and (49), respectively.

Proof. The proof of this theorem is straightforward. First, if $\alpha = \Psi/H \leq \kappa_1$, then $R_1 = \kappa_1/\alpha \geq 1$. Thus, $\alpha R \leq \kappa_1$ for all $R \in [0, 1]$, and such points are points of convexity of $\overline{W}(\cdot)$ by the Constitutive Hypotheses. Therefore, by Theorem 2 the field $\tilde{\chi} = \chi_0$, where χ_0 is given by (28), is the unique minimizer of problem (10). Now, if $\alpha = \Psi/H > \kappa_2$, then $R_2 = \kappa_2/\alpha < 1$, and the mixture-phase domain of $\mathcal{B}$ lies in the interval $(R_1, R_2) \subset [0, 1]$ which is interior to $\mathcal{B}$. Thus, in addition to the single-phase central core of $\mathcal{B}$, there is a second annular single-phase domain in the region $(R_2, 1]$ which is adjacent to the outer lateral boundary. We may conclude, for this case, that the deformation fields χ_N , $N = 1, 2, \dots$, as indicated in this theorem, form a minimizing sequence for the problem by using an argument similar to that given in Theorem

5. The main difference here, over the situation considered in Theorem 5, is that now there is a second annular single-phase domain which borders the mixture-phase domain on its outside nominally at the radius R_2 , and a corresponding transition domain and match function that must be introduced as in (49). We shall not carry out the details of the construction and the limiting behavior of the sequence here, as the line of argument is analogous to that which was given in Section 4 for the case $\alpha = \Psi/H \in (\kappa_1, \kappa_2]$. Δ

Finally, we offer the following

Closing Remarks and Discussion. Let us call those parts of $\mathcal{B}$ for which $\kappa \leq \kappa_1$ and $\kappa \geq \kappa_2$ the low and the high strain domains, respectively. From Theorems 5 and 6, we see that for a given cylinder $\mathcal{B}$ with a stored energy function which satisfies the associated Constitutive Hypotheses of Section 2 the deformed configuration of the cylinder will contain (i) only a low strain domain when the angle of twist Ψ is small (i.e., $\Psi \leq \kappa_1 H$); (ii) a cylindrical core of low strain bordered by an annular mixture-phase domain of both high and low strains out to the lateral surface of the cylinder when the angle of twist is moderate (i.e., $\kappa_1 H < \Psi \leq \kappa_2 H$); and (iii) an annular mixture-phase domain of both high and low strains bordered on the inside by a cylindrical core of low strain, and on the outside by an annular domain of high strain which extends out to the lateral boundary of the cylinder when the angle twist is large (i.e., $\Psi > \kappa_2 H$). Suppose that such a cylinder is subjected to a static sequence of torsional deformations such that the height of the cylinder is kept fixed and the angle of the twist Ψ increases from zero to a value greater than $\kappa_2 H$. Then, according to the above analysis, there will be the development of an annular mixture-phase domain of high and low strains, whose relative fractions are radial dependent, which starts to appear at the outer lateral boundary of the cylinder when the angle of twist Ψ reaches the value $\kappa_1 H$, and which grows inwardly with increasing Ψ . As the angle of twist increases, the deformed state of the cylinder will undergo a transition from a single-phase state of completely low strain, to a complex field of single-phase low strain in a central core which is surrounded by an outer annular mixture-phase, to a field which consists of a mixture-phase state in an annulus which is bordered on the inside by a single-phase low strain core, and on the outside by another single-phase annulus of high strain which persists out to the lateral boundary of the cylinder.

As we mentioned in the Introduction, the motivation for this work was based on certain old experimental observations concerning the torsion of bars, as reported by

Nadai [13]. In these experiments, it is important to note that much care was taken to eliminate the possible misinterpretation of the microstructures of melt-freeze grain boundaries, often present due to the solidification process in manufacturing bars, with the morphologies that were induced by twisting. In Figure 4, we give some pictures from a series of torsion experiments from Nadai [13] which show the cross sections of twisted and deformed soft-iron circular bars. In these photographs one can observe, as the angle of twist increases, the development of “plastic” zones of large shear which are, roughly, sector shaped with an apparent periodic circumferential distribution. When the angle of twist is small, e.g., $\Psi/H = 0.007$ deg/length, there appears to be no sector shaped zones of large shear. When the angle of twist is large, e.g., $\Psi/H = 1.84$ or 3.02 deg/length, there appears to be a transition domain which separates the purely low strain inner core and the purely high strain outer annulus. In comparison, recall that our minimizing sequence for the problem (10) gave rise to the existence of an annular mixture-phase domain which separates the low and high strain single-phase domains when the angle of twist Ψ is relatively large, and we showed that the low strain single-phase core (bounded by the radius R_1) gets smaller as Ψ increases.

It is fairly clear from Figure 4 that the mixture-phase region which separates the low and high strain regions consists of a mixed distribution of zones of both low and high strains. Each of the separate low strain or high strain zones of this mixed distribution appears to occupy a finite volume and the mixture in the cross section appears to be reasonably periodic. An interesting more recent study of the scale effect and periodic order of the plastic zone pattern has been reported by Shioya, Machida, Ishida and Fujiura [20]. Our analysis gives rise to a minimizing sequence which has the property that in the mixture-phase domain each material point is, roughly, a mixture of both low and high strains whose volume fraction ratio of low to high strain is given by $\frac{R_2-R}{R_2-R_1}$ for any $R \in (R_1, R_2)$. This difference between the discreteness of the morphology in the experimental observations and the homogeneity of the theoretical result (which predicts, nevertheless, a radial dependent density field distribution of phases) might be resolved by taking into account the interfacial energy in the formulation of the total potential energy. For example, by appending an additional term proportional to $|\nabla(\nabla\chi)|^2$ to the stored energy function, another length scale naturally enters the problem and this may relate to the periodicity question.

In Figure 5, 6, and 7 we simply show examples of experimental observations from

Nadai [13] for the cross sections of steel bars, of square, rectangular, and triangular cross section which are subject to large torsion. In these photographs we see the development of wedge shaped plastic zones of large shear, which have a periodical circumferential distribution. The corresponding general minimization problem (10) is open.

Finally, it is with caution that we note that while certain important features of the minimizing sequence developed in this work are indeed similar to experimental observations, the theory that we used is, after all, elasticity, and does not contain fundamental ideas of dissipation, loading, unloading and yield.

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Figure 4: Cross sections of soft-iron bars of circular cross section subjected to a torsion deformation (from [13]).

- a. $\frac{\Psi}{H}=0.007 \text{ deg/cm}$, b. $\frac{\Psi}{H}=0.04 \text{ deg/cm}$, c. $\frac{\Psi}{H}=0.22 \text{ deg/cm}$,
d. $\frac{\Psi}{H}=1.84 \text{ deg/cm}$, e. $\frac{\Psi}{H}=3.02 \text{ deg/cm}$.

Figure 5: Cross sections of soft-iron bars with a square cross section subjected to a torsion deformation (from [13]).

- a. $\frac{\Psi}{H}=0.38 \text{ deg/cm}$, b. $\frac{\Psi}{H}=0.68 \text{ deg/cm}$, c. $\frac{\Psi}{H}=0.90 \text{ deg/cm}$.

Fig. 36-12.

Fig. 36-13.

Fig. 36-14.

Figure 6: Cross sections of soft-iron bars with a rectangular cross section subjected to a torsion deformation (from [13]).

Figure 7: Cross sections of soft-iron bars with a triangular cross section subjected to a torsion deformation (from [13]).

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