**ON AN ELLIPTIC-PARABOLIC PROBLEM RELATED
TO PHASE TRANSITIONS IN SHAPE MEMORY ALLOYS**

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ON AN ELLIPTIC-PARABOLIC PROBLEM RELATED TO PHASE TRANSITIONS IN SHAPE MEMORY ALLOYS*

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Abstract. We consider the problem of phase transitions in shape memory alloys in the case of a given strain. The resulting energy conservation equation is shown to admit a weak solution. Uniqueness and regularity results are also proved. The assumptions needed upon the Helmholtz free energy and the strain tensor are brought to the fore. A numerical approximation method is proposed. The fully discrete scheme is given by C^0 piecewise finite elements in space combined with a semi-implicit scheme in time. Error estimates are derived.

Key words. elliptic-parabolic, phase transitions, shape memory alloys, finite elements, error estimates

AMS(MOS) subject classifications (1985 revision). 35K55, 35M10, 65M15, 65M60, 73B30

1. Introduction

We are interested in the following unusual behavior of some metallic alloys : they can be permanently deformed and may, by thermal means, recover their initial shape. Such a phenomenon is called shape memory and can be interpreted as the effect of an austenite-martensite phase transition (see e.g. Kinderlehrer [7]).

In the sequel, we use the following notations

$$\begin{array}{ll} \psi & : \text{Helmholtz free energy,} \\ \vartheta & : \text{absolute temperature,} \\ u & : \text{displacement,} \\ \varepsilon & : \text{strain tensor } (\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})). \end{array} \quad \begin{array}{ll} k & : \text{heat conductivity,} \\ f & : \text{heat source,} \\ g & : \text{body force,} \end{array}$$

For the sake of simplicity, we assume that the material we consider has a constant density $\rho = 1$. Therefore the leading equations governing the dynamics of the phase transition arise from the two following general conservation principles : conservation of energy and conservation of linear momentum.

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We assume that the Helmholtz free energy has the form $\psi = \psi(\vartheta, \varepsilon)$, and admits the representation

$$(1.1) \quad \psi = \mathcal{E} - \vartheta s,$$

where $\mathcal{E}$ and s are the densities of internal energy and entropy respectively.

If σ denotes the stress tensor, the conservation of linear momentum is given by

$$(1.2) \quad \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + g_i,$$

and the energy conservation equation may be written down as

$$(1.3) \quad \frac{\partial}{\partial t} \mathcal{E} + q_{i,i} - \sigma_{ij} v_{i,j} = f.$$

Our problem is now completed with the following constitutive laws

$$(1.4) \quad \frac{\partial \psi}{\partial \varepsilon_{ij}} = \sigma_{ij} \quad \text{and} \quad q_i = -k \vartheta_{,i}.$$

Let us here point out that the problem (1.2), (1.3) is wide open, even in the one-dimensional case ! The reasons are the following. As a consequence of the memory effect, we cannot assume any convexity property upon $\psi(\vartheta, \cdot)$, therefore equation (1.2) is of a very special (mixed) type. The one-dimensional uncoupled equation is a difficult problem in itself. For some theoretical results, the reader can refer to James [6], Slemrod [15] or the recent paper by Hattori and Mischaikow [5]; see also Truskinovsky [17] for a more general discussion of the problem as a whole.

With respect to the coupled problem, many papers have been published. Niezgódka et al. [9] studied the one-dimensional problem. They obtained some existence and uniqueness results. However, they assume, instead of (1.4), constitutive relations of the following type

$$\sigma = \frac{\partial \psi}{\partial \varepsilon} + \mu \frac{\partial \varepsilon}{\partial t} \quad \text{and} \quad q = -k \frac{\partial \vartheta}{\partial x} - \alpha k \frac{\partial^2}{\partial x \partial t} \vartheta, \quad \mu, \alpha > 0,$$

(see also Sprekels and Zheng [16]).

Unfortunately, the stabilizing terms in the above relations are more mathematically convenient than physically meaningful, and are usually introduced in order to define, by taking the limit, some admissibility criterion, see e.g. Shearer [13] or Slemrod [14].

A more general three-dimensional model has been studied by Colli et al. [2]. Authors prove existence and uniqueness of a weak solution, however the mechanical equation is quasi-static and a stabilizing term is also considered.

The goal of this paper is to study the thermal equation, which is by far the simplest one in our problem. We consider equation (1.3) in the case of a given strain ε . Relations (1.1)–(1.4) as well as the symmetry of the stress tensor σ lead to the following problem. Let $\Omega \subset \mathbf{R}^d$ be a smooth, bounded, material domain and let $T > 0$ be some positive number. To find $\vartheta : \Omega \times (0, T) \rightarrow \mathbf{R}$ such that

$$(1.5) \quad -\vartheta \frac{\partial}{\partial t} \left(\frac{\partial}{\partial \vartheta} \psi(\vartheta, \varepsilon) \right) - \Delta \vartheta = f \quad \text{in } Q,$$

$$(1.6) \quad \vartheta = 0 \quad \text{on } \Sigma,$$

$$(1.7) \quad \vartheta(\cdot, 0) = \vartheta_0 \quad \text{in } \Omega,$$

where $\varepsilon : \Omega \rightarrow \mathbf{R}^{d^2}$ and $\vartheta_0 : \Omega \rightarrow \mathbf{R}$ are given functions and where $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$.

In the statement of problem (1.5)–(1.7), we have assumed, for the sake of simplicity, that $k = 1$ and that ϑ satisfies homogeneous Dirichlet boundary conditions. However, with the same methods as those used below, more general situations may be considered.

The motivations for studying problem (1.5)–(1.7) are the following.

- It can be viewed as a first part of a study of problem (1.2), (1.3) with constitutive laws (1.4).
- Unlike what is done in all the references quoted above (in a way or another), we do not assume that equation (1.5) is everywhere parabolic. More precisely, the coefficient of the time derivative $\frac{\partial \vartheta}{\partial t}$ in (1.5) is only supposed to be non-negative (and not strictly positive as in Sprekels and Zheng [16]), i.e. $-\vartheta \frac{\partial^2 \psi}{\partial \vartheta^2}(\vartheta, \varepsilon) \geq 0$.
- As a consequence of the previous remark, equation (1.5) is of mixed elliptic-parabolic type. In order to prove existence of a solution, we use a compactness result due to Alt and Luckhaus [1]; however, we do not have the same kind of non-linearities as in [1].
- Equation (1.5) may be written as

$$(1.8) \quad \frac{\partial u}{\partial t} - \Delta \vartheta + e_\varepsilon(\vartheta) = f \quad u = b_\varepsilon(\vartheta),$$

where $b_\varepsilon(\vartheta) = - \int_0^\vartheta \lambda \frac{\partial^2 \psi}{\partial \vartheta^2}(\lambda, \varepsilon) d\lambda$ and $e_\varepsilon(\vartheta) = \vartheta \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t}$. In this relation, u may be viewed as a “generalized enthalpy”. However, unlike the Stefan problem, on the one hand the “enthalpy” is uniquely determined by the temperature, and on the other hand the converse is not true, since the inverse of b_ε is here a multi-valued function. Such a situation also arises in the so-called nonstationary filtration problem, (see e.g. [1] or van Duyn and Peletier [3]).

• The former remark allows us, with some modifications studied below, to apply numerical results obtained by Nochetto and Verdi [10] in the case of the nonstationary filtration problem. The nonlinear term e_ε we consider depends on the temperature ϑ and not on u as in the classical statement of the nonstationary filtration problem (see [10]).

The outline of the paper is as follows. In section 2, we prove existence of a weak solution. Section 3 is devoted to regularity and uniqueness results. In each case, the assumptions required upon ψ are brought to the fore. Finally in section 4, we study a numerical approximation scheme based on C^0 piecewise linear finite elements in space together with a semi-implicit finite difference scheme in time. Error estimates are derived.

2. Existence of a weak solution

In this section, we state and prove an existence result for weak solutions to problem (1.5)–(1.7). The assumptions on the free energy function ψ and on the given strain function ε are brought to the fore. Let us first introduce some notations.

Hereafter $H^m(\Omega)$, $m \in \mathbf{Z}$, stands for the classical Sobolev space; its norm and semi-norm are respectively denoted by $\|\cdot\|_{m,\Omega}$, $m \in \mathbf{Z}$, and $|\cdot|_{m,\Omega}$, $m \geq 1$.

Assuming that ψ , ϑ and ε are smooth enough, relation (1.1) leads to

$$(2.1) \quad -\vartheta \frac{\partial^2 \psi}{\partial \vartheta^2}(\vartheta, \varepsilon) \frac{\partial \vartheta}{\partial t} - \Delta \vartheta - \vartheta \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} = f \quad \text{in } \Omega \times (0, T),$$

where we use the summation convention.

REMARK 2.1. The term $-\vartheta \frac{\partial^2 \psi}{\partial \vartheta^2}(\vartheta, \varepsilon)$ appearing in (2.1) as factor of the time derivative $\frac{\partial \vartheta}{\partial t}$ is of paramount importance in the sequel. In order to avoid changes of the “time direction”, we shall suppose that this term is non-negative (see hypothesis (2.2) below). However equation (2.1) can still be degenerated in an elliptic equation with time as parameter in the case $-\vartheta \frac{\partial^2 \psi}{\partial \vartheta^2}(\vartheta, \varepsilon) = 0$. $\square$

According to remark 2.1, we assume that

$$(2.2) \quad \psi \in C^2(\mathbf{R} \times \mathbf{R}^{d^2}) \quad \text{and} \quad -s \frac{\partial^2 \psi}{\partial \vartheta^2}(s, \varepsilon) \geq 0 \quad \forall (s, \varepsilon) \in \mathbf{R} \times \mathbf{R}^{d^2}.$$

For any $\varepsilon \in \mathbf{R}^{d^2}$, let $b_\varepsilon : \mathbf{R} \rightarrow \mathbf{R}$ and $\Phi_\varepsilon : \mathbf{R} \rightarrow \mathbf{R}$, a primitive of b_ε , defined by

$$b_\varepsilon(s) = - \int_0^s \lambda \frac{\partial^2 \psi}{\partial \vartheta^2}(\lambda, \varepsilon) d\lambda \quad \text{and} \quad \Phi_\varepsilon(s) = - \int_0^s d\mu \int_0^\mu \lambda \frac{\partial^2 \psi}{\partial \vartheta^2}(\lambda, \varepsilon) d\lambda.$$

Taking into account assumption (2.2), we notice that $\Phi_\varepsilon \in C^4(\mathbf{R})$ is a convex function. Defining $E_\varepsilon(s) = \sup_{\mu \in \mathbf{R}} \int_0^1 (s - b_\varepsilon(\lambda\mu))\mu d\lambda$, we obtain, since $b_\varepsilon(s) = \Phi'_\varepsilon(s)$

$$E_\varepsilon(s) = \sup_{\mu \in \mathbf{R}} (s\mu - \Phi_\varepsilon(\mu) + \Phi_\varepsilon(0)).$$

We now set $B_\varepsilon(s) = E_\varepsilon(b(s)) = \sup_{\mu \in \mathbf{R}} (b_\varepsilon(s)\mu - \Phi_\varepsilon(\mu) + \Phi_\varepsilon(0))$. The convexity of Φ_ε implies

$$B_\varepsilon(s) = b_\varepsilon(s)s - \Phi_\varepsilon(s) + \Phi_\varepsilon(0) = \int_0^1 (b_\varepsilon(s) - b_\varepsilon(\lambda s))s d\lambda = \int_0^s (b_\varepsilon(s) - b_\varepsilon(\lambda))d\lambda.$$

Let us remark that the convexity of Φ_ε also implies the useful relations

$$(2.3) \quad B_\varepsilon(s) - B_\varepsilon(s_0) \geq (b_\varepsilon(s) - b_\varepsilon(s_0))s_0 \quad \forall s, s_0 \in \mathbf{R},$$

$$(2.4) \quad B_\varepsilon(s) \leq sb_\varepsilon(s) \quad \forall s \in \mathbf{R}.$$

Finally if ε is now a sufficiently smooth function in Q , we define e_ε as being the Nemytskiĭ's operator associated to $e_\varepsilon(\cdot) = -\cdot \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\cdot, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t}$, i.e.

$$e_\varepsilon(\vartheta)(x, t) = -\vartheta(x, t) \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta(x, t), \varepsilon(x, t)) \frac{\partial \varepsilon_{ij}}{\partial t}(x, t) \quad \text{a.e. } (x, t) \in Q,$$

where ϑ is an arbitrary measurable function defined in Q . Using the above notations, equation (2.1) may be written down as

$$(2.5) \quad \frac{\partial}{\partial t} b_\varepsilon(\vartheta) - \Delta \vartheta + e_\varepsilon(\vartheta) = f \quad \text{in } Q.$$

Let us now define $b_{\varepsilon 0}$ as follows $b_{\varepsilon 0} \equiv b_\varepsilon(\vartheta_0) = -\int_0^{\vartheta_0} \lambda \frac{\partial^2 \psi}{\partial \vartheta^2}(\lambda, \varepsilon) d\lambda$. We assume that the initial condition ϑ_0 and the heat source f are such that

$$(2.6) \quad b_{\varepsilon 0} \in L^1(\Omega) \quad \text{and} \quad f \in L^2(Q).$$

All this leads to the following definition.

DEFINITION. We call $\vartheta \in L^2(0, T; H_0^1(\Omega))$ a weak solution of (1.5)- (1.7) if $b_\varepsilon(\vartheta) \in L^\infty(0, T; L^1(\Omega))$, $\frac{\partial}{\partial t} b_\varepsilon(\vartheta) \in L^2(0, T; H^{-1}(\Omega))$ and $e_\varepsilon(\vartheta) \in L^2(Q)$ and if the following relations are satisfied

$$(2.7) \quad \int_0^T \left\langle \frac{\partial}{\partial t} b_\varepsilon(\vartheta), \zeta \right\rangle dt + \int_0^T \int_\Omega (b_\varepsilon(\vartheta) - b_{\varepsilon 0}) \frac{\partial \zeta}{\partial t} dx dt = 0$$

$$\forall \zeta \in L^2(0, T; H_0^1(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega)), \quad \zeta(T) = 0,$$

$$(2.8) \quad \int_0^T \left\langle \frac{\partial}{\partial t} b_\varepsilon(\vartheta), \zeta \right\rangle dt + \int_0^T \int_\Omega (\nabla \vartheta \nabla \zeta + e_\varepsilon(\vartheta) \zeta) dx dt = \int_0^T \int_\Omega f \zeta dx dt$$

$$\forall \zeta \in L^2(0, T; H_0^1(\Omega)). \quad \square$$

Using the above properties of b_ε and B_ε , it is possible to prove a compactness result.

LEMMA 2.1. Let $\{\vartheta_\delta\}_{\delta>0}$ be a weakly convergent sequence in $L^2(0, T; H_0^1(\Omega))$ with limit u . If the following properties hold

$$\int_\tau^T \int_\Omega (b_\varepsilon(\vartheta_\delta(t)) - b_\varepsilon(\vartheta_\delta(t - \tau))) (\vartheta_\delta(t) - \vartheta_\delta(t - \tau)) \leq C\tau,$$

and

$$\int_\Omega B_\varepsilon(\vartheta_\delta(t)) \leq C \quad \text{for } 0 < t < T,$$

then $b_\varepsilon(\vartheta_\delta) \rightarrow b_\varepsilon(\vartheta)$ in $L^1(Q)$ and $B_\varepsilon(\vartheta_\delta) \rightarrow B_\varepsilon(\vartheta)$ almost everywhere.

Proof. See Alt and Luckhaus [1], lemma 1.9. $\square$.

We also have the following property.

LEMMA 2.2. Let $\vartheta \in L^2(0, T; H_0^1(\Omega))$ satisfy (2.7), then $B_\varepsilon(\vartheta) \in L^\infty(0, T; L^1(\Omega))$ and

$$\int_0^t \int_\Omega \left\langle \frac{\partial}{\partial t} b_\varepsilon(\vartheta), \vartheta \right\rangle dx dt = \int_\Omega B_\varepsilon(\vartheta(t)) dx - \int_\Omega B_\varepsilon(\vartheta_0) dx \quad \text{a.e. } t \in (0, T).$$

Proof. We drop the index ε in the proof. Let $\tau > 0$ be some time increment. Taking into account relation (2.3), we have

$$B(\vartheta(t)) - B(\vartheta(t - \tau)) \leq \left(b(\vartheta(t)) - b(\vartheta(t - \tau)) \right) \vartheta(t) \quad \text{a.e. } t \in (0, T), \text{ a.e. in } \Omega,$$

$$B(\vartheta(t)) - B(\vartheta(t - \tau)) \geq \left(b(\vartheta(t)) - b(\vartheta(t - \tau)) \right) \vartheta(t - \tau) \quad \text{a.e. } t \in (0, T) \text{ a.e. in } \Omega,$$

where $\vartheta(t) \equiv \vartheta_0$ for $-\tau < t < 0$ and thus $b(\vartheta(t - \tau)) = b_0$ if $0 < t < \tau$.

Let us consider the first equation. After integration over $\Omega \times (0, t^*)$, $t^* \in (0, T)$, we get

$$\int_{t^*-\tau}^{t^*} \int_{\Omega} B(\vartheta(t)) - \int_{-\tau}^0 \int_{\Omega} B(\vartheta(t)) \leq \int_0^{t^*} \int_{\Omega} (b(\vartheta(t)) - b(\vartheta(t-\tau))) \vartheta(t),$$

and thus

$$\frac{1}{\tau} \int_{t^*-\tau}^{t^*} \int_{\Omega} B(\vartheta(t)) - \int_{\Omega} B(\vartheta_0) \leq \int_0^{t^*} \int_{\Omega} \partial_{\tau} b(\vartheta(t)) \vartheta(t),$$

where ∂_{τ} denotes the discrete backward time derivation operator.

Similarly, we obtain for the second equation

$$\frac{1}{\tau} \int_{t^*-\tau}^{t^*} \int_{\Omega} B(\vartheta(t)) - \frac{1}{\tau} \int_0^{\tau} \int_{\Omega} B(\vartheta(t)) \geq \int_0^{t^*-\tau} \int_{\Omega} \partial_{-\tau} b(\vartheta(t)) \vartheta(t).$$

Taking the limit $\tau \rightarrow 0$ in the last two relations achieves the proof. $\square$

We now establish an existence result. We approximate the differential equation by a backward finite difference method in time (Euler backward scheme) and solve the resulting elliptic problems using a Galerkin procedure.

THEOREM 2.1. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ and let assumptions (2.2) and (2.6) hold. If $\frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s, \lambda)$ is bounded with respect to s and if $s \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s, \lambda)$ is Lipschitz continuous with respect to s , more precisely if, for all $s, s_1, s_2 \in \mathbb{R}$, $\lambda \in \mathbb{R}^{d^2}$, $i, j = 1, \dots, d^2$*

$$(2.9) \quad \begin{aligned} & \left| \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s, \lambda) \right| \leq C_1 |\lambda|^q, \quad q \geq 0, \\ & \left| s_1 \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s_1, \lambda) - s_2 \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s_2, \lambda) \right| \leq C_2(\lambda) |s_1 - s_2|, \end{aligned}$$

where C_1 and C_2 are independent of s, s_1, s_2, i, j , then for any $\varepsilon \in W^{1,\infty}(0, T; L^{\infty}(\Omega)^{d^2})$, with $\|\varepsilon\|_{W^{1,\infty}(0, T; L^{\infty}(\Omega)^{d^2})}$ small enough, the problem (1.5)–(1.7) admits a weak solution.

Proof. Let $\{v_i\} \subset H_0^1(\Omega) \cap L^{\infty}(\Omega)$ be a set of linearly independent functions such that $\text{span}\{v_i\}$ is dense in $H_0^1(\Omega)$. We set $V_m = \text{span}\{v_1, \dots, v_m\}$. Given a time step $\tau > 0$, we are looking for a function $\vartheta_{\tau m}(x, t) = \sum_{i=1}^m \Theta_i(t) v_i(x)$, where $\Theta_i \in L^{\infty}(0, T)$ is such that

$$(2.10) \quad \begin{aligned} & \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}(t)) \zeta + \int_{\Omega} \left(\nabla \vartheta_{\tau m}(t) \nabla \zeta + e(\vartheta_{\tau m}(t)) \zeta \right) \\ & \quad - \int_{\Omega} f \zeta = 0 \quad \forall \zeta \in V_m, \text{ a.e. } t \in (0, T), \end{aligned}$$

with initial condition $\vartheta_{\tau m}(t) = \vartheta_{0\tau}$, $-\tau < t < 0$, where $\vartheta_{0\tau}$ is some bounded approximation of ϑ_0 .

It can be proved in a standard way that relation (2.10) is a system of nonlinear algebraic equations associated with a continuous and uniformly monotone operator. This leads to the existence of a unique function $\vartheta_{\tau m}$ (see Ortega and Rheinboldt [11], th.6.4.4, p.167).

We now have to prove the convergence of $\vartheta_{\tau m}$ as $(\tau, m) \rightarrow (0, \infty)$.

Let us first consider $\zeta = \vartheta_{\tau m}$ in (2.10). After integration over t from 0 to t^* and using relation (2.3), we get for the parabolic part of the equation

$$\int_0^{t^*} \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) \vartheta_{\tau m} \geq \frac{1}{\tau} \int_0^{t^*} \int_{\Omega} (B(\vartheta_{\tau m}(t)) - B(\vartheta_{\tau m}(t - \tau))),$$

and thus

$$\int_0^{t^*} \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) \vartheta_{\tau m} \geq \frac{1}{\tau} \int_{t^* - \tau}^{t^*} \int_{\Omega} B(\vartheta_{\tau m}) - \int_{\Omega} B(\vartheta_{0\tau}).$$

For the elliptic part, we have by (2.9)

$$\begin{aligned} I &\equiv \int_0^{t^*} \int_{\Omega} (\nabla \vartheta_{\tau m})^2 + \int_0^{t^*} \int_{\Omega} e(\vartheta_{\tau m}) \vartheta_{\tau m} \\ &\geq \int_0^{t^*} \int_{\Omega} (\nabla \vartheta_{\tau m})^2 - C \|\varepsilon\|_{W^{1,\infty}(0,T;L^{\infty}(\Omega)^{d^2})}^{1+q} \int_0^{t^*} \int_{\Omega} \vartheta_{\tau m}^2, \end{aligned}$$

where, if $q > 0$, we use the Young inequality $ab^q \leq \frac{1}{p}a^p + \frac{1}{p'}b^{p'q}$, $a \geq 0$, $b \geq 0$, with $p = 1 + q$ and $p' = \frac{1}{q} + 1$ and where C is a constant independent of ε , ϑ , τ and m . Thus, by Poincaré inequality, we obtain

$$I \geq \left(1 - C \|\varepsilon\|_{W^{1,\infty}(0,T;L^{\infty}(\Omega)^{d^2})}^{1+q}\right) \int_0^{t^*} \int_{\Omega} (\nabla \vartheta_{\tau m})^2.$$

Therefore, if $\|\varepsilon\|$ is small enough in the above relation, we get finally for both elliptic and parabolic parts

$$(2.11) \quad \operatorname{esssup}_{0 < t < T} \int_{\Omega} B(\vartheta_{\tau m}) + \int_0^T \int_{\Omega} \nabla \vartheta_{\tau m}^2 \leq c(\varepsilon),$$

where we have used the fact that t^* is arbitrary in $(0, T)$.

Consequently, there exists a subsequence still denoted $\{\vartheta_{\tau m}\}$ such that

$$(2.12) \quad \vartheta_{\tau m} \rightharpoonup \vartheta \quad \text{in } L^2(0, T; H_0^1(\Omega)) \text{ weak.}$$

Now, let ϕ be any function in the space $L^2(0, T; H_0^1(\Omega))$, we have

$$\int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) \phi = \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) \Pi_m \phi + \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) (\phi - \Pi_m \phi),$$

where Π_m is the orthogonal projection operator onto the space V_m defined with respect to the $H_0^1(\Omega)$ -inner product, i.e.

$$g \in H_0^1(\Omega) \quad (\nabla g, \nabla \zeta)_{0, \Omega} = (\nabla \Pi_m g, \nabla \zeta)_{0, \Omega} \quad \forall \zeta \in V_m,$$

where $(\cdot, \cdot)_{0, \Omega}$ stands for the $L^2(\Omega)$ -inner product. Clearly, it is possible to construct V_m such that

$$\lim_{m \rightarrow \infty} \|g - \Pi_m g\|_{1, \Omega} = 0 \quad \forall g \in H_0^1(\Omega).$$

By (2.10) and (2.11) together with the properties of Π_m , we have

$$\begin{aligned} \int_0^T \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) \phi &= - \int_0^T \int_{\Omega} \left(\nabla \vartheta_{\tau m} \nabla \Pi_m \phi + e(\vartheta_{\tau m}) \Pi_m \phi + f \Pi_m \phi \right) \\ &\quad + \int_0^T \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}) (\phi - \Pi_m \phi) \\ &\leq \int_0^T \left(\|\nabla \vartheta_{\tau m}\|_{0, \Omega} + C \|f\|_{0, \Omega} \right) \|\Pi_m \phi\|_{1, \Omega} + \int_0^T \int_{\Omega} |e(\vartheta_{\tau m}) \Pi_m \phi| \\ &\quad + \int_0^T \|\partial_{\tau} b(\vartheta_{\tau m})\|_{-1, \Omega} \|\phi - \Pi_m \phi\|_{1, \Omega}, \end{aligned}$$

and thus for m large enough

$$(2.13) \quad \int_0^T \|\partial_{\tau} b(\vartheta_{\tau m})\|_{-1, \Omega} \leq c(\varepsilon).$$

It follows that for any $\chi \in \mathcal{D}(\Omega)$

$$\int_{\tau}^T \int_{\Omega} \chi \left(b(\vartheta_{\tau m}(t)) - b(\vartheta_{\tau m}(t - \tau)) \right) \left(\vartheta_{\tau m}(t) - \vartheta_{\tau m}(t - \tau) \right) \leq c(\varepsilon) \|\nabla \chi\|_{L^{\infty}(\Omega)} \tau.$$

By (2.11) and the latter relation, the assumptions of lemma 2.1 are fulfilled, and we obtain

$$b(\vartheta_{\tau m}) \longrightarrow b(\vartheta) \quad \text{in } L^1(Q), \quad \text{and} \quad B(\vartheta_{\tau m}) \longrightarrow B(\vartheta) \quad \text{a.e. in } \Omega.$$

Moreover by (2.13), there exists $\lambda_{\tau m} \in L^2(0, T; H^{-1}(\Omega))$ such that

$$\int_0^T \langle \lambda_{\tau m}, \phi \rangle = \int_0^T \int_{\Omega} \partial_{\tau} b(\vartheta_{\tau m}(t)) \phi = - \int_0^T \int_{\Omega} \left(b(\vartheta_{\tau m}(t)) - b(\vartheta_{0\tau}) \right) \partial_{-\tau} \phi,$$

for any $\phi \in L^2(0, T; H_0^1(\Omega))$ with $\phi(t) = 0$ for $t > T - \tau$.

Therefore, there exists a subsequence still denoted $\{\lambda_{\tau m}\}$ such that

$$\lambda_{\tau m} \rightharpoonup \lambda \quad \text{weak in } L^2(0, T; H^{-1}(\Omega)).$$

We obtain that $b(\vartheta)$ satisfies (2.7) and thus

$$(2.14) \quad b(\vartheta_{\tau m}) \rightharpoonup b(\vartheta) \quad \text{weak in } H^1(0, T; H^{-1}(\Omega)),$$

and we have $\lambda = \frac{\partial b(\vartheta)}{\partial t}$. If we now wish to consider the limit $(\tau, m) \rightarrow (0, \infty)$ in (2.10), it remains to study the term

$$\int_{\Omega} e(\vartheta_{\tau m}) \zeta = - \int_{\Omega} \vartheta_{\tau m} \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta_{\tau m}, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} \zeta.$$

We have by (2.9)

$$\begin{aligned} & - \int_0^T \int_{\Omega} \vartheta_{\tau m} \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta_{\tau m}, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} \zeta + \int_0^T \int_{\Omega} \vartheta \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} \zeta \\ & \leq C \|\varepsilon\|_{W^{1,\infty}(0,T;L^\infty(\Omega)^{d^2})}^{1+q} \int_0^T \int_{\Omega} |\zeta| |\vartheta_{m\tau} - \vartheta|. \end{aligned}$$

Since by (2.12) $\{\vartheta_{\tau m}\}$ converges strongly in $L^2(Q)$, we conclude

$$\begin{aligned} & - \int_0^T \int_{\Omega} \vartheta_{\tau m} \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta_{\tau m}, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} \zeta \, dx dt \longrightarrow - \int_0^T \int_{\Omega} \vartheta \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(\vartheta, \varepsilon) \frac{\partial \varepsilon_{ij}}{\partial t} \zeta \, dx dt \\ & \quad \forall \zeta \in L^2(0, T; H_0^1(\Omega)). \end{aligned}$$

Finally, taking into account this latter convergence property, together with (2.12) and (2.14), we may consider the limit $(\tau, m) \rightarrow (0, \infty)$ in (2.10) and obtain in this way existence of a weak solution. $\square$

3. Regularity and uniqueness of a weak solution

In this section, we establish sufficient conditions for the weak solution of theorem 2.1 to be regular and unique.

We first prove a regularity result.

THEOREM 3.1. *Let the assumptions of theorem 2.1 hold and assume that*

$$(3.1) \quad \vartheta_0 \in H_0^1(\Omega) \quad \text{and } f \in L^\infty(0, T; L^2(\Omega)),$$

and

$$(3.2) \quad \left| s \frac{\partial^2 \psi}{\partial \vartheta^2}(s, \lambda) \right| \leq C_3(\lambda) \quad \forall s \in \mathbb{R}, \forall \lambda \in \mathbb{R}^{d^2},$$

where the constant C_3 is independent of s , then the problem (1.5)–(1.7) admits a weak solution ϑ with $\frac{\partial}{\partial t} b_\varepsilon(\vartheta) \in L^2(Q)$.

Proof. We establish an a priori majoration for the approximating solution $\vartheta_{\tau m}$ considered in section 2. Let us choose $\zeta = \partial_\tau \vartheta$ in (2.10) and integrate with respect to time in $(0, t_\tau)$ where t_τ is a multiple of τ converging to $t \in (0, T]$. We obtain

$$\int_0^{t_\tau} \int_\Omega \left(\partial_\tau b(\vartheta_{\tau m}) \partial_\tau \vartheta_{\tau m} + \nabla \vartheta_{\tau m} \nabla \partial_\tau \vartheta_{\tau m} + e(\vartheta_{\tau m}) \partial_\tau \vartheta_{\tau m} + f \partial_\tau \vartheta_{\tau m} \right) = 0.$$

By (3.2), we remark that b is uniformly Lipschitz continuous, and thus we get for the parabolic term

$$\int_0^{t_\tau} \int_\Omega \partial_\tau b(\vartheta_{\tau m}) \partial_\tau \vartheta_{\tau m} \geq C \int_0^{t_\tau} (\partial_\tau b(\vartheta_{\tau m}))^2.$$

By (2.9), we obtain after some manipulations

$$\begin{aligned} & C \int_0^{t_\tau} \int_\Omega (\partial_\tau b(\vartheta_{\tau m}))^2 + \frac{1}{2} \int_\Omega (\nabla \vartheta_{\tau m}(t_\tau))^2 - \frac{1}{2} \int_\Omega (\nabla \vartheta_{\tau m}(0))^2 + \frac{\tau}{2} \int_0^{t_\tau} \int_\Omega (\nabla \partial_\tau \vartheta_{\tau m})^2 \\ & \leq C \|\varepsilon\|_{W^{1,\infty}(0,T;L^\infty(\Omega)^{d^2})}^{1+q} \left(\frac{1}{2} \int_\Omega (\vartheta_{\tau m}(t_\tau))^2 - \frac{1}{2} \int_\Omega (\vartheta_{\tau m}(0))^2 + \frac{\tau}{2} \int_0^{t_\tau} \int_\Omega (\partial_\tau \vartheta_{\tau m})^2 \right) \\ & \quad + \left| \int_0^{t_\tau} \int_\Omega f \partial_\tau \vartheta_{\tau m} \right|, \end{aligned}$$

and thus by Poincaré inequality, we have for $\|\varepsilon\|$ small enough

$$\int_0^{t_\tau} \int_\Omega (\partial_\tau b(\vartheta_{\tau m}))^2 + \int_\Omega (\nabla \vartheta_{\tau m}(t_\tau))^2 + \tau \int_0^{t_\tau} \int_\Omega (\nabla \partial_\tau \vartheta_{\tau m})^2 \leq c,$$

where $\{\vartheta_{\tau m}(0)\} = \{\vartheta_{0\tau}\}$ is a bounded sequence which converges to ϑ_0 in $H_0^1(\Omega)$. The theorem is now a consequence of the latter estimate, since we already know that $\partial_\tau b(\vartheta_{\tau m}) \rightharpoonup \frac{\partial}{\partial t} b(\vartheta)$ in $L^2(0, T; H^{-1}(\Omega))$ weak. $\square$

Let us now turn to a uniqueness result.

THEOREM 3.2. *Under the assumptions of theorem 2.1 and if*

$$(3.3) \quad (e(s_2) - e(s_1))^2 \leq c(b(s_2) - b(s_1))(s_2 - s_1) \quad \forall s_1, s_2 \in \mathbb{R},$$

there exists a unique solution to problem 1.5–1.7.

Proof. Let ϑ_1 and ϑ_2 be two weak solutions. By (2.7), we have

$$b(\vartheta_2) - b(\vartheta_1) = \beta \in L^2(0, T; H^{-1}(\Omega)).$$

Therefore, there is a function $v \in L^2(0, T; H_0^1(\Omega))$ such that

$$\int_0^T \int_{\Omega} \nabla v \nabla \zeta = \int_0^T \langle \beta, \zeta \rangle \quad \forall \zeta \in L^2(0, T; H_0^1(\Omega)).$$

This yields

$$\begin{aligned} 2 \int_{\tau}^{t+\tau} \langle \partial_{\tau} \beta, v \rangle + \frac{1}{\tau} \int_0^{\tau} \langle \beta, v \rangle &= \int_0^t \langle \partial_{-\tau} \beta, v(s+\tau) \rangle - \int_0^t \langle \beta, \partial_{-\tau} v \rangle + \frac{1}{\tau} \int_t^{t+\tau} \langle \beta, v \rangle \\ &= \frac{1}{\tau} \int_0^t \langle \beta(s+\tau) - \beta(s), v(s+\tau) - v(s) \rangle + \frac{1}{\tau} \int_0^t \langle \beta(s+\tau), v(s) \rangle \\ &\quad - \frac{1}{\tau} \int_0^t \langle \beta(s), v(s+\tau) \rangle + \frac{1}{\tau} \int_t^{t+\tau} \langle \beta, v \rangle \\ &= \frac{1}{\tau} \int_0^t \int_{\Omega} (\nabla v(s+\tau) - \nabla v(s))^2 - \frac{1}{\tau} \int_t^{t+\tau} \int_{\Omega} \nabla v^2. \end{aligned}$$

Letting $\tau \rightarrow 0$, we obtain

$$\int_0^t \langle \frac{\partial \beta}{\partial t}, v \rangle = \frac{1}{2} \int_{\Omega} \nabla v^2(t) \quad \text{a.e. } t \in (0, T).$$

Moreover, we have

$$\int_0^t \int_{\Omega} \nabla v \nabla (\vartheta_2 - \vartheta_1) = \int_0^t \langle \beta, \vartheta_2 - \vartheta_1 \rangle = \int_0^t \int_{\Omega} (b(\vartheta_2) - b(\vartheta_1))(\vartheta_2 - \vartheta_1).$$

Using v as test function in (2.8) leads to

$$\frac{1}{2} \int_{\Omega} \nabla v^2(t) + \int_0^t \int_{\Omega} (b(\vartheta_2) - b(\vartheta_1))(\vartheta_2 - \vartheta_1) + \int_0^t \int_{\Omega} (e(\vartheta_2) - e(\vartheta_1))v = 0.$$

Assumption (3.3) and Young and Poincaré inequalities lead to

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \nabla v^2(t) + (1 - \frac{\eta}{2}) \int_0^t \int_{\Omega} (b(\vartheta_2) - b(\vartheta_1))(\vartheta_2 - \vartheta_1) \\ \leq \frac{1}{2\eta} C \int_0^t \int_{\Omega} \nabla v^2 \quad \forall \eta \in \mathbf{R}^+. \end{aligned}$$

For η small enough, we prove the theorem by Gronwall inequality. $\square$

Let us now discuss the “uniqueness condition” (3.3)

$$(3.4) \quad \left| s_1 \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s_1, \varepsilon) - s_2 \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon_{ij}}(s_2, \varepsilon) \right|^2 \\ \leq C(\varepsilon) \left(- \int_{s_1}^{s_2} \lambda \frac{\partial^2 \psi}{\partial \vartheta^2}(\lambda, \varepsilon) d\lambda \right) (s_2 - s_1) \quad i, j = 1, \dots, d^2.$$

We first remark that if we consider the purely parabolic case, i.e. if $-s \frac{\partial^2 \psi}{\partial \vartheta^2}(s, \varepsilon) \geq C > 0$ $\forall (s, \varepsilon) \in \mathbf{R} \times \mathbf{R}^{d^2}$ holds instead of (2.2), then the Lipschitz property assumed in (2.9) is a direct consequence in (3.4) (see also remark 3.1 below).

We now analyze the case of some classical free energy expressions. For the sake of simplicity and since we are here interested in the thermal equation, we consider the above relation in the one-dimensional case, i.e. $\varepsilon \in \mathbf{R}$.

We assume that ψ has the Landau-Devonshire form

$$(3.5) \quad \psi(\vartheta, \varepsilon) = \psi_0(\vartheta) + a(\vartheta - \vartheta^*)\varepsilon^2 - b\varepsilon^4 + d\varepsilon^6,$$

where ψ_0 is a smooth function, a, b, d are positive constants and $\vartheta^* \in \mathbf{R}$.

We have

$$\frac{\partial^2 \psi}{\partial \vartheta^2} = \psi_0''(\vartheta) \quad \text{and} \quad \frac{\partial^2 \psi}{\partial \vartheta \partial \varepsilon} = 2a\varepsilon.$$

If $\psi_0 \in C^2(\mathbf{R})$ is such that

$$(3.6) \quad -s\psi_0''(s) \geq 0 \quad \forall s \in \mathbf{R},$$

then the assumptions of theorem 2.1 hold. Moreover, relation (3.4) leads to

$$4a^2\varepsilon^2(s_2 - s_1)^2 \leq C \left(- \int_{s_1}^{s_2} \lambda \psi_0''(\lambda) d\lambda \right) (s_2 - s_1),$$

and thus if $s_2 > s_1$

$$(3.7) \quad 4a^2\varepsilon^2(s_2 - s_1) \leq -C \int_{s_1}^{s_2} \lambda \psi_0''(\lambda) d\lambda.$$

Let us study this latter inequality.

In (3.5), the terms $\psi_0(\vartheta)$, $a(\vartheta - \vartheta^*)\varepsilon^2$ and $-b\varepsilon^4 + d\varepsilon^6$ represent respectively pure heat conduction, shape memory and nonlinear effects due to temperature-dependent elastic phenomena. By definition of Helmholtz free energy, we have

$$(3.8) \quad \psi_0(\vartheta) = \psi_{00} + \int_0^\vartheta c(\lambda) d\lambda - \vartheta \int_0^\vartheta \frac{c(\lambda)}{\lambda} d\lambda,$$

where ψ_{00} is the value of ψ_0 at $\vartheta = 0$ and c is the specific heat capacity at constant pressure (see Landau and Lifshitz [8], §23).

By (3.8), we have $\psi_0''(\vartheta) = -\frac{c(\vartheta)}{\vartheta}$ and thus relation (3.7) leads to

$$4a^2\varepsilon^2(s_2 - s_1) \leq C \int_{s_1}^{s_2} c(\lambda) d\lambda.$$

Sufficient conditions for this relation (together with (3.6)) to be fulfilled are

$$c(\lambda) \geq c_0 > 0 \quad \forall \lambda \in \mathbf{R}, \quad \text{and} \quad 4a^2\varepsilon^2 \leq Cc_0.$$

REMARK 3.1. If ψ is assumed in Landau-Devonshire form (or more generally in Ginzburg-Landau form, see [16]), then

$$-\vartheta\psi''(\vartheta) = -\vartheta\psi_0''(\vartheta) \geq c(\vartheta) > 0,$$

by positivity of the specific heat capacity. Moreover, if we suppose $c(\lambda) \geq c_0 > 0, \forall \lambda \in \mathbf{R}$, then equation (1.5) is purely parabolic. However, we do not have any positivity results or a priori majorations in the L^∞ -norm for ϑ . Therefore the latter relation has to be a priori assumed, since physically $c(\vartheta) \rightarrow 0$ as $\vartheta \downarrow 0$. $\square$

Finally, we remark that for ε large enough, we are not able to prove a uniqueness result if we consider ψ in Landau-Devonshire form.

4. Numerical approximations

In this section, we study a fully discrete approximation scheme to our problem. The algorithm is based on C^0 piecewise finite elements in space and, for numerical convenience, the time discretization we consider here is semi-implicit (instead of fully implicit as in section 2).

In this section, the domain $\Omega \subset \mathbf{R}^d$, $d = 2$ or 3 , is supposed to be polyhedral and convex.

Let $\{\mathcal{T}_h\}_h$ be a family of decomposition of Ω into closed d -simplices; h stands for the mesh size. We assume that

$$(4.1) \quad \text{the family } \{\mathcal{T}_h\}_h \text{ is regular and satisfies } \cup_{K \in \mathcal{T}_h} K = \bar{\Omega} \quad \forall h > 0.$$

We set as usual $V_h = \{\phi \in C^0(\bar{\Omega}); \phi|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h, \phi = 0 \text{ on } \partial\Omega\}$, where $\mathcal{P}_1(K)$ denotes the set of polynomials of degree 1 over K .

In the sequel, we use the following numerical integration formula

$$(\phi, \psi)_h = \sum_{K \in \mathcal{T}_h} \int_K r_h(\phi\psi) dx \quad \|\phi\|_h = (\phi, \phi)_h^{1/2} \quad \phi, \psi \in V_h,$$

where r_h denotes the piecewise linear interpolation operator.

It is well known that under assumption (4.1) the topology defined in V_h by $\|\cdot\|_h$ is uniformly equivalent to the one induced by the classical $L^2(\Omega)$ -topology, i.e.

$$(4.2) \quad C_1 \|\phi\|_{0,\Omega} \leq \|\phi\|_h \leq C_2 \|\phi\|_{0,\Omega} \quad \forall \phi \in V_h,$$

where C_1 and C_2 are two constants independent of h (see e.g. Raviart [12]).

Moreover, we will need the elementary relation

$$(4.3) \quad \int_a^b \lambda f(\lambda) d\lambda \leq b \int_a^b f(\lambda) d\lambda, \quad a, b \in \mathbf{R}, \quad f : \mathbf{R} \rightarrow \mathbf{R}, f \geq 0.$$

The discrete problem is defined as follows : for $1 \leq n \leq N$, find $\Theta^n \in V_h$ such that

$$(4.4) \quad \left(\frac{b(\Theta^n) - b(\Theta^{n-1})}{\tau}, \phi \right)_h + (\nabla \Theta^n, \nabla \phi) + (e(\Theta^{n-1}), \phi)_h - (f, \phi)_h = 0, \\ \forall \phi \in V_h, \quad n = 1, \dots, N,$$

where $\Theta^0 = \Pi_h \vartheta_0$, Π_h being the orthogonal projection operator into V_h defined with respect to the inner product $(\nabla \cdot, \nabla \cdot)$, and where $\tau = t/N$ is the time step and $(\cdot, \cdot)$ the $L^2(\Omega)$ inner product.

Once again existence and uniqueness of solution Θ^n , $n = 1, \dots, N$, are easily proved, since relations (4.4) define a system of linear algebraic equations associated with a continuous and uniformly monotone operator (see [11], th.6.4.4, p.167).

We first prove the following stability results.

LEMMA 4.1. *Let assumptions (2.2), (2.6), (2.9), (3.1) and (4.1) hold and let ε belong to $W^{1,\infty}(0, T; L^\infty(\Omega)^{d^2})$ be a given function. Then if $\|\varepsilon\|_{W^{1,\infty}(0, T; L^\infty(\Omega)^{d^2})}$ is small enough, there exists a constant C , independent of h and τ , such that*

$$(4.5) \quad \sum_{n=1}^N \tau \|\Theta^n\|_{1,\Omega}^2 \leq C,$$

$$(4.6) \quad \max_{n=1, \dots, N} \|U^n\|_{0,\Omega} + \sum_{n=1}^N \|U^n - U^{n-1}\|_{0,\Omega}^2 \leq C,$$

where $U^n = r_h b(\Theta^n)$, $n = 1, \dots, N$.

Proof. We take $\phi = \tau\Theta^n$ in (4.4) and sum over n from 1 to n_0 , $1 \leq n_0 \leq N$. We get

$$\begin{aligned} \sum_{n=1}^{n_0} (b(\Theta^n) - b(\Theta^{n-1}), \Theta^n)_h + \sum_{n=1}^{n_0} \tau(\nabla\Theta^n, \nabla\Theta^n) + \sum_{n=1}^{n_0} \tau(e(\Theta^{n-1}), \Theta^n)_h \\ + \sum_{n=1}^{n_0} \tau(f, \Theta^n)_h := I + II + III + IV = 0. \end{aligned}$$

Let us estimate each of these terms separately.

Let $\{\phi_i\}_{i=1}^M$ be the canonical basis in V_h . We set $\Theta^n = \sum_{i=1}^M \Theta_i^n \phi_i$. Then we have

$$\sum_{n=1}^{n_0} (b(\Theta_i^n) - b(\Theta_i^{n-1}))\Theta_i^n = \sum_{n=1}^{n_0} \Theta_i^n \int_{\Theta_i^{n-1}}^{\Theta_i^n} b'(\lambda) d\lambda \geq \sum_{n=1}^{n_0} \int_{\Theta_i^{n-1}}^{\Theta_i^n} \lambda b'(\lambda) d\lambda,$$

where we have used the elementary relation (4.3). Consequently by (2.2) we get $I \geq 0$. Moreover Poincaré inequality leads to

$$II \geq C \sum_{n=1}^{n_0} \tau \|\Theta\|_{1,\Omega}^2.$$

On the other hand, using (2.9) and (4.2), we obtain as in the demonstration of theorem 2.1

$$III \leq C \|\varepsilon\|_{W^{1,\infty}(0,T;L^\infty(\Omega)^{d^2})}^{1+q} \sum_{n=0}^{n_0} \tau \|\Theta\|_{0,\Omega}^2.$$

Then, if $\|\varepsilon\|$ is small enough, Young inequality for the term IV leads to estimate (4.5).

Let us now consider $\phi = \tau U^n = \tau r_h b(\Theta^n)$ in (4.4). We have after summation

$$\begin{aligned} \frac{1}{2} \|U^{n_0}\|_h^2 - \frac{1}{2} \|U_0\|_h^2 + \frac{1}{2} \sum_{n=1}^{n_0} \|U^n - U^{n-1}\|_h^2 \\ \leq \sum_{n=1}^{n_0} \tau \|e(\Theta^{n-1})\|_h \|U^n\|_h + \sum_{n=1}^{n_0} \tau \|f\|_h \|U^n\|_h, \end{aligned}$$

where we have used the fact that $b' \geq 0$, and where $U_0 = r_h b(\Theta^0)$. Using again Young inequality together with relations (4.2) and (4.5), we get

$$\|U^{n_0}\|_{0,\Omega}^2 + \sum_{n=1}^{n_0} \|U^n - U^{n-1}\|_{0,\Omega}^2 \leq C + \tau \sum_{n=1}^{n_0} \|U^n\|_{0,\Omega}^2.$$

Estimates (4.6) are then direct consequences of the discrete Gronwall inequality. $\square$

These stability results allow us to use the error estimates theorem obtained by Nochetto and Verdi [10] in the case of general degenerate problems. We set the following notations

$$\begin{aligned}\theta_{h\tau}(\cdot, t) &= \Theta^n(\cdot) & t \in ((n-1)\tau, n\tau], \\ U_{n\tau}(\cdot, t) &= U^n(\cdot) = r_h b(\Theta^n(\cdot)) & n = 1, \dots, N.\end{aligned}$$

Moreover, let $G : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$ be the inverse of the Laplace operator with Dirichlet boundary conditions. Our problem is said to be regular if

$$(4.7) \quad \{f \in L^2(\Omega)\} \Rightarrow \{Gf \in H^2(\Omega) \text{ and } \|Gf\|_{H^2(\Omega)} \leq c\|f\|_{L^2(\Omega)}\}.$$

The above conditions are for instance fulfilled in the case of convex polyhedral domains (see Grisvard [4]).

In this way, we obtain the following result.

THEOREM 4.1. *Let assumptions (2.2), (2.6), (2.9), (3.1)–(3.3) and (4.1) be satisfied and let $\varepsilon \in W^{1,\infty}(0, T; L^\infty(\Omega)^{d^2})$ be a given function. Then, if $\|\varepsilon\|_{W^{1,\infty}(0, T; L^\infty(\Omega)^{d^2})}$ is small enough and if $\tau = Ch$, where C is an arbitrary positive constant, the following estimate is satisfied for any $\delta > 0$, δ small enough*

$$(4.8) \quad \|b(\vartheta) - r_h b(\vartheta_{h\tau})\|_{L^2(Q)} + \left\| \int_0^t (\vartheta - \vartheta_{h\tau}) \right\|_{L^\infty(0, T; H_0^1(\Omega))} = \mathcal{O}(h^{1/2-\delta}),$$

where ϑ is the unique solution of problem (1.5)–(1.7).

Moreover if the problem is regular in the sense of (4.7), then for $\|\varepsilon\|_{W^{1,\infty}(0, T; L^\infty(\Omega)^{d^2})}$ small enough and $\tau = Ch^2$, where C is an arbitrary positive constant, the following estimate holds

$$(4.9) \quad \|b(\vartheta) - r_h b(\vartheta_{h\tau})\|_{L^2(Q)} + \left\| \int_0^t (\vartheta - \vartheta_{h\tau}) \right\|_{L^\infty(0, T; H_0^1(\Omega))} = \mathcal{O}(h).$$

Proof. Once the stability results in lemma 4.1 proved, the demonstration can be borrowed almost verbatim from Nochetto and Verdi [10], th.3. $\square$

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