**REGIONAL BLOW-UP IN A SEMILINEAR HEAT EQUATION
WITH CONVERGENCE TO A HAMILTON-JACOBI EQUATION**

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BY VICTOR A. GALAKTIONOV* AND JUAN L. VAZQUEZ†

ABSTRACT

We investigate the asymptotic behaviour of blowing-up solutions $u = u(x, t) \geq 0$ to the semilinear parabolic equation with source

$$u_t = u_{xx} + (1 + u) \log^2(1 + u) \quad \text{for } x \in \mathbf{R}, t > 0,$$

with nonnegative and radial symmetric initial data $u_0(|x|)$ which are nonincreasing in $|x|$. Any nontrivial solution u to this problem blows up in a finite time $T > 0$. It is remarkable that the blow-up behaviour of u as t approaches T can be described by the exact blow-up solutions of the quasilinear Hamilton-Jacobi equation

$$U_t = \frac{(U_x)^2}{1 + U} + (1 + U) \log^2(1 + U),$$

with the same blow-up time T . These explicit profiles are only approximate solutions for our problem. We prove that the blow-up set B of the solution satisfies $\text{meas}(B) \geq 2\pi$, and under some additional hypothesis on the initial function we show that B is just the interval $[-\pi, \pi]$ and the rescaled blow-up shape consists of one hump with formula $\cos^2(x/2)$. The proofs rely on the knowledge of a family of explicit solutions, the method of intersection comparison, some dynamical systems ideas and a stability analysis for solutions of the Hamilton-Jacobi equation.

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Key words. Semilinear heat equation, regional blow-up, asymptotic behaviour, Hamilton-Jacobi equation.

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1. Introduction. In this paper we consider the Cauchy problem for the semilinear heat equation

$$(1.1) \quad u_t = u_{xx} + (1+u)\log^2(1+u) \quad \text{for } x \in \mathbf{R}, t > 0,$$

$$(1.2) \quad u(x, 0) = u_0(x) \quad \text{for } x \in \mathbf{R}.$$

We assume that the initial function $u_0 \in L^1_{\text{loc}}(\mathbf{R}) \cap L^\infty(\mathbf{R})$ is radially symmetric, $u_0 = u_0(|x|)$, nonnegative, $u_0 \geq 0$, and nonconstant. Our main results also assume that u_0 is nonincreasing as a function of $|x| > 0$. The final section deals with periodic solutions.

Problem (1.1)–(1.2) has a unique classical solution $u = u(|x|, t) > 0$ defined in some maximal time interval, [Fr1]. Since the heat source $Q(u) = (1+u)\log^2(1+u)$ in the right-hand side of (1.1) satisfies $Q(u) > 0$ for $u > 0$, $Q(u) = u^2 + o(u^2)$ as $u \rightarrow 0$ and $\int_1^\infty dz/Q(z) < \infty$, the solution blows up in a finite time T , $0 < T < \infty$ (cf. [Fu]): $u(x, t)$ is a classical solution of (1.1) in $\mathbf{R} \times (0, T)$ and

$$(1.3) \quad \sup_{x \in \mathbf{R}} u(x, t) \rightarrow \infty \text{ as } t \rightarrow T.$$

Much work has been devoted in recent years to understanding finite-time blow-up for equations of the form $u_t = u_{xx} + Q(u)$. Four major problems arise in this context and have been studied by different authors, see the books [BE] and [SGKM] for extensive references. They are: (i) existence of blow-up in terms of the initial data, (ii) the form of the *blow-up set*

$$(1.4) \quad B = B(u_0) \equiv \{x \in \mathbf{R} : \exists x_n \rightarrow x \text{ and } t_n \rightarrow T \text{ such that } u(x_n, t_n) \rightarrow \infty \text{ as } n \rightarrow \infty\},$$

(iii) the rate of divergence of $u(x, t)$ as $t \rightarrow T$ for points $x \in B$, and (iv) the shape of the solution at $t = T$, after scaling out this rate. The aim of the paper is to answer these questions for the solutions of equation (1.1) for different choices of initial data.

The simplest form of blow-up is *global flat blow-up*, i.e. the solution blows up in the whole space with no spatial structure. This blow-up form can always be obtained by starting with a constant initial function, and the precise profile $u = u(t)$ is obtained by integrating the ODE: $u_t = Q(u)$. It is remarkable that precisely the opposite situation occurs for many blow-up equations for suitably concentrated initial data, say when u_0 is bell-shaped with one maximum and compact support: then blow up occurs at an isolated point, *single-point blow-up*. Other solutions may blow up at a finite number of points. This happens for instance in the possibly most studied cases $Q(u) = u^p$ with $p > 1$ and $Q(u) = \exp(u)$.

Since no blow-up occurs for $Q(u) = u^p$ with $p = 1$, it is interesting to understand the limit situation $p \approx 1$. In this direction the family of equations

$$(1.5) \quad u_t = u_{xx} + (1+u)\log^\beta(1+u), \quad \beta > 0$$

was introduced in [GKMS] in 1979 (see also [Sam], generalizations in [G1] and the references of [SGKM, Chapter IV]). The case $\beta = 2$ has a particular interest since it serves as a limit case for the blow-up behaviour as we explain below.

Indeed, qualitative and numerical methods described in the above references had suggested that for suitably concentrated initial data blow-up occurs in problem (1.1)-(1.2) in a bounded domain, i.e. we have the so-called *regional blow-up*. Moreover, these studies pointed out that the behaviour of the solution to (1.1)-(1.2) as it approaches blow-up should be described by some solution of the quasilinear Hamilton-Jacobi equation

$$(1.6) \quad U_t = \frac{(U_x)^2}{1+U} + (1+U)\log^2(1+U) \quad \text{for } x \in \mathbf{R}, t \in (0, T),$$

of the form

$$(1.7) \quad U(x, t) = \exp\{(T-t)^{-1}\bar{g}(x)\} - 1.$$

By substituting (1.7) into equation (1.6) one can obtain that the function $\bar{g} \geq 0$ solves the first-order quadratic ordinary differential equation

$$(1.8) \quad (\bar{g}_x)^2 + \bar{g}^2 - \bar{g} = 0 \quad \text{in } \mathbf{R}.$$

In this paper we give a rigorous proof of these facts for the class of solutions which are radially symmetric in the space variable x and nonincreasing in $|x| > 0$. Let us briefly describe the main ideas and results: in studying the blow-up behaviour of equation (1.1) it is convenient to introduce the transformation

$$(1.9) \quad u(x, t) = e^{v(x, t)} - 1,$$

which yields the following equation for the function $v(x, t)$

$$(1.10) \quad v_t = v_{xx} + (v_x)^2 + v^2 \quad \text{for } x \in \mathbf{R}, t > 0.$$

In view of the expected behaviour (1.7), we also introduce the rescaled function

$$(1.11) \quad \theta(x, \tau) = (T-t)v(x, t),$$

where $\tau = -\log(T-t)$ (so that, in particular, the new time $\tau \rightarrow \infty$ as $t \rightarrow T$). The function $\theta(x, \tau)$ solves the Cauchy problem

$$(1.12) \quad \theta_\tau = e^{-\tau}\theta_{xx} + (\theta_x)^2 + \theta^2 - \theta \quad \text{for } x \in \mathbf{R}, \tau > \tau_0 = -\log T,$$

$$(1.13) \quad \theta(x, \tau_0) = \theta_0(x) \equiv T \log(1 + u_0(x)) \quad \text{for } x \in \mathbf{R}.$$

After describing in Section 2 a family of 2π -periodic blow-up solutions constructed in [GP3], which satisfy (1.10) but blow up in the whole space and are essential in later comparison arguments, we proceed in Sections 3 to 8 to study solutions with radially symmetric and nonincreasing initial data. Thus, in Section 3 we establish a sharp estimate for the supremum of θ as $\tau \rightarrow \infty$: $\sup \theta(\cdot, \tau) \rightarrow 1$ (Propositions 3.1 and 3.2). Section 4 is devoted to prove a semiconvexity result, of the form $\theta_{xx} \geq -C$ for τ large (Proposition

4.1), which together with the previous estimate implies that single point blow-up cannot occur.

Regional blow up means that $u(x, t)$ goes to infinity as $t \rightarrow T$ in a set of nonzero finite measure. We have already introduced the blow-up set, where u diverges. A more precise divergence is obtained on a possibly smaller set,

$$(1.14) \quad B_* = B_*(u_0) \equiv \{x \in \mathbf{R} : \exists t_n \rightarrow T \text{ such that } u(x, t_n) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

Obviously, $B_* \subseteq B$. By the monotonicity results of [GP1] and [GP4], B_* can also be defined as follows

$$(1.15) \quad B_* \equiv \{x \in \mathbf{R} : u(x, t) \rightarrow \infty \text{ as } t \rightarrow T\}.$$

B_* is called the *monotone blow-up set*, see [G3]. We establish in Section 5 that the minimal configuration for blow-up corresponds to the particular profile

$$(1.16) \quad G(x) = \begin{cases} \cos^2(x/2) & \text{if } |x| \leq \pi, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the blow-up set consists of an interval of length at least 2π . Moreover, $B_* \supseteq (-\pi, \pi)$. Conversely, we obtain in Section 6 localization of the blow-up set in terms of the number of intersections of u_0 with respect to the explicit solutions of Section 2 (Theorem 6.1).

In order to better understand the blow-up phenomenon we have to study the stabilization of the function $\theta(x, \tau)$ as $\tau \rightarrow \infty$ to a solution $\bar{g}(x)$ of the stationary equation (1.8). This is done in Sections 7 and 8 where we prove that under a strict condition (which is satisfied for instance by suitably bell-shaped initial functions with support in $(-\pi, \pi)$, see condition (7.26)) the blow-up set is exactly $[-\pi, \pi]$ and θ converges as $\tau \rightarrow \infty$ to the function G . In these proofs we consider equation (1.12) as a perturbation of the Hamilton-Jacobi equation

$$(1.17) \quad g_\tau = (g_x)^2 + g^2 - g \quad \text{in } \mathbf{R} \times (0, \infty),$$

and apply the asymptotic results for small perturbations of autonomous dynamical systems of [GV] and some stability analysis of the viscosity solutions of (1.17). In fact, we will need an improvement of the result of [GV] which simplifies its application.

Many solutions have non-minimal blow-up. Solutions with more than one hump can have a nonconnected blow-up set. As an example we consider in Section 9 x -periodic solutions with period $2m\pi$, m an integer, since we can use the techniques developed in the radially symmetric case. We construct solutions whose blow-up set consists of the union of intervals of the form $[(2km - 1)\pi, (2km + 1)\pi]$, $k \in \mathbf{Z}$, with a corresponding union of copies of $G(x)$ as asymptotic profile. We have refrained to consider in this paper more general situations for the sake of brevity. We hope to treat some of them, in particular examples of two-hump blow-up, in a forthcoming article.

As a final comment on related equations, let us mention that the exponent $\beta = 2$ of our equation is critical for the blow-up behaviour of equation (1.5). Thus, the Cauchy-Dirichlet

Problem has been studied in [GP2] for $\beta > 2$, and *single-point blow-up* is shown to occur using the method of [FM], while [L] shows *global blow-up* for $1 < \beta < 2$ for the same problem. This leaves $\beta = 2$ as the only case with *regional blow-up*. On the other hand, it is interesting to remark that in the case $\beta \in (0, 1)$ studied in [GKS1], though solutions are global in time, the asymptotic profiles as $t \rightarrow \infty$ are given as the solutions of the Hamilton-Jacobi equation, $U_t = (U_x)^2/(1+U) + (1+U)\log^\beta(1+U)$, which parallels (1.6).

Let us also recall that regional blow-up has been observed for several models of quasi-linear heat equation, like

$$(1.18) \quad u_t = (u^m)_{xx} + u^p$$

for $m > 1$ and $p = m$, see [SGKM, Chapter IV] and [G3]. The present model seems to be the only known case of a *semilinear* heat equation with regional blow-up.

2. Explicit Solutions. Equation (1.10) admits an explicit solution, 2π -periodic in x , of the form (cf. [GP3], [G2])

$$(2.1) \quad v_*(x, t) = \varphi(t)[\psi(t) + \cos(x)],$$

where the functions $\varphi(t)$ and $\psi(t)$ satisfy the system of ordinary differential equations

$$(2.2) \quad \varphi' = -\varphi + 2\varphi^2\psi, \quad \psi' = \varphi + \psi - \varphi\psi^2 \quad \text{for } t > 0.$$

There exists a one-parameter family of nonnegative blowing-up solutions (2.1) having a given fixed blow-up time $T \in (0, \infty)$. To see this we observe that system (2.2) is equivalent to the following first order differential equation

$$(2.3) \quad \frac{d\psi}{d\varphi} = \frac{\varphi + \psi - \varphi\psi^2}{2\varphi^2\psi - \varphi}, \quad \varphi > 0, \psi \geq 1.$$

Equation (2.3) admits a one-parameter family of trajectories with the asymptotic behaviour (see [G2])

$$(2.4) \quad \psi = 1 + \frac{\log \varphi}{2\varphi} + \frac{\mu}{\varphi} + o\left(\frac{1}{\varphi}\right) \quad \text{as } \varphi \rightarrow \infty,$$

where $\mu \in \mathbf{R}$ is a parameter describing this family. Given a certain μ , namely on a given trajectory, fixing the blow-up time $T > 0$ is equivalent to fixing an initial point (φ_0, ψ_0) on the trajectory corresponding to $t = 0$. It follows from the detailed study of the system (2.2) that for fixed $T > 0$ there exists a constant μ_T such that for any $\mu \geq \mu_T$ the explicit solution $v_* = v_*(x, t; T, \mu)$ with blow-up time T is well-defined and nonnegative in $\mathbf{R} \times (0, T)$, while for $\mu < \mu_T$ the initial value of v_* has changing sign. In terms of the functions $\varphi(t)$, $\psi(t)$ the constant μ_T is such that $\psi_0 = 1$ and $\varphi_0 > 0$. This implies that the corresponding initial function has the form

$$(2.5) \quad v_*(x, 0; T, \mu_T) \equiv \varphi_0[1 + \cos(x)] \geq 0 \quad \text{in } \mathbf{R}.$$

Clearly, for $\mu > \mu_T$ we have $\psi_0 > 1$, hence

$$(2.6) \quad v_*(x, 0) \equiv \varphi_0[\psi_0 + \cos x]$$

is positive in $\mathbf{R}$. In any case $v_*(x, t; T, \mu) > 0$ in $\mathbf{R} \times (0, T)$ for $\mu \geq \mu_T$. Integrating the first equation of (2.2) with the help of expansion (2.4) for $\mu \geq \mu_T$ we obtain the following asymptotic behaviour of the explicit solutions near a fixed blow-up time $t = T$:

$$\begin{aligned} \varphi(t) &= \frac{1}{2}(T-t)^{-1} \left[1 - \frac{1}{2}(T-t)|\log(T-t)|(1+o(1)) \right], \\ \psi(t) &= 1 + (T-t)|\log(T-t)|(1+o(1)) \text{ as } t \rightarrow T. \end{aligned}$$

Therefore, as $t \rightarrow T$

$$(2.7) \quad \begin{aligned} v_*(x, t; T, \mu) &= (T-t)^{-1} \left[\cos^2\left(\frac{x}{2}\right) + \frac{1}{2} \sin^2\left(\frac{x}{2}\right)(T-t)|\log(T-t)| \right. \\ &\quad \left. + \left(\frac{1}{4} - \frac{1+2\log 2 - 4\mu}{4} \sin^2\left(\frac{x}{2}\right) \right) (T-t)(1+o(1)) \right]. \end{aligned}$$

At the maxima $x = 2\pi k$, $k = 0, \pm 1, \dots$, where $\cos^2(x/2) = 1$, we have as $t \rightarrow T$

$$(2.8) \quad \begin{aligned} \sup_{x \in \mathbf{R}} v_*(x, t; T, \mu) &\equiv v_*(2\pi k, t; T, \mu) \\ &= (T-t)^{-1} \left[1 + \frac{1}{4}(T-t)(1+o(1)) \right]. \end{aligned}$$

At the minima $x = \pi + 2\pi k$, $k = 0, \pm 1, \dots$, where $\cos^2(x/2) = 0$, the behaviour as $t \rightarrow T$ is quite different:

$$(2.9) \quad \inf_{x \in \mathbf{R}} v_*(x, t; T, \mu) \equiv v_*(\pi(2k+1), t; T, \mu) = \frac{1}{2}|\log(T-t)|(1+o(1)).$$

For the rescaled explicit solution

$$(2.10) \quad \theta_*(x, \tau; T, \mu) \equiv (T-t)v_*(x, t; T, \mu), \quad \tau = -\log(T-t),$$

we have as $\tau \rightarrow \infty$

$$(2.11) \quad \theta_*(x, \tau; T, \mu) \equiv \cos^2\left(\frac{x}{2}\right) + \frac{1}{2} \sin^2\left(\frac{x}{2}\right)\tau e^{-\tau}(1+o(1)),$$

$$(2.12) \quad \theta_*(0, \tau; T, \mu) \equiv 1 + \frac{1}{4}e^{-\tau}(1+o(1)) \rightarrow 1,$$

$$(2.13) \quad \theta_*(\pm\pi, \tau; T, \mu) \equiv \frac{1}{2}\tau e^{-\tau}(1+o(1)) \rightarrow 0.$$

Notice that $\theta_*(x, \tau; T, \mu)$ converges to $G(x)$ as $\tau \rightarrow \infty$ for $|x| \leq \pi$, cf. [G2].

3. Lower and Upper Bounds. As a first step in controlling the blow-up behaviour of the solution, we derive in this section upper and lower bounds for the supremum in x of the rescaled solution θ defined by (1.11). We begin with the lower bound.

PROPOSITION 3.1. *For every $\tau > \tau_0$ there holds*

$$(3.1) \quad \sup_{x \in \mathbf{R}} \theta(x, \tau) \equiv \theta(0, \tau) > 1.$$

PROOF: Equation (1.10) admits a blowing-up solution of the form

$$(3.2) \quad \bar{v}(t) = (T - t)^{-1},$$

with the same blow-up time, T , as u . Then for any $t \in [0, T)$ $\bar{v}(t)$ intersects in x the solution $v(x, t)$. Indeed, if for some $t = t_0 \in [0, T)$ we have $\bar{v}(t_0) \geq v(x, t_0)$ in $\mathbf{R}$, then by using the Strong Maximum Principle [Fr1] we arrive at the conclusion that $\bar{v}$ must blow up before v does, in contradiction with our assumption. Hence,

$$(3.3) \quad \sup_{x \in \mathbf{R}} v(x, t) > \bar{v}(t) \quad \text{for } t \in [0, T),$$

thus completing the proof. ■

Notice that we do not use any special assumptions on u_0 in proving this result. The proof is based on intersection comparison with a solution which is constant as a function of x , and so inequality (3.3) remains valid in much more general situations, e.g. for any arbitrary nonnegative initial function satisfying $u_0 \rightarrow 0$ as $|x| \rightarrow \infty$. Then $u(x, t) \rightarrow 0$ as $|x| \rightarrow \infty$ for any fixed $t \in (0, T)$, and (3.3) yields the lower estimate $\sup_{x \in \mathbf{R}} \theta(x, \tau) > 1$ for $\tau \in [\tau_0, \infty)$.

We proceed now with the upper bound. This estimate is much more difficult to obtain.

PROPOSITION 3.2. *Let u_0 be radially symmetric and nonincreasing as a function of $|x|$. As $\tau \rightarrow \infty$ there holds*

$$(3.4) \quad \sup_{x \in \mathbf{R}} \theta(x, \tau) \equiv \theta(0, \tau) < 1 + 2\tau e^{-\tau}.$$

PROOF: It consists in several steps. We shall use some ideas of the method of stationary states (cf. [SGKM, Chapter VII] and also [GKS2]).

Step 1. Stationary solutions. We begin with some simple properties of the stationary solutions of equation (1.1). Thus, we look at solutions $U(x; \lambda)$ of

$$(3.5) \quad U_{xx} + (1 + U) \log^2(1 + U) = 0 \quad \text{for } x > 0 \text{ and } x < 0,$$

$$(3.6) \quad U_x(0; \lambda) = 0, \quad U(0; \lambda) = \lambda,$$

where $\lambda > 0$ is an arbitrary constant. One easily shows that a positive solution of (3.5), (3.6) exists in a finite interval $\{|x| < x_0(\lambda)\}$, where the endpoint $x_0(\lambda) > 0$ is the point x at which $U(\pm x; \lambda)$ reaches the zero value. We let $U(x; \lambda) \equiv 0$ for $|x| \geq x_0(\lambda)$. Since $U(x; \lambda)$ is monotone for $x > 0$ and $x < 0$, we have the inequality

$$(3.7) \quad U_{xx} \geq -(1 + U) \log^2(1 + U) \geq -(1 + \lambda) \log^2(1 + \lambda).$$

By integrating twice we obtain the lower bound

$$(3.8) \quad U(x; \lambda) \geq U_-(x; \lambda) \equiv \left(\lambda - \frac{1}{2}(1 + \lambda) \log^2(1 + \lambda)x^2 \right)_+$$

and, in particular,

$$(3.9) \quad x_0(\lambda) \geq x_*(\lambda) \equiv \left(\frac{2\lambda}{(1 + \lambda) \log^2(1 + \lambda)} \right)^{1/2}.$$

Multiplying (3.5) by U_x and integrating over $(0, x)$, $0 \leq x < x_0(\lambda)$, yield the identity

$$(3.10) \quad \begin{aligned} (U_x)^2 &= \frac{1}{2}(1 + \lambda)^2 \left[\log^2(1 + \lambda) - \log(1 + \lambda) + \frac{1}{2} \right] \\ &\quad - \frac{1}{2}(1 + U)^2 \left[\log^2(1 + U) - \log(1 + U) + \frac{1}{2} \right]. \end{aligned}$$

By integrating (3.10) we obtain that

$$(3.11) \quad x_0(\lambda) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty.$$

It follows also from (3.10) that for every fixed $m > 0$ and $x' = x'(\lambda, m) > 0$ is the positive on the level set $\{x \in \mathbf{R} : U(x; \lambda) = m\}$, then

$$(3.12) \quad U_x(x'; \lambda) \rightarrow \infty \quad \text{as} \quad \lambda \rightarrow \infty.$$

Step 2. Intersection comparison. The result given below is true for a wide class of quasi-linear heat equations with source, see [SGKM, Chapters VI, VII] and [GKS2].

LEMMA 3.3. *There exists $\lambda_* > 0$ large enough such that for any $\lambda > \lambda_*$*

$$(3.13) \quad u(x, t) \geq U(x; \lambda) \quad \text{in } \mathbf{R} \quad \text{if } u(0, t) \geq \lambda.$$

PROOF: For a fixed $t \in [0, T)$ denote by $N(t; \lambda)$ the number of intersections in $\{|x| \leq x_0(\lambda)\}$ of solutions $u(x, t)$ and $U(x; \lambda)$, or, in other words, the number of sign changes in $\{|x| \leq x_0(\lambda)\}$ of the difference $w(x, t; \lambda) \equiv u(x, t) - U(x; \lambda)$.

By a standard smoothness result for semilinear heat equations we may conclude that $u(x, t)$ is smooth enough for arbitrarily small $t > 0$. It then follows from (3.11) and (3.12) that for every $\lambda > \lambda_*$ large enough there exist exactly two intersections for $t = t_1$ small enough, i.e.,

$$(3.14) \quad N(t_1; \lambda) = 2 \quad \text{for} \quad \lambda > \lambda_*.$$

Since $U(\pm x_0(\lambda); \lambda) \equiv 0$ and $u(x, t) > 0$ in $\mathbf{R} \times (0, T)$, by a well-known intersection property we conclude that $N(t; \lambda)$ does not increase with time (cf. [A], [GP1], [M], [S]). Hence by (3.14) we deduce that

$$(3.15) \quad N(t; \lambda) \leq 2 \quad \text{for} \quad t \in (t_1, T), \lambda > \lambda_*.$$

We now prove that if $t_\lambda \in (t_1, T)$ is such that $u(0, t_\lambda) = \lambda$, then

$$(3.16) \quad N(t_\lambda; \lambda) = 0.$$

If on the contrary, $N(t_\lambda; \lambda) > 0$ (more exactly, $N(t_\lambda; \lambda) \geq 2$ by symmetry), then one can see that for any small $\delta > 0$ the inequality $N(t_\lambda; \lambda - \delta) \geq 4$ holds contradicting (3.15).

Thus (3.16) implies that $u(x, t_\lambda) \geq U(x; \lambda)$ in $\mathbf{R}$, and hence (3.13) holds by comparison. ■

As a consequence of Lemma 3.3 we obtain a control of the L^1 -norm near the blow-up time.

LEMMA 3.4. *As $t \rightarrow T$ the integral $\|u(\cdot, t)\|_{L^1(\mathbf{R})}$ diverges. More precisely, we have*

$$(3.17) \quad \begin{aligned} \|u(\cdot, t)\|_{L^1(\mathbf{R})} &\geq \|U(\cdot; u(0, t))\|_{L^1(\mathbf{R})} \geq \|U_-(\cdot; u(0, t))\|_{L^1(\mathbf{R})} \\ &= \frac{2}{3} \lambda x_*(\lambda)|_{\lambda=u(0, t)} = \frac{2^{2/3}}{3} \frac{u(0, t)}{\log u(0, t)} (1 + o(1)) \rightarrow \infty. \end{aligned}$$

Though relation (3.17) is not exact in view of the precise asymptotic results that follow, it gives a correct idea of the relative growth of the L^1 and sup-norms.

Step 3. Ordinary differential inequality for an energy function. We now introduce the local weighted energy

$$(3.18) \quad E(t) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(x) u(x, t) dx,$$

where

$$(3.19) \quad \phi(x) = \frac{1}{2} \cos(x) > 0 \text{ in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

so that $\phi'' \equiv -\phi$ and

$$(3.20) \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(x) dx = 1.$$

An energy estimate is obtained as follows. We multiply equation (1.1) by $\phi(x)$ and integrate over $(-\pi/2, \pi/2)$ to obtain

$$(3.21) \quad \frac{dE}{dt} \geq -E + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(x) (1+u) \log^2(1+u) dx,$$

valid for $t \in (0, T)$. Using now Jensen's inequality for the convex function $(1+u) \log^2(1+u)$ and (3.20) we have

$$(3.22) \quad \frac{dE}{dt} \geq -E + (1+E) \log^2(1+E)$$

(Inequalities of the type (3.22) have been used by many authors for proving global (in time) nonexistence of solutions, see e.g. the first such result in [K] and references in [SGKM]).

By integrating (3.22) over (t, T) and using the fact that $E(T) = \infty$ (see (3.17)) we arrive at the estimate

$$(3.23) \quad T - t \leq \int_{E(t)}^{\infty} \frac{dz}{(1+z) \log^2(1+z) - z} \quad \text{as } t \rightarrow T.$$

Using now the fact that

$$\int_p^{\infty} \frac{dz}{(1+z) \log^2(1+z) - z} = \frac{1}{\log p} \left[1 + \frac{1}{3} \frac{1}{\log^2 p} (1 + o(1)) \right]$$

as $p \rightarrow \infty$, we obtain the following estimate for E

LEMMA 3.5. *As $t \rightarrow T$ we have*

$$(3.24) \quad T - t \leq \frac{1}{\log E(t)} \left[1 + \frac{1}{3} \frac{1}{\log^2 E(t)} (1 + o(1)) \right].$$

Step 4. End of proof of Proposition 3.2. By using (3.13), (3.8) and (3.9) we conclude that as $t \rightarrow T$

$$\begin{aligned} E(t) &\geq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \phi(x) U(x; u(0, t)) dx \geq \int_{-x_*(\lambda)}^{x_*(\lambda)} \phi(x) U_-(x; u(0, t)) dx \\ &\geq \frac{1}{2} \phi(0) \|U_-(\cdot; u(0, t))\|_{L^1(\mathbf{R})}. \end{aligned}$$

It then follows from (3.17) that as $t \rightarrow T$

$$(3.25) \quad E(t) \geq \frac{1}{4} \frac{2^{2/3}}{3} \frac{u(0, t)}{\log u(0, t)} (1 + o(1)).$$

Together (3.24) and (3.25) yield the following inequality

$$(3.26) \quad T - t \leq \frac{1}{\log u(0, t)} \left[1 + \frac{\log \log u(0, t)}{\log u(0, t)} (1 + o(1)) \right]$$

as $u(0, t) \rightarrow \infty$. This inequality can be transformed into

$$(3.27) \quad \log u(0, t) \leq (T - t)^{-1} [1 + (T - t) |\log(T - t)| (1 + o(1))]$$

as $t \rightarrow T$. The last estimate yields (3.4). $\#$

4. Semiconvexity. In the next five sections u_0 is radially symmetric and nonincreasing in $|x|$. We may also assume that u_0 is not constant. The analysis of the asymptotic behaviour is best done in terms of the variable θ introduced in (1.11). As is usual in Dynamical Systems we introduce the ω -limit set of the solution $\theta(x, \tau)$ as follows

$$(4.1) \quad \omega(\theta_0) = \{f \in C(\mathbf{R}) : \exists \tau_j \rightarrow \infty \text{ such that } \theta(\cdot, \tau_j) \rightarrow f(\cdot) \\ \text{as } j \rightarrow \infty \text{ uniformly in any compact subset of } \mathbf{R}\}.$$

A key estimate in controlling the ω -limit and in preventing single-point blow-up is the following

PROPOSITION 4.1. *For any $\varepsilon > 0$ there exists a constant $a_\varepsilon > 0$ such that*

$$(4.2) \quad \theta_{xx}(x, \tau) \geq -\left(\frac{1}{2} + \varepsilon\right) - \frac{a_\varepsilon}{(\tau - \tau_1)}$$

in $\mathbf{R} \times (\tau_1, \infty)$, where $\tau_1 = \tau_0 + 1$.

PROOF: The function $z = \theta_{xx}$ solves a semilinear parabolic equation in $\mathbf{R} \times (\tau_0, \infty)$:

$$(4.3) \quad z_\tau = e^{-\tau} z_{xx} + 2\theta_x z_x + 2z^2 + (2\theta - 1)z + 2(\theta_x)^2.$$

By Proposition 3.2 there exists a constant $\tau_\varepsilon > \tau_1$ such that

$$(4.4) \quad \|\theta(\cdot, \tau)\|_\infty \leq 1 + \varepsilon \text{ for } \tau > \tau_\varepsilon.$$

Then it is easily seen that the function

$$(4.5) \quad \mathbf{z}(\tau) = -\left(\frac{1}{2} + \varepsilon + \frac{a_\varepsilon}{(\tau - \tau_1)}\right) < 0$$

will be a subsolution of equation (4.3) in $\mathbf{R} \times (\tau_1, \infty)$, namely

$$(4.6) \quad \mathbf{z}' \leq 2\mathbf{z}^2 + (2\theta - 1)\mathbf{z} \quad \text{for } \tau > \tau_1$$

if

$$(4.7) \quad \frac{a_\varepsilon(2a_\varepsilon - 1)}{(\tau - \tau_1)^2} - 2\mathbf{z}(1 + \varepsilon - \|\theta(\cdot, \tau)\|_\infty) + (1 + 2\varepsilon)\frac{a_\varepsilon}{\tau - \tau_1} \geq 0$$

for $\tau > \tau_1$. By using (4.4) we have that (4.7) holds for $\tau > \tau_\varepsilon$, and (4.7) will be also valid for $\tau_1 < \tau \leq \tau_\varepsilon$ if $a_\varepsilon > 0$ is large enough. ■

As a straightforward consequence of Proposition 3.2 and Lemma 4.1 we obtain a control on the derivative θ_x .

COROLLARY 4.2. *There exists a constant $C > 0$ such that*

$$(4.8) \quad |\theta_x(x, \tau)| \leq C \quad \text{in } \mathbf{R} \times (\tau_1, \infty).$$

5. Lower Bound for the Blow-Up Set and the Asymptotic Profile. We begin here the analysis of the asymptotic profile of the solution as $t \rightarrow T$. This analysis starts with a simple consequence of our semiconvexity result, Proposition 4.1.

LEMMA 5.1. *If $f \in \omega(\theta_0)$, then*

$$(5.1) \quad f_{xx} \geq -\frac{1}{2} \quad \text{a.e. in } \mathbf{R},$$

and, in particular,

$$(5.2) \quad f(x) \geq \left(1 - \frac{x^2}{4}\right)_+ \quad \text{a.e. in } \mathbf{R}.$$

Notice that, since (5.1) is true for every $f \in \omega(\theta_0)$, then necessarily

$$v(x, t) \geq (T - t)^{-1} \left[\left(1 - \frac{x^2}{4}\right)_+ + o(1) \right]$$

in $\mathbf{R}$ as $t \rightarrow T$. This implies a first lower estimate of the blow-up set:

$$(5.3) \quad [-2, 2] \subset B, \quad \text{hence} \quad \text{meas}(B) \geq 4.$$

Although, according to the Introduction, these estimates will not be optimal in predicting the asymptotic shape, it is interesting to note that exactly (5.3) has been proved in [L] for the initial-boundary value problem by a completely different approach.

A sharp lower estimate is as follows

THEOREM 5.2. *If $f \in \omega(\theta_0)$, then*

$$(5.4) \quad f(x) \geq \cos^2\left(\frac{x}{2}\right) \quad \text{for} \quad |x| \leq \pi,$$

so that

$$(5.5) \quad v(x, t) \geq (T - t)^{-1} \left(\cos^2\left(\frac{x}{2}\right) + o(1) \right) \quad \text{in} \quad \{|x| \leq \pi\}.$$

It follows that $(-\pi, \pi) \subseteq B_*$, so that $\text{meas}(B_*) \geq 2\pi$.

PROOF: *Step 1.* By the results of Section 3 we know that the maximum value of the variable $\theta(\cdot, t)$ tends to 1 as $t \rightarrow T$. By Lemma 5.1 we also have a first estimate from below for all elements of the ω -limit, namely

$$f(0) = 1, \quad f_{xx} \geq -\frac{1}{2} \quad \text{and} \quad f \geq \left(1 - \frac{x^2}{4}\right)_+ \quad \text{a.e. in } \mathbf{R}.$$

Step 2. The Hamilton-Jacobi equation. We next analyze the equation satisfied by the solution θ after some limit process. This turns out to be a Hamilton-Jacobi equation.

Fix an arbitrary sequence $\tau_j \rightarrow \infty$ such that $\theta(\cdot, \tau_j) \rightarrow f(\cdot) \in \omega(\theta_0)$ as $j \rightarrow \infty$. Using Propositions 3.2, 4.1 and Corollary 4.2 in passing to the limit as $\tau = \tau_j + s \rightarrow \infty$ in the linear and nonlinear terms of equation (1.12) we conclude that $\theta(\cdot, \tau_j + s) \rightarrow g(\cdot, s)$

as $j \rightarrow \infty$ locally in $L^\infty([0, \infty) : C(\mathbf{R}))$, where $g(x, s)$ satisfies the following nonlinear Hamilton-Jacobi equation

$$(5.6) \quad g_s = (g_x)^2 + g^2 - g \quad \text{for } x \in \mathbf{R}, s > 0,$$

with initial data

$$(5.7) \quad g(x, 0) = f(x) \quad \text{in } \mathbf{R}.$$

Step 3. Explicit solutions. Equation (5.6) with quadratic nonlinearities admits a family of explicit classical solutions, 2π -periodic in x , given by

$$(5.8) \quad g_*(x, s; \alpha) = \left(2 \cos^2 \left(\frac{x}{2}\right) + \alpha e^{-s}\right) (2 + \alpha e^{-s})^{-1}$$

for $x \in \mathbf{R}$, $s > 0$, where $\alpha > -2$ is a fixed constant. If $\alpha = 0$, then g_* is the classical stationary solution of equation (5.6):

$$(5.9) \quad g_*(x, s; 0) = \cos^2 \left(\frac{x}{2}\right) \equiv G_*(x),$$

where $G_*(x) = G(x)$ for $|x| \leq \pi$, see (1.16). For $\alpha > 0$ we have $g_*(x, s; \alpha) \geq G_*(x)$ and

$$(5.10) \quad g_*(x, s; \alpha) \rightarrow G_*(x) \quad \text{as } s \rightarrow \infty$$

from above, uniformly in $\mathbf{R}$. On the other hand, it is easy to see that when $\alpha \in (-2, 0)$ then $g_*(x, s; \alpha) = 0$ for $x = \pm x_*(s; \alpha)$ and $g_*(x, s; \alpha) > 0$ if $|x| < x_*(s; \alpha)$, where

$$x_*(s; \alpha) = 2 \arccos \left(|\alpha|^{1/2} 2^{-1/2} e^{-s/2} \right) < \pi$$

and $x_*(s; \alpha) \rightarrow \pi$ as $s \rightarrow \infty$. We also notice that if $\alpha \in (-2, 0)$, then $g_*(x, s; \alpha) \leq G_*(x)$ and

$$(5.11) \quad g_*(x, s; \alpha) \rightarrow G_*(x) \quad \text{as } s \rightarrow \infty$$

from below uniformly in $\mathbf{R}$.

Step 4. Final estimate. It follows from (5.2) that there exists $\alpha \in (-2, 0)$, which does not depend on f , such that

$$(5.12) \quad f(x) \geq g_-(x, 0; \alpha) \quad \text{in } \mathbf{R},$$

where

$$(5.13) \quad g_-(x, s; \alpha) = [g_*(x, s; \alpha)]_+ \quad \text{for } |x| \leq \pi,$$

$$(5.14) \quad g_-(x, s; \alpha) = 0 \quad \text{for } |x| > \pi, s > 0,$$

is a viscosity subsolution of the Hamilton-Jacobi equation (5.6) (cf. [CL], [CEL]) and hence

$$(5.15) \quad g(x, s) \geq g_-(x, s; \alpha) \quad \text{in } \mathbf{R} \times (0, \infty).$$

Then by (5.11) we have that for any small $\varepsilon > 0$ there exists $s_\varepsilon > 0$, which is independent of f , such that

$$(5.16) \quad g(x, s_\varepsilon) \geq G(x) - \varepsilon \quad \text{for } |x| \leq \pi$$

(recall that $G \equiv G_*$ in the interval $|x| \leq \pi$). Since $\{\tau_j\}$ is an arbitrary sequence, we conclude that (5.16) holds for arbitrary $f \in \omega(\theta_0)$, whence (5.4) follows. ■

REMARK: The lower estimate (5.4) can be also proved by an analysis similar to one given in Section 4. This can be done by using instead of (4.3) the equation for the function $w = \theta_{xx} + \theta$. This equation has the form

$$(5.17) \quad w_\tau = e^{-\tau} w_{xx} + 2\theta_x w_x + (w - \theta)(2w - 1) + (\theta_x)^2 + \theta^2 - \theta.$$

To derive a lower bound of $w(x, \tau)$ as $\tau \rightarrow \infty$ we need a special estimate of the term $(\theta_x)^2 + \theta^2 - \theta$ in the right-hand side of (5.17).

6. Localization. In order to obtain upper bounds on the size of the blow-up set we shall make the additional assumption that v_0 has a finite number of intersections with an initial function of the form (2.6): $v_*(x, 0) = v_*(x, 0; T, \mu)$, with the same blow-up time $T > 0$ as u and $\mu \geq \mu_T$ (which means that $v_*(x, 0)$ is nonnegative). It is easily seen that there exists a wide class of initial functions with this property, in particular among the functions with compact support (see also Proposition 7.4 below). For a fixed $t \in [0, T)$ we denote by $N(t)$ the number of intersections of the solution $v(x, t)$ and the particular solution $v_*(x, t) = v_*(x, t; T, \mu) \geq 0$ with the above initial data. By symmetry $N(t)$ is an even number (if the maxima do not coincide). We then get the following estimate.

THEOREM 6.1. *Let T be the blow-up time of a solution u , and, with the above assumptions and notation, let*

$$(6.1) \quad N(0) = 2k < \infty.$$

Then the solution is uniformly bounded in sets of the form

$$(6.2) \quad K = \{(x, t) : |x| \geq x_1, 0 \leq t < T\} \quad \text{with } x_1 > L_k,$$

where $L_k = \pi(k + 2)$ if k is an odd number, and $L_k = \pi(k + 1)$ if k is even (L_k is a precise minimum for the function v_). Consequently,*

$$(6.3) \quad B \subseteq [-L_k, L_k].$$

PROOF: To begin with, we need the following result.

LEMMA 6.2. Assume that as $t \rightarrow T$

$$(6.4) \quad v(x, t) \geq (T - t)^{-1} [1 + \vartheta(t)]$$

for $|x| \leq \pi/2$. Then

$$(6.5) \quad \vartheta(t) < (T - t)^2 \text{ as } t \rightarrow T.$$

PROOF: By (3.24) we know that

$$(6.6) \quad T - t \leq \frac{1}{\log E(t)} \left[1 + \frac{1}{3 \log^2 E(t)} (1 + o(1)) \right]$$

as $t \rightarrow T$. On the other hand, it follows from (6.4) and the definition of E , (3.18), that

$$(6.7) \quad E(t) \geq \exp[(T - t)^{-1} (1 + \vartheta(t))] - 1 \text{ as } t \rightarrow T.$$

Then (6.5) is a direct consequence of (6.6) and (6.7). ■

As a consequence of Lemma 6.2 and (2.8) we have

COROLLARY 6.3. For $t \approx T$ there holds

$$(6.8) \quad v(\pm 2\pi, t) < v_*(\pm 2\pi, t).$$

We continue the proof of Theorem 6.1 with

LEMMA 6.4. For $t \approx T$ we have

$$(6.9) \quad v(\pm L_k, t) \leq v_*(\pm L_k, t) \equiv \inf_{x \in \mathbf{R}} v_*(x, t).$$

PROOF: Hypothesis (6.1) implies that

$$(6.10) \quad N(t) \leq 2k \quad \text{for any } t \in (0, T).$$

By Corollary 6.3 we have that if $|x| \geq 2\pi$ and $t \approx T$, then

$$(6.11) \quad v(x, t) < \sup_{x \in \mathbf{R}} v_*(x, t) \equiv v_*(\pm 2\pi, t).$$

We conclude that (6.9) holds. Indeed, if it is not valid and for $t \rightarrow T$

$$(6.12) \quad v(\pm L_k, t) > v_*(\pm L_k, t),$$

then from (6.11) and (6.12) and using the spatial 2π -periodic structure of the explicit solution (2.1) we have the estimate $N(t) \geq 2(k + 1)$ contradicting (6.10). ■

We are now ready to consider what happens in the domain $D = \{(x, t) : x > L_k, t_0 < t < T\}$ with $t_0 \approx T$. Observe first that since $v(|x|, t)$ is nonincreasing in $|x|$, estimates (6.9) and (2.9) imply that for $t_0 \approx T$ we have

$$(6.13) \quad v(x, t) \leq \frac{1}{2} |\log(T - t)| (1 + o(1))$$

in D . Consider the function $u(x, t) = e^{v(x, t)} - 1$. It solves in D the problem consisting of equation (1.1) with initial condition

$$(6.14) \quad u(x, t_0) = a(x) \quad \text{for } x > L_k,$$

where a is bounded and nonnegative: $0 \leq a(x) \leq C$. It also satisfies a boundary condition for $t_0 \leq t < T$ of the form

$$(6.15) \quad u(x_0, t) = h(t) \leq \frac{1}{(T - t)} - 1.$$

Without loss of generality we may shift the axes and take $L_k = t_0 = 0$. The outer analysis of our solution relies on the following result

LEMMA 6.5. *Let $u(x, t)$ be a solution of (1.1) in $D = \{(x, t) : x > 0, 0 < t < T\}$ which is nonincreasing as a function of x for all $t > 0$. If $u(x, 0)$ is bounded for $x \geq 0$, and there exists a constant $c > 0$ such that*

$$(6.16) \quad u(0, t) \leq \frac{c}{T - t} - 1,$$

for $0 < t < T$, then $u(x, t)$ is bounded uniformly on sets of the form $K = \{(x, t) : x \geq x_1, 0 \leq t < T\}$ with $x_1 > 0$ arbitrary.

PROOF: It proceeds via a two-step comparison argument. We obtain a first bound from above for $u(x, t)$ by comparing it with the solution $\bar{u}$ of the equation

$$(6.17) \quad \bar{u}_t = \bar{u}_{xx} + \frac{c}{(T - t)} \log^2\left(\frac{c}{(T - t)}\right)$$

with initial data C , a bound for $u(x, 0)$ in $(0, \infty)$, and boundary data $c/(T - t) - 1$. Moreover, we may apply the superposition principle to the linear equation (6.17) and split $\bar{u}$ into $\bar{u}_1 + \bar{u}_2$, where $\bar{u}_1$ is a solution of (6.17) with zero initial and boundary data and $\bar{u}_2$ solves the homogeneous heat equation $(\bar{u}_2)_t = (\bar{u}_2)_{xx}$ with same initial and boundary data as $\bar{u}$. By comparison with a solution which depends only on t , the function $\bar{u}_1$ can be easily estimated as

$$(6.18) \quad \bar{u}_1(x, t) \leq \frac{c}{3} \{\log^3(c/(T - t)) - \log^3(c/T)\}.$$

As for $\bar{u}_2$, by the results of [SGKM, Chapter III] we can control the profile near the time T in the form

$$(6.19) \quad \bar{u}_2(x, t) \leq C_1 + C_2 x^{-2}, \quad \text{for some } C_1, C_2 > 0.$$

It follows that for $x \geq x' > 0$ and as $t \rightarrow T$

$$(6.20) \quad u(x, t) + 1 < c \log^3 \left(\frac{c}{T-t} \right).$$

The second step of the iteration consists in using (6.20) to modify equation (6.17) for the majorant into

$$(6.21) \quad \tilde{u}_t = \tilde{u}_{xx} + c \log^3 \left(\frac{c}{T-t} \right) \log^2 \left(c \log^3 \left(\frac{c}{T-t} \right) \right)$$

posed in $D' = \{(x, t) : x > x', 0 < t < T\}$ with corresponding initial and boundary conditions. We now get a finite estimate for both components of $\tilde{u}$ defined as before. Indeed, $\tilde{u}_1$ is uniformly bounded in D' since the last term in (6.21) is now integrable; $\tilde{u}_2$ is bounded for $x > x_1$ if $x_1 > x'$ by the same reason as in (6.19). ■

With this the proof of Theorem 6.1 is complete.

7. Minimal Asymptotic Behaviour. We are interested in giving a precise description of the shape of the solution at blow-up. Under a stricter intersection condition on the data we establish in this section that the blow-up profile of our solution is just the minimal configuration (1.16). Our result is the following

THEOREM 7.1. *Let as above v_0 be radially symmetric, nonincreasing in $|x|$, and assume moreover that it has two transversal intersections with a member $v_*(x, 0; T, \mu) \geq 0$ of the explicit family (2.1) which has the same blow-up time T as v_0 . Then as $\tau \rightarrow \infty$*

$$(7.1) \quad \theta(x, \tau) \rightarrow G(x) = \begin{cases} \cos^2(x/2) & \text{for } |x| \leq \pi, \\ 0 & \text{for } |x| > \pi, \end{cases}$$

uniformly in $\mathbf{R}$.

REMARK: We will give below in Proposition 7.4 direct conditions on v_0 so that this intersection hypothesis holds.

PROOF: Step 1. Perturbation analysis. As explained in the Introduction the variable θ introduced in (1.11) satisfies equation (1.12):

$$\theta_\tau = e^{-\tau} \theta_{xx} + (\theta_x)^2 + \theta^2 - \theta$$

for $x \in \mathbf{R}$, $\tau > \tau_0 = -\log T$. This can be considered as a perturbation of the Hamilton-Jacobi equation

$$(HJ) \quad g_s = (g_x)^2 + g^2 - g \quad \text{for } x \in \mathbf{R}, s > 0.$$

We want to describe the asymptotic behaviour of our solution of (1.12) in terms of the asymptotic behaviour of equation (HJ). More precisely, we want to apply Theorem 3 of [GV], which has to be conveniently adapted. [GV]'s result says that, if certain 3 conditions

to be discussed below are satisfied, then the ω -limit of the solution θ to equation (1.12) is contained in some part of the ω -limit of equation (HJ). In fact, we will need an improvement of that result and will show that, under our assumptions, such part is just the function G .

The first two steps in [GV]'s method consist in proving that the orbits under consideration for equation (1.12) are compact in an appropriate topology and that, whenever a sequence $\theta(\cdot, \tau_j + s)$ converges locally in s to a function $g(\cdot, s)$ as $\tau_j \rightarrow \infty$, this g is a solution of equation (HJ). By the estimates proved in Section 4 we conclude that such facts are true and that the convergence takes place in $L_{\text{loc}}^\infty([0, \infty) : C(\mathbf{R}))$.

Step 2. Stationary ω -limits for (HJ). The last hypothesis in Theorem 3 of [GV] consists in showing

(i) that the set Ω of all ω -limits of solutions g of (HJ) obtained in this way consists of stationary solutions, and

(ii) that it is nonvoid, compact and uniformly stable in the sense of Lyapunov for the flow generated by (HJ).

We first observe that the solutions of (HJ) we are considering take as initial data

$$(7.2) \quad g(x, 0) = f(x) \quad \text{in} \quad \mathbf{R},$$

where $f(\cdot) = \lim_{\tau_j \rightarrow \infty} \theta(\cdot, \tau_j)$ is an element of $\omega(\theta_0)$ satisfying

$$(7.3) \quad f = f(|x|) \text{ is nonincreasing in } |x|,$$

$$(7.4) \quad |f_x| \leq C \quad \text{in} \quad \mathbf{R},$$

$$(7.5) \quad f(0) = 1, \quad f_{xx} \geq -\frac{1}{2} \quad \text{in} \quad \mathbf{R}.$$

It is easily seen the whole set of stationary and symmetric solutions $g = g(|x|)$ of (HJ), which are nonincreasing and tend to zero at infinity, is the one-parametric family of functions $S = \{\bar{g}(x; a), a \geq 0\}$, where for a fixed $a \geq 0$

$$(7.6) \quad \begin{cases} \bar{g}(x, a) = 1 & \text{for } |x| \leq a, \\ \bar{g}(x, a) = \cos^2\left(\frac{|x|-a}{2}\right) & \text{for } a \leq |x| \leq a + \pi, \\ \bar{g}(x, a) = 0 & \text{for } |x| \geq a + \pi. \end{cases}$$

Note that $G(x) \equiv \bar{g}(x; 0)$. We have

LEMMA 7.2. *The ω -limit of g , solution of (HJ), with initial data f satisfying (7.3)-(7.5), consists of stationary solutions, i.e.*

$$(7.7) \quad \omega_*(f) \subseteq S = \{\bar{g}(x; a), a \geq 0\},$$

where ω_* denotes the ω -limit along the flow of equation (HJ).

PROOF: We consider an arbitrary solution $g(x, s)$ of problem (HJ), (7.2), where the initial function $f(x)$ satisfies (7.3)-(7.5). Set

$$(7.8) \quad a = \sup\{x | f(x) = 1\} \geq 0.$$

By Theorem 6.1 we have $a < \infty$, $f(x) = 1$ for $|x| \leq a$ and $f(x) < 1$ for $|x| > a$. We shall compare $g(x, s)$ with the explicit solutions g_* of the Hamilton-Jacobi equation given in Section 5.

We now introduce a modification, g_a , of the function g_- given by (5.13)-(5.14). It is defined for $a \geq 0$ and $\alpha \in (-2, 0)$ as

$$\begin{aligned} g_a(x, s; \alpha) &= 1 && \text{for } |x| \leq a, \\ g_a(x, s; \alpha) &= [g_*(|x| - a, s; \alpha)]_+ && \text{for } a \leq |x| \leq a + \pi, \\ g_a(x, s; \alpha) &= 0 && \text{for } |x| \geq a + \pi, s \geq 0. \end{aligned}$$

For any value of the parameter $\alpha \in (-2, 0)$, the function $g_a(x, t) = g_a(x, t; \alpha)$ thus defined is a subsolution of (HJ). By using estimates (7.5), which imply that

$$(7.9) \quad f(x) \geq \left[1 - \frac{1}{4}(|x| - a)^2 \right]_+ \quad \text{for } |x| \geq a,$$

we deduce that there exists a value of the parameter $\alpha \in (-2, 0)$, which does not depend on f , such that, fixing this α ,

$$(7.10) \quad f(x) \geq g_a(x, 0) \quad \text{in } \mathbf{R}.$$

Then by comparison (cf. [CEL], [CL])

$$(7.11) \quad g(x, s) \geq g_a(x, s) \quad \text{in } \mathbf{R} \times (0, \infty).$$

Similarly, we find a strictly positive super-solution g^a of equation (HJ) of the form

$$\begin{aligned} g^a(x, s; \alpha, \lambda) &= 1 && \text{for } |x| \leq a + \lambda, \\ g^a(x, s; \alpha, \lambda) &= g_*(|x| - (a + \lambda), s; \alpha) && \text{for } |x| \geq a + \lambda, s \geq 0, \end{aligned}$$

with parameters α and $\lambda > 0$. One can see that for any arbitrarily small $\lambda > 0$ there exists $\alpha_+(\lambda) > 0$ such that

$$f(x) \leq g^a(x, 0) \quad \text{in } \mathbf{R},$$

and therefore by comparison

$$g(x, s) \leq g^a(x, s) \quad \text{in } \mathbf{R} \times (0, \infty).$$

It follows from (5.10) and (5.11) that

$$(7.12) \quad g_a(x, s) \rightarrow \bar{g}(x; a), \quad g^a(x, s) \rightarrow \bar{g}(x; a + \lambda)$$

as $s \rightarrow \infty$ uniformly in $\mathbf{R}$. Using the fact that $\bar{g}(x; a + \lambda) \rightarrow \bar{g}(x; a)$ as $\lambda \rightarrow 0$ uniformly in $\mathbf{R}$ completes the proof of the Lemma. ■

Step 3. The reduced Ω -limit set. The last condition we still have to check in order to apply Theorem 3 of [GV] is the uniform stability (in the sense of Lyapunov) of the Ω -limit set of (HJ). Unfortunately, such an assertion is false for the set S and the current topology. However, a careful inspection of the proof in [GV] shows that we only work with a *reduced Ω -limit set*, which we define as “the set of ω -limits which actually occur as solutions of (HJ), (7.2) when the initial datum f is an ω -limit of (1.12)”. Consequently, we *only have to prove that this reduced set has the desired properties*.

LEMMA 7.3. *The reduced Ω -limit set consists only of the special profile G .*

PROOF: Let $v_*(x, t) = v_*(x, t; T, \mu)$ be the special solution referred to in the statement of the Theorem. The proof of this Lemma is based on a careful study of the comparison between our solution v and the special solution slightly delayed in time, $v_*(x, t + \varepsilon)$, $\varepsilon > 0$ small. We carefully control the relative situation of both solutions at $x = 0$ and $x = \pm\pi$. If an ω -limit is not G then we show that the intersection count is violated as $t \rightarrow T$.

Let $N(t; \varepsilon)$ be the number of intersections in x at time $t \geq 0$ of the solutions $v(x, t)$ and $v_*(x, t + \varepsilon)$ with initial data $v_0(x)$ and $v_*(x, \varepsilon)$. Our assumption is that $N(0; 0) = 2$. By transversality and continuous dependence, the same is valid for the functions $v_0(x)$ and $v_*(x, \varepsilon)$ provided ε is small enough. This means that

$$(7.13) \quad N(0; \varepsilon) = 2 \quad \text{for any small } \varepsilon \geq 0,$$

and hence by the Strong Maximum Principle we have

$$(7.14) \quad N(t; \varepsilon) \leq 2 \quad \text{for } t \in [0, T - \varepsilon),$$

if $\varepsilon > 0$ is small enough. Notice that (7.14) with $\varepsilon = 0$ automatically implies the lower bound

$$(7.15) \quad v(0, t) > v_*(0, t) \quad \text{for } t \in [0, T).$$

Assume now that the assertion of the Lemma is not true and that there exists a sequence $t_k \rightarrow T$ such that

$$\theta(x, t_k) \equiv (T - t_k)v(x, t_k) \rightarrow f(x)$$

as $k \rightarrow \infty$ uniformly in $\mathbf{R}$, and the function g , solution of (HJ) starting from f , tends in its turn as $s \rightarrow \infty$ to some $h \not\equiv G$. By Lemma 7.2 we have $h = \bar{g}(x; a)$ for some $a > 0$. Therefore, we conclude from a simple triangular argument that there exists another sequence, that we denote by t_j , such that

$$(7.16) \quad \theta(x, t_j) \equiv (T - t_j)v(x, t_j) \rightarrow \bar{g}(x; a).$$

Notice that the corresponding explicit solution $v_*(x, t)$ satisfies (cf. (2.11))

$$(7.17) \quad \theta_*(x, t_j) \equiv (T - t_j)v_*(x, t_j) \rightarrow \bar{g}(x; 0)$$

as $j \rightarrow \infty$ uniformly in $\mathbf{R}$, and

$$(7.18) \quad \bar{g}(x; a) > \bar{g}(x; 0) \quad \text{for } 0 < |x| < a + \pi,$$

so that

$$(7.19) \quad \bar{g}(\pi; a) > 0 = \bar{g}(\pi; 0),$$

and

$$(7.20) \quad \bar{g}(0; a) = \bar{g}(0; 0) = 1.$$

Fix arbitrary j such that $s = T - t_j > 0$ is small enough and set $\varepsilon = s/2$. Then by (7.17), (7.16) and (7.20) we have that

$$(7.21) \quad \begin{aligned} v_*(0, t_j + \varepsilon) &\simeq (T - t_j - \varepsilon)^{-1} \equiv 2s^{-1} > \\ &> s^{-1} \simeq v(0, t_j). \end{aligned}$$

The time displacement ε is used here. Using (7.17), (7.19) and (2.9) we get if $s > 0$ is small enough the inequality in the converse direction at $x = \pm\pi$:

$$(7.22) \quad \begin{aligned} v_*(\pm\pi, t_j + \varepsilon) &\simeq \frac{1}{2} |\log(T - t_j - \varepsilon)| \equiv \frac{1}{2} |\log \frac{s}{2}| \\ &< s^{-1} \bar{g}(\pm\pi; a) \simeq v(\pm\pi, t_j). \end{aligned}$$

One can see also that

$$(7.23) \quad v_*(\pm 2\pi, t_j + \varepsilon) > v(\pm 2\pi, t_j).$$

Estimates (7.21)–(7.23) imply that for the choice of $t = t_j$ and $\varepsilon > 0$ given above we have that

$$N(t_j; \varepsilon) \geq 4,$$

which contradicts (7.14) for $t = t_j$. Hence $a = 0$ which completes the proof. ■

Step 4. Stability of G . To end the proof we have to show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $g(\cdot, s)$ is a solution of (HJ), (7.2) such that

$$(7.24) \quad d(f, G) < \delta,$$

then

$$(7.25) \quad d(g(\cdot, s), G) < \varepsilon \quad \text{for every } s > 0$$

(we denote by $d(\cdot, \cdot)$ the distance associated to $L^\infty(\mathbf{R})$). We can repeat the comparison arguments mentioned in Lemma 7.2, using the subsolution g_a and supersolution g_a . This completes the proof of Theorem 7.1. ■

To end the section we give explicit conditions under which the intersection assumption of Theorem 7.1 is fulfilled, as promised above.

PROPOSITION 7.4. *Let $v_0 = v_0(|x|)$ be a radially symmetric function, decreasing in $|x|$. Assume moreover that there exists a constant $m \in [0, v_0(0))$ such that*

$$(7.26) \quad v'_0(x) \leq -\frac{1}{2}(v_0(0) - m) \sin x$$

for any $x \in [0, \pi]$ such that $v_0(x) > m$. Then the number of intersections of $v_0(x)$ with any positive function $v_(x, 0)$ of the form (2.6) and satisfying*

$$(7.27) \quad v_0(0) > \max_x \{v_*(x, 0)\} > \min_x \{v_*(x, 0)\} \geq m$$

is two and they are transversal. Moreover, if $v_0(0)$ is large and m small enough we can choose the parameters φ_0 and ψ_0 in (2.6) so that $v_*(x, t)$ has the same blow-up time as v and (7.26) holds.

PROOF: The initial function $\bar{v}_*(x) \equiv v_*(x, 0; T, \mu)$ has the form (2.6) for $\mu \geq \mu_T$, i.e.

$$(7.28) \quad \bar{v}_*(x) = \varphi_0(\psi_0 + \cos x),$$

where $\varphi_0 > 0, \psi_0 \geq 1$ are constants. According to (7.27)

$$(7.29) \quad M_1 = \inf_x \bar{v}_*(x) \equiv \varphi_0(\psi_0 - 1) \geq m,$$

and

$$(7.30) \quad M_2 = \sup_x \bar{v}_*(x) \equiv \varphi_0(\psi_0 + 1) < v_0(0),$$

where m is as in (7.26). One can see that under hypothesis (7.26) the function $v_0(x)$ intersects an arbitrary function of the family $\bar{v}_*(x)$ exactly at two points. In fact, at any intersection point $x > 0$ we have

$$\bar{v}'_*(x) = -\varphi_0 \sin x > v'_0(x),$$

since $v_0(0) - m > M_2 - M_1 = 2\varphi_0$.

Finally, in order for v_* to have the same blow-up time T as u , and according to the analysis of Section 2, the data φ_0 and ψ_0 have to satisfy a certain condition, namely the point (φ_0, ψ_0) must lie on some curve situated in the region $\{0 < \varphi_0 < \infty, 1 < \psi_0 < \infty\}$. This condition is clearly compatible with (7.29)-(7.30) if m is small and $v_0(0)$ large. Then there are infinitely many choices. ■

8. Minimal Blow-up Set. We have established minimal asymptotic behaviour in terms of the variable θ in the preceding section. Unfortunately, due to the factor $(T - t)$ in the change of variables (1.11), this does not automatically imply that u remains bounded outside of the minimal set $[-\pi, \pi]$. This section is devoted to establish such a result.

THEOREM 8.1. *Assume that v_0 satisfies the conditions of Proposition 7.4. Then $B = [-\pi, \pi]$.*

REMARK: In view of Theorem 5.2, it follows that $\text{meas}(B_*) = 2\pi$, so that the solution actually converges to infinity as $t \rightarrow T$ at any point of the interval $-\pi < x < \pi$, see (8.1) below. The actual asymptotic behaviour at $x = \pm\pi$ is an open question.

PROOF: The proof consists of a preliminary analysis of what happens near $x = \pm\pi$ (*boundary analysis*) plus an analysis of the situation for $|x| > \pi$. This latter outer analysis follows Lemma 6.5. To begin with, by Theorem 7.1 we know that as $t \rightarrow T$

$$(8.1) \quad \theta(x, t) \equiv (T - t)v(x, t) \rightarrow G(x)$$

uniformly in $\mathbf{R}$. The special solution with the same blow-up time selected as in Proposition 7.4 satisfies

$$(8.2) \quad \theta_*(x, t) \equiv (T - t)v_*(x, t) \rightarrow G(x)$$

as $t \rightarrow T$ uniformly in $\mathbf{R}$. For a fixed $t \in [0, T)$ and arbitrary $\lambda \geq 0$ small enough we denote by $N(t; \lambda)$ the number of intersections in x of the solutions $v(x, t)$ and this $v_*(x - \lambda, t)$ having the same blow-up time T as v . Then, since by Proposition 7.4 each intersection of the initial functions $v_0(x)$ and $v_*(x, 0)$ is transversal, by continuity of the function $v_*(x - \lambda, 0)$ with respect to λ we conclude that there holds (cf. (7.13))

$$(8.3) \quad N(0; \lambda) = 2 \quad \text{for any small } \lambda \geq 0.$$

Hence (cf. (7.14))

$$(8.4) \quad N(t; \lambda) \leq 2 \quad \text{for } t \in [0, T),$$

if $\lambda \geq 0$ is small enough (notice that (7.26) yields that (8.4) is valid for arbitrary $\lambda \geq 0$).

We now perform the boundary analysis. We prove that the solution $v(x, t)$ is small enough as $t \rightarrow T$ in a right neighborhood of the point $x = \pi$. (By symmetry, the same is true for a left neighborhood of $x = -\pi$).

LEMMA 8.2. *Let $\lambda_0 > 0$ be small enough. Then, as $t \rightarrow T$*

$$(8.5) \quad v(\pi + \lambda_0, t) \leq v_*(\pi, t) \equiv \frac{1}{2} |\log(T - t)| (1 + o(1)).$$

PROOF: We use an interesting technique of intersection comparison with shifting in x . Let $x = x_0(t) > 0$ be the unique positive intersection point of the solutions $v(x, t)$ and $v_*(x, t)$. Notice that since both solutions are analytic functions in the x variable, each intersection point is isolated for $0 < t < T$, [Fr2]. If $x_0(t) \leq \pi$ as $t \rightarrow T$, then using (7.15) and (8.4) with $\lambda = 0$ we deduce that $v(x, t) \leq v_*(x, t)$ for $x \geq \pi$ and hence (8.5) is valid even with $\lambda_0 = 0$.

Suppose on the contrary that there exists a monotone sequence $t_j \rightarrow T$, such that

$$(8.6) \quad x_0(t_j) > \pi \quad \text{for } j = 1, 2, \dots$$

Then by the Strong Maximum Principle [Fr1] we have that

$$(8.7) \quad v(x, t_j) > v_*(x, t_j) \quad \text{for } |x| \leq \pi.$$

Hence,

$$(8.8) \quad \delta_j = \delta(t_j) \equiv \sup\{\lambda > 0 : v(x, t_j) \geq v_*(x - \lambda, t_j) \text{ for } |x| \leq \pi\} > 0.$$

It follows from (8.1) and (8.2) that $\delta_j \rightarrow 0$ as $j \rightarrow \infty$.

Compare now for a fixed j large enough the functions $v(x, t_j)$ and $v_*(x - \delta_j, t_j)$. First, we conclude that

$$(8.9) \quad v(x, t_j) > v_*(x - \delta_j, t_j) \quad \text{for } x \in [0, \pi).$$

Otherwise, if (8.9) is not valid so that, by the definition of δ_j , there exists some interior tangency point $x = x_1 \in (0, \pi)$ of these functions, at which

$$(8.10) \quad \begin{aligned} v(x_1, t_j) &= v_*(x_1 - \delta_j, t_j), \\ v_x(x_1, t_j) &= (v_*)_x(x_1 - \delta_j, t_j), \end{aligned}$$

we arrive at the contradiction with (8.4). Indeed, under conditions (8.10) by additionally “shifting to the right in x ” the function $v_*(x - \delta_j, t_j)$, we deduce that for arbitrary $0 < \nu \ll \delta_j$ the number of intersections grows,

$$(8.11) \quad N(t_j; \delta_j + \nu) \geq 3,$$

contradicting (8.4) with $t = t_j, \lambda = \delta_j + \nu$.

Thus, (8.9) holds. From the definition of δ_j given by (8.8) we conclude then that necessarily

$$(8.12) \quad v_*(\pi - \delta_j, t_j) = v(\pi, t_j),$$

and therefore (8.4) implies that

$$(8.13) \quad v(\pi + \delta_j, t_j) < v_*(\pi, t_j).$$

Since $\{t_j\}$ is arbitrary, by using the fact that $\delta(t) \rightarrow 0$ as $t \rightarrow T$ we have that for any small $\lambda_0 > 0$ there exists $t_0 \in (0, T)$ such that (8.5) holds for $t \in (t_0, T)$. ■

It then follows from (2.9) that for $t \rightarrow T$

$$(8.14) \quad v(x_0, t) \leq -\frac{1}{2} \log(T - t)(1 + o(1)),$$

where $x_0 = \pi + \lambda_0$. The proof that u is uniformly bounded on sets of the form $K = \{(x, t) : x \geq x_1, 0 \leq t < T\}$ with $x_1 > x_0$ is done by applying Lemma 6.5. Since $\lambda > 0$ is arbitrarily small the proof of Theorem 9.1 is thus complete. ■

9. Periodic solutions. The techniques developed in the preceding sections can be easily adapted to some other situations. We consider in this section the case of periodic initial data with period a multiple of 2π , namely

$$(9.1) \quad u_0(x + 2m\pi) = u_0(x),$$

with m an integer. Let us take the interval $I_m = [-m\pi, m\pi]$ as the basic period. We also assume that u_0 is symmetric in I_m , nonnegative and nonincreasing for $0 \leq x \leq m\pi$. We also ask u_0 not to be constant in order to avoid trivial cases whose behaviour is different from the one we want to study here. Notice that we can think of the solution of the Cauchy Problem for (1.1) in $Q = \mathbf{R} \times (0, T)$ under these initial data as the solution of the Neumann Problem posed in $Q_m = I_m \times (0, T)$ with boundary data

$$(9.2) \quad u_x(\pm m\pi, t) = 0 \quad \text{for } t > 0.$$

To begin with, the explicit solutions u_* considered in Section 2 are still solutions of this problem. Moreover, for any solution of the Neumann Problem we have:

- i) If t is fixed the solution $u(x, t)$ is a symmetric function of x , decreasing for $0 < x < m\pi$. Hence, the maximum is always taken at $x = 0$. The Maximum Principle applies and the intersection number in I_m is still a nonincreasing function of time.
- ii) Estimates (3.1) and (3.4) of Section 3 hold. The proofs are unchanged.
- iii) The same is true for semiconvexity results of Section 4 and the minimal profiles of Section 5.
- iv) Theorem 6.1 and the other results of Section 6 are valid, though statements (6.2), (6.3) are void if $m\pi \leq L_k$.

Thus, we obtain

THEOREM 9.1. *Let u be a solution of (1.1), (9.1), (9.2) with $m \geq 4$. Assume that u_0 satisfies the intersection condition $N(0) = 2$ with some nonnegative initial data $u_*(x, 0; T, \mu)$, $\mu \geq \mu_T$. Then*

$$(9.3) \quad B \cap I_m \subseteq [-3\pi, 3\pi].$$

PROOF: Our assumption is that $v_0 = \log(1 + u_0)$ satisfies

$$(9.4) \quad N(0) = 2,$$

where $N(t)$ denotes number of intersections in I_m of the functions $v(x, t)$ and $v_*(x, t)$. As before it follows that

$$(9.5) \quad N(t) \leq 2$$

for every $0 < t < T$. As in Corollary 6.3 we have $v(\pm 2\pi, t) < v_*(\pm 2\pi, t)$ for $t \approx T$. This and (9.5) imply that

$$(9.6) \quad v(3\pi, t) < v_*(3\pi, t) = \frac{1}{2} |\log(T - t)| (1 + o(1))$$

as $t \rightarrow T$. Since the solution $v(x, t)$ is nonincreasing in $x > 0$ we conclude that for $t \approx T$ and $3\pi \leq x \leq m\pi$

$$(9.7) \quad v(x, t) \leq |\log(T - t)|.$$

We now apply Lemma 6.5 to conclude that v is uniformly bounded for $m\pi \geq x \geq x_1 > 3\pi$ and $0 < t < T$. ■

v) Under stricter conditions on the initial data we obtain as in Theorem 7.1

THEOREM 9.2. *Let $m \geq 3$ and let $v_0 = \log(1 + u_0)$ satisfy the conditions of Proposition 7.4 in I_m . Then*

$$(9.8) \quad B \cap I_m = [-\pi, \pi].$$

PROOF: We have (9.4) and then (9.5). Arguments similar to those given in Sections 7 and 8 show that, similar to (9.6) in the preceding result, the inequality $v(\pi, t) \leq v_*(\pi, t)$ is valid after slightly shifting in x and t the explicit solution $v_*(x, t)$. In this way we can apply the arguments of Theorems 7.1 and 8.1. ■

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