**BIFURCATION ANALYSIS OF MINIMIZING HARMONIC
MAPS DESCRIBING THE EQUILIBRIUM OF
NEMATIC PHASES BETWEEN CYLINDERS**

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Introduction

Consider a nematic liquid crystal confined between two coaxial circular cylinders centered on the z -axis. Suppose that each cylinder imposes a strong anchoring boundary condition requiring the director field to be normal to it. In experiments on a nematic material[#] in such a configuration, Williams, Cladis, and Kléman [1973, 2]

[#] The material was MBBA, *i.e.*, methoxy-benzilidene-butylaniline, a frequently studied low molecular weight nematic.

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observed that, for the range of cylinder diameters studied, at equilibrium the director lies along radial lines in the (x, y) -plane with no component in the z -direction. We shall here show that such is the case in the one-constant theory, *i.e.*, for minimizing harmonic maps into S^2 , provided the ratio ρ of the radius of the inner cylinder to that of the outer cylinder is equal to or greater than $e^{-\pi}$; however, if ρ is less than $e^{-\pi}$, then the free-energy has a minimizer that preserves radial symmetry in the sense that the projection of u onto the (x, y) -plane lies along radial lines, but this minimizer has a component along the z -axis at every point off the bounding surfaces.

It is a consequence of an elementary remark, which is here called “Theorem 0” and whose proof is given in Appendix I, that the present minimization problems for functions on a three-dimensional region can be reduced to problems for functions on two-dimensional regions.

Let Ω be a smooth bounded domain in $\mathbb{R}^2$, put $D = \Omega \times [0, 1]$, and let φ be a given smooth map from $\partial\Omega$ to S^2 . Define the sets $\mathcal{F}$ and $\mathcal{E}$ by

$$\mathcal{F} = \{ u \in H^1(D; S^2); u(x, y, z) = \varphi(x, y) \text{ for } (x, y, z) \in \partial\Omega \times [0, 1] \},$$

$$\mathcal{E} = \{ u \in H^1(\Omega; S^2); u(x, y) = \varphi(x, y) \text{ for } (x, y) \in \partial\Omega \}.$$

Theorem 0. *If u is a minimizer for*

$$\min_{u \in \mathcal{F}} \int_D |\nabla u|^2 dx dy dz ,$$

then u is independent of z and is a minimizer for

$$\min_{u \in \mathcal{E}} \int_{\Omega} |\nabla u|^2 dx dy .$$

In Section 1, where we identify the minimizers for the coaxial case, the domain Ω will be $B_1 \setminus \overline{B_\rho}$, where ρ is in $(0, 1)$ and

$$B_t = \{ (x, y) \in \mathbb{R}^2; x^2 + y^2 < t^2 \} ,$$

and the strong-anchoring boundary condition mentioned above will be expressed by the following choice for the function φ :

$$\varphi(x, y) = (x/r, y/r, 0), \quad (r^2 = x^2 + y^2) .$$

Our main result, Theorem 1, asserts: (i) If $\rho \geq e^{-\pi}$, the energy

$$E(u) = \int_{\Omega} |\nabla u|^2 dx dy$$

is minimized on $\mathcal{E}$ by the function u_0 defined on Ω by

$$u_0(x, y) = (x/r, y/r, 0) .$$

(ii) If $\rho < e^{-\pi}$, u_0 does not minimize E ; there is then a minimizer with radial symmetry

whose range does not intersect the equator of S^2 , i.e., the set

$$S^1 = \{ (x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, z = 0 \} ;$$

to within a reflexion through the (x, y) -plane, the minimizer of E is unique in the class of radially symmetric maps.

As our discussion will show, for $\rho < e^{-\pi}$, the magnitude of the z -component, u_3 , of the radially symmetric minimizer u of E has its maximum value at $r = \rho^{1/2}$. Because $|u_3(\rho^{1/2})|$ approaches zero as ρ approaches $e^{-\pi}$ from below, experiments to test our new form of “escape to the third-dimension” will require care: when the maximum of $|u_3|$ is large, it occurs near a bounding surface.[#]

[#] This and related matters will be discussed in a paper on physical applications of the present analysis.

In Section 2, for the general case in which the hole B_ρ is not centered at the origin, we study the behavior of the minimizer and its energy as ρ approaches zero. We there describe the sense in which the minimizing map from $B_1 \setminus \overline{B_\rho}$ to S^2 approaches the radially symmetric minimizing map from B_1 to S^2 whose importance to liquid crystal theory was brought out Cladis & Kléman [1972, 1] and Meyer [1973, 1].

Several results of independent interest are given in the appendices. Among these is the following: any harmonic map from a smooth bounded domain in $\mathbb{R}^2$ to S^2 whose range lies in the northern hemisphere,

$$S^2_+ = \{ (x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1, z \geq 0 \} ,$$

but is not a subset of the equator S^1 , is minimizing.

1. Identification of the Minimal Energy Maps

Here $\Omega = B_1 \setminus \overline{B_\rho}$, and

$$\mathcal{E} = \{ u \in H^1(\Omega, S^2); u_0(x, y) = (x/r, y/r, 0) \text{ on } \partial\Omega \}.$$

Recall that the function $u_0(x, y) = (x/r, y/r, 0)$ satisfies the equation

$$-\Delta u_0 = u_0 |\nabla u_0|^2$$

on Ω and hence is a harmonic map from Ω to S^2 . Of course u_0 is in $\mathcal{E}$, but, as we show

below, this critical point of the energy E on $\mathcal{E}$ is not necessarily a minimizer.

Let $\mathcal{E}_0$ be set of functions in $\mathcal{E}$ which have *radial symmetry*, i.e., have the form

$$u(x, y) = c(r)u_0(x, y) + s(r)(0, 0, 1)$$

with $c^2 + s^2 = 1$.

Theorem 1.

(1) If $\rho \geq e^{-\pi}$, then u_0 is a minimizer of E on $\mathcal{E}$; if $\rho > e^{-\pi}$, u_0 is the unique

minimizer in $\mathcal{E}$.

(2) If $\rho < e^{-\pi}$, then u_0 is not a minimizer of E on $\mathcal{E}$, but there is a minimizer of E

on $\mathcal{E}$ that belongs to $\mathcal{E}_0$ and has $s(r) > 0$ for every r in $(\rho, 1)$; this minimizer is

unique in $\mathcal{E}_0$ to within replacement of $s(r)$ by $-s(r)$.

Remark 1. For $\rho < e^{-\pi}$ Meyer [1973, 1, pp. 409-411] found a harmonic map in $\mathcal{E}_0$ for which $|u_3|$ has a non-zero maximum at $r = \sqrt{\rho}$; as the proof of Theorem 1 shows, for each $\rho < e^{-\pi}$ that radially symmetric harmonic map gives a minimizer of E on $\mathcal{E}$ and, to within a reflection through the (x, y) -plane, is the unique such minimizer in $\mathcal{E}_0$. After the present study was completed, Sandier [1991, 2] proved that for each ρ in $(0, 1)$ the minimizer of E on $\mathcal{E}$ is unique (to within reflection) and is radially symmetric.

To prove Theorem 1 we employ two lemmas.

Lemma 1. *For each ρ in $(0, 1)$, $\min_{u \in \mathcal{E}} E(u)$ is attained at a map in $\mathcal{E}_0$.*

Proof. For an $\varepsilon > 0$, put

$$\Psi_\varepsilon(x, y) = \left(\frac{x}{r} \cos \varepsilon, \frac{y}{r} \cos \varepsilon, \sin \varepsilon \right)$$

for (x, y) in $\partial\Omega$, and define

$$\mathcal{E}_\varepsilon = \{ u \in H^1(\Omega, S^2); u = \Psi_\varepsilon \text{ on } \partial\Omega \}.$$

Let u^ε be a minimizer for E on $\mathcal{E}_\varepsilon$. By a standard argument, we may assume that

$u_3^\varepsilon \geq \sin \varepsilon$ in Ω . [First we may replace (u_1, u_2, u_3) by $(u_1, u_2, |u_3|)$ which is again a

minimizer, and as we then have

$$-\Delta |u_3| = |u_3| |\nabla u|^2 \geq 0,$$

[1979, 1], there is a unique harmonic map from Ω into $S^2 \cap \{u_3 \geq \sin \varepsilon\}$ with a given boundary value. This map u^ε is radially symmetric, since any minimizer of E in the class $\mathcal{E}_0$ is harmonic. We claim that for some sequence $\varepsilon_n \rightarrow 0$, u^{ε_n} converges strongly in H^1 to a minimizer of E on $\mathcal{E}$, and this minimizer is radially symmetric. To prove this assertion it suffices to check that

$$\limsup_{\varepsilon \rightarrow 0} \min_{\mathcal{E}_\varepsilon} E \leq \min_{\mathcal{E}} E .$$

Given any v in $\mathcal{E}$, consider $v^\varepsilon = R_\varepsilon(v)$, where $R_\varepsilon : S^2 \rightarrow S^2$ is defined by

$$R_\varepsilon(\xi_1, \xi_2, \xi_3) = (a_\varepsilon \xi_1, a_\varepsilon \xi_2, \max\{|\xi_3|, \sin \varepsilon\})$$

with

$$a_\varepsilon = \left[\frac{1 - \max(|\xi_3|^2, \sin^2 \varepsilon)}{1 - |\xi_3|^2} \right]^{1/2} .$$

Clearly v^ε is in $\mathcal{E}_\varepsilon$, $E(v^\varepsilon) \geq E(u^\varepsilon)$, and $E(v^\varepsilon) \rightarrow E(v)$.

Lemma 2. *For each ρ in $(0, 1)$ there is a unique minimizer of E in the class $\mathcal{E}_0$ (modulo a reflection through the (x, y) -plane). This minimizer is u_0 if and only if $\rho \geq e^{-\pi}$.*

Proof of Lemma 2. Let $u = (u_1, u_2, u_3)$ be a minimizer of E in $\mathcal{E}_0$. We claim that either $u_3 \geq 0$ on Ω or $u_3 \leq 0$ on Ω . Suppose, to the contrary, that u_3 takes both positive and negative values in Ω . Then $\tilde{u} = (u_1, u_2, |u_3|)$ is also a minimizer of E on $\mathcal{E}_0$ and coincides with u on some open set. As both u and $\tilde{u}$ are minimizers, they are analytic functions from Ω to $\mathbb{R}^3$, and hence $u = \tilde{u}$ on Ω , i.e., $u_3 \geq 0$ on Ω , which is a contradiction.

Without loss of generality we assume $u_3 \geq 0$ and we write

$$u(r \cos \theta, r \sin \theta) = (\cos \theta \cos \varphi(r), \sin \theta \cos \varphi(r), \sin \varphi(r))$$

for some smooth function $\varphi : [\rho, 1] \rightarrow [0, \pi/2]$ with $\varphi(\rho) = \varphi(1) = 0$.

The energy $E(u)$ is given by

$$E(u) = 2\pi \int_{\rho}^1 \left[\left(\frac{d\varphi}{dr} \right)^2 + \frac{\cos^2 \varphi}{r^2} \right] r \, dr.$$

It is convenient to change variables and let $t = \log r$, with t in $[\log \rho, 0]$, so that

$$E(u) = 2\pi \int_{\log \rho}^0 \left[\left(\frac{d\varphi}{dt} \right)^2 + \cos^2 \varphi \right] dt.$$

As u_0 corresponds to $\varphi \equiv 0$,

$$E(u) - E(u_0) = 2\pi \int_{\log \rho}^0 \left[\left(\frac{d\varphi}{dt} \right)^2 - \sin^2 \varphi \right] dt .$$

We now distinguish two cases.

(1) In the case $\rho \geq e^{-\pi}$, i.e., $|\log \rho| \leq \pi$, we have

$$E(u) - E(u_0) = 2\pi \int_{\log \rho}^0 \left[\left(\frac{d\varphi}{dt} \right)^2 - \varphi^2 \right] dt + 2\pi \int_{\log \rho}^0 \left[\varphi^2 - \sin^2 \varphi \right] dt \geq 0 ,$$

where the inequality is strict unless $\varphi \equiv 0$, because, for φ in $H_0^1(\log \rho, 0)$,

$$\int_{\log \rho}^0 \left(\frac{d\varphi}{dt} \right)^2 dt \geq \lambda_1 \int_{\log \rho}^0 \varphi^2 dt ,$$

where $\lambda_1 = \pi^2(\log \rho)^{-2}$, which is not less than 1, is the first eigenvalue of $-\varphi''$. It follows

that u_0 is the unique minimizer of E on $\mathcal{E}_0$ when $\rho \geq e^{-\pi}$.

(2) In the case of $\rho < e^{-\pi}$, i.e., $|\log \rho| > \pi$, u_0 is not a minimizer, for the choice

$$\varphi(t) = \beta \sin \frac{\pi t}{|\log \rho|}$$

with β sufficiently small yields $E(u) < E(u_0)$. For each minimizer of E on $\mathcal{E}_0$, φ obeys the variational equation

$$-\varphi'' = \sin \varphi \cos \varphi,$$

and hence $\psi = 2\varphi$ obeys the following boundary-value problem for the pendulum equation:

$$\left. \begin{aligned} -\psi'' &= \sin \psi && \text{in } (\log \rho, 0), \\ \psi(\log \rho) &= \psi(0) = 0. \end{aligned} \right\} \quad (1)$$

As $0 \leq \varphi \leq \pi/2$, we have $0 \leq \psi \leq \pi$, and thus $\sin \psi \geq 0$. By the strong maximum principle, either $\psi \equiv 0$ or $\psi > 0$ on $(\log \rho, 0)$. As u_0 is not a minimizer, $\psi \equiv 0$ is excluded. The uniqueness of the positive solution of (1) is a classical result and follows from the concavity of the sine function on $[0, \pi]$ (see, e.g., Krasnoselskii [1964, 1] or Brezis & Oswald [1987, 1] and the references therein).

We remark in passing that the unique positive solution of (1) attains its maximum at $(\log \rho)/2$ and is symmetric about that point. Thus,

$$\int_{B_{\sqrt{\rho}} \setminus B_{\rho}} |\nabla u|^2 dx dy = \int_{B_1 \setminus B_{\sqrt{\rho}}} |\nabla u|^2 dx dy$$

and the maximum value of $|u_3| = |\sin \varphi(r)|$ is attained at $r = \rho^{1/2}$.

Proof of Theorem 1.

(1) By Lemma 1, there is a minimizer u^* of E on $\mathcal{E}$ that belongs to $\mathcal{E}_0$, and since u^* is in $\mathcal{E}_0$, Lemma 2 yields $E(u^*) \geq E(u_0)$; thus u_0 is a minimizer of E not just on $\mathcal{E}_0$, but on all of $\mathcal{E}$. We now show that u_0 is the unique minimizer of E on $\mathcal{E}$ if $\rho > e^{-\pi}$.

When $\rho > e^{-\pi}$, we may choose $R > 1$ with $\rho/R > e^{-\pi}$. For any minimizer v of E on $\mathcal{E}$ we may consider the map $w: B_R \setminus B_\rho \rightarrow S^2$ defined by

$$w(x, y) = \begin{cases} v(x, y) & \text{on } B_1 \setminus B_\rho, \\ u_0(x, y) & \text{on } B_R \setminus B_1. \end{cases}$$

Clearly w is in $H^1(B_R \setminus B_\rho; S^2)$ and

$$\int_{B_R \setminus B_\rho} |\nabla w|^2 dx dy = \int_{B_R \setminus B_\rho} |\nabla u_0|^2 dx dy .$$

Recall that u_0 is a minimizer for the energy

$$\tilde{E}(u) = \int_{B_R \setminus B_\rho} |\nabla u|^2 dx dy ,$$

as $\rho/R > e^{-\pi}$. Thus w is also a minimizer of $\tilde{E}$, and hence is analytic on $B_R \setminus B_\rho$.

Because w coincides with u_0 on $B_R \setminus B_1$, it coincides with u_0 everywhere, i.e., $v = u_0$ on $B_1 \setminus B_\rho$.

(2) The assertion in Theorem 1 for the case $\rho \leq e^{-\pi}$ follows directly from the two Lemmas.

Remark 2. There is an alternative proof of assertion (2) in Theorem 1. One can use Lemma 1 together with a general fact, proved in Appendix II: Any harmonic map to S^2 from a smooth bounded domain in $\mathbb{R}^2$ whose range is a subset of S_+^2 but not a subset of S^1 is minimizing.

Remark 3. Theorem 1 has an extension to dimension n for $1 \leq n \leq 6$. More precisely, put

$$B_t = \{ x \in \mathbb{R}^n ; r = |x| < t \},$$

$\Omega = B_1 \setminus B_\rho$, and

$$\mathcal{E} = \{ u \in H^1(\Omega, S^n); u(x) = (x/r, 0) \text{ on } \partial\Omega \},$$

and let $\mathcal{E}_0$ be the set of functions in $\mathcal{E}$ with radial symmetry, which here means that

$$u(x) = c(r)u_0(x) + s(r)(0, 0, \dots, 1)$$

where $c^2 + s^2 = 1$ and $u_0(x) = (x/r, 0)$. We recall (see Jäger and Kaul [1983, 2]) that, for $n \geq 7$ and $\Omega = B_1$, u_0 is a minimizer of E and that it is not a minimizer of E for $3 \leq n \leq 6$.

Using the same arguments as in the proof of Theorem 1 we have

Theorem 1'. Assume $2 \leq n \leq 6$, so that $P(n) = -\frac{1}{4}n^2 + 2n - 2$ is nonnegative.

(1) If $\rho \geq e^{-\pi/P(n)}$, then u_0 is a minimizer of E on $\mathcal{E}_0$. Moreover u_0 is the unique minimizer of E in $\mathcal{E}$ if $\rho > e^{-\pi/P(n)}$.

(2) If $\rho < e^{-\pi/P(n)}$, then u_0 is not a minimizer of E on $\mathcal{E}_0$. In this case there is a minimizer of E on $\mathcal{E}$ which belongs to $\mathcal{E}_0$ and has $s(r) > 0$ for all r in $(\rho, 1)$. Moreover, to within replacement of $s(r)$ by $-s(r)$, this minimizer is unique in $\mathcal{E}_0$.

2. Asymptotic Analysis for Small ρ

We consider here the more general case in which the hole B_ρ need not be centered at the origin and thus

$$\Omega = \Omega_\rho(a) = B_1 \setminus \overline{B_\rho}(a) ,$$

where $a = (a_1, a_2)$ is a given point in B_1 and

$$B_\rho(a) = \{ (x, y) \in \mathbb{R}^2; (x - a_1)^2 + (y - a_2)^2 = r_a^2 < \rho^2 \} .$$

We let $\rho \rightarrow 0$, and are concerned with the behavior of minimizers of the energy in Ω_ρ

subject to the boundary condition that u be normal to the boundary of Ω_ρ , i.e.,

$$\left. \begin{aligned} u(x, y) &= (x, y, 0) \quad \text{on } \partial B_1, \\ u(x, y) &= ((x - a_1)/\rho, (y - a_2)/\rho, 0) \quad \text{on } \partial B_\rho(a). \end{aligned} \right\} \quad (2)$$

As it is natural to expect that these minimizers converge to a minimizer of the energy in the full disc B_1 , it is useful to recall some known facts about the problem in B_1 . On the set

$$\mathcal{E}_1 = \{ u \in H^1(B_1; S^2); u = u_0 \text{ on } \partial B_1 \} ,$$

with u_0 as in Section 1, the minimum,

$$\min_{u \in \mathcal{E}_1} \int_{B_1} |\nabla u|^2 dx dy ,$$

is attained uniquely, modulo a reflection through the (x, y) -plane, and the minimizer is

given by[#]

$$\bar{u}(x, y) = \frac{2}{1+r^2}(x, y, 1) + (0, 0, -1).$$

For the convenience of the reader we sketch the proof following the method of Brezis &

Coron [1983, 1]. We have

$$E(\bar{u}) = 4\pi,$$

which may be verified by direct computation or by use of the conformality of $\bar{u}$. Let $\tilde{u}$ be

a minimizer, and assume, without loss of generality, $\tilde{u}_3 \geq 0$. Put

$$v(x, y) = \begin{cases} \tilde{u}(x, y) & \text{for } r \leq 1, \\ \bar{u}(x/r^2, y/r^2) & \text{for } r > 1, \end{cases}$$

where

[#] This map $\bar{u}$ is the same as that studied by Cladis & Kléman [1972, 1] and Meyer

[1973, 1]; i.e., in terms of polar coordinates r, θ on B_1 ,

$\bar{u}(r \cos \theta, r \sin \theta) = (\cos \theta \cos \varphi(r), \sin \theta \cos \varphi(r), \sin \varphi(r))$ where

$\cot(\frac{\pi}{4} + \frac{\varphi}{2}) = r$. Alternatively, $\bar{u}$ is the stereographic projection of B_1 onto S_+^2 with vertex

at the south pole $(0, 0, -1)$.

$$\underline{u}(x,y) = \frac{2}{1+r^2} (x, y, -1) + (0, 0, 1).$$

The degree of v (as a map from $\mathbb{R}^2 \cup \{\infty\} \simeq S^2$) is 1, because each point in the southern hemisphere has only one counterimage. This implies that

$$\begin{aligned} 1 &= \frac{1}{4\pi} \int_{\mathbb{R}^2} v \cdot v_x \wedge v_y \, dx \, dy \leq \frac{1}{8\pi} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx \, dy \\ &= \frac{1}{8\pi} [E(\bar{u}) + E(\underline{u})] = \frac{1}{8\pi} [E(\bar{u}) + 4\pi]; \end{aligned}$$

hence $E(\bar{u}) \geq 4\pi$, and therefore $E(\bar{u}) = 4\pi$; i.e., $\underline{u}$ and $\bar{u}$ are minimizers.

Consequently, v is a minimizer of the energy on $\mathbb{R}^2 \cup \{\infty\}$ among maps of degree 1, and thus v is analytic. This implies that $\bar{u} = \underline{u}$ and completes the proof.

With $\Omega_\rho(a)$ as above, let $\mathcal{E}_\rho(a)$ be the set of functions in $H^1(\Omega_\rho(a); S^2)$ that obey the boundary condition (2), and let u_ρ be a minimizer of

$$E_{\rho,a}(u) = \int_{\Omega_\rho(a)} |\nabla u|^2$$

on $\mathcal{E}_\rho(a)$. Without loss generality we may assume that u_ρ takes its values in the northern hemisphere S_+^2 .

Theorem 2. *For each fixed a in B_1 , as $\rho \rightarrow 0$, u_ρ converges to $\bar{u}$ uniformly on every compact subset of $B_1 \setminus \{a\}$; moreover,*

$$\lim_{\rho \rightarrow 0} E_{\rho, a}(u_\rho) = 8\pi = \int_{B_1} |\nabla \bar{u}|^2 + 4\pi .$$

Remark 4. It is an obvious consequence of this theorem that, for any location $a = (a_1, a_2)$ of the center of the hole $B_\rho(a)$, if ρ is small enough, u_ρ “escapes to the third dimension,” i.e., has values off the equator S^1 .

The proof of Theorem 1 is constructed in stages requiring proof of the following four assertions about limits as $\rho \rightarrow 0$:

- (1) $\limsup E_{\rho, a}(u_\rho) \leq 8\pi$
- (2) $u_\rho \rightarrow \bar{u}$ weakly in $H_{\text{loc}}^1(B_1 \setminus \{a\})$;
- (3) $\liminf E_{\rho, a}(u_\rho) \geq 8\pi$;
- (4) $u_\rho \rightarrow \bar{u}$ uniformly on every compact subset of $B_1 \setminus \{a\}$.

Proof. (1) To show that

$$\limsup_{\rho \rightarrow 0} \int_{\Omega_\rho(a)} |\nabla u_\rho|^2 \leq 8\pi$$

we seek, for each (small) $\rho > 0$, a map v from Ω_ρ to S^2 such that

$$\int_{\Omega_\rho(a)} |\nabla v|^2 \leq 8\pi + o(1). \quad (3)$$

To this end, let

$$v = \begin{cases} \bar{u}((x-a_1)\rho/r_a^2, (y-a_2)\rho/r_a^2) & \text{in } B_{\sqrt{\rho}}(a) \setminus \overline{B}_\rho(a), \\ w(x,y)/|w(x,y)| & \text{in } B_\delta(a) \setminus \overline{B}_{\sqrt{\rho}}(a), \\ \bar{u}(x,y) & \text{in } B_1 \setminus B_\delta(a), \end{cases}$$

where $\delta > \sqrt{\rho}$ will be determined later and w is the solution of

$$\left. \begin{aligned} \Delta w &= 0 \quad \text{in } B_\delta(a) \setminus \bar{B}_{\sqrt{\rho}}(a), \\ w &= \bar{u}(x-a_1, y-a_2) \quad \text{on } \partial B_{\sqrt{\rho}}(a), \\ w &= \bar{u}(x, y) \quad \text{on } \partial B_\delta(a). \end{aligned} \right\}$$

Now,

$$E_{\rho, a}(v) = I + II + III$$

with

$$I = \int_{B_{\sqrt{\rho}}(a) \setminus B_\rho(a)} |\nabla v|^2, \quad II = \int_{B_\delta(a) \setminus B_{\sqrt{\rho}}(a)} |\nabla v|^2, \quad III = \int_{B_1(a) \setminus B_\delta(a)} |\nabla v|^2.$$

Clearly,

$$I \leq \int_{B_1} |\nabla \bar{u}|^2 = 4\pi \quad (4)$$

and

$$III \leq 4\pi. \quad (5)$$

We claim that when δ is chosen appropriately,

$$II \rightarrow 0 \quad \text{as } \rho \rightarrow 0. \quad (6)$$

By the maximum principle, w has its values outside of a fixed neighborhood of 0 that depends only on a , and, therefore, there is a number C , independent of ρ , for which

$$\int_{B_\delta(a) \setminus B_{\sqrt{\rho}}(a)} |\nabla v|^2 \leq C \int_{B_\delta(a) \setminus B_{\sqrt{\rho}}(a)} |\nabla w|^2.$$

To prove (6) it suffices to show that as $\rho \rightarrow 0$

$$\int_{B_\rho(a) \setminus \bar{B}_{\sqrt{\rho}}(a)} |\nabla w|^2 \rightarrow 0. \quad (7)$$

To this end we observe that

$$w = w_1 + w_2 + w_3,$$

where w_1 is defined by

$$\left. \begin{aligned} \Delta w_1 &= 0 && \text{in } B_\delta(a) \setminus \bar{B}_{\sqrt{\rho}}(a), \\ w_1 &= (0, 0, 1) && \text{on } \partial B_{\sqrt{\rho}}(a), \\ w_1 &= \bar{u}(a) && \text{on } \partial B_\delta(a), \end{aligned} \right\}$$

w_2 by

$$\left. \begin{aligned} \Delta w_2 &= 0 && \text{in } B_\delta(a) \setminus \bar{B}_{\sqrt{\rho}}(a), \\ w_2 &= 0 && \text{on } \partial B_{\sqrt{\rho}}(a), \\ w_3 &= \bar{u}(x, y) - \bar{u}(a) && \text{on } \partial B_\delta(a), \end{aligned} \right\}$$

and w_3 by

$$\left. \begin{aligned} \Delta w_3 &= 0 && \text{in } B_\delta(a) \setminus \bar{B}_{\sqrt{\delta}}(a), \\ w_3 &= \bar{u}(x-a_1, y-a_2) - (0, 0, 1) && \text{on } \partial B_{\sqrt{\rho}}(a), \\ w_3 &= 0 && \text{on } \partial B_\delta(a). \end{aligned} \right\}$$

Let us now choose $\delta = \delta(\rho)$ so that $\delta/\sqrt{\rho} \rightarrow \infty$ as $\rho \rightarrow 0$, (for example, we may put

$\delta = \rho^{1/3}$). By an explicit calculation,

$$\int_{B_\delta(a) \setminus B_{\sqrt{\rho}}(a)} |\nabla w_1|^2 = \frac{2\pi |(0,0,1) - \bar{u}(a)|^2}{|\log(\delta/\sqrt{\rho})|} \rightarrow 0.$$

Moreover, there is c , independent of δ , for which

$$\begin{aligned} \int_{B_\delta(a) \setminus B_{\sqrt{\rho}}(a)} |\nabla w_1|^2 &\leq c \|w_2\|_{H^{1/2}(\partial B_\delta(a))}^2 \\ &\leq c \|w_2\|_{L^2(\partial B_\delta(a))}^2 \|w_2\|_{H^1(\partial B_\delta(a))} \\ &\leq c \delta^2, \end{aligned}$$

and a similar result holds for w_3 . Thus (7) follows, and hence v obeys (6). In view of

(4), (5), and (6), the assertion (1) is proven.

(2) Consider a sequence of minimizers u_ρ of $E_{\rho,a}$ with $\rho \rightarrow 0$. As $E_{\rho,a}(u_\rho)$ is bounded there is a subsequence u_{ρ_n} converging weakly in $H_{\text{loc}}^1(B_1 \setminus \{a\})$ to a limit $\hat{u}$ in $H^1(B_1)$ with $\hat{u} = u_0$ on ∂B_1 . Fix $r > 0$ arbitrarily small. In view of a result of Schoen [1984, 2], u_{ρ_n} converges to $\hat{u}$ in the C^2 -norm on every compact subset of

$B_1 \setminus (B_r(a) \cap S)$ where S is a finite set, and it then follows from a result of Sacks & Uhlenbeck [1988, 1] that $\hat{u}$ is a smooth harmonic map on B_1 . In view of Theorem A.1, proven in Appendix II, to show that $\hat{u}$ is a minimizing harmonic map on B_1 , it suffices to show that its range is not in S^1 . If, to the contrary, $\hat{u}(B_1) \subset S^1$, then $\hat{u}$ is a continuous map from B_1 to S^1 that equals the identity map on ∂B_1 , which is impossible by the Brouwer fixed point theorem. As $\hat{u}$ is the unique minimizing S^2_+ -valued harmonic map in $\mathcal{E}_1$, $\hat{u} = \bar{u}$, and thus we have subsequence u_{ρ_n} converging weakly H^1_{loc} to $\bar{u}$. By the uniqueness of $\bar{u}$, the full sequence converges weakly in H^1_{loc} to $\bar{u}$.

(3) To show that the number $L = \liminf E_{\rho, a}(u_\rho)$ is not less than 8π , we note that there is a sequence of minimizers u_{ρ_n} for which $L = \lim E_{\rho_n, a}(u_{\rho_n})$ and, by (2), $u_{\rho_n} \rightarrow \bar{u}$ weakly in $H^1_{\text{loc}}(B_1 \setminus \{a\})$. Employing again the result of Schoen mentioned above, we can assert (after a second selection of a subsequence if necessary) that $u_{\rho_n} \rightarrow \bar{u}$ in the C^2 -norm except at isolated points of $B_1 \setminus \{a\}$. We now fix a circle Σ of small radius α centered at a with

$$\|u_{\rho_n} - \bar{u}\|_{L^\infty(\Sigma)} \rightarrow 0.$$

This can be done for arbitrary α , with the possible exception of a countable set of values of

α . In the following discussion c denotes various constants independent of α .

As $|\bar{u}(x, y) - \bar{u}(a)| \leq c\alpha$ for all (x, y) in Σ , there holds

$$\|u_{\rho_n} - \bar{u}(a)\|_{L^\infty(\Sigma)} \leq c\alpha + o(1). \quad (8)$$

Because $u_{\rho_n} \rightarrow \bar{u}$ weakly in $H^1(B_1 \setminus \bar{B}_\alpha(a))$,

$$\liminf E_{\alpha, a}(u_{\rho_n}) \geq E_{\alpha, a}(\bar{u}) \geq 4\pi - c\alpha^2. \quad (9)$$

In view of (8),

$$u_{\rho_n}(\Sigma) \subset B_{c\alpha + o(1)}(\bar{u}(a)).$$

From the observation that

$$u_{\rho_n}(x, y) = ((x-a_1)/\rho_n, (y-a_2)/\rho_n, 0) \quad \text{on } \partial B_{\rho_n}(a)$$

and a degree argument, it follows that if we put

$$\Gamma_n = B_\alpha(a) \setminus B_{\rho_n}(a)$$

then

$$u_{\rho_n}(\Gamma_n) \supset S_+^2 \setminus B_{c\alpha + o(1)}(\bar{u}(a)),$$

and this yields

$$\begin{aligned} \frac{1}{2} \int_{\Gamma_n} |\nabla u_{\rho_n}|^2 &\geq \int_{\Gamma_n} \left| \frac{\partial u_{\rho_n}}{\partial x} \wedge \frac{\partial u_{\rho_n}}{\partial y} \right| = \text{area } u_{\rho_n}(\Gamma_n) \\ &\geq 2\pi - c\alpha^2. \end{aligned} \quad (10)$$

As $\Omega_{\rho_n}(a) = \Gamma_n \cup (B_1 \setminus B_\alpha(a))$, (9) and (10) yield

$$L = \lim_{n \rightarrow \infty} \int_{\Omega_{\rho_n}(a)} |\nabla u_{\rho_n}|^2 \geq 8\pi - c\alpha^2,$$

and as this holds for a set of α with 0 as a limit point, we conclude that $L \geq 8\pi$.

(4) To prove that $u_\rho \rightarrow \bar{u}$ uniformly on every compact subset of $B_1 \setminus \{a\}$, we show that $u_\rho \rightarrow \bar{u}$ strongly in $H_{\text{loc}}^1(B_1 \setminus \{a\})$ and employ a result of Schoen [1984, 2].

In view of (10)

$$\int_{\Gamma_n} |\nabla u_{\rho_n}|^2 \geq 4\pi - 2c\alpha^2$$

and by (1),

$$\lim_{n \rightarrow \infty} \int_{\Omega_{\rho_n}(a)} |\nabla u_{\rho_n}|^2 \leq 8\pi.$$

Thus,

$$\begin{aligned} \lim_{\rho_n \rightarrow 0} \sup \int_{B_1 \setminus B_\alpha(a)} |\nabla u_{\rho_n}|^2 &\leq 4\pi + 2c\alpha^2, \\ &= \int_{B_1} |\nabla \bar{u}|^2 + 2c\alpha^2. \end{aligned}$$

Fix a compact subset K of $B_1 \setminus \{a\}$. For every $\alpha < \text{dist}(a, K)$ we have

$$\int_K |\nabla u_{\rho_n} - \nabla \bar{u}|^2 \leq \int_{B_1 \setminus B_\alpha(a)} |\nabla u_{\rho_n} - \nabla \bar{u}|^2.$$

Therefore,

$$\begin{aligned} \lim_{\rho_n \rightarrow 0} \sup_K \int |\nabla u_{\rho_n} - \nabla \bar{u}|^2 &\leq \int_{B_1} |\nabla \bar{u}|^2 + 2c\alpha^2 - \int_{B_1 \setminus B_\alpha(a)} |\nabla \bar{u}|^2 \\ &= \int_{B_\alpha(a)} |\nabla \bar{u}|^2 + 2c\alpha^2. \end{aligned}$$

Finally, we let α tend to zero and conclude that $u_{\rho_n} \rightarrow \bar{u}$ strongly in $H^1(K)$.

Remark 5. Rivière and Shafir [1991, 1] recently investigated the behavior of minimizers of

$E_{\rho, a}$ as ρ approaches 0 and a does not remain constant but approaches the boundary of

B_1 . They show that $\min E_{\rho, a}$ remains bounded and identify the limit in terms of

$$\lim \rho / (\text{dist}(a, \partial B_1)).$$

Appendix I. Proof of Theorem 0

Proof. Let

$$\alpha = \min_{v \in \mathcal{E}} \int_{\Omega} |\nabla v|^2 dx dy$$

be attained at v_0 in $\mathcal{E}$. For each u in $\mathcal{F}$, we have

$$\int_D |\nabla_{x,y} u|^2 dx dy dz \geq \alpha, \quad (\text{I.1})$$

and since we can choose u in $\mathcal{F}$ so that $u(x, y, z) = v_0(x, y)$, we have

$$\alpha = \min_{u \in \mathcal{F}} \int_D |\nabla u|^2 dx dy dz.$$

Moreover, if a given u in $\mathcal{F}$ is a minimizer, *i.e.*, obeys

$$\int_D |\nabla u|^2 dx dy dz = \alpha,$$

then, by (I.1),

$$\int_D |\nabla_{x,y} u|^2 dx dy dz \geq \alpha,$$

which yields $\partial u / \partial z = 0$ and proves the theorem.

Remark 6. Sandier [1991, 3] has recently shown that if one replaces the energy,

$$E(u) = \int_D |\nabla u|^2 dx dy dz ,$$

by the Oseen-Frank energy of liquid-crystal theory, *i.e.*, by

$$\hat{E}(u) = \int_D (k_1 (\operatorname{div} u)^2 + k_2 (u \cdot \operatorname{curl} u)^2 + k_3 |u \wedge \operatorname{curl} u|^2) dx dy dz$$

with the k_i positive constants, then the conclusion of Theorem 0 does not hold when

$k_2 = k_3$. In particular, if $k_1 = k_2 = k_3 = 1$, then minimizers of $\hat{E}$ depend on z . This does

not contradict Theorem 0, for in the equiconstant case $\hat{E}$ reduces to

$$\hat{E}(u) = E(u) - \int_D \operatorname{div} ((\nabla u)u - (\operatorname{div} u)u) dx dy dz ,$$

where the added integral depends only on the values of u on the boundary of D . For the

minimization problem considered here and by Sandier, values of u are not prescribed on

the surfaces $z = 0$ and $z = 1$, and thus the minimizers of E and $\hat{E}$ differ.

Appendix II. Harmonic Maps with Values in S_+^2

Let Ω be a smooth bounded domain in $\mathbb{R}^2$.

Theorem A.1. *Let u be a harmonic map from Ω into S^2 with $u(\Omega) \subset S_+^2$. If $u(\Omega)$ is not contained in S^1 , then u is minimizing.*

Remark 7. When Ω is not simply connected, the assumption that $u(\Omega)$ is not contained in S^1 is essential. There are harmonic maps $u : \Omega \rightarrow S^2$ which do not minimize the energy in $H^1(\Omega; S^1)$ and therefore in $H^1(\Omega; S^2)$. To have an example, let $\Omega = B_1 \setminus \bar{B}_{1/2}$ and

$$u(x, y) = (\cos f(x, y), \sin f(x, y), 0).$$

This map $u : \Omega \rightarrow S^1$ is harmonic whenever $\Delta f = 0$. If one chooses for f the boundary conditions

$$\left. \begin{aligned} f &= 0 && \text{on } \partial B_1, \\ f &= 2\pi k && \text{on } \partial B_{1/2}, \end{aligned} \right\}$$

with k a nonzero integer, then $|\nabla u|$ is not zero on Ω , i.e., the energy corresponding to u is positive. But the minimum energy subject to the condition that u equal the constant $(1, 0, 0)$ on ∂B_1 and $\partial B_{1/2}$ is zero, and hence u is not minimizing in $H^1(\Omega; S^1)$.

Corollary to Theorem A.1 *If u is a harmonic map from B^1 to S^2 that reduces to the identity map on ∂B_1 and has its range, $u(B_1)$, in S_+^2 , then u is the stereographic projection with vertex $(0, 0, -1)$, i.e., $u = \bar{u}$ where*

$$\bar{u}(x, y) = \frac{2}{1 + r^2} (x, y, 1) + (0, 0, -1) .$$

Proof. As $\bar{u}$ is the unique S_+^2 -valued minimizer of the energy in the class $\mathcal{E}_1$, i.e., in the class of H^1 -functions that reduce to the identity on ∂B_1 , it here suffices to show that u is a minimizer. By Theorem A.1, it suffices to check that $u(B_1)$ is not contained in S^1 .

Suppose, to the contrary, that $u(B_1) \subset S^1$; then, as u can be deformed continuously into a constant map, the degree of $u|_{\partial B_1}$ is zero, which is impossible, since u is the identity map on ∂B_1 .

Remark 8.[#] It is an open problem whether an arbitrary harmonic map u from

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[#] See [1983, 1].

B^1 to S^2 that reduces to the identity on ∂B_1 must equal either $\bar{u}$ or $\underline{u} = (\bar{u}_1, \bar{u}_2, -\bar{u}_3)$.

Proof of Theorem A.1. In view of a result of Hélein [1990, 1], we know that u is smooth in Ω . We have

$$\Delta u_3 = u_3 |\nabla u|^2 \geq 0 \quad \text{on } \Omega,$$

and $u_3 \geq 0$ on Ω . As $u_3 \not\equiv 0$, it follows from the maximum principle that $u_3 > 0$ in Ω .

Let

$$\Omega_\delta = \{ (x, y) \in \Omega; \text{dist}((x, y), \partial\Omega) > \delta \}.$$

Let u° be a minimizer for

$$\min_{\substack{v \in H^1(\Omega; S^2) \\ v=u \text{ on } \partial\Omega}} \int |\nabla v|^2.$$

We shall prove that $\int |\nabla u|^2 = \int |\nabla u^\circ|^2$.

Clearly,

$$\|u - u^\circ\|_{H^{1/2}(\partial\Omega_\delta)} \rightarrow 0 \quad \text{as } \delta \rightarrow 0. \quad (\text{II.1})$$

Let φ_δ be the solution of

$$\left. \begin{aligned} \Delta \varphi_\delta &= 0 && \text{in } \Omega_\delta, \\ \varphi_\delta &= u - u^\circ && \text{on } \partial\Omega_\delta. \end{aligned} \right\}$$

By (II.1),

$$\int_{\Omega_\delta} |\nabla \varphi_\delta|^2 \rightarrow 0. \quad (\text{II.2})$$

Set

$$u_\delta = \frac{u^\circ + t\varphi_\delta}{|u^\circ + t\varphi_\delta|} \quad \text{in } \Omega_\delta.$$

As $u_\delta = u$ on $\partial\Omega_\delta$, and $u(\Omega_\delta)$ is in a compact subset of S_+^2 , it follows from a result of

Jäger & Kaul [1979, 1] that

$$\int_{\Omega_\delta} |\nabla u_\delta|^2 \geq \int_{\Omega_\delta} |\nabla u|^2.$$

By (II.2),

$$\int_{\Omega_\delta} |\nabla u_\delta|^2 \rightarrow \int_{\Omega} |\nabla u^\circ|^2 \quad \text{as } \delta \rightarrow 0$$

and hence

$$\int_{\Omega} |\nabla u|^2 \leq \int_{\Omega} |\nabla u^0|^2,$$

which, as u^0 is a minimizer, proves our assertion.

Appendix III. Uniqueness and Nonuniqueness of Minimizers

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and let $\varphi: \partial\Omega \rightarrow S_+^2$ be given.

Consider the problem of finding

$$\min_{u \in H_\varphi^1} \int |\nabla u|^2, \quad (\text{III.1})$$

where

$$H_\varphi^1 = \{ u \in H^1(\Omega, S^2); u = \varphi \text{ on } \partial\Omega \}.$$

Note that, by the maximum principle, each minimizer takes all its values in S_+^2 (or in S_-^2 if

$u(\partial\Omega) \subset S^1$). Let u be a minimizer for (III.1) such that $u(\Omega) \subset S_+^2$.

Theorem A.2 *If $u(\Omega)$ is not a subset of S^1 , we have the following alternative.*

Either

(i) *u is a strict local minimizer in which case it is the unique minimizer,*

or

(ii) *or there is a continuum of minimizers containing u .*

Remark 9. Both cases occur. Here are two examples:

1. $\Omega = B_1$ and $\varphi(x, y) = (x, y, 0)$. Then in accord with the discussion given early in Section 2, the unique minimizer is $u = \bar{u}$.

2. $\Omega = B_1 \setminus \overline{B}_{1/2}$ and φ is constant on each component of the boundary, namely

$$\left. \begin{aligned} \varphi &= (1, 0, 0) && \text{on } \partial B_1, \\ \varphi &= (-1, 0, 0) && \text{on } \partial B_{1/2}. \end{aligned} \right\}$$

In this case, we have a family of minimizers. If f is a harmonic function on Ω with

$$\left. \begin{aligned} f &= 0 && \text{on } \partial B_1, \\ f &= \pi && \text{on } \partial B_{1/2}, \end{aligned} \right\}$$

then

$$u_\theta = (\cos f, \sin f \cos \theta, \sin f \sin \theta)$$

is a minimizer for every θ in $(0, \pi)$.

Proof of Theorem A.2. We use the notation of the proof of Theorem A.1. It suffices to

show that if there is another minimizer u_1 (besides u), u_1 and u can be connected by a

branch of minimizers. Let

$$u_\delta = \frac{u_1 + \varphi_\delta}{|u_1 + \varphi_\delta|} \quad \text{on } \Omega_\delta.$$

Note that $u_\delta = u$ on $\partial\Omega_\delta$. The flow,

$$\left. \begin{aligned} \frac{\partial v}{\partial t} - \Delta v &= v|\nabla v|^2 \quad \text{in } \Omega_\delta \times [0, \infty), \\ v(x, 0) &= u_\delta(x) \quad \text{on } \Omega_\delta, \\ v(x, t) &= u(x) \quad \text{on } \partial\Omega_\delta \times [0, \infty), \end{aligned} \right\}$$

has a smooth global solution, as $u_\delta(\Omega_\delta)$ is contained in a compact subset of S_+^2 (see, e.g., [1982, 1] [1985, 1] [1989, 1].) Moreover, as $t \rightarrow \infty$, $v(t) \rightarrow u|_{\Omega_\delta}$ in $H^1(\Omega_\delta)$; indeed u restricted to Ω_δ is the only harmonic map that has values in S_+^2 and the same boundary data. A standard estimate gives

$$E(v(T_2)) + 2 \int_{T_1}^{T_2} \left| \frac{\partial v}{\partial t} \right|^2 \leq E(v(T_1)), \quad T_1 \leq T_2,$$

where all integrals are evaluated on Ω_δ . Thus, for all $t > 0$,

$$E(u) \leq E(v(t)) \leq E(u_\delta). \quad (\text{III.2})$$

Clearly we have, as $\delta \rightarrow 0$,

$$\|u - u_\delta\|_{H^1(\Omega_\delta)} = \alpha(\delta) \rightarrow \|u - u_1\|_{H^1(\Omega)} = \alpha > 0.$$

Fix ε such that $0 < \varepsilon < \alpha$; for δ sufficiently small, there is, by continuity, a

$T = T(\varepsilon, \delta)$ such that

$$\|u - v(T)\|_{H^1(\Omega_\delta)} = \varepsilon. \quad (\text{III.3})$$

Note that as $\delta \rightarrow 0$ (at fixed ε), we have by (III.2)

$$E(v(T)) \rightarrow E(u),$$

and $v(T) \rightarrow v_\varepsilon$ weakly in H^1 for a subsequence $\delta_n \rightarrow 0$ and some $v_\varepsilon \in H^1$. The

assertion that

$$v(T) \rightarrow v_\varepsilon \text{ strongly in } H^1$$

follows from the observation that

$$E(v(T)) \rightarrow E(u) \leq E(v_\varepsilon).$$

By passing to the limit of small δ in (III.2), we obtain

$$E(v_\varepsilon) = E(u),$$

and hence v_ε is a minimizer; by passing to the same limit in (III.3), we find that

$$\|u - v_\varepsilon\|_{H^1} = \varepsilon.$$

We now present an application of Theorem A.2: There is an $\alpha < e^{-\pi}$ such that the minimizer in Theorem 1 is unique for every $\rho \in [\alpha, e^{-\pi})$.

It suffices to check that the second variation of the energy is positive definite at the (unique) radial minimizer. Let u be the radial minimizer in $\Omega = B_1 \setminus \overline{B_\rho}$, and write

$$u(r \cos \theta, r \sin \theta) = (\cos \theta \cos \varphi(r), \sin \theta \cos \varphi(r), \sin \varphi(r)).$$

Let u_t be a perturbation of u of the form

$$u_t(r \cos \theta, r \sin \theta) = (\cos(\theta + t\tau) \cos(\varphi + t\psi), \sin(\theta + t\tau) \cos(\varphi + t\psi), \sin(\varphi + t\psi)),$$

where $\tau(r, \theta)$ and $\psi(r, \theta)$ are smooth admissible perturbations, i.e.,

$$\tau(1, \theta) = \tau(\rho, \theta) = \psi(1, \theta) = \psi(\rho, \theta) = 0.$$

We have

$$\begin{aligned} E(u_t) &= E(u) \\ &+ t^2 \int_{\rho}^1 \int_0^{2\pi} \left[|\nabla \psi|^2 - \frac{\psi^2}{r^2} \cos 2\varphi + |\nabla \tau|^2 \cos^2 \varphi - \frac{2}{r^2} \psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi \right] r dr d\theta \\ &+ o(t^2); \end{aligned}$$

that is,

$$\delta^2 E = \int_{\rho}^1 \int_0^{2\pi} \left[|\nabla \psi|^2 - \frac{\psi^2}{r^2} \cos 2\varphi + |\nabla \tau|^2 \cos^2 \varphi - \frac{2}{r^2} \psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi \right] r dr d\theta.$$

On taking $t = \log r$ as a new variable we obtain

$$\delta^2 E = \int_{\log \rho}^0 \int_0^{2\pi} \left[|\nabla \psi|^2 - \psi^2 \cos 2\varphi + |\nabla \tau|^2 \cos^2 \varphi - 2\psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi \right] dt d\theta.$$

To show that for an α in $(0, e^{-\pi})$ (to be determined later), $\delta^2 E$ is positive definite for each ρ in $(\alpha, e^{-\pi})$, we split $\delta^2 E$ into three integrals:

$$\delta^2 E = I + II + III,$$

where

$$I = - \int_0^{2\pi} \int_{\log \rho}^0 2\psi \frac{\partial \tau}{\partial \theta} \sin 2\varphi dt d\theta,$$

$$II = \int_0^{2\pi} \int_{\log \rho}^0 \left[\left(\frac{\partial \psi}{\partial \theta} \right)^2 + |\nabla \tau|^2 \cos^2 \varphi \right] dt d\theta,$$

$$III = \int_0^{2\pi} \int_{\log \rho}^0 \left[\left(\frac{\partial \psi}{\partial t} \right)^2 - \psi^2 \cos 2\varphi \right] dt d\theta.$$

Put

$$\bar{\psi}(t) = \frac{1}{2\pi} \int_0^{2\pi} \psi(t, \theta) d\theta.$$

We have

$$\begin{aligned} I &= - \int_0^{2\pi} \int_{\log \rho}^0 2(\psi - \bar{\psi}) \frac{\partial \tau}{\partial \theta} \sin 2\varphi dt d\theta \\ &= -4 \int_0^{2\pi} \int_{\log \rho}^0 [(\psi - \bar{\psi}) \sin \varphi] \left[\frac{\partial \tau}{\partial \theta} \cos \varphi \right] dt d\theta \\ &\geq - \int_0^{2\pi} \int_{\log \rho}^0 \left[4(\psi - \bar{\psi})^2 \sin^2 \varphi + \left(\frac{\partial \tau}{\partial \theta} \right)^2 \cos^2 \varphi \right] dt d\theta. \end{aligned}$$

We now choose α such that if $\alpha < \rho < e^{-\pi}$ then for $\log \rho \leq t \leq 0$,

$$\sin \varphi(t) \leq \frac{1}{2}.$$

We obtain

$$\begin{aligned}
I &\geq - \int_0^{2\pi} \int_{\log \rho}^0 \left[(\psi - \bar{\psi})^2 + \left(\frac{\partial \tau}{\partial \theta} \right)^2 \cos^2 \varphi \right] dt d\theta \\
&\geq - \int_0^{2\pi} \int_{\log \rho}^0 \left[\left(\frac{\partial \psi}{\partial \theta} \right)^2 + \left(\frac{\partial \tau}{\partial \theta} \right)^2 \cos^2 \varphi \right] dt d\theta
\end{aligned}$$

and

$$I + II \geq \int_0^{2\pi} \int_{\log \rho}^0 \left(\frac{\partial \tau}{\partial \theta} \right)^2 \cos^2 \varphi dt d\theta .$$

From the choice of α it follows that $\cos^2 \varphi \geq 3/4$. Hence,

$$\delta^2 E \geq \frac{3}{4} \int_0^{2\pi} \int_{\log \rho}^0 \left(\frac{\partial \tau}{\partial t} \right)^2 dt d\theta + \int_0^{2\pi} \int_{\log \rho}^0 \left[\left(\frac{\partial \psi}{\partial t} \right)^2 - \psi^2 \cos 2\varphi \right] dt d\theta . \quad (\text{III.4})$$

As the second integral in the right hand side of (III.4) is precisely the second variation of the energy E when u is restricted to the set of radially symmetric maps in $\mathcal{E}_0$, our

conclusion follows from (III.4) and from the fact that the second variation of

$$E_0(\varphi) = 2\pi \int_{\log \rho}^0 \left[\left(\frac{d\varphi}{dt} \right)^2 - \sin^2 \varphi \right] dt$$

is positive definite at a minimizer, which is a consequence of the concavity of the function

$\psi \mapsto \sin \psi$ for $\psi \in [0, \pi]$ (e.g., [1992,2]).

Remark 10. The above argument fails when $\rho = e^{-\pi}$, because $\delta^2 E$ is not positive

definite. To see this, note that for $(\tau, \psi) = (0, \sin t)$, $\delta^2 E = 0$.

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