**AN APPROXIMATE BOUNDARY INTEGRAL METHOD
FOR ACOUSTIC SCATTERING IN SHALLOW OCEANS**

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IMA Preprint Series # 923

February 1992

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Abstract

The problem of a time-harmonic acoustic wave scattering from an cylindrical object in shallow oceans is solved by an approximate boundary integral method. In the method we employ a Green's function of the Helmholtz equation with sound soft sea level and sound hard sea bottom conditions, and reformulate the problem into a boundary integral equation on the surface of the scattering object. The kernel of the integral equation is given by a infinite series, and is approximated by an appropriate truncation. The integral equation is then fully discretized by applying a quadrature rule. The method has an $O(N^{-3})$ rate of convergence. Various numerical examples are presented.

1 Introduction

This paper studies an approximate boundary integral method of the scattering problem which describes the scattering of acoustic waves from an cylindrical object with a sound soft boundary in a shallow ocean. This scattering problem is essentially a two dimensional problem, and is modelled as a boundary value problem in a waveguide, where the governing equation is the Helmholtz equation.

Let $\mathbf{R}_d^2 = \{(x_1, x_2); x_1 \in \mathbf{R}, 0 \leq x_2 \leq d\}$ be a region corresponding to the finite depth ocean, where d is the ocean depth. Consider an object Ω imbedded in $\mathbf{R}_d^2$, which is a bounded, simply connected domain with a C^2 boundary $\partial\Omega$. An incoming wave u^i is incident on $\partial\Omega$, and is scattered to produce a propagating wave u as well as its far-field pattern. If the object has a sound soft boundary $\partial\Omega$, this problem can be formulated as a Dirichlet boundary value problem for the scattering of time-harmonic acoustic waves in $\Omega_e := \mathbf{R}_d^2 \setminus \overline{\Omega}$, namely to find a solution $u \in C^2(\Omega_e) \cap C(\overline{\Omega_e})$ for the Helmholtz equation

$$\Delta_2 u + k^2 u = 0, \text{ in } \mathbf{R}_d^2 \setminus \overline{\Omega}, \quad (1.1)$$

such that u satisfies the boundary conditions

$$u = 0, \text{ as } x_2 = 0, \quad (1.2)$$

$$\frac{\partial u}{\partial x_2} = 0, \text{ as } x_2 = d, \quad (1.3)$$

$$u = -u^i, \text{ on } \partial\Omega. \quad (1.4)$$

Here k is a positive constant known as the wave number. We assume in this paper that $k \neq (2n-1)\pi/(2d)$

For $(x_1, x_2) \in \mathbf{R}_d^2$ such that $|x_1|$ large enough, the scattered wave has the modal representation

$$u = \sum_{n=1}^{\infty} \phi_n(x_2) u_n(x_1), \quad (1.5)$$

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This author's research was supported in part by the Institute for Mathematics and its Applications with funds provided by the National Science Foundation, the Minnesota Supercomputer Institute and the Alliant Techsystem Co.

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This author's work was supported in part by the National Science Foundation Grant RII-8610671 and the Commonwealth of Kentucky through the University of Kentucky's Center for Computational Sciences.

where

$$\phi_n(x_2) = \sin[k(1 - a_n^2)^{\frac{1}{2}} x_2] \quad (1.6)$$

with

$$a_n = [1 - \frac{(2n-1)^2 \pi^2}{4k^2 d^2}]^{\frac{1}{2}}, \quad (1.7)$$

and the n^{th} mode of u , $u_n(x_1)$, satisfies the radiation condition

$$\lim_{r \rightarrow \infty} (\frac{\partial u_n}{\partial r} - ika_n u_n) = 0, \quad r = |x_1|, \quad n = 1, 2, \dots, \infty. \quad (1.8)$$

For the radiation problem, the condition (1.4) is replaced by the condition $u = f$ on $\partial\Omega$, where f is a known function defined on $\partial\Omega$. We call the problem (1.1)-(1.4) and (1.8) as problem D.

This problem, or more generally the direct and inverse scattering problems in a shallow ocean, has been discussed in a series papers of Gilbert and Xu (ref. [8],[7],[11],[12]) due to their importance in ocean acoustics. A fine analysis for similar problems in parallel waveguides has also been given in Morgenrother and Werner [10]. A difficulty in solution of the scattering problems in such a waveguide comes from the two parallel unbounded boundaries. The usual boundary integral (BIE) method uses the fundamental solution of (1.1) and reformulates the solution u as a layer potential. The BIE method has the spirit of reducing the problem to a problem in a lower dimensional space, but leads to an integral equation on both the boundary $\partial\Omega$ and the two unbounded boundaries. As result, a complicated Wiener-Hopf integral equation system needs to be solved. To avoid the integral equation on the two unbounded boundaries, instead of using the fundamental solution, we may use the Green's function of the Helmholtz equation in $\mathbf{R}_d^2$

$$G(x, y) = G(x_1, x_2; y_1, y_2) = \sum_{n=1}^{\infty} \frac{i}{\pi k a_n} \phi_n(x_2) \phi_n(y_2) e^{i k a_n |x_1 - y_1|} \quad (1.9)$$

which satisfies the boundary conditions (1.2), (1.3) and the radiation condition (1.8). Let $\nu_x = (\nu_1, \nu_2)$ denote the outward normal vector (toward the interior of Ω_e) at the point $x = (x_1, x_2)$. By Green's formula we have that

$$\int_{\partial\Omega} \{u(y) \frac{\partial G}{\partial \nu_y}(x, y) - \frac{\partial u}{\partial \nu_y}(y) G(x, y)\} d\sigma_y = \begin{cases} 0 & \text{if } x \in \Omega, \\ u(x) & \text{if } x \in \mathbf{R}_d^2 \setminus \bar{\Omega}, \end{cases} \quad (1.10)$$

for any function u satisfying (1.1), (1.2), (1.3) and (1.8).

For problem D, we know that (see [8]) if $x_1 \nu_1 \geq 0$ holds for any $(x_1, x_2) \in \partial\Omega$, then problem D has a unique solution. Moreover, defining a double layer potential

$$u(x) = \int_{\partial\Omega} \frac{\partial G(x, y)}{\partial \nu_y} \psi(y) d\sigma_y, \quad \text{for } x \in \Omega_e, \quad (1.11)$$

we know that the solution of problem D is given by (1.11) if ψ is the solution of the boundary integral equation

$$\psi(x) + 2 \int_{\partial\Omega} \frac{\partial G}{\partial \nu_y}(x, y) \psi(y) d\sigma_y = -2u^i(x), \quad \text{for } x \in \partial\Omega. \quad (1.12)$$

Equation (1.12) has a unique solution when k is not an eigenvalue of the interior Neumann problem in Ω .

Numerical methods for solving the boundary integral equations arising from the Helmholtz equation (1.1) with only the boundary condition (1.4) have been presented and studied by many authors, for example, see [3], [4], [2], [1], [9]. Recently Yan [13] [14] developed a fast method to solve these boundary integral equations. This method is fully discrete, involves no integrals, and has a polynomial order of convergence, which demonstrates a high computational efficiency. However, it

is a challenge to develop an efficient numerical method for solving the boundary integral equation (1.12) where its kernel function is given in an infinite series while a ready method for efficient evaluation of this series does not exist. A good method for evaluation of the infinite series requires an appropriate truncation which preserves a certain accuracy on one hand, and minimizes arithmetic operations on the other. In this paper, we shall focus on a good method for evaluation of the kernel of the boundary integral equation (1.12), and then present a quadrature method for solving this equation. The method is fully discrete, and is estimated to have a $O(N^{-3})$ rate of convergence, where N is the number of the quadrature knots distributed along the boundary. Our numerical experiments show that the method has a good accuracy and involves a fine CPU time.

In the next section, we investigate with a great care the approximation of the kernel in (1.12). Using the approximation results we discretize in section 3 the integral equation (1.12) by a quadrature method. In the quadrature method we manage to get with possibly least arithmetic operations for the kernel approximated within $O(N^{-3})$ error estimates, and consequently to get a $O(N^{-3})$ truncation errors. Some numerical results and convergence discussion are presented in the last section.

2 Approximation of the Integral Kernel in (1.12)

Numerical solution of equation (1.12) involves inevitably the evaluation of the kernel $\frac{\partial G(x,y)}{\partial \nu_y}$. Since $G(x,y)$ is given only as a sum of the infinite series, this evaluation can only be done approximately. In this section we shall provide an approximation to the kernel $\frac{\partial G(x,y)}{\partial \nu_y}$. The evaluation of the kernel is the most expensive part in numerical solution of equation (1.12), so requires a careful treatment in the approximation.

We shall split G into $G = G_0 + M$, where G_0 is the Green's function for the Laplace equation ((1.1) with $k = 0$) and the conditions (1.2)-(1.3). $G_0(x,y)$ is singular at $x = y$, and $M(x,y)$ is continuous, and hence G_0 is the dominant term in the splitting. It should be emphasized that the Green's function (1.9) is given only for wave number $k \neq (2n-1)\pi/(2d)$. Otherwise one of the coefficients in the expression (1.9), $i/(\pi k a_n)$, will become infinite. For simplicity, we assume that the depth $d = \pi$, and use the same notation ϕ_n as in (1.6) for $d = \pi$, i.e.,

$$\phi_n(x_2) = \sin \left[\left(n - \frac{1}{2} \right) x_2 \right].$$

The function G_0 is defined by

$$G_0(x,y) := G_0(x_1, x_2; y_1, y_2) = \sum_{n=1}^{\infty} \frac{1}{\pi(n - \frac{1}{2})} \phi_n(x_2) \phi_n(y_2) e^{-(n - \frac{1}{2})|x_1 - y_1|}, \quad (2.1)$$

and the function M by

$$\begin{aligned} M(x,y) : &= G(x,y) - G_0(x,y) \\ &= \sum_{n=1}^{\infty} \frac{1}{\pi} \phi_n(x_2) \phi_n(y_2) \left(\frac{i}{k a_n} e^{i k a_n |x_1 - y_1|} - \frac{1}{n - \frac{1}{2}} e^{-(n - \frac{1}{2})|x_1 - y_1|} \right), \end{aligned} \quad (2.2)$$

where $a_n = [1 - \frac{(2n-1)^2}{4k^2}]^{\frac{1}{2}}$. $G_0(x,y)$ has a simple expression (see, for example, 1.448.4 in [6])

$$G_0(x,y) = -\frac{1}{4\pi} \{ F(x_1 - y_1, x_2 - y_2) - F(x_1 - y_1, x_2 + y_2) \}, \quad (2.3)$$

where

$$F(s,t) = \log \frac{\cosh \frac{s}{2} - \cos \frac{t}{2}}{\cosh \frac{s}{2} + \cos \frac{t}{2}}.$$

Now we split the kernel into

$$\frac{\partial G}{\partial \nu_y} = \frac{\partial G_0}{\partial \nu_y} + \frac{\partial M}{\partial \nu_y}. \quad (2.4)$$

Because G_0 has an analytic expression (2.3), $\frac{\partial G_0}{\partial \nu_y}$ can be evaluated analytically. A straightforward calculation leads to useful properties

$$\int_{\partial\Omega} \frac{\partial G_0(x, y)}{\partial \nu_y} d\sigma_y = \begin{cases} 0 & \text{if } x \in \Omega_e \\ -1 & \text{if } x \in \Omega \\ -\frac{1}{2} & \text{if } x \in \partial\Omega, \end{cases} \quad (2.5)$$

and, for $(x, y) \in \mathbf{R}_\pi^2 \times \partial\Omega$,

$$\left| \frac{\partial G_0(x, y)}{\partial \nu_y} \right| \leq \frac{1}{8\pi} \frac{|(x - y) \cdot \nu_y|}{\sinh^2 \frac{x_1 - y_1}{2} + \sin^2 \frac{x_2 - y_2}{2}} + C, \quad (2.6)$$

where C is a constant.

Now we focus on the approximation of $\frac{\partial M}{\partial \nu_y}$. Let

$$a_{k,n} = \left(n - \frac{1}{2}\right) + ika_n = \frac{k^2}{n - \frac{1}{2} - ika_n}.$$

Then

$$e^{ika_n|x_1 - y_1|} - e^{-(n - \frac{1}{2})|x_1 - y_1|} = e^{-(n - \frac{1}{2})|x_1 - y_1|} \left(e^{a_{k,n}|x_1 - y_1|} - 1 \right).$$

A direct calculation follows

$$\nabla M := \left(\frac{\partial M}{\partial y_1}, \frac{\partial M}{\partial y_2} \right) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$I_1 = \frac{k^2|x_1 - y_1|}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})} \phi_n(x_2) e^{-(n - \frac{1}{2})|x_1 - y_1|} \left(\operatorname{sgn}(x_1 - y_1) \phi_n(y_2), \frac{\phi'_n(y_2)}{n - \frac{1}{2}} \right),$$

$$I_2 = \frac{|x_1 - y_1|}{2\pi} \sum_{n=1}^{\infty} \frac{a_{k,n}^2}{(n - \frac{1}{2})} \phi_n(x_2) e^{-(n - \frac{1}{2})|x_1 - y_1|} \left(\operatorname{sgn}(x_1 - y_1) \phi_n(y_2), \frac{\phi'_n(y_2)}{n - \frac{1}{2}} \right),$$

$$I_3 = \frac{1}{\pi} \sum_{n=1}^{\infty} \left(e^{a_{k,n}|x_1 - y_1|} - a_{k,n}|x_1 - y_1| - 1 \right) \phi_n(x_2) e^{-(n - \frac{1}{2})|x_1 - y_1|} \left(\operatorname{sgn}(x_1 - y_1) \phi_n(y_2), \frac{\phi'_n(y_2)}{n - \frac{1}{2}} \right),$$

$$I_4 = \frac{k^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^2} \phi_n(x_2) \phi'_n(y_2) e^{-(n - \frac{1}{2})|x_1 - y_1|} (0, 1),$$

$$I_5 = \frac{k^2}{2\pi} \sum_{n=1}^{\infty} \frac{1}{(n - \frac{1}{2})^2} \phi_n(x_2) \phi'_n(y_2) e^{-(n - \frac{1}{2})|x_1 - y_1|} \left(e^{a_{k,n}|x_1 - y_1|} - 1 \right) (0, 1),$$

and

$$I_6 = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{ia_{k,n}[(n - \frac{1}{2})a_{k,n} + k^2]}{(n - \frac{1}{2})^3 ka_n} \phi_n(x_2) \phi'_n(y_2) e^{ika_n|x_1 - y_1|} (0, 1).$$

A good exercise in calculus gives the following lemmas.

Lemma 1

$$I_1 = \frac{k^2}{2} ((x_1 - y_1)G_0(x, y), |x_1 - y_1|G_1(x, y)), \quad (2.7)$$

where $G_0(x, y)$ is given by (2.3), and

$$G_1(x, y) := G_1(x_1, x_2; y_1, y_2) = \frac{1}{2\pi} \left\{ \arctan \frac{\sin \frac{x_2 - y_2}{2}}{\sinh \frac{|x_1 - y_1|}{2}} + \arctan \frac{\sin \frac{x_2 + y_2}{2}}{\sinh \frac{|x_1 - y_1|}{2}} \right\}$$

(see, for example, 1.448.3 in [6]).

Lemma 2

$$I_4 = \frac{k^2}{4\pi} [\phi(x_1 - y_1, x_2 - y_2) + \phi(x_1 - y_1, x_2 + y_2)](0, 1), \quad (2.8)$$

where

$$\phi(s, t) = -\frac{t}{2} F(s, t) + \frac{\cosh \frac{s}{2}}{2} \int_0^t \frac{z \sin \frac{z}{2}}{\sinh^2 \frac{s}{2} + \sin^2 \frac{z}{2}} dz.$$

Lemma 3 Define

$$I_2^p = \frac{|x_1 - y_1|}{2\pi} \sum_{n=1}^{p-1} \frac{a_{k,n}^2}{(n - \frac{1}{2})} \phi_n(x_2) e^{-(n-\frac{1}{2})|x_1 - y_1|} \left(\operatorname{sgn}(x_1 - y_1) \phi_n(y_2), \frac{\phi'_n(y_2)}{n - \frac{1}{2}} \right),$$

and

$$R_2 = I_2 - I_2^p.$$

Then

$$|R_2| \leq \begin{cases} \frac{k^4}{2\pi(p-1)^3} e^{-(p-1)|x_1 - y_1|} & \text{if } |x_1 - y_1| \neq 0, \\ 0 & \text{if } |x_1 - y_1| = 0. \end{cases} \quad (2.9)$$

Lemma 4 Define

$$I_3^p = \frac{1}{\pi} \sum_{n=1}^{p-1} \left(e^{a_{k,n}|x_1 - y_1|} - a_{k,n}|x_1 - y_1| - 1 \right) \phi_n(x_2) e^{-(n-\frac{1}{2})|x_1 - y_1|} \left(\operatorname{sgn}(x_1 - y_1) \phi_n(y_2), \frac{\phi'_n(y_2)}{n - \frac{1}{2}} \right),$$

and

$$R_3 = I_3 - I_3^p.$$

Then

$$|R_3| \leq \frac{k^4}{\pi(p-1)^2} |x_1 - y_1| e^{(\kappa^2/(p-1) - (p-1))|x_1 - y_1|}. \quad (2.10)$$

Lemma 5 Define

$$I_5^p = \frac{k^2}{2\pi} \sum_{n=1}^{p-1} \frac{1}{(n - \frac{1}{2})^2} \phi_n(x_2) \phi'_n(y_2) e^{-(n-\frac{1}{2})|x_1 - y_1|} \left(e^{a_{k,n}|x_1 - y_1|} - 1 \right) (0, 1),$$

and

$$R_5 = I_5 - I_5^p.$$

Then

$$|R_5| \leq \begin{cases} \frac{k^4}{\pi(p-1)^3} e^{(\kappa^2/(p-1) - (p-1))|x_1 - y_1|} & \text{if } |x_1 - y_1| \neq 0, \\ 0 & \text{if } |x_1 - y_1| = 0. \end{cases} \quad (2.11)$$

Lemma 6 Define

$$I_6^p = \frac{1}{2\pi} \sum_{n=1}^{p-1} \frac{ia_{k,n}[(n - \frac{1}{2})a_{k,n} + k^2]}{(n - \frac{1}{2})^3 ka_n} \phi_n(x_2) \phi'_n(y_2) e^{ika_n|x_1 - y_1|} (0, 1),$$

and

$$R_6 = I_6 - I_6^p.$$

Then, for $p > \sqrt{3}\kappa + 1$,

$$|R_6| \leq \begin{cases} \frac{2k^4}{\pi(p-1)^4} \frac{e^{(\kappa^2/(p-1)-(p-1))|x_1-y_1|}}{|x_1-y_1|} & \text{if } |x_1 - y_1| \neq 0, \\ \frac{k^4}{\pi(p-1)^3} & \text{if } |x_1 - y_1| = 0. \end{cases} \quad (2.12)$$

Using these lemmas, we approximate $\frac{\partial M}{\partial \nu_y}$ in the way that ∇M is approximated by $I_1 + I_2^p + I_3^p + I_4 + I_5^p + I_6^p$, where I_1 is evaluated analytically, and I_4 is evaluated by Gaussain-Legendre quadrature rule. The choice of p relies on the size of $|x_1 - y_1|$, which will be discussed in detail in next section.

3 Numerical Solution of BIE (1.12)

By use of the splitting (2.4), euqation (1.12) can be written in the form

$$\psi(x) + 2 \int_{\partial\Omega} \frac{\partial G_0}{\partial \nu_y}(x, y) \psi(y) d\sigma_y + 2 \int_{\partial\Omega} \frac{\partial M}{\partial \nu_y}(x, y) \psi(y) d\sigma_y = -2u^i(x), \text{ for } x \in \partial\Omega. \quad (3.1)$$

In this section we discretize this equation by a quadrature method, and replace the kernel $\frac{\partial M}{\partial \nu_y}$ by a good approximation.

We assume that the boundary $\partial\Omega$ is given by a 2π -periodic parametric representation

$$\gamma(s) = (\gamma_1(s), \gamma_2(s)), \quad s \in \mathbf{R},$$

with $|\gamma'(s)| \neq 0$ for all s . Furthermore, we assume that γ is a C^∞ function. Denote the kernel of the integral equation (3.1) by

$$K_0(x, y) = 2 \frac{\partial}{\partial \nu_y} G_0(x, y), \quad K_1(x, y) = 2 \frac{\partial}{\partial \nu_y} M(x, y). \quad (3.2)$$

Set

$$\begin{aligned} w(s) &= \psi(\gamma(s)), \quad g(s) = -2u^i(\gamma(s)), \\ L_0(s, \sigma) &= K_0(\gamma(s), \gamma(\sigma)) |\gamma'(\sigma)|, \quad L_1(s, \sigma) = K_1(\gamma(s), \gamma(\sigma)) |\gamma'(\sigma)|. \end{aligned}$$

Thus equation (3.1) reduces to

$$w(s) + \int_{-\pi}^{\pi} w(\sigma) L_0(s, \sigma) d\sigma + \int_{-\pi}^{\pi} w(\sigma) L_1(s, \sigma) d\sigma = g(s), \quad s \in [-\pi, \pi]. \quad (3.3)$$

Recall from [5] that for a C^2 boundary $\partial\Omega$, there is a constant $C > 0$ such that

$$|(\nu_y, x - y)| \leq C|x - y|^2, \quad x, y \in \partial\Omega.$$

Therefore, $L_0(s, \sigma)$ is continuous for $(s, \sigma) \in [-\pi, \pi] \times [-\pi, \pi]$ by (2.5). The continuity of $L_1(s, \sigma)$ is obvious since, from Lemma 1 - Lemma 6, we know that $I_j, j = 1, \dots, 6$ are all uniformly convergent serieses. Moreover, letting

$$J(x, y) = I_2 + I_3 + I_5 + I_6,$$

it can be showed that $\frac{\partial^j J}{\partial y_2^j}$, for $j=1,2,3$, and $\frac{\partial J}{\partial y_1}$ are continuous functions of x and y , and that $\frac{\partial^j J}{\partial y_1^j}$, $j=2,3$, are continous except at points where $y_1 = x_1$. In addition, for $x = \gamma(s)$ and $y = \gamma(\sigma)$, the function $(I_1 + I_4)(x, y)$ can be splitted as

$$\begin{aligned} I_1 + I_4 &= -\frac{k^2}{4\pi}(x_1 - y_1, x_2 - y_2) \log |2 \sin \frac{s - \sigma}{2}| \\ &\quad - (0, 1) \frac{k^2}{4\pi}(x_1 - y_1) \left(\arctan \cot \frac{s + \sigma}{2} + \operatorname{sgn}(s^2 - \sigma^2) \frac{\pi}{2} \right) \\ &\quad + I_{1,4}(s, \sigma), \end{aligned}$$

where $I_{1,4}(s, \sigma)$ is a smooth function of (s, σ) . Then

$$L_1(s, \sigma) = -a(s, \sigma) \log |2 \sin \frac{s - \sigma}{2}| + b(s, \sigma) \left(\arctan \cot \frac{s + \sigma}{2} + \operatorname{sgn}(s^2 - \sigma^2) \frac{\pi}{2} \right) + L_2(s, \sigma),$$

where

$$a(s, \sigma) = \frac{k^2}{2\pi}(x_1 - y_1, x_2 - y_2) \cdot \nu_y |\gamma'(\sigma)|,$$

$$b(s, \sigma) = -\frac{k^2}{2\pi}(0, x_1 - y_1) \cdot \nu_y |\gamma'(\sigma)|$$

and

$$L_2(s, \sigma) = 2(I_{1,4}(s, \sigma) + J(\gamma(s), \gamma(\sigma))) \cdot \nu_y |\gamma'(\sigma)|.$$

For this reason, we shall use the ordinary rectangular formula

$$\int_{-\pi}^{\pi} v(\sigma) d\sigma \approx h \sum_{k=-N/2+1}^{N/2} v(t_k), \quad (3.4)$$

the weighted quadrature formula

$$-\int_{-\pi}^{\pi} v(\sigma) \log |2 \sin \frac{s - \sigma}{2}| d\sigma \approx h \sum_{k=-N/2+1}^{N/2} R^1(s - t_k) v(t_k), \quad (3.5)$$

and the weighted quadrature formula

$$\int_{-\pi}^{\pi} v(\sigma) \left(\arctan \cot \frac{s + \sigma}{2} + \operatorname{sgn}(s^2 - \sigma^2) \frac{\pi}{2} \right) d\sigma \approx h \sum_{k=-N/2+1}^{N/2} R^2(s, t_k) v(t_k), \quad (3.6)$$

where $t_k = kh$ with $h = \frac{2\pi}{N}$ and N an even integer are the equidistant quadrature knots and the weights are given by

$$R^1(s) = \sum_{l=1}^{N/2-1} \frac{1}{l} \cos ls + \frac{1}{N} e^{i\frac{N}{2}s}$$

and

$$R^2(s, t_k) = \sum_{\substack{l=-N/2+1 \\ l \neq 0}}^{N/2} \left(\frac{\sin l|s|}{l} + \frac{ie^{-ils}}{2l} \right) e^{-ilt_k} + |s| - \frac{\pi}{2}.$$

Applying the quadrature formula (3.4), (3.5) and (3.6) to the integrals in (3.3), we replace the integral equation (3.3) by the linear system

$$w_j + h \sum_{k=-N/2+1}^{N/2} (R_1(t_{j-k})a(t_j, t_k) + R_2(t_j, t_k)b(t_j, t_k) + L_0(t_j, t_k) + L_2(t_j, t_k))w_k = g_j, \quad (3.7)$$

$$j = -\frac{N}{2} + 1, \dots, \frac{N}{2},$$

for the approximate values w_j to $w(t_j)$, where $g_j = g(t_j)$. This linear system has an $O(h^3)$ truncation, and so can produce an $O(h^3)$ rate of convergence for w_j to $w(t_j)$.

The linear system (3.7) involve the calculation of functions a , b , L_0 and L_2 at the points (t_j, t_k) , and the calculation of the weights. The evaluations of $a(t_j, t_k)$ and $b(t_j, t_k)$ are direct. $L_0(t_j, t_k)$ can be evaluated using the explicit formula (2.3). $R_1(t_{j-k})$ and $R_2(t_j, t_k)$ can be evaluated using FFT with only $O(N \log N)$ arithmetic operations. A difficulty and expensive part is the evaluation of $L_2(t_j, t_k)$, because $J(x, y)$ is an infinite series function which has to be truncated properly.

The following theorem provides an $O(\frac{1}{N^3})$ approximation to L_2 .

Theorem 1 Let

$$J^p(x, y) = \begin{cases} I_2^p + I_3^p + I_5^p + I_6^p & \text{if } |x_1 - y_1| \neq 0, \\ I_6^p & \text{if } |x_1 - y_1| = 0, \end{cases} \quad (3.8)$$

and

$$L_2^*(s, \sigma) = 2(I_{1,4}(s, \sigma) + J^p(\gamma(s), \gamma(\sigma))) \cdot \nu(\gamma(\sigma))|\gamma'(\sigma)|, \quad (3.9)$$

where I_2^p, I_3^p, I_5^p and I_6^p are defined as in last section, and p is chosen as

$$p = \begin{cases} N + 1 & \text{if } |x_1 - y_1| = 0 \text{ or } \left| |x_1 - y_1| - \frac{1}{N} \right| \leq \frac{e^{-1}}{N}, \\ 2 + \left\lceil \kappa + \frac{3}{|x_1 - y_1|} |\log(N|x_1 - y_1|)| \right\rceil & \text{otherwise.} \end{cases} \quad (3.10)$$

Then for any $s, \sigma \in [-\pi, \pi]$, there is a constant C independent of N , s and σ such that

$$|(L_2 - L_2^*)(s, \sigma)| \leq CN^{-3}. \quad (3.11)$$

Proof: It is clear that we only need to prove

$$|(J - J^p)(x, y)| \leq CN^{-3}, \quad \text{for } x, y \in \partial\Omega. \quad (3.12)$$

For $|x_1 - y_1| \geq (1 + e^{-1})/N$, we have from (2.9), (2.10), (2.11) and (2.12) that

$$\begin{aligned} |J - J^p| &\leq |I_2 - I_2^p| + |I_3 - I_3^p| + |I_5 - I_5^p| + |I_6 - I_6^p| \\ &\leq \left(\frac{k^4}{54\pi \log^3(N|x_1 - y_1|)} + \frac{k^4}{9\pi \log^2(N|x_1 - y_1|)} \right. \\ &\quad \left. + \frac{k^4}{27\pi \log^3(N|x_1 - y_1|)} + \frac{2k^4}{81\pi \log^4(N|x_1 - y_1|)} \right) \frac{1}{N^3} \\ &\leq C \frac{1}{N^3}. \end{aligned}$$

The proof for other cases follows similarly. These then complete the proof of the lemma. $\square$

The choice (3.10) of p is made in order to save arithmetic operations when the truncation J^p is used to approximate J . Employing the approximation $L_2^*(s, \sigma)$ of $L_2(s, \sigma)$ into (3.7), we arrive at a linear system

$$w_j + h \sum_{k=-N/2+1}^{N/2} (R_1(t_{j-k})a(t_j, t_k) + R_2(t_j, t_k)b(t_j, t_k) + L_0(t_j, t_k) + L_2^*(t_j, t_k))w_k = g_j, \quad (3.13)$$

$$j = -\frac{N}{2} + 1, \dots, \frac{N}{2},$$

for the approximation values w_j to $w(t_j)$. From Theorem 1, the system (3.13) has a $O(\frac{1}{N^3})$ rate of convergence.

4 Numerical Examples

We consider an object Ω centered at $(0, z_0)$. in the finite depth ocean with $d = \pi$. The boundary of Ω is given by

$$\partial\Omega = \left\{ (x_1, x_2) : \frac{x_1^2}{\beta_1^2} + \frac{(x_1 - z_0)^2}{\beta_2^2} = 1 \right\}.$$

The incident wave $u^i(x)$ is given for test by

$$u^i(x) = -G(x_1, x_2; 0, z_0),$$

so the solution of the boundary value problem is

$$u(x) = G(x_1, x_2; 0, z_0).$$

We approximate the scattering wave represented as the double layer potential (1.11) by

$$u_h(x) = \frac{h}{2} \sum_{j=-N/2+1}^{N/2} w_j (K_0(x, \gamma(t_j)) + K_1^*(x, \gamma(t_j))) |\gamma'(t_j)| \quad (4.1)$$

for x away from the boundary, where $K_1^*(x, y)$ is an approximation to $K_1(x, y)$, given by

$$K_1^*(x, y) = 2(I_1 + I_2^p + I_3^p + I_4 + I_5^p + I_6^p) \cdot \nu(y).$$

For x near the boundary, (4.1) is replaced by

$$u_h(x) = -u^i(x), \quad x \in \partial\Omega \quad (4.2)$$

and

$$u_h(x) = \frac{h}{2} \sum_{j=-N/2+1}^{N/2} \{w_j (K_0(x, \gamma(t_j)) + K_1^*(x, \gamma(t_j))) - w_r K_0(x, \gamma(t_j))\}, \quad (4.3)$$

$$\text{for } 1 < \frac{x_1^2}{\beta_1^2} + \frac{(x_2 - z_0)^2}{\beta_2^2} \leq 1.1,$$

where r is chosen such that

$$|x - \gamma(t_r)| = \min_j |x - \gamma(t_j)|.$$

We calculated the error $e_h(x) = u(x) - u_h(x)$, where $u(x)$ is approximately evaluated by

$$u(x) = G_0(x_1, x_2; 0, z_0) + M(x_1, x_2; 0, z_0)$$

with M truncated at its 30th term in (2.2). The boundary $\partial\Omega$ is subdivided uniformly with respect to the parameter σ by N quadrature knots, where N is chosen respectively as $N = 2^l$, $l = 2, \dots, 8$. The CPU time with respect to the values of N and the points are reported. All calculations were performed in double precision on the University of Kentucky's IBM 3090-600J using vectorization.

In Tables 1.1, 1.2 and 1.3, we choose $z_0 = \pi/2$ and $(\beta_1, \beta_2) = (1, 1)$, and report the results at the points $x = (1.01, \pi/2)$, $(2, \pi/2)$, and $(4, \pi/2)$ for wave number $\kappa = 2, 5, 10$, respectively. The potential points $x = (1.01, \pi/2)$, $(2, \pi/2)$, $(4, \pi/2)$ are chosen so that the distance between x and Ω are 0.01, 1, 3, respectively.

Table 1.1. $z_0 = \pi/2$, $(\beta_1, \beta_2) = (1, 1)$, $\kappa = 2$.

N	$ e_h(1.01, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
4	2.26E -2	0.03	1.11E -1	0.03	1.57E -1	0.02
8	2.58E -1	0.09	6.90E -4	0.08	2.01E -3	0.07
16	7.32E -2	0.44	4.34E -4	0.39	8.37E -4	0.38
32	1.41E -2	2.45	1.50E -4	2.26	3.27E -4	2.24
64	1.31E -3	14.88	3.20E -5	14.37	6.65E -5	14.32
128	2.26E -5	98.20	2.45E -6	96.86	2.03E -6	96.68
256	1.18E -6	691.59	6.97E -7	687.46	6.97E -7	687.83

Table 1.2. $z_0 = \pi/2$, $(\beta_1, \beta_2) = (1, 1)$, $\kappa = 5$.

N	$ e_h(1.01, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
16	1.69E -1	0.45	4.61E -3	0.40	3.92E -3	0.39
32	9.43E -3	2.47	6.50E -4	2.29	1.51E -3	2.27
64	5.52E -4	14.95	3.27E -4	14.58	4.27E -4	14.49
128	5.19E -5	98.58	7.60E -5	98.08	4.06E -5	97.67
256	1.85E -5	694.05	4.60E -5	694.77	4.43E -5	693.19

Table 1.3. $z_0 = \pi/2$, $(\beta_1, \beta_2) = (1, 1)$, $\kappa = 10$.

N	$ e_h(1.01, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
64	1.68E -1	15.19	1.53E -2	14.67	1.02E -2	14.64
128	7.02E -2	99.58	4.51E -3	98.38	3.39E -3	99.51
256	2.20E -3	696.83	9.22E -4	694.84	5.34E -4	694.99

In Tables 2.1, 2.2 and 2.3, we choose $z_0 = 1$ and $(\beta_1, \beta_2) = (0.5, 0.5)$, and report the results at the points $x = (0.51, 1)$, $(2, 1)$, and $(4, 1)$ for wave number $\kappa = 2, 5, 10$, respectively. The potential points $x = (0.51, 1)$, $(2, 1)$, $(4, 1)$ are chosen so that the distance between x and Ω are 0.01, 1.5, 3.5, respectively.

Table 2.1. $z_0 = 1$, $(\beta_1, \beta_2) = (0.5, 0.5)$, $\kappa = 2$.

N	$ e_h(0.51, 1) $	CPU(sec)	$ e_h(2, 1) $	CPU(sec)	$ e_h(4, 1) $	CPU(sec)
4	2.43E -2	0.03	1.89E -2	0.03	1.74E -2	0.02
8	5.21E -2	0.10	1.07E -3	0.08	1.07E -3	0.08
16	6.19E -3	0.50	4.31E -4	0.42	3.86E -4	0.41
32	8.04E -4	2.81	1.30E -4	2.53	1.17E -4	2.53
64	6.70E -5	17.29	2.52E -5	16.48	2.24E -5	16.54
128	1.86E -6	113.24	9.01E -7	110.86	7.72E -7	111.27
256	1.40E -6	778.61	6.85E -7	774.12	9.25E -7	775.04

Table 2.2. $z_0 = 1, (\beta_1, \beta_2) = (0.5, 0.5), \kappa = 5$.

N	$ e_h(0.51, 1) $	CPU(sec)	$ e_h(2, 1) $	CPU(sec)	$ e_h(4, 1) $	CPU(sec)
16	2.90E -2	0.51	2.96E -3	0.43	1.71E -3	0.42
32	3.51E -3	2.84	8.71E -4	2.56	4.97E -4	2.55
64	1.39E -4	17.41	1.72E -4	16.64	8.10E -5	16.59
128	1.83E -5	113.51	1.24E -5	111.46	2.00E -5	111.05
256	2.11E -6	781.31	2.45E -5	776.88	2.38E -5	776.52

Table 2.3. $z_0 = 1, (\beta_1, \beta_2) = (0.5, 0.5), \kappa = 10$.

N	$ e_h(0.51, 1) $	CPU(sec)	$ e_h(2, 1) $	CPU(sec)	$ e_h(4, 1) $	CPU(sec)
16	6.16E -2	0.52	1.09E -3	0.44	6.59E -3	0.44
32	4.09E -3	2.88	4.15E -4	2.61	1.94E -3	2.60
64	3.66E -4	17.59	3.32E -5	16.82	4.95E -4	16.81
128	3.68E -5	114.12	9.09E -5	112.09	1.56E -4	112.39
256	1.22E -5	785.44	1.67E -5	778.49	6.78E -5	778.92

In Tables 3.1, 3.2 and 3.3, we choose $z_0 = \pi/2$ and $(\beta_1, \beta_2) = (0.25, 1)$, and report the results at the points $x = (0.27, \pi/2)$, $(2, \pi/2)$, and $(4, \pi/2)$ for wave number $\kappa = 2, 5, 10$, respectively. The potential points $x = (0.27, \pi/2)$, $(2, \pi/2)$, $(4, \pi/2)$ are chosen so that the distance between x and Ω are 0.02, 1.75, 3.75, respectively.

Table 3.1. $z_0 = \pi/2, (\beta_1, \beta_2) = (0.25, 1), \kappa = 2$.

N	$ e_h(0.27, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
16	6.96E -1	0.58	6.18E -4	0.48	6.58E -4	0.47
32	1.10E -1	3.43	3.20E -3	3.10	2.65E -3	3.10
64	1.10E -2	21.76	7.11E -5	20.79	8.46E -5	20.88
128	7.18E -4	142.65	7.08E -5	139.46	8.44E -5	140.08
256	3.63E -5	960.53	7.08E -5	950.90	8.44E -5	953.76

Table 3.2. $z_0 = \pi/2, (\beta_1, \beta_2) = (0.25, 1), \kappa = 5$.

N	$ e_h(0.27, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
16	4.43E -1	0.59	1.73E -3	0.49	1.77E -3	0.48
32	5.01E -2	3.46	1.17E -3	3.14	1.39E -3	3.10
64	6.47E -3	21.83	4.32E -4	20.98	3.33E -4	20.76
128	3.96E -4	142.38	4.39E -4	140.27	3.35E -4	139.81
256	6.17E -5	961.18	4.39E -4	956.35	3.35E -4	952.31

Table 3.3. $z_0 = \pi/2, (\beta_1, \beta_2) = (0.25, 1), \kappa = 10$.

N	$ e_h(0.27, \pi/2) $	CPU(sec)	$ e_h(2, \pi/2) $	CPU(sec)	$ e_h(4, \pi/2) $	CPU(sec)
32	5.61E -1	3.52	1.00E -2	3.20	4.67E -3	3.17
64	6.20E -2	21.99	6.09E -3	21.09	3.83E -3	21.06
128	2.76E -3	143.65	5.52E -3	141.16	3.43E -3	140.94
256	1.65E -3	966.38	5.42E -3	958.63	3.37E -3	955.96

Based on our observation of these numerical results, the quadrature method introduced in Section 3 works well for the wave number at least within the arrange $k \leq 10$. The method is convergent, and has a good accuracy, though the accuracy is affected when the wavenumber k becomes large. The involved CPU time is reasonable and affordable. The majority of the CPU time is spent on the matrix generation of equation (3.13). It is possible to improve the approximation method in the matrix generation, and then to reduce the CUP time. This, however, requires more rigorous research.

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