**WEAK AND CLASSICAL SOLUTIONS OF SOME NONLINEAR
VOLTERRA INTEGRODIFFERENTIAL EQUATIONS**

By

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**Weak and Classical Solutions of Some Nonlinear
Volterra Integrodifferential Equations ¹**

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Abstract

This paper deals with the following Volterra integrodifferential equation

$$\begin{aligned} & u_t - \operatorname{div} \vec{A}(x, t, u, u_x) - a(x, t, u, u_x) \\ &= \int_0^t [\operatorname{div} \vec{B}(x, t, \tau, u, u_x) + b(x, t, \tau, u, u_x)] d\tau. \end{aligned}$$

We first investigate weak solvability. Under certain structure conditions, the global existence of a weak solution is established by means of the Galerkin method. We then study the classical solvability for the above type of equation with a linear principal part. Using the energy method and exploiting the classical maximum principle, we derive some *a priori* estimates and prove the existence of a classical solution globally. Finally, a qualitative property of solutions is presented.

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1. Introduction

In the study of heat conduction in materials with memory, one often encounters the following Volterra integrodifferential equation of parabolic type (VIDEP) (cf. Nohel [13] and Nunziato [14]):

$$u_t = \nabla[a\nabla u(x, t)] + \int_0^t [\nabla b(x, t, \tau)\nabla u(x, \tau)]d\tau. \quad (1.1)$$

There are also some other physical and engineering problems which can be described by such an equation. As an example, one considers the motion of a viscoelastic material whose stress depends on the history of the motion, the equation can be written in the following form: (cf. Dafermos [4], Greenberg [7], e.g.)

$$u_{tt} = cu_{xxt} + \sigma(u_x)_x, \quad (1.2)$$

or the more general equation in higher space dimension (linear version):

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_i} [b_{ij}(x, t) \frac{\partial u}{\partial x_j} + a(x, t)u] - \frac{\partial^2}{\partial t \partial x_i} [a_{ij} \frac{\partial u}{\partial x_j}] = f(x, t). \quad (1.3)$$

The equation (1.3) can also be used to model the propagation of sound in a viscous medium (cf. Suveika [16]), where the term with the third order derivative serves as a small viscosity. Obviously, it is the same type of equation as (1.1), if one performs the integration over $[0, t]$ on the equation (1.2) or (1.3).

The above physical models motivate us to investigate general parabolic Volterra integrodifferential equations. These equations have been studied for several decades (see the recent proceeding [3]). A common approach is to set the equation in an abstract form and then the theory of analytic semigroups is applied to establish existence. In [2], Crandall, Londen and Nohel considered the abstract equation

$$u'(t) + Bu(t) + \int_0^t a(t-s)Au(s)ds \ni F(t),$$

where A and B are maximal monotone operators. Under certain assumptions on A and B , a weak solution was obtained. As the application of their theory, the equations

$$u_t - \Delta u + \int_0^t a(t-s)g(u(x, s))ds = F(t, x)$$

and

$$u_t - u_{xx} - \int_0^t a(t-s)(\sigma(u_x(x, s)))_x ds = F(t, x)$$

were discussed (also see [11]). Heard and Rankin [8]-[9] considered the following problem

$$u'(t) + Au(t) = \int_0^t \frac{\partial}{\partial x_i} g_i(x, t, s, u, u_x) ds,$$

where A is a linear elliptic operator and the function g satisfies a uniform Lipschitz condition. With these assumptions, a global weak solution was established via the semigroup theory. Asymptotic behavior of solutions was studied in [9] (also see Fitzgibbon [6]). Other work in this category can be found in [3] and the references therein. The other popular way is to regard the equation (1.2) as a mixed type of equation and to establish existence via the energy method (cf. [1], [4], [7], [16], etc.). However, these previous arguments are hardly be applied to a VIDEP with a nonlinear principal part. Moreover, regularity of the weak solution becomes a challenging question. Recently, the author of [17]-[18] considered the equation:

$$u_t = a(x, t, u, u_x)u_{xx} + \int_0^t b(x, t, \tau, u, u_x, u_{xx})d\tau.$$

With the conditions $a \geq a_0$ and

$$|b(x, t, u, p, r)| + |b_t(x, t, \tau, u, p, r)| \leq C[1 + |u| + |p| + |r|]$$

a classical solution was established globally via the theory of parabolic equations. Moreover, in [19] we derived the classical Schauder and L_p estimates for a linear VIDEP, which are similar to those for parabolic equations. As an application, we studied the equation

$$u_t = \frac{\partial}{\partial x_i} [a_i(x, t, u, \int_0^t u d\tau)u_{x_i}] + \int_0^t \frac{\partial}{\partial x_i} b_i(x, t, u) d\tau.$$

The global solution in the classical sense was obtained.

The present work is a continuation of the recent paper [19]. In this paper we shall deal with the following equation

$$\begin{aligned} & u_t - \frac{\partial}{\partial x_i} a_i(x, t, u, u_x) - a(x, t, u, u_x) \\ &= \int_0^t \left[\frac{\partial}{\partial x_i} b_i(x, \tau, u, u_x) + b(x, t, \tau, u, u_x) \right] d\tau \end{aligned} \tag{1.4}$$

subject to appropriate initial and boundary conditions. We shall define weak solutions and prove the existence of a weak solution by applying the Galerkin method. Then we will study classical solvability for the equation with a linear principal term:

$$u_t - \Delta u = \int_0^t [b(t - \tau) \Delta u] d\tau + f(u). \quad (1.5)$$

It is well known that for a parabolic equation

$$u_t = \Delta u + f(u),$$

the one-sided growth condition

$$f(u)u \leq C[u^2 + 1] \quad (1.6)$$

implies the global existence of a classical solution. This is, however, not so obvious if an integral term is involved in the equation (1.5). By exploiting the maximum principle, we are able to obtain a similar result as that for a parabolic equation. Moreover, using the energy method, we derive some *a priori* estimates for a semilinear equation with lower order terms and establish the classical solvability. Finally, we present a certain type of maximum principle which we call the quasi-maximum principle.

2. Weak Solutions

Let $T > 0$ and $Q_T = \Omega \times (0, T]$, where Ω is an open bounded region in R^n with a smooth boundary $S = \partial\Omega$, and $S_T = S \times [0, T]$. In this section we study weak solutions of the following problem:

$$\begin{aligned} u_t - \frac{\partial}{\partial x_i} a_i(x, t, u, u_x) - a(x, t, u, u_x) \\ = \int_0^t \left[\frac{\partial}{\partial x_i} b_i(x, t, \tau, u, u_x) + b(x, t, \tau, u, u_x) \right] d\tau, \quad (x, t) \in Q_T, \end{aligned} \quad (2.1)$$

$$u(x, t) = g(x, t), \quad (x, t) \in S_T, \quad (2.2)$$

$$u(x, 0) = u_0(x), \quad x \in \Omega. \quad (2.3)$$

Without loss of generality, we may assume that $g(x, t) = 0$. We list some structure conditions for the equation.

H(1): The functions a_i, a and b_i, b are differentiable with respect to all of their arguments in $Q_T \times R \times R^n$ and $Q_T \times R^+ \times R \times R^n$, respectively.

H(2): The following ellipticity assumption and the growth conditions hold:

$$\begin{aligned} \sum_{i=1}^n [a_i(x, t, u, p_1) - a_i(x, t, u, p_2)](p_1 - p_2) &\geq a_0(p_1 - p_2)^2; \\ \sum_{i=1}^n [|a_i(x, t, u, p)| + |b_i(x, t, \tau, u, p)|] + |a| + |b| &\leq A_0[1 + |u| + |p|]; \\ \sum_{i=1}^n |b_{ip}(x, t, u, p)| &\leq B_0, \end{aligned}$$

where a_0, A_0 and B_0 are positive constants.

H(3): $u_0(x) \in H_0^1(\Omega)$.

Remark 1: One could relax the growth condition a little as in [10]. However, it is not our attention to state the most general theorem.

Definition: A function $u(x, t)$ in $V_2(Q_T) = L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ is said to be a weak solution of (2.1)-(2.3), if $u(x, t)$ satisfies

$$\begin{aligned} \int_0^T \int_\Omega [-u\phi_t + (a_i + \int_0^t b_i d\tau)\phi_{x_i} - (a + \int_0^t b d\tau)\phi] dx dt \\ = \int_\Omega u_0(x)\phi(x, 0)dx - \int_\Omega u(x, T)\phi(x, T)dx \end{aligned} \quad (2.4)$$

for any $\phi(x, t) \in H^1(0, T; H_0^1(\Omega))$.

We shall use the Galerkin method to show the existence of a weak solution. Let $\psi_k(x)$ be the normal fundamental basis of the space $H_0^1(\Omega)$. For any $f(x), g(x)$ in $L^2(\Omega)$, we denote

$$\langle f(x), g(x) \rangle = \int_\Omega f(x)g(x)dx.$$

Let

$$u_m(x, t) = \sum_{k=1}^m c_{km}(t)\psi_k(x),$$

where $\{c_{km}(t)\}$ is the solution of the system:

$$\frac{d}{dt}c_{km}(t) + \sum_{i=1}^n \langle a_i(x, t, u_m, u_{mx}) + \int_0^t b_i(x, t, \tau, u_m, u_{mx})d\tau, \psi_{kx_i} \rangle$$

$$- < a(x, t, u_m, u_{mx}) + \int_0^t b(x, t, \tau, u_m, u_{mx}) d\tau, \psi_k > = 0; \quad (2.5)$$

$$c_{km}(0) = < u_0(x), \psi_k >, \quad k = 1, 2, \dots, m. \quad (2.6)$$

We have the following

Lemma 2.1: The system (2.5)-(2.6) admits a unique solution on $[0, T]$ and the solution satisfies the following uniform estimate:

$$\sum_{k=1}^m |c_{km}(t)| \leq C,$$

where C depends only on the known data and the upper bound of T .

Proof: Local existence is clear. To show global existence, we only need to derive an a priori bound of $\{c_{km}(t)\}$. To this end, we multiply (2.5) by $c_{km}(t)$ and take the summation on k . Noting that $c_{km}(t) = < u_m, \psi_k >$, we have

$$\begin{aligned} & \frac{d}{dt} \|u_m(\cdot, t)\|_{L^2(\Omega)}^2 + \sum_{i=1}^n < a_i(x, t, u_m, u_{mx}) + \int_0^t b_i(x, t, \tau, u_m, u_{mx}) d\tau, u_{mx_i} > \\ & - < a(x, t, u_m, u_{mx}) + \int_0^t b(x, t, \tau, u_m, u_{mx}) d\tau, u_m > = 0. \end{aligned}$$

Using the growth assumptions on a_i, b_i, a, b and Cauchy's inequality, we obtain

$$\begin{aligned} & \frac{d}{dt} \|u_m(\cdot, t)\|_{L^2(\Omega)}^2 + a_0 \int_{\Omega} u_{mx}^2 dx \\ & \leq C + \varepsilon \int_{\Omega} u_{mx}^2 dx + C(\varepsilon) \int_{\Omega} u_m^2 dx + C \int_0^t \int_{\Omega} [u_m^2 + u_{mx}^2] dx dt. \end{aligned}$$

By choosing ε properly and using Gronwall's inequality, we have

$$\|u_m\|_{V^2(Q_T)} \leq C.$$

Thus, the desired result follows from the identity

$$\max_{0 \leq t \leq T} \sum_{k=1}^m c_{km}^2(t) = \max_{0 \leq t \leq T} \|u_m\|_{L^2(\Omega)}^2 \leq \|u_m\|_{V^2(Q_T)}^2.$$

By the compactnes results (cf. [15]), we see that (a subsequence if necessary)

$$u_m(x, t) \rightarrow u(x, t) \text{ in } L^2(Q_T) \text{ strongly, a.e. in } Q_T, \text{ and } u_{mx_i} \rightharpoonup u_{x_i}(x, t) \text{ weakly in } L_2(Q_T).$$

Moreover, the weak compactness of $L^2(Q_T)$ implies

$a_i(x, t, u_m, u_{mx}), b_i(x, t, \tau, u_m, u_{mx}), a(x, t, u_m, u_{mx}), b(x, t, \tau, u_m, u_{mx})$

are also convergent weakly in $L^2(Q_T)$ to certain functions

$A_i(x, t), B_i(x, t, \tau), A(x, t), B(x, t, \tau)$, respectively, $i = 1, \dots, n$.

Let $\phi(x, t) \in H^1(0, T; H_0^1(\Omega))$, then

$$\phi(x, t) = \sum_{j=1}^{\infty} \beta_j(t) \psi_j(x).$$

We multiply (2.5) by $\beta_j(t)$ and then sum over j . After integrating over $[0, T]$ and taking limits, we have

$$\begin{aligned} & \int_0^T \int_{\Omega} [-u\phi_t + (A_i(x, t) + \int_0^t B_i d\tau) \phi_{x_i} - (A(x, t) + \int_0^t B d\tau) \phi] dx dt \\ &= \int_{\Omega} u_0(x) \phi(x, 0) dx - \int_{\Omega} u(x, T) \phi(x, T) dx. \end{aligned} \quad (2.7)$$

To show that $u(x, t)$ is a weak solution, we need to show

$$\begin{aligned} & \int_0^T \int_{\Omega} [A_i(x, t) + \int_0^t B_i d\tau] \phi_{x_i} dx dt = \int_0^T \int_{\Omega} [a_i(x, t, u, u_x) + \int_0^t b_i d\tau] \phi_{x_i} dx dt; \\ & \int_0^T \int_{\Omega} [A(x, t) + \int_0^t B d\tau] \phi dx dt = \int_0^T \int_{\Omega} [a + \int_0^t b d\tau] \phi dx dt. \end{aligned}$$

Lemma 2.2: For any $u(x, t), v(x, t)$ in $V_2(Q_T)$, the following inequality holds for $T \leq T_0$:

$$\begin{aligned} & \int_0^T \int_{\Omega} [a_i(x, t, u, u_x) - a_i(x, t, u, v_x)] (u_{x_i} - v_{x_i}) dx dt \\ &+ \int_0^T \int_{\Omega} \left\{ \int_0^t [b_i(x, t, \tau, u, u_x) - b_i(x, t, \tau, u, v_x)] d\tau \right\} (u_{x_i} - v_{x_i}) dx dt \geq 0, \end{aligned}$$

where T_0 depends only on a_0 and B_0 .

Proof: The proof is straightforward by employing the assumption H(2).

From now on, we always restrict $T \leq T_0$. We multiply the equation (2.5) by $\beta_j(t)$, take the summation on j and then integrate with respect to t over $[0, T]$ to obtain

$$\begin{aligned} & \int_0^T \int_{\Omega} \left\{ -u_m \phi_t + [a(x, t, u_m, u_{mx}) + \int_0^t b_i(x, t, \tau, u_m, u_{mx}) d\tau] \phi_{x_i} \right. \\ & \left. - [a(x, t, u_m, u_{mx}) + \int_0^t b(x, t, \tau, u_m, u_{mx}) d\tau] \phi \right\} dx dt \\ &+ \int_{\Omega} [u_m(x, T) \phi(x, T) - u_0(x) \phi(x, 0)] dx = 0. \end{aligned} \quad (2.8)$$

By choosing $\phi(x, t)$ properly and using Lemma 2.2 (cf. [10]), we have for any $\eta \in H^1(0, T; H_0^1(\Omega))$ that

$$\int_0^T \int_{\Omega} \{[A_i(x, t) - a_i(x, t, u, \eta_x)] + \int_0^t [B_i(x, t, \tau) - b_i(x, t, \tau, u, \eta_x)] d\tau\} (u - \eta)_x dx dt \geq 0.$$

It follows by taking $\eta(x, t) = u(x, t) - \delta \xi(x, t)$ for any $\xi(x, t) \in H^1(0, T; H_0^1(\Omega))$ (see Remark 2 below) and then letting $\delta \rightarrow 0$ that

$$\int_0^T \int_{\Omega} \{[A_i(x, t) - a_i(x, t, u, u_x)] + \int_0^t [B_i(x, t, \tau) - b_i(x, t, \tau, u, u_x)] d\tau\} \xi_x dx dt \geq 0.$$

This implies

$$\begin{aligned} & \int_0^T \int_{\Omega} [A_i(x, t) + \int_0^t B_i(x, t, \tau) d\tau] \xi_x dx dt \\ &= \int_0^T \int_{\Omega} [a_i(x, t, u, u_x) + \int_0^t b_i(x, t, \tau, u, u_x) d\tau] \xi_x dx dt. \end{aligned}$$

Remark 2: In the above derivation, although $u(x, t)$ does not belong to $H^1(0, T; H_0^1(\Omega))$, one can use the well-known Steklov averging $u_h(x, t)$ (cf. [10]) to approximate $u(x, t)$ and then take the limit. With the above equality and the strong convexity condition H(2), one can actually show that $u_{mx_i}(x, t)$ converges to u_{x_i} in $L^2(Q_T)$ strongly. Indeed, if we choose $\phi(x, t)$ in (2.8) to be $u_m(x, t)$ and $u(x, t)$, respectively, noting that $u_m(x, t)$ converges $u(x, t)$ in $L^2(Q_T)$ strongly, we see

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} [a(x, t, u_m, u_{mx}) + \int_0^t b_i(x, t, \tau, u_m, u_{mx}) d\tau] [u_{mx_i} - u_{x_i}] dx dt = 0. \quad (2.9)$$

On the other hand, the weak convergence implies

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} [a(x, t, u, u_x) + \int_0^t b_i(x, t, \tau, u, u_x) d\tau] [u_{mx_i} - u_{x_i}] dx dt = 0. \quad (2.10)$$

It follows by (2.9) and (2.10) that

$$\lim_{m \rightarrow \infty} \int_0^T \int_{\Omega} [I_i + J_i] [u_{mx_i} - u_{x_i}] dx dt = 0,$$

where

$$\begin{aligned} I_i &= a_i(x, t, u_m, u_{mx}) - a_i(x, t, u, u_x); \\ J_i &= \int_0^t [b_i(x, t, \tau, u_m, u_{mx}) - b_i(x, t, \tau, u, u_x)] d\tau. \end{aligned}$$

We split I_i into two terms:

$$\begin{aligned} I_i &= [a(x, t, u_m, u_{mx}) - a_i(x, t, u_m, u_x)] + [a_i(x, t, u_m, u_x) - a_i(x, t, u, u_x)] \\ &\equiv I_i^* + I_i^{**}. \end{aligned}$$

The strong convergence of $u_m(x, t)$ implies

$$\int_0^T \int_{\Omega} I_i^{**} [u_{mx_i} - u_{x_i}] dx dt \rightarrow 0$$

as m tends to ∞ . The assumption H(2) yields

$$\int_0^T \int_{\Omega} I_i^* [u_{mx_i} - u_{x_i}] dx dt \geq a_0 \int_0^T \int_{\Omega} (u_{mx_i} - u_{x_i})^2 dx dt.$$

Similarly, we split J_i into two terms $J_i = J_i^* + J_i^{**}$ and have

$$\int_0^T \int_{\Omega} J_i^{**} [u_{mx_i} - u_{x_i}] dx dt \rightarrow 0$$

as m tends to ∞ .

By the Cauchy-Schwartz inequality and H(2), we see

$$\begin{aligned} &\int_0^T \int_{\Omega} |J_i^*| |u_{mx_i} - u_{x_i}| dx dt \\ &\leq \delta \int_0^T \int_{\Omega} (u_{mx_i} - u_{x_i})^2 dx dt + C(\delta)T \int_0^T \int_{\Omega} [u_{mx_i} - u_{x_i}]^2 dx dt. \end{aligned}$$

It follows that

$$[a_0 - \delta - C(\delta)T] \int_0^T \int_{\Omega} (u_{mx_i} - u_{x_i})^2 dx dt \leq \int_0^T \int_{\Omega} [I_i^{**} + J_i^{**}] [u_{mx_i} - u_{x_i}] dx dt \rightarrow 0$$

as m approaches ∞ . By choosing $\delta = \frac{a_0}{4}$ and restricting $T \leq T_0 = \frac{a_0}{4C(\delta)}$ we see that $u_{mx_i}(x, t)$ converges to u_{x_i} in $L^2(Q_T)$ strongly. Consequently, the second equality follows immediately.

Theorem 2.1: Under the hypotheses H(1)-H(3), the problem (2.1)-(2.3) admits a unique weak solution.

Proof: The uniqueness can be proved by the standard energy method. We have shown the existence of a weak solution on the interval $[0, T_0]$. However, since T_0 depends only on a_0 and B_0 we can repeat the above procedure and obtain a solution on $[T_0, 2T_0]$. After a finite number of steps, we can obtain a weak solution on the whole interval $[0, T]$ for any $T > 0$.

3. Classical Solutions

In this section we consider the following initial-boundary value problem:

$$u_t - \Delta u - f(u) = \int_0^t [b(t, \tau) \Delta u] d\tau, \quad (x, t) \in Q_T, \quad (3.1)$$

$$u(x, t) = 0, \quad (x, t) \in S_T, \quad u(x, 0) = u_0(x), \quad x \in \Omega. \quad (3.2)$$

We shall show the global existence of a classical solution to (3.1)-(3.2) under the assumption of a one-sided growth condition which is a little stronger than (1.6). The argument is to exploit the classical maximum principle.

$H^*(1)$: The function $b(t, \tau)$ is differentiable on R^2 ;

$H^*(2)$: $f(u)$ is differentiable and there exist positive constants A_0 and M such that

$$f'(u) \leq A_0, \quad \text{for } |u| \geq M;$$

$H^*(3)$: $u_0(x) \in C^{2+\alpha}(\bar{\Omega})$ and the following compatibility conditions hold:

$$u_0(x) = 0, \quad \Delta u_0(x) + f(u_0) = 0 \quad \text{on } S.$$

Theorem 2: Under the conditions $H^*(1) - H^*(3)$, the problem (3.1)-(3.2) admits a unique classical solution on $[0, T]$ for any $T > 0$.

The proof of local existence is standard. To obtain the global existence, we only need to derive an *a priori* estimate in $C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{Q}_T)$. Throughout this section various constants will be denoted by the same symbol C , it may be different from one line to the next. Their dependency will be specified in the text.

Lemma 3.1 There exists a constant C which depends on u_0, Ω and the upper bound of T such that

$$\|u\|_{L^\infty(Q_T)} \leq C. \quad (3.3)$$

Proof: To prove the result, we introduce a new function:

$$w(x, t) = u(x, t) + \int_0^t b(t, \tau) u(x, \tau) d\tau, \quad (x, t) \in \bar{Q}_T.$$

Then using the Neumann series, we see that there exists a small number $T_0 > 0$ such that for $t \in [0, T_0]$

$$u(x, t) = w(x, t) + \int_0^t B(t, \tau) w(x, \tau) d\tau, \quad (3.4)$$

where $B(t, \tau)$ is a smooth function and T_0 depends only on the function $b(t, \tau)$.

It is easy to see that $w(x, t)$ satisfies

$$w_t - \Delta w + B(t, t)w - f(u) = \int_0^t [-B_t(t, \tau)w]d\tau, \quad (x, t) \in Q_{T_0}. \quad (3.5)$$

Moreover, $w(x, t) = 0$ on S_{T_0} and $w(x, 0) = u_0(x)$ for $x \in \Omega$.

Let $w(x, t) = e^{\beta t}V(x, t)$, the elementary calculation gives

$$V_t - \Delta V = [-B(t, t) - \beta]V - \int_0^t B_t(t, \tau)e^{-\beta(t-\tau)}Vd\tau + f(u)e^{-\beta t}.$$

Now

$$\begin{aligned} f(u)e^{-\beta t} &= \left[\int_0^1 f'(\theta u)d\theta u + f(0) \right] e^{-\beta t} \\ &= \int_0^1 f'(\theta u)d\theta [V(x, t) + \int_0^t B(t, \tau)e^{-\beta(t-\tau)}Vd\tau] + f(0)e^{-\beta t}. \end{aligned}$$

Now we require that T_0 is so small such that $\int_0^t B(t, \tau)d\tau \leq \frac{1}{2}$ for $t \in [0, T_0]$. Assume that $V(x, t)$ attains its positive maximum which is greater than M in Q_{T_0} . Then at this extreme point, say (x_0, t_0) , $V(x_0, t_0) = M^*$ and

$$V_t - \Delta V \geq 0.$$

Moreover,

$$V(x_0, t_0) + \int_0^{t_0} B(t_0, \tau)e^{-\beta(t_0-\tau)}V(x_0, \tau)d\tau \geq M^* - \frac{1}{2}M^* > 0.$$

The condition $H^*(2)$ implies

$$\begin{aligned} &\int_0^1 f'(\theta u)d\theta [V(x_0, t_0) + \int_0^{t_0} B(t_0, \tau)e^{-\beta(t_0-\tau)}V(x_0, \tau)d\tau] \\ &\leq A_0[V(x_0, t_0) + \int_0^{t_0} B(t_0, \tau)e^{-\beta(t_0-\tau)}V(x_0, \tau)d\tau] \\ &\leq A_0[M^* + B_0T_0M^*], \end{aligned}$$

where

$$B_0 = \max_{[0, T] \times [0, T]} |B(t, \tau)|.$$

It follows by combining the above inequalities that at (x_0, t_0)

$$[\beta + B(t_0, t_0) - A_0 - B_0T_0]M^* \leq \max\{M, f(0)e^{-\beta t_0}\}.$$

Consequently, we have

$$M^* \leq 2\max\{M, f(0)e^{-\beta T}\},$$

if we choose $\beta + B(t, t) - A_0 - B_0 T_0 \geq \frac{1}{2}$.

This yields

$$\max_{Q_T} V(x, t) \leq \max\{\max_{\Omega} |u_0(x)|, 2M, 2f(0)e^{-\beta T}\}.$$

Similarly, we can obtain the lower bound. Finally, since T_0 depends only on $b(t, \tau)$, we can repeat the above procedure to obtain the above estimate on $[T_0, 2T_0]$. After a finite number of steps, we can obtain the estimate (3.6) on $[0, T]$ for any fixed $T > 0$. Thus, the desired estimate in Lemma 3.1 follows from the relation (3.4).

Lemma 3.2 : There exists a constant C with the same dependency as in Lemma 3.1 such that

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)} \leq C.$$

Proof: To show estimates, we employ the $W_p^{2,1}(Q_T)$ -estimate (cf. [19]) for any $p > 1$ to yield

$$\|u\|_{W_p^{2,1}(Q_T)} \leq C.$$

The interpolation inequality implies

$$\|u\|_{C^{1+\alpha, \frac{1+\alpha}{2}}(Q_T)} \leq C.$$

Finally, the Schauder estimate (cf. [19]) infers

$$\|u\|_{C^{2+\alpha, 1+\frac{\alpha}{2}}(Q_T)} \leq C,$$

where the constant C depends only on the known data, Ω and the upper bound of T .

With the above a priori estimate at hand, we can use the method of continuation to establish the existence of a classical solution on the whole interval $[0, T]$ for any given $T > 0$ (cf. [19]). Hence we obtain Theorem 2.

The proof of Lemma 3.1 is based on the classical maximum principle. Here we present a proof for a particular nonlinear function, namely,

$$f(u) = u - u^{2m+1},$$

where m is a positive integer. The method relies on various energy estimates instead of using the maximum principle, although $H^*(2)$ holds for this $f(u)$. This argument allows us to deal with a more general class of semilinear integrodifferential equations.

Lemma 3.3: For the above particular $f(u)$, the estimate (3.3) holds.

Proof: It is easy to see that

$$\int_{\Omega} u^2 dx + \int_0^t \int_{\Omega} u_x^2 dx dt \leq C.$$

As in the proof of Lemma 3.2, let

$$w(x, t) = u(x, t) + \int_0^t b(t, \tau) u(x, \tau) d\tau.$$

Then $w(x, t)$ satisfies

$$w_t - \Delta w + B(t, t)w = \int_0^t [-B_t(t, \tau)] w d\tau + f(u).$$

We calculate the following integral for $p \geq 2$ even:

$$\begin{aligned} \int_0^T \frac{d}{dt} \int_{\Omega} w^p dx dt &= \int_0^T \int_{\Omega} p w^{p-1} w_t dx dt \\ &= \int_0^T \int_{\Omega} p w^{p-1} \{ [\Delta w - B(t, t)w + f(u)] + \int_0^t [-B_t(t, \tau)w] d\tau \} dx dt \end{aligned}$$

After executing the integration by parts, we have

$$\begin{aligned} \int_{\Omega} w(x, T)^p dx + p(p-1) \int_0^T \int_{\Omega} w^{p-2} w_x^2 dx dt \\ \leq \int_{\Omega} u_0(x)^p dx + I, \end{aligned}$$

where

$$I = \int_0^T \int_{\Omega} p w^{p-1} [-B(t, t)w + f(u) + \int_0^t [-B_t(t, \tau)w] d\tau] dx dt.$$

Now we split I into three terms denoted by I_1, I_2 and I_3 , respectively. By using the Hölder and Young inequalities, we see

$$\begin{aligned} |I_1| + |I_3| &\leq Cp \int_0^T \int_{\Omega} w^p dx dt. \\ I_2 &= \int_0^T \int_{\Omega} p w^{p-1} [u - u^{2m+1}] dx dt \equiv J_1 + J_2. \end{aligned}$$

It is clear that

$$\begin{aligned}
|J_1| &\leq C \int_0^T \int_{\Omega} p w^p dx dt. \\
J_2 &= - \int_0^T \int_{\Omega} p w^{p-1} [w + \int_0^t B(t, \tau) w d\tau]^{2m+1} dx dt \\
&= - \int_0^T \int_{\Omega} p w^{p-1} \{ w^{2m+1} + \sum_{i=0}^{2m} \binom{2m+1}{i} w^i [\int_0^t B(t, \tau) w d\tau]^{2m+1-i} \} dx dt \\
&= - \int_0^T \int_{\Omega} p w^{p+2m} dx dt - \int_0^T \int_{\Omega} p w^{p-1} \sum_{i=0}^{2m} \binom{2m+1}{i} w^i [\int_0^t B(t, \tau) w d\tau]^{2m+1-i} dx dt \\
&= K_1 + K_2.
\end{aligned}$$

We use the Hölder and Young inequalities to obtain

$$|K_2| \leq \delta \int_0^T \int_{\Omega} p w^{p+2m} dx dt + C(\delta) T \int_0^T \int_{\Omega} p w^{p+2m} dx dt.$$

However, both terms on the right-hand side of the above inequality can be dominated by K_1 , by restricting δ and $T \leq T_0$ to be small enough. Therefore, in a small interval $[0, T_0]$, we have

$$\begin{aligned}
&\int_{\Omega} w(x, T)^p dx + p(p-1) \int_0^T \int_{\Omega} w^{p-2} w_x^2 dx dt \\
&\leq \int_{\Omega} u_0(x)^p dx + Cp \int_0^T \int_{\Omega} w^p dx dt.
\end{aligned}$$

Using the iteration technique (cf. [19]), we have

$$\|w\|_{L^\infty(Q_T)} \leq C.$$

As T_0 depends only on the known data, we can repeat the process to achieve the desired estimate in the whole interval $[0, T]$ for any given $T > 0$. Thus, we have

Theorem 3: Assume that $f(u) = u - u^{2m+1}$ and the other conditions in Theorem 2, then the problem (3.1)-(3.2) admits a unique classical solution globally.

Remark 3: The energy method can be used to deal with the following more general equation:

$$u_t - \Delta u + a(x, t, u, u_x) = \int_0^t [B(t, \tau) \Delta u d + \frac{\partial}{\partial x_i} b_i(x, t, u) + c(x, t, \tau, u)] \tau + f(u),$$

where the functions $a(x, t, u, p)$, $b_i(x, t, \tau, u)$ and $c(x, t, \tau, u)$ satisfy the joint linear growth condition

$$|a| + |b| + |c| \leq C[1 + |u| + |p|]$$

and $f(u)$ is the same as in Theorem 3. Then we have the global existence for the above equation. We omit to state the result here (cf.[19]).

Remark 4: It would be very interesting to use the energy method, instead of the maximum principle, to derive the maximum norm of solutions for general $f(u)$ with one-sided condition (1.6).

4. Quasi-maximum Principle

In this section we prove a theorem which provides an a priori bound of solutions of a semilinear equation. More specifically, we consider the following equation

$$u_t = \Delta u + \int_0^t b(t, \tau) \Delta u d\tau + c(x, t)u, (x, t) \in Q_T \quad (4.1)$$

subject to the initial and boundary conditions:

$$u(x, t) = g(x, t), \quad (x, t) \in S_T, \quad (4.2)$$

$$u(x, 0) = u_0(x), \quad (x, t) \in \Omega. \quad (4.2)$$

It is easy to see the classical maximum principle can not be directly applied in general. However, we have seen that a modified principle still holds.

Theorem 4.1: (Quasi-maximum Principle) Assume that $c(x, t) \leq C_0$, then there exists a constant $K > 0$ which depends only on C_0 and $b(t, \tau)$ such that

$$e^{KT} \min\{\min_{S_T} G(x, t), \min_{\Omega} u_0(x)\} \leq w(x, t) \leq e^{KT} \max\{\max_{S_T} G(x, t), \max_{\Omega} u_0(x)\},$$

where $w(x, t) = u(x, t) + \int_0^t b(t, \tau) u d\tau$ and $G(x, t) = g(x, t) + \int_0^t b(t, \tau) g(x, \tau) d\tau$.

Proof: As in Section 3, $w(x, t)$ satisfies

$$w_t = \Delta w + [c(x, t) - B(t, t)]w + \int_0^t [c(x, t)B(t, \tau) - B_t(t, \tau)]w d\tau$$

Let $V(x, t) = e^{-Kt}w$, where K is a large positive constant. For the equation of V , we are able to use the classical maximum principle to obtain the desired result.

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