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ON MULTIPARTITE TOURNAMENT MATRICES WITH CONSTANT TEAM SIZE*

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Abstract

We consider spectral properties of the class, $\mathcal{T}_{d,\ell}$, of $(0,1)$-matrices M which satisfy

$$M + M^T = (J_d - I_d) \otimes J_\ell,$$

where J_k denotes the all ones matrix, and I_k the identity matrix, of order k. These matrices generalize tournament matrices. This paper establishes some of the basic properties for the eigenvalues of matrices in $\mathcal{T}_{d,\ell}$.

1. Introduction. A *tournament* of order n is a directed graph, T, obtained by orienting each edge of the complete graph K_n. The name derives from considering a round-robin tournament with n players $1, \ldots, n$ in which no tie games are allowed. If the edge between i and j is oriented from i to j whenever i beats j, then the resulting digraph is a tournament. Tournaments have been extensively studied and the books by Moon [1968] and Beineke and Reid [1978] provide excellent surveys of the subject.

Associated with a tournament T with vertices $1, \ldots, n$ is the (0,1)-matrix $:M = [m_{ij}]$ of order n where $m_{ij} = 1$ if there is an arc from i to j in T and $m_{ij} = 0$ otherwise. M is the *adjacency matrix* of T. The matrix M satisfies

$$M + M^T = J_n - I_n, \tag{1}$$

where J_n and I_n denote the all 1's matrix and the identity matrix of order n, respectively. Clearly, there is a one-to-one correspondence between labelled tournaments with n vertices and tournament matrices of order n.

The tournament matrices are an interesting class of nonnegative matrices. One of the basic concepts in the study of nonnegative matrices is the notion of irreducibility. Recall that a square nonnegative matrix, A, of order $n \geq 2$ is *reducible* if there exists a permutation matrix P such that

$$P^T A P = \left[\begin{array}{c|c} A_1 & O \\ \hline X & A_2 \end{array}\right],$$

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where A_1 and A_2 are square (nonvacuous) matrices. If no such P exists, then the matrix A is *irreducible*. If for some positive integer m each entry of A^m is positive, then A is *primitive*. Thus a primitive matrix is necessarily irreducible. The Perron–Frobenius Theorem (see Minc [1988]) implies that the spectral radius, ρ, of a primitive matrix is an algebraically simple eigenvalue and that the eigenspace corresponding to the *Perron-value* ρ is spanned by an eigenvector, the *Perron-vector*, each of whose entries is positive. Thompson [1958] has shown that a tournament matrix of order $n \geq 4$ is primitive if and only if it is irreducible.

Consider the results of a round-robin tournament with $n \geq 2$ players and the corresponding tournament matrix M. Then M is reducible if and only if the players can be partitioned into two nonempty sets S_1 and S_2 such that whenever a player $i \in S_1$ and a player $j \in S_2$ competed, player i beat player j. Any ranking of the players should rank a player in S_1 as better than a player in S_2. Hence in studying methods for ranking the players it is often assumed that the tournament has an irreducible adjacency matrix. Suppose M is an irreducible tournament matrix of order $n \geq 4$. A ranking method due to Kendall [1962] and Wei [1952] proposes that the strength of player i be proportional to the i^{th} entry in a Perron-vector $v = (v_1, \ldots, v_n)^T$ of M. The quantity

$$\mathrm{var}(v) = \sum_{i<j} (v_i - v_j)^2$$

is a measure of how evenly this scheme ranks the players. Namely smaller values of $\mathrm{var}(v)$ correspond to more evenly ranked tournaments. It follows directly from (1) that if M has Perron-value ρ, then

$$\frac{2\rho}{n-1} + \frac{\mathrm{var}(v)}{(n-1)v^*v} = 1 \ . \tag{2}$$

Thus ρ also provides a measure of how evenly the Kendall–Wei scheme ranks players, since large values of ρ necessitate small values of $\mathrm{var}(v)$ and hence more evenly ranked players. We are led quite naturally to the study of eigenvalues of tournament matrices. Spectral properties of tournament matrices have been examined in a number of papers, including Brauer and Gentry [1968], deCaen, Gregory, Kirkland, Maybee and Pullman [1991], Kirkland [1991] and Shader[1991].

Let $p_1, \ldots, p_d$ be positive integers. A directed graph obtained by orienting each edge of the complete d-partite graph $K_{p_1,\ldots,p_d}$ is a *d-partite tournament*. Evidently a d-partite tournament with $p_1 = \cdots = p_d = 1$ is just an (ordinary) tournament, while the 2-partite tournaments are also known as *bipartite tournaments*. The d-partite tournaments have been the object of some study, including a recent paper of Goddard, Kubicki, Gellerman and Tian [1991] which looks at their cycle structure.

In this paper we focus on the case where $p_1 = p_2 = \cdots = p_d = \ell$ for some positive integer ℓ. A digraph which arises from orienting the edges of the complete d-partite graph $K_{\ell,\ldots,\ell}$ is a *d-partite tournament with team size ℓ*. The name derives from considering a competition between d teams of ℓ players each, in which each pair of players from different teams compete in exactly one game. Associated with a d-partite tournament with team size ℓ is a (0,1)-matrix $M = [m_{ij}]$ of order $d\ell$ where $m_{ij} = 1$ if there is an arc from i to j and $m_{ij} = 0$ otherwise. Without loss of generality we may assume that the k^{th} team consists of the players in the set

$$t_k = \{(k-1)\ell + 1, \ldots, k\ell\} \qquad (1 \leq k \leq d).$$

Thus M satisfies

$$M + M^T = (J_d - I_d) \otimes J_\ell \tag{3}$$

where $\otimes$ denotes the Kronecker product of matrices. A $(0,1)$-matrix which satisfies (3) is a *d-partite tournament matrix with team size ℓ.*

Suppose M is a primitive d-partite tournament matrix with team size ℓ and let ρ and v be its Perron-value and Perron-vector, respectively. As in the Kendall–Wei ranking scheme for tournaments, one could propose that the strength of player i be given by the i^{th} entry of v. In Section 2 we provide a relation between ρ and $\text{var}(v)$ analogous to (2). Once again we are led inexorably towards the study of eigenvalues. This paper establishes some of the basic spectral properties of d-partite tournament matrices with team size ℓ.

Let d, ℓ, and n be positive integers with $n = d\ell$. We denote the set of all d-partite tournament matrices with team size ℓ by $\mathcal{T}_{d,\ell}$. Suppose $M \in \mathcal{T}_{d,\ell}$. In Section 2 we show that with two types of exceptions M is primitive whenever M is irreducible. Further an eigenvalue λ of M is shown to satisfy

$$-\ell/2 \leq \text{Re}(\lambda) \leq \frac{n-\ell}{2}\,.$$

Section 3 discusses the geometric multiplicity of an eigenvalue λ of M. We show that if $\text{Re}(\lambda) = -\ell/2$ then the geometric and algebraic multiplicities of λ coincide, and that if $\text{Re}(\lambda) \neq -1/2$ then the geometric multiplicity of λ is at most $n - d + 1$. In Section 4 the *regular* matrices in $\mathcal{T}_{d,\ell}$, that is those whose row sums are equal, are studied. We show that such a matrix is irreducible and has rank at least d. In addition the normal matrices in $\mathcal{T}_{d,\ell}$ are characterized as the regular matrices which have $\frac{n-\ell}{2}$ as an eigenvalue, $d-1$ eigenvalues with real part $-\ell/2$ and $n-d$ purely imaginary eigenvalues. In Section 5 we conclude by proposing directions for future research.

2. Spectral bounds. Throughout the remainder of this paper we let d, ℓ and n be positive integers with $d \geq 2$ and $n = d\ell$. If $v = [v_1, \ldots, v_n]^T$ then for $1 \leq i \leq d$ we let

$$v^{(i)} = [v_{(i-1)\ell+1}, \ldots, v_{i\ell}]^T.$$

The i^{th} *team* consists of the players in $t_i = \{(i-1)\ell + 1, \ldots, i\ell\}$. Finally the $k \times 1$ column vector of all 1's is denoted by $\mathbb{1}_k$.

Suppose $M \in \mathcal{T}_{d,\ell}$ is a primitive matrix with Perron–value ρ and Perron–vector $v = [v_1, \ldots, v_n]^T$. As previously noted the entries of v provide a measure of the strengths of the players. With this point of view, it is natural to define the strength, w_i of the i^{th} team to be the sum of the strengths of its players. Thus $w_i = \mathbb{1}_\ell^T v^{(i)}$. Let $w = (w_1, \ldots, w_d)^T$ be the *strength vector of the teams.* The quantities

$$\operatorname{var}(v) = \sum_{1 \le i < j \le n} (v_i - v_j)^2$$

and

$$\operatorname{var}(w) = \sum_{1 \le i < j \le d} (w_i - w_j)^2$$

measure how evenly the players and the teams are ranked, respectively. Pre- and post-multiplying the equation

$$M + M^T = J_n - (I_d \otimes J_\ell) \tag{4}$$

by v^* and v, respectively, gives

$$2\rho v^* v = v^* J_n v - v^*(I_d \otimes J_\ell) v.$$

Since

$$\begin{aligned} v^* J_n v &= d w^* w - \operatorname{var}(w), \\ v^*(I_d \otimes J_\ell) v &= w^* w, \end{aligned}$$

and

$$w^* w = \ell v^* v - \sum_{i=1}^{d} \operatorname{var}(v^{(i)}) \ ,$$

we have

$$2\left(\frac{n-\ell}{2} - \rho\right) v^* v = \operatorname{var}(w) + (d-1) \sum_{i=1}^{d} \operatorname{var}(v^{(i)}). \tag{5}$$

Thus $\dfrac{n-\ell}{2} - \rho$ indicates how evenly the teams and the players on the same team are ranked. This analogue of the Kendall–Wei scheme piques our interest in the spectral properties of matrices in $\mathcal{T}_{d,\ell}$. We observe that (5) implies that the spectral radius of M is bounded above by $\dfrac{n-\ell}{2}$ and that equality holds if and only if $\operatorname{var}(w) = 0$ and $\operatorname{var}(v^{(i)}) = 0$ for each $1 \le i \le d$. It follows that $\rho \le \dfrac{n-\ell}{2}$ with equality if and only if M is a regular matrix.

We now establish the analogue of Thompson's [1958] result on irreducible tournament matrices.

THEOREM 1. *Assume $M \in \mathfrak{T}_{d,\ell}$ is an irreducible matrix. Then exactly one of the following holds:*

(i) $d = 2$,

(ii) $d = 3$, *and M or M^T equals*

$$\begin{bmatrix} O & J_\ell & O \\ O & O & J_\ell \\ J_\ell & O & O \end{bmatrix},$$

(iii) *M is primitive.*

Proof. If X and Y are subsets of $\{1, \ldots, n\}$ we let $M[X, Y]$ denote the submatrix of M whose rows are indexed by the elements of X and columns by the elements of Y. We write $M[X, Y] = J$ if each entry of $M[X, Y]$ is a 1.

Assume $d > 2$ and M is not primitive. Then there exists an integer $k \geq 2$ and a partition $S_1 \cup \cdots \cup S_k$ of $\{1, \ldots, n\}$ such that

$$M[S_i, S_j] \neq O \text{ if and only if } (i, j) \in \{(1, 2), \ldots, (k-1, k), (k, 1)\}.$$

Hence for each i $(1 \leq i \leq k)$ there exists a j $(1 \leq j \leq d)$ with $S_i \subseteq t_j$. Without loss of generality assume that $S_1 \subseteq t_1$. If S_j is not contained in t_1, then

$$M[S_1, S_j] + M[S_j, S_1]^T = J \ ,$$

and hence at least one of $M[S_1, S_j]$ or $M[S_j, S_1]$ is nonzero. Since there exists a unique j such that $M[S_1, S_j] \neq O$ and a unique j' such that $M[S_{j'}, S_1] \neq O$, there are at most two m's such that S_m is not contained in t_1 and hence there are at most 2 teams besides t_1. Thus $d = 3$ and each of t_2 and t_3 consists of a unique S_m. Applying a similar argument to t_2 we conclude t_1 consists of a unique S_m. It now follows that (ii) holds.

□

Next we establish some basic spectral inequalities for matrices in $\mathfrak{T}_{d,\ell}$. Recall that, as in the tournament case, a matrix in $\mathfrak{T}_{d,\ell}$ is regular if its row sums are equal.

THEOREM 2. *Suppose $M \in \mathfrak{T}_{d,\ell}$ and that λ is an eigenvalue of M with corresponding eigenvector v. Then*

(i) $-\dfrac{\ell}{2} \leq \mathrm{Re}(\lambda) \leq \dfrac{n-\ell}{2}$,

(ii) $\mathrm{Re}(\lambda) = \dfrac{n-\ell}{2}$ *if and only if M is regular,*

(iii) $\mathrm{Re}(\lambda) = -\dfrac{\ell}{2}$ *if and only if $v^* J_n v = 0$ and for $1 \leq i \leq d$ each $v^{(i)}$ is a multiple of $\mathbb{1}_\ell$.*

Proof. We have already shown that the Perron–value, ρ, of M satisfies $\rho \leq \dfrac{n-\ell}{2}$. Since $|\lambda| \leq \rho$, $\quad \mathrm{Re}(\lambda) \leq \dfrac{n-\ell}{2}$ with equality holding if and only if $\rho = \lambda$ and M is regular. By pre- and post-multiplying (4) by v^* and v, respectively, and then applying the Cauchy–Schwarz inequality we conclude that

$$\begin{aligned} 2\,\mathrm{Re}(\lambda) v^* v &= v^* J v - \sum_{i=1}^{d} |v^{(i)^*} \mathbb{1}_\ell|^2 \\ &\geq v^* J v - \sum_{i=1}^{d} \ell v^{(i)^*} v^{(i)} \\ &= v^* J v - \ell v^* v \\ &\geq -\ell v^* v \ . \end{aligned}$$

It now follows that $\mathrm{Re}(\lambda) \geq -\ell/2$ and that (iii) holds.

□

COROLLARY 3. *Suppose $M \in \mathcal{T}_{d,\ell}$, λ is an eigenvalue of M and v is a corresponding eigenvector. Assume $\mathrm{Re}(\lambda) = -\ell/2$ and that u is an eigenvector of M which corresponds to a different eigenvalue γ. Then $u^* v = 0$.*

Proof. Pre-and post-multiplying (4) by v^* and u, respectively we have

$$v^* M u + v^* M^T u = v^* J_n u - \sum_{i=1}^{d} v^{(i)^*} J_\ell u^{(i)} \ .$$

Simplifying and using Theorem 2 we obtain

$$\begin{aligned} (\gamma + \overline{\lambda}) v^* u &= -\sum_{i=1}^{d} v^{(i)} J_\ell u^{(i)} \\ &= -\ell v^* u \ . \end{aligned}$$

Thus $(\gamma + \overline{\lambda} + \ell) v^* u = 0$ and we conclude that $v^* u = 0$.

□

By considering the trace, it is easy to see that a regular tournament matrix of order n has $n-1$ eigenvalues with real part $-1/2$. However if $\ell \geq 2$ and $M \in \mathcal{T}_{d,\ell}$ is regular then the trace argument only guarantees that M has an eigenvalue with real part at least $-\dfrac{(n-\ell)}{2(n-1)}$. However, there are examples of matrices in $\mathcal{T}_{d,\ell}$, which give simultaneous equality in the bounds of Theorem 2. Let R be a regular tournament matrix of order d. Then $M = R \otimes J_\ell$ is a regular d-partite tournament matrix of order n. It is easily verified that M has a simple eigenvalue $\frac{n-\ell}{2}$, $d-1$ eigenvalues with real part $-\ell/2$ and 0 as an eigenvalue with algebraic multiplicity $n-d$. There are other constructions that yield simultaneous equality in Theorem 2. Consider the matrix

$$M = S \otimes (J_\ell - I_\ell) + S^T \otimes I_\ell,$$

where

$$S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

Then $M \in \mathcal{T}_{d,\ell}$. Since $S \otimes (J_n - I_\ell)$ and $S^T \otimes I_\ell$ are commuting normal matrices, they are simultaneously diagonalizable. It can now be easily verified that the eigenvalues of M are $\frac{n-\ell}{2} = \ell$, $-\frac{\ell \pm i\sqrt{3}\,(\ell-2)}{2}$, 0 with multiplicity $\ell-1$ and $\pm i\sqrt{3}$ both with multiplicity $\ell-1$.

3. Multiplicity of Eigenvalues. In this section we discuss the geometric multiplicity, and to a lesser extent the algebraic multiplicity, of d-partite tournament matrices with team size ℓ. We begin with a lower bound on the number of distinct eigenvalues of an irreducible matrix in $\mathcal{T}_{d,\ell}$. We are indebted to D. Gregory for a simplification of the following proof.

THEOREM 4. *Suppose $\ell \geq 2$ and that M is an irreducible matrix $\mathcal{T}_{d,\ell}$. Then M has at least 4 distinct eigenvalues.*

Proof. Since M is a nonnegative, irreducible matrix and $\operatorname{tr}(M^2) = 0$, the matrix M has a real positive eigenvalue ρ and a complex conjugate pair of nonreal eigenvalues $\alpha \pm \beta i$. Suppose that these are the only eigenvalues of M.

First assume M is primitive. Then ρ has algebraic multiplicity 1 and $\alpha \pm \beta i$ each have algebraic multiplicity $\frac{n-1}{2}$. Since ρ and $\alpha + \beta i$ are roots of

$$\det(\lambda I - M) = (\lambda - \rho)(\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)^{(n-1)/2},$$

ρ and 2α are algebraic integers. Since a minimal polynomial over the rationals has distinct roots, the minimal polynomial of $\alpha+\beta i$ over the rationals is either $(\lambda-\rho)(\lambda^2-2\alpha\lambda+\alpha^2+\beta^2)$ or $(\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)$. Thus either $2\alpha + \rho$ or 2α is rational. Because

$$0 = \operatorname{tr}(M) = \rho + \left(\frac{n-1}{2}\right)(2\alpha)$$

and $n > 3$, both ρ and 2α are rational. We conclude that ρ and 2α are integers and that $\left(\frac{n-1}{2}\right)$ divides ρ. In particular $\rho \geq \frac{n-1}{2}$, a contradiction.

Now assume M is imprimitive. By Theorem 1 either $d = 2$, or one of M and M^T equals

$$\begin{bmatrix} O & J_\ell & O \\ O & O & J_\ell \\ J_\ell & O & O \end{bmatrix}.$$

In the latter cases M has 4 distinct eigenvalues. In the former case, the Perron–Frobenius Theorem implies that $-\rho$ is an eigenvalue of M, and hence ρ, $-\rho$ and $\alpha \pm \beta$ are 4 distinct eigenvalues of M.

□

An immediate consequence is an upper bound on the algebraic multiplicity of an eigenvalue.

COROLLARY 5. *Let M be an irreducible matrix in $\mathfrak{T}_{d,\ell}$ with $\ell \geq 2$, and let λ be a nonreal eigenvalue of M. Then the algebraic multiplicity of λ is at most $\frac{n-2}{2}$.*

A *Hadamard tournament matrix* of order n is a tournament matrix H which satisfies

$$HH^T = H^TH = \left(\frac{n+1}{4}\right) I_n + \left(\frac{n-3}{4}\right) J_n$$

(note that n is necessarily congruent to 3 mod 4). It is shown in deCaen, Gregory, Kirkland, Maybee and Pullman [1991] that an irreducible tournament matrix, M, of order $n \geq 3$ has at least 3 distinct eigenvalues and has exactly 3 distinct eigenvalues if and only if M is a Hadamard tournament matrix. It is easy to show that the eigenvalues of a Hadamard tournament matrix of order n are $\frac{n-1}{2}$ of multiplicity 1, and $-\frac{1}{2} \pm \frac{i\sqrt{n}}{2}$ each of multiplicity $\frac{n-1}{2}$. We now use Hadamard tournament matrices to construct two classes of d-partite tournament matrices with exactly 4 distinct eigenvalues. First suppose that $d \equiv 3 \bmod 4$ and that H is a Hadamard tournament matrix of order d. Then $H \otimes J_\ell \in \mathfrak{T}_{d,\ell}$ and has 4 eigenvalues: $\frac{n-\ell}{2}$, $-\frac{\ell}{2} \pm \frac{i\ell\sqrt{d}}{2}$ $\left(\text{each with algebraic multiplicity } \frac{d-1}{2}\right)$, and 0 (with algebraic multiplicity $n-d$). Now suppose that $d = 2$, $\ell \equiv 3 \bmod 4$ and that H is a Hadamard tournament matrix of order ℓ. Let

$$M = \left[\begin{array}{c|c} O & \mathrm{H} \\ \hline J_\ell - H^T & O \end{array}\right].$$

Then $M \in \mathfrak{T}_{d,\ell}$ and since

$$M^2 = \left[\begin{array}{c|c} \left(\frac{\ell+1}{4}\right)(J_\ell - I_\ell) & O \\ \hline O & \left(\frac{\ell+1}{4}\right)(J_\ell - I_\ell) \end{array}\right].$$

the eigenvalues of M are: $\pm\sqrt{\frac{\ell^2-1}{2}}$ (each with multiplicity 1) and $\pm\frac{i\sqrt{\ell+1}}{2}$ (each with multiplicity $\ell-1$). Thus M has exactly 4 distinct eigenvalues. In addition the eigenvalues $\pm\frac{i\sqrt{\ell+1}}{2}$ yield equality in the upper bound for the algebraic multiplicity of a nonreal eigenvalue given in Corollary 5.

It is known (see deCaen, Gregory, Kirkland, Maybee and Pullman [1991]) that the geometric and algebraic multiplicities of an eigenvalue λ with $\mathrm{Re}(\lambda) = -1/2$ of a tournament matrix are equal. We extend this result to d-partite tournament matrices.

THEOREM 6. *Let $M \in \mathfrak{T}_{d,\ell}$ and suppose λ is an eigenvalue of M with* $\mathrm{Re}(\lambda) = -\ell/2$. *Then the geometric and algebraic multiplicities of λ are equal.*

Proof. Since the eigenvalue λ lies on the boundary of the field of values of M, the theorem follows from standard results on such eigenvalues (see Theorem 1.6.6 of Horn and Johnson [1991]). However we also give a more direct proof. Suppose that the geometric multiplicity of λ is less than its algebraic multiplicity. Then there exists an eigenvector u and a generalized eigenvector v such that $Mu = \lambda u$ and $Mv = \lambda v + u$. Without loss of generality we may assume that $v^*u = 0$. By Theorem 2, $u^*\mathbb{1}_n = 0$ and each $u^{(i)}$ $(1 \leq i \leq d)$ is a multiple of $\mathbb{1}_\ell$. Pre- and post-multiplying (4) by u^* and v, respectively, yields

$$u^*u + (\lambda + \overline{\lambda})u^*v = u^*J_nv - \sum_{i=1}^{d} u^{(i)*}J_\ell v^{(i)}.$$

Since $u^{(i)*}\mathbb{1}_\ell = \ell u^{(i)*}$, $u^*J_nv = 0$ and $u^*v = 0$, we conclude that $u^*u = 0$. This contradicts our assumption and hence the multiplicities of λ are equal.

□

Next we address the geometric multiplicity of an eigenvalue of a matrix in $\mathfrak{T}_{d,\ell}$. The result rests upon a simple lemma which we state without proof (see Theorem 1.4.9 of Horn and Johnson[1985]).

LEMMA 7. *Let A be a matrix of order n and suppose that λ is an eigenvalue of A with geometric multiplicity at least t. Then λ is an eigenvalue with geometric multiplicity at least $m - n + t$ of each principal submatrix of A of order $m \geq n - t + 1$.*

THEOREM 8. *Let $M \in \mathcal{T}_{d,\ell}$ and suppose that λ is an eigenvalue of M with geometric multiplicity at least $n-d+2$. Then* $\mathrm{Re}(\lambda) = -1/2$.

Proof. By Lemma 7, each principal submatrix of M of order d has λ as an eigenvalue of geometric multiplicity at least 2. The submatrix

$$M[\{1, \ell+1, 2\ell+1, \ldots, (d-1)\ell+1\}, \{1, \ell+1, 2\ell+1, \ldots, (d-1)\ell+1\}]$$

is a tournament matrix of order d. Since the geometric multiplicity of an eigenvalue β of a tournament matrix is greater than 1 only if $\mathrm{Re}(\beta) = -1/2$ (Maybee and Pullman[1990]), we conclude that $\mathrm{Re}(\lambda) = -1/2$.

□

COROLLARY 9. *A matrix in $\mathcal{T}_{d,\ell}$ has rank at least $d-1$.*

Irreducible matrices in $\mathcal{T}_{d,\ell}$ $(d \geq 6)$ with minimum rank can be constructed as follows. Let S be a singular irreducible tournament matrix of order d (Maybee and Pullman [1991] have shown that such matrices exist for any $d \geq 6$). Let $M = S \otimes J_\ell$. Let $v = (v_1, \ldots, v_d)^T$ be a nonzero null vector of S and let $\{u_1, \ldots, u_{\ell-1}\}$ be a basis for the nullspace of J_ℓ. Then it is easy to verify that

$$\{e_i \otimes u_j : \quad 1 \leq i \leq d \quad \text{and} \quad 1 \leq j \leq \ell - 1\} \cup \{v \otimes \mathbb{1}_\ell\},$$

is a linearly independent set of vectors in the nullspace of M (where $\{e_1, \ldots, e_d\}$ are the standard basis vectors for the $d \times 1$ vectors). Hence the rank of M equals $d-1$.

The final result of this section bounds the real part of an eigenvalue in terms of its geometric multiplicity.

THEOREM 10. *Suppose $M \in \mathcal{T}_{d,\ell}$ and that λ is an eigenvalue of M. If the geometric multiplicity of λ is at least $n - ad + 1$ for some integer $1 \leq a \leq \ell$, then*

$$-a/2 \leq \mathrm{Re}(\lambda) \leq a\left(\frac{n-\ell}{2\ell}\right).$$

Proof. Suppose the geometric multiplicity of λ is at least $n - ad + 1$. By Lemma 7, λ is an eigenvalue of any principal submatrix of M of order ad. M has principal submatrices of order ad that are d-partite tournament matrices with team size a. Theorem 2 now implies that

$$-\frac{a}{2} \leq \mathrm{Re}(\lambda) \leq \frac{ad-a}{2}$$

and the result follows.

□

4. Regular matrices in $\mathcal{T}_{d,\ell}$. In this section we discuss the regular matrices in $\mathcal{T}_{d,\ell}$. Let $\mathcal{R}_{d,\ell}$ denote the set of regular matrices in $\mathcal{T}_{d,\ell}$ and suppose $M \in \mathcal{R}_{d,\ell}$. Since a matrix in $\mathcal{T}_{d,\ell}$ has exactly $\frac{n(n-\ell)}{2}$ ones, each row and hence each column of M has exactly $\frac{n-\ell}{2}$ ones. Because $\frac{n-\ell}{2} = \frac{\ell(d-1)}{2}$ is an integer, either ℓ is even or d is odd. Suppose d is odd and let R be a regular tournament matrix of order d. Then $R \otimes J_\ell \in \mathcal{R}_{d,\ell}$. Now suppose ℓ is even and let P be a (0,1)-matrix of order ℓ with exactly $\ell/2$ ones in each row and column. Then

$$M = \begin{bmatrix} O & P & \dots & P \\ J_\ell - P^T & O & & \vdots \\ \vdots & \ddots & O & P \\ J_\ell - P^T & \dots & J_\ell - P^T & O \end{bmatrix} \in \mathcal{R}_{d,\ell}.$$

Thus $\mathcal{R}_{d,\ell}$ is nonempty if and only if d is odd or ℓ is even. Throughout the remainder of this section, unless otherwise stated, we assume d is odd or ℓ is even. The following lemma will be useful.

LEMMA 11. *Let $M \in \mathcal{R}_{d,\ell}$ and suppose v is an eigenvector of M corresponding to an eigenvalue λ different from $\frac{n-\ell}{2}$. Then $v^*\mathbb{1}_n = 0$.*

Proof. Pre- and post-multiplying (4) by v^* and $\mathbb{1}_n$, respectively, yields

$$\begin{aligned}\left(\frac{n-\ell}{2} + \overline{\lambda}\right)(v^*\mathbb{1}_n) &= v^* J_n \mathbb{1}_n - \sum_{i=1}^{d} v^{(i)^*} J_\ell \mathbb{1}_\ell \\ &= n v^* \mathbb{1}_n - \ell \sum_{i=1}^{d} v^{(i)^*} \mathbb{1}_\ell.\end{aligned}$$

Thus $\left(\overline{\lambda} - \frac{(n-\ell)}{2}\right) v^*\mathbb{1}_n = 0$ and the result follows.

□

In Moon [1968] it was shown that a regular tournament matrix is irreducible. We now establish the analogous result for regular d-partite tournament matrices.

PROPOSITION 12. *Suppose $M \in \mathcal{R}_{d,\ell}$. Then M is irreducible.*

Proof. Suppose $S_1 \cup S_2$ is a partition of $\{1,\dots,n\}$ with both S_1 and S_2 nonempty such that $M[S_1, S_2] = O$. Since M is regular, $M[S_1, S_1]$ and $M[\{1,\dots,n\}, S_i]$ each contain exactly $\frac{|S_1|(n-\ell)}{2}$ ones. It follows that $M[S_2, S_1] = O$. Since $M[S_1, S_2] = O$, $M[S_2, S_1] =$

O and $S_1 \cup S_2$ partitions the rows of M, the matrix M can not be in $\mathcal{T}_{d,\ell}$, a contradiction. We conclude that M is irreducible.

□

It is well-known (see Brauer and Gentry [1968]) that a tournament matrix is normal if and only if it is regular. Our next result provides a characterization of the normal matrices in $\mathcal{T}_{d,\ell}$.

THEOREM 13. *Let $M \in \mathcal{T}_{d,\ell}$. Then M is normal if and only if M is regular, M has $d-1$ eigenvalues with real part $-\ell/2$, and M has $n-d$ purely imaginary eigenvalues.*

Proof. First suppose that M is normal. Then M and M^T commute. It follows from (4) that M, M^T and $J_n - (I_d \otimes J_\ell)$ are pairwise commuting normal matrices. Thus M, M^T and $J_n - (I_d \otimes J_\ell)$ can be simultaneously diagonalized by an orthogonal matrix. The eigenvalues of $J_n - (I_d \otimes J_\ell)$ are $n-\ell$ of multiplicity 1, $-\ell$ of multiplicity $d-1$, and 0 of multiplicity $n-d$. It now follows that M has the desired eigenvalues.

Conversely, suppose M is regular, M has $d-1$ eigenvalues $\lambda_1, \dots, \lambda_{d-1}$ with real part $-\ell/2$ and $n-d$ purely imaginary eigenvalues $\mu_1, \dots, \mu_{n-d}$. Then $\mathbb{1}_n$ is an eigenvector corresponding to $\frac{n-\ell}{2}$ and by Lemma 11 is orthogonal to any eigenvector corresponding to another eigenvalue. By Theorem 6, the geometric and algebraic multiplicities of $\lambda_j \quad (1 \le j \le d-1)$ coincide, and by Corollary 3 an eigenvector corresponding to $\lambda_j \quad (1 \le j \le d-1)$ is orthogonal to an eigenvector corresponding to any other eigenvalue.

Suppose $1 \le j \le n-d$ and that v is an eigenvector of M corresponding to μ_j. Then

$$(\mu_j + \overline{\mu}_j) v^* v = v^* J_n v - \sum_{i=1}^{d} v^{(i)} J_\ell v^{(i)} \tag{6}$$

and it follows from Lemma 11 that $v^{(i)} \mathbb{1}_\ell = 0$ for $1 \le i \le d$. A straight forward calculation shows that v is orthogonal to any eigenvector corresponding to another purely imaginary eigenvalue. If there exists a vector w such that $Mw = \mu_j w + v$ then

$$w^* M v + w^* M^T v = w^* J v - \sum_{i=1}^{d} w^{(i)} J_\ell v^{(i)} \ ,$$

which implies that $v^* v = 0$. We conclude that the geometric and algebraic multiplicities of μ_j coincide.

From the above considerations there is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of M, and hence M is normal.

□

We give another characterization of the normal matrices in $\mathcal{T}_{d,\ell}$.

THEOREM 14. *A matrix $M \in \mathcal{T}_{d,\ell}$ is normal if and only if M is regular and each $M[t_i, t_j] \quad (1 \le i, j \le d)$ has constant row sums.*

Proof. Since $M + M^T = J_n - (I_d \otimes J_\ell)$, M is normal if and only if M commutes with $J_n - (I_d \otimes J_\ell)$. By Theorem 13 normal matrices in $\mathcal{T}_{d,\ell}$ are regular. Hence $M \in \mathcal{T}_{d,\ell}$ is normal if and only if M is regular and M commutes with $I_d \otimes J_\ell$. The result now follows.

□

COROLLARY 15. *A regular bipartite tournament matrix is normal.*

Proof. Suppose $M \in \mathcal{T}_{d,\ell}$ is regular. Then

$$M = \begin{bmatrix} O & P \\ J_\ell - P^T & 0 \end{bmatrix}$$

where P is of order ℓ and has $\ell/2$ ones in each row and column. By Theorem 14, M is normal.

□

Suppose $M \in \mathcal{T}_{d,\ell}$ is normal. By Theorem 14 each $M[t_i, t_j]$ has constant row sums, say r_{ij}. Let $S = [s_{ij}]$ be the nonnegative matrix of order d with $s_{ij} = \frac{r_{ij}}{\ell}$. Then

$$S + S^T = J_d - I_d \; . \tag{7}$$

A nonnegative matrix S satisfying (7) is a *generalized tournament matrix* of order d (see Maybee and Pullman [1991]). It is not difficult to show that if λ is an eigenvalue of S, then $\ell\lambda$ is an eigenvalue of M.

A regular tournament matrix of order n has Perron-value $\frac{n-1}{2}$ and has $n-1$ other eigenvalues with real part $-1/2$. Thus a regular tournament matrix has full rank. We now discuss the rank of matrices in $\mathcal{R}_{d,\ell}$.

PROPOSITION 16. *Let $M \in \mathcal{R}_{d,\ell}$. Then M has at most $n-d$ purely imaginary eigenvalues.*

Proof. Since M is regular, $\frac{n-\ell}{2}$ is an eigenvalue of M. Suppose M has k purely imaginary eigenvalues and let the remaining eigenvalues of M be $\frac{n-\ell}{2}$ and $\lambda_1, \ldots, \lambda_{n-1-k}$. Since $\mathrm{tr}(M) = 0$ and the purely imaginary eigenvalues of M occur in conjugate pairs,

$$0 = \frac{n-\ell}{2} + \sum_{j=1}^{n-1-k} \mathrm{Re}(\lambda_j).$$

By Theorem 2, $\mathrm{Re}(\lambda_j) \ge -\ell/2 \quad (1 \le j \le n-1-k)$, and it follows that $k \le n-d$.

□

A consequence of Proposition 16 is that a matrix in $\mathcal{R}_{d,\ell}$ has rank at least d. If d is odd and R is a regular tournament matrix of order d, then $R \otimes J_\ell \in \mathcal{R}_{d,\ell}$ has rank equal to d. We now prove the converse.

THEOREM 17. *Let $M \in \mathcal{R}_{d,\ell}$ and suppose M has rank d. Then d is odd and $M = R \otimes J_\ell$ for some regular tournament matrix of order d.*

Proof. Since M is regular, $\dfrac{n-\ell}{2}$ is an eigenvalue of M. Because 0 is an eigenvalue of M with algebraic multiplicity at least $n-d$ and $\mathrm{tr}(M) = 0$, Theorem 2 implies that 0 has algebraic multiplicity exactly $n-d$ and M has $d-1$ eigenvalues with real part $-\ell/2$. Thus by Theorem 14, M is normal. Pre- and postmultiplying (4) by w^* and w, respectively, where w is a a vector in the nullspace of M, and applying Lemma 11, we conclude that the nullspace of M lies in the subspace

$$W = \{w : {w^{(i)}}^T \mathbb{1}_\ell = 0\}.$$

Since W has dimension $n-d$, the nullspace of M equals W. Let e_i denote the $n \times 1$ column vector with a 1 in position i and 0's elsewhere. Suppose $i, j \in t_k$. Then $(e_i - e_j) \in W$. It follows that column i and j of M are equal. We conclude from Theorem 14 that for each i and j, $M[t_i, t_j]$ is either the matrix of all 0's or the matrix of all 1's. The result now follows.

□

COROLLARY 18. *Let $M \in \mathcal{R}_{d,\ell}$ and suppose d is even. Then M has rank at least $d+1$, and if M is normal then M has a nonzero purely imaginary eigenvalue.*

Proof. The result follows immediately from Theorem 17.

□

A regular tournament matrix (i.e. $\ell = 1$) has eigenvalues with real part $-\ell/2$. As the next proposition shows, this is not necessarily the case when $\ell \geq 2$.

PROPOSITION 19. *Assume $\ell \geq 2$ and d is odd. Then there exists a matrix $M \in \mathcal{R}_{d,\ell}$ none of whose eigenvalues has real part $-\ell/2$.*

Proof. Let $C = [C_{ij}]$ be the permutation matrix of order d whose 1's occur in the positions $(1,2), \ldots, (d-1, d)$ and $(d, 1)$. Let N be the (0,1)-matrix of order d with a 1 in position (1,1) and 0's elsewhere. Define the matrices A and B of order n by

$$A = \left(C^1 + \cdots + C^{\frac{d-1}{2}}\right) \otimes J_\ell$$

and

$$B = \left[-\left(C^1 + \cdots + C^{\frac{d-1}{2}}\right) + \left(C^{\frac{d+1}{2}} + \cdots + C^{d-1}\right)\right] \otimes N.$$

The matrix A is in $\mathcal{R}_{d,\ell}$. Since B is a skew-symmetric $(0, 1, -1)$-matrix with line sums 0 and $A + B$ is a nonnegative matrix, $A + B \in \mathcal{R}_{d,\ell}$.

Suppose that $A+B$ has an eigenvalue with real part $-\ell/2$ and a corresponding eigenvector v. By Theorem 2 each $v^{(i)}$ is a multiple of $\mathbb{1}_\ell$ and $v^*\mathbb{1} = 0$. Thus each $(Av)^{(i)}$ is a multiple of $\mathbb{1}_\ell$. It follows that each $(Bv)^{(i)}$ is a multiple of $\mathbb{1}_\ell$. Let $v^{(i)} = a_i\mathbb{1}_\ell$ and $a = [a_1, \ldots, a_d]^T$. Then

$$\left[-\left(C^1 + \cdots + C^{\frac{d-1}{2}}\right) + \left(C^{\frac{d+1}{2}} + \cdots + C^{d-1}\right)\right] a = 0.$$

It follows that $a_1 = \cdots = a_d$ and that v is a multiple of $\mathbb{1}_n$. Since $v^*\mathbb{1}_n = 0$, we conclude that $A+B$ has no eigenvalues with real part $-\ell/2$.

□

5. Future Research. In this section we pose a few problems which arise from the material in the preceding sections. Throughout M will be a matrix in $\mathcal{T}_{d,\ell}$ and λ an eigenvalue of M.

Problem 1. If $\text{Re}(\lambda) \neq -1/2$ and the algebraic multiplicity of λ is at most $n-d+1$, are the geometric and algebraic multiplicities of λ necessarily the same?

Problem 2. Suppose N is a tournament matrix of order n. Then an eigenvalue μ of N with $\text{Re}(\mu) \neq -1/2$ is geometrically simple. In particular, the generalized eigenspace of N corresponding to λ is spanned by the generalized eigenvectors in a single Jordan chain. If $\text{Re}(\lambda) > -\ell/2$, what is the structure of the eigenspace of M corresponding to λ?

Problem 3. For which values of d and ℓ does there exist a matrix in $\mathcal{T}_{d,\ell}$ with exactly 4 distinct eigenvalues?

Problem 4. Determine the values of d and ℓ such that $\mathcal{T}_{d,\ell}$ contains a matrix with a nonreal eigenvalue whose algebraic multiplicity equals $\dfrac{n-2}{2}$?

Problem 5. What is the maximum possible geometric multiplicity of an eigenvalue with real part $-1/2$ for a matrix in $\mathcal{T}_{d,\ell}$?

Problem 6. If d and ℓ are both even, and $M \in \mathcal{R}_{d,\ell}$, does M necessarily have an eigenvalue whose real part equals $-\ell/2$?

Problem 7. Find necessary and sufficient spectral conditions for a matrix $M \in \mathcal{T}_{d,\ell}$ to be diagonalizable.

Problem 8. If d and ℓ are both even, what is the minimum possible rank of a matrix in $\mathcal{R}_{d,\ell}$?

Problem 9. For tournament matrices there is a connection between the row sum vector and both the nonsingularity of the matrix and its Perron-value (see Katzenberger and Shader [1990], Kirkland[1991]). Is there a similar relationship between these properties of M and the vectors $M\mathbb{1}_n, (M\mathbb{1}_n)^{(1)}, \ldots, (M\mathbb{1}_n)^{(d)}$?

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