**A FREE BOUNDARY PROBLEM MODELING
LOOP DISLOCATIONS IN CRYSTALS**

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In this paper we consider a model of loop dislocations which arise in rapid thermal annealing of impurities into a crystal. The loop is a torus $D(R(t))$ generated by rotation of a disc of unit radius about an axis, a distance $R(t)$ from the center of the circle. The function $R(t)$ varies according to the flux of the interstitial density c across the boundary $\partial D(R(t))$ of the torus. Outside $D(R(t))$ the function c satisfies a parabolic equation, and on the boundary $\partial D(R(t))$ c is a function of the space variable and of $R(t)$. The problem is to solve the system for c and R . This is a free boundary problem where the free boundary is $\partial D(R(t))$. It is shown that a unique local solution does exist; however, for some class of data it “blows up” in finite time. For other classes of data we prove that a global solution exists, and study its asymptotic behavior.

§1. The model. During rapid thermal annealing of impurities in a crystal, dislocations develop. An (impurity) atom A squeezed into an interstitial position in a crystal lattice is called *interstitial*. It forms a strong center of repulsion, producing compressive stresses in the lattice. Similarly when atom B of the lattice is removed from the crystal, it creates a *vacancy*; this becomes a center of attraction, producing tension stress. Both interstitials and vacancies are formed during rapid thermal annealing of impurities. Interstitials cause volume expansion, whereas vacancies cause volume shrinkage. Both are responsible for deforming the geometry of the lattice, giving rise to various shapes of irregular formations in the crystal; these are called *dislocations*. The most commonly studied types are circular, line, screw and loop dislocations. For background material see Hirth–Lothe [8], Ballough–Neuman [2], Kroupa [9] and Eshelby [4].

Recently L. Borucki [3] has studied numerically a mathematical model for a loop dislocation, that is, a dislocation whose shape is a torus in infinite space. The model is based on the papers by Eshelby, Hirth–Lothe and Kroupa. It consists of two parabolic equations for the concentrations c_i and c_v of the interstitials and vacancies. These equations hold outside the core (i.e., the loop)

$$D(R(t)) = \{(r - R(t))^2 + z^2 \leq 1\} \quad (r = \sqrt{x^2 + y^2})$$

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where $R(t)$ varies with time. The rate of change of $R(t)$ is roughly given by the flux of the difference $c_i - c_v$ across $\partial D(t)$.

In this paper we study a simplified model whereby $c_i \equiv 0$; we shall denote c_v simply by c . We shall also assume that the pressure produced by the interstitials is negligible outside the core (In fact, it delays to zero very fast away from the loop; see [3]). Then the parabolic equation for c becomes the heat equation

$$(1.1) \quad c_t - \Delta c = 0 \quad \text{in } \mathbb{R}^3 \setminus D(R(t)), \quad t > 0 .$$

In addition, c satisfies the boundary condition

$$(1.2) \quad c = e^{\frac{A \log R}{R}} h(\theta) \quad \text{on } \partial D(R(t))$$

where A is a positive constant and

$$r - R = \cos \theta , \quad z = \sin \theta \quad \text{on } \partial D(R(t));$$

the function $h(\theta)$ is equal approximately to

$$(1.3') \quad B_0 e^{-B \cos \theta} \quad \text{for some positive constants } B_0, B .$$

In this paper we shall only need the general properties

$$(1.3) \quad h \in C^{2+\alpha} , \quad h'(\theta) > 0 \quad \text{if } 0 < \theta < \pi , \quad h(-\theta) = h(\theta) > 0 .$$

For simplicity we shall henceforth take (except in Section 7) $A = 1$ in (1.2).

The rate of change of $R(t)$ is given by

$$(1.4) \quad R(t) \dot{R}(t) = \iint_{\partial D(R(t))} \frac{\partial c}{\partial N} dS$$

where N is the normal pointing into $D(R(t))$. Finally, we also prescribe initial conditions

$$(1.5) \quad c(r, z, 0) = c_0(r, z), \quad R(0) = R_0 \quad (c_0 \geq 0, R_0 > 1) .$$

The physical problem makes sense only as long as $D(R(t))$ remains a torus, that is, as long as

$$(1.6) \quad R(t) > 1 .$$

In case $c_v \equiv 0$ we get for $c = c_i$ a system similar to (1.1)–(1.5) with

$$h(0) = B_0 e^{B \cos \theta}$$

and with negative sign in front of the integral in (1.4). All our results can be extended to this case.

In this paper we study the problem (1.1)–(1.6). Actually, (1.2) should be replaced by the slightly more complicated relation (see [3])

$$c = e^{\frac{A \log R + A_0}{R}} h(\theta) ,$$

but this does not affect our results. We have also lumped here together, as well as non-dimensionalized, various physical constants and variables; if all these quantities are chosen as in [3], we get, in our normalized situation, B_0 small and B approximately 1. There is however uncertainty about some of the physical constants, and this results in smaller values of B . The constant B varies of course also with the particular crystal and its processing. Some of the results of the present paper require that B is “small,” and thus do not directly apply to the range of parameters used in the computations in [3]. The variable t is related to the real time variable by $t \sim D_0 t_{\text{real}}$ where $D_0 \gg 1$; this motivates the asymptotic study of the solution as $t \rightarrow \infty$.

Problem (1.1)–(1.6) has a 1-dimensional analog whereby

$$c_t = c_{xx} \quad \text{if} \quad x < R(t) \quad \text{or} \quad x > R(t+1)$$

and

$$\dot{R}(t) = -c_x(R(t)+1, t) + c_x(R(t), t) .$$

The study of this problem offers some insight into the (physical) 3-dimensional dislocation problem.

If $R(t)$ is large, the 3-dimensional problem might be approximated by replacing the torus by a cylinder. This leads to a 2-dimensional problem whereby

$$c_t = c_{xx} + c_{zz} \quad \text{if} \quad (x - R(t))^2 + z^2 > 1 .$$

In Section 2 we prove that problem (1.1)–(1.6) has a unique solution for small time. The solution can be extended to all time provided (1.6) holds and provided, for any $T > 0$,

$$(1.7) \quad R(t) \leq C , \quad |\dot{R}(t)| \leq C \quad \text{if} \quad 0 < t < T$$

where C is finite constant (depending on T).

It turns out that these a priori bounds are not always satisfied. Indeed, in Section 4 we prove that for any small T there is a class of initial data c_0 for which the solution (c, R)

does not exist beyond some time $T^* \leq T$ and, in fact, $\limsup_{t \rightarrow T^*} \dot{R}(t) = \infty$. This is the case, for instance, if

$$(1.8) \quad \begin{aligned} & 4\pi \left[\int_0^{\theta_0} h(\theta) \cos \theta d\theta \right] \cos \theta_0 > 1 \quad \text{for some } \theta_0 \in \left(0, \frac{\pi}{2}\right), \\ & \text{and } \lim_{r^2+z^2 \rightarrow \infty} c_0(r, z) \text{ is sufficiently small} \end{aligned}$$

For the one dimensional problem the first condition in (1.8) can be replaced by $h(0) > 1$; moreover,

$$\dot{R}(t) \rightarrow \infty \quad \text{if } t \rightarrow T^*,$$

although $R(T^* - 0)$ remains finite. The proofs of these statements are given in Section 3.

The one dimensional case is also considered in Section 5, where we prove that for a large class of initial data with $h(0) < 1$ the solution (c, R) exists for all t (although $R(t)$ may become smaller than 1).

The remaining part of the paper is concerned with establishing existence to the 3-dimensional problem for some classes of data h, c_0 . We shall consider roughly three cases:

Case 1. (Section 6): $h(0)$ and c_0 are “small.”

Case 2. (Section 7–9) $h - h(0)$ is “small” and c_0 is “close” to the data of a travelling wave solution for the 2-dimensional problem, with small velocity λ .

Case 3. (Sections 10, 11): h and c_0 are “close” to the data of a stationary solution, i.e., a solution with $\dot{R}(t) \equiv 0$.

In Section 6 we assume that

$$(1.9) \quad \|h\|_{C^{2+\alpha}} \ll 1 \quad \text{and} \quad \|c_0\|_{C^{2+\alpha}} \ll 1, \quad \int_{\mathbf{R}^3 \setminus D(R(0))} c_0 \ll 1$$

for some $0 < \alpha < 1$. We establish global existence of a solution, and the crude estimates

$$(1.10) \quad ct^{1/2} \leq R(t) \leq Ct \quad (c > 0, C > 0)$$

for t large enough. This result immediately extends to the case where

$$(1.11) \quad \begin{aligned} & \|h - c^*\|_{C^{2+\alpha}} \ll 1, \quad \|c_0 - c^*\|_{C^{2+\alpha}} \ll 1 \quad (c^* \text{ constant}), \\ & \int_{\mathbf{R}^3 \setminus D(R(0))} |c_0 - c^*| \ll 1, \quad \text{and} \quad R(0) \gg 1, \end{aligned}$$

by simply applying the previous result to $c - c^*$.

In Sections 7–9 we establish global existence and derive sharp asymptotic estimates for data h and $R(0)$ as in (1.11) and c_0 “close” to a travelling wave solution G ($|c_0 - c^*|$ is “small”, but need not satisfy the integral estimate in (1.11)).

We begin (in Section 7) by constructing a travelling wave solution, with velocity λ , for the 2-dimensional problem, in the case where $A = 0$ in (1.2). In Section 8 we consider the case where λ is small and study the 3-dimensional problem linearized about the traveling wave solution. Finally, in Section 9, we consider the complete 3-dimensional problem with data “close” to the travelling wave solution for λ small, and prove the existence of a solution which is asymptotically close to the travelling wave solution provided $R(0) \gg 1$; in particular

$$R(t) \sim R(0) + \lambda t \quad \text{as } t \rightarrow \infty .$$

In Section 10 we show that any $\bar{R} > 1$ and for general h there exists a unique ξ such that there exists a stationary solution $c_0(r, z)$ to the 3-dimensional problem with $R(t) \equiv \bar{R}$ for which $c_0 \rightarrow \xi$ at ∞ . We then study the 3-dimensional problem, linearized about this stationary solution, under a “stability” condition (see (10.12), (10.14)). The stability condition is satisfied if

$$|h'(\theta)| \ll h(\theta) \quad \text{and} \quad \bar{R} > e .$$

In Section 11 we establish existence of a global solution (c, R) of the 3-dimensional problem for data “close” to the data of the stationary problems, assuming that the “stability” condition holds. We also prove that

$$R(t) \rightarrow \bar{R} \quad \text{as } t \rightarrow \infty .$$

There is clearly a gap between the condition (1.8) which ensures nonexistence for some class of initial data, and the assumptions in Cases 1–3 under which global existence does hold. For the 1-dimensional problem there is no such gap (cf. Sections 3, 5), and this suggests that more general global existence theorems might hold for the 3-dimensional problem.

§2. Local existence and uniqueness. The concentration $c(r, z, t)$ satisfies

$$(2.1) \quad c_t = c_{zz} + c_{rr} + \frac{1}{r} c_r \quad \text{in } \{t > 0, (r - R(t))^2 + z^2 > 1, r > 0\} ,$$

$$(2.2) \quad c = e^{\frac{\log R(t)}{R(t)}} h(\theta) \quad \text{on } \{t \geq 0, r = R(t) + \cos \theta, z = \sin \theta\},$$

$$(2.3) \quad c_r = 0 \quad \text{on } r = 0 ,$$

$$(2.4) \quad c(r, z, 0) = c_0(r, z) \geq 0 ,$$

and

$$(2.5) \quad R(0) = R_0 > 1 ,$$

$$R(t)\dot{R}(t) = \iint_{\{(r-R(t))^2+z^2=1\}} \frac{\partial c}{\partial N} r \, ds$$

where $N = (-\cos \theta, -\sin \theta) = (-(r-R), -z)$ is the normal vector pointing into $\{(r-R)^2 + z^2 \leq 1\}$; here $rds = dS =$ surface element on the torus.

We assume:

$$(2.6) \quad \|c_0\|_{C^{2+\alpha}\{(r-R_0)^2+z^2 \geq 1, \, r \geq 0\}} < \infty , \quad c_0 = e^{\frac{\log R(0)}{R(0)} h} \quad \text{on} \quad \{(r-R(0))^2 + z^2 = 1\} .$$

THEOREM 2.1. *If τ is sufficiently small then there exists a unique solution (c, R) of (2.1)–(2.6) with*

$$(2.7) \quad \dot{R} \in C^{\alpha/2}[0, \tau] \cap C^\infty(0, \tau] .$$

Actually the proof of uniqueness does not require that $\dot{R}$ belong to $C^\infty(0, \tau]$.

THEOREM 2.2. *If a solution (c, R) of (2.1) – (2.6) exists for $0 \leq t < T$ and*

$$(2.8) \quad |\dot{R}(t)| \leq C \quad \forall \quad 0 \leq t < T ,$$

$$(2.9) \quad R(t) \geq 1 + \delta \quad \forall \quad 0 \leq t < T$$

where $\delta > 0$, then there exists a $\tau > 0$ such that the solution exists and is unique for $0 < t < T + \tau$, and

$$(2.10) \quad \dot{R} \in C^{\alpha/2}[0, T + \tau] \cap C^\infty(0, T + \tau] .$$

It follows that if the priori bounds in (2.8), (2.9) can be established for any $T > 0$, then the solution exists for all t .

Proof of Theorem 2.1. Let

$$(2.11) \quad u(x, z, t) = c(r, z, t), \quad x = r - R(t) .$$

Then

$$(2.12) \quad u_t = u_{zz} + u_{xx} + \frac{1}{x+R} u_x + \dot{R} u_x$$

$$\text{in} \quad \{t > 0, \, x^2 + z^2 > 1, \, x > -R\},$$

$$(2.13) \quad u = e^{\frac{\log R}{R}} h(\theta) \quad \text{on} \quad \{t \geq 0, x = \cos \theta, z = \sin \theta\} ,$$

$$(2.14) \quad u_x = 0 \quad \text{on} \quad x = -R ,$$

$$(2.15) \quad u(x, z, 0) = c_0(x + R_0, z) ,$$

and

$$(2.16) \quad R(t)\dot{R}(t) = \iint_{\{x^2+z^2=1\}} \frac{\partial u}{\partial N} r ds .$$

Let $\beta = \alpha/2$ and

$$K = \{R \in C^{1,\beta}[0, \tau]; R(0) = R_0, \dot{R}(0) = R_1, [\dot{R}]_{C^\beta[0, \tau]} \leq R_{1,\alpha}\} .$$

where R_0 is given by (2.5) and R_1 is given by

$$R_0 R_1 = \iint_{\{(r-R_0)^2+z^2=1\}} \frac{\partial c_0}{\partial N} r ds ;$$

$R_{1,\alpha}$ will be determined later. Here $[\cdot]_{C^\beta}$ is the β -Hölder coefficient semi-norm.

Let $\sigma = R_0 - 1$ and choose τ to satisfy

$$(2.17) \quad R_{1,\alpha} \tau^\beta \leq 1 , \quad (|R_1| + 1)\tau \leq \frac{\sigma}{2} .$$

Then, for all $R \in K$,

$$(2.18) \quad \|\dot{R}\|_{L^\infty[0, \tau]} \leq |R_1| + 1 ,$$

$$(2.19) \quad R(t) \geq R_0 - \frac{\sigma}{2} = 1 + \frac{\sigma}{2} \quad \forall \quad 0 \leq t \leq \tau .$$

Given $R \in K$ we solve u from (2.12)–(2.15). Then we define a map $T : K \rightarrow C^{1,\alpha}[0, \tau]$ by

$$(2.20) \quad \frac{d}{dt} \left\{ \frac{1}{2} (TR)^2(t) \right\} = \iint_{\{x^2+z^2=1\}} \frac{\partial u}{\partial N} r ds ,$$

$$(2.21) \quad TR(0) = R_0 .$$

By Schauder's interior-boundary estimate [8]

$$(2.22) \quad \|u\|_{C^{2+\alpha, 1+\alpha/2}(Q_\tau)} \leq C \quad \forall \quad R \in K$$

where $Q_\tau = \left\{ 1 \leq x^2 + z^2 \leq 1 + \frac{\sigma}{4}, 0 \leq t \leq \tau \right\}$, and, by imbedding [10],

$$\|u_X\|_{C_t^{(1+\alpha)/2}} \leq C \quad \text{in } Q_\tau ,$$

where $X = (x, z)$. It follows that

$$(2.23) \quad \left\| \frac{d}{dt} (TR) \right\|_{C^{\frac{1+\alpha}{2}}[0, \tau]} \leq C .$$

Hence T is compact. T is clearly also continuous. We next show that T maps K into itself.

For each $R \in K$ we introduce an extension

$$\tilde{R}(t) = \begin{cases} R(t) & \text{if } 0 \leq t \leq \tau \\ R(\tau) & \text{if } \tau \leq t \leq 1 . \end{cases}$$

and denote the corresponding solution of (2.11) – (2.15) by $\tilde{u}$. Clearly $\tilde{u} = u$ if $0 \leq t \leq \tau$.

By L^p interior-boundary estimates [10]

$$(2.24) \quad \|\tilde{u}\|_{W_p^{2,1}(Q_1)} \leq C_p$$

where the constant C_p is independent of $R_{1,\alpha}$ and τ . By imbedding,

$$\|\tilde{u}_X\|_{C^{2\beta, \beta}(Q_1)} \leq C$$

provided p is large enough so that

$$1 - \frac{4}{p} > 2\beta = \alpha .$$

The same estimate then holds for u , for $0 < t < \tau$, and, consequently,

$$(2.25) \quad \left\| \frac{d}{dt} (TR) \right\|_{C^\beta[0, \tau]} \leq C^*$$

where C^* is a constant independent of $R_{1,\alpha}$ and τ .

If we now choose $R_{1,\alpha} = C^*$ then we find that T maps K into itself.

Using the Schauder fixed point theorem, we conclude that T has a fixed point R . From (2.23) we see that

$$\dot{R} \in C^{\frac{1+\alpha}{2}} [0, \tau] .$$

Recalling (2.22) we then deduce that

$$\dot{R}u_x \in C^{1+\alpha, (1+\alpha)/2}(Q) .$$

We can now apply Schauder's estimates to equation (2.12) and conclude that

$$u_X \in C^{2+\alpha, (1+\alpha)/2}(\{1 \leq x^2 + z^2 \leq 1 + \frac{\sigma}{8}, \varepsilon \leq t \leq \tau\})$$

for any $\varepsilon > 0$. This implies that

$$\dot{R} \in C^{1+\frac{\alpha}{2}}[\varepsilon, \tau] \quad \text{for any } \varepsilon > 0.$$

We can repeat this process to get increasingly higher regularity for R ; thus $R \in C^\infty(0, \tau]$.

We shall now prove uniqueness. Suppose there are two solutions $(u_1, R_1), (u_2, R_2)$ such that

$$u_1, u_2 \in C^{2+\alpha, 1+\frac{\alpha}{2}} ,$$

$$R_1, R_2 \in C^{1+\frac{\alpha}{2}}[0, \tau] .$$

Let

$$L = \min_{0 \leq t \leq \tau_1} \min\{R_1(t), R_2(t)\} .$$

Then by continuity

$$L \geq 1 + \frac{\sigma}{2}$$

provided τ_1 is small enough.

We first consider the corresponding solutions c_1 and c_2 . It is clear that the difference $c_1 - c_2$ in their commonly defined region satisfies

$$|c_1(r, z, t) - c_2(r, z, t)| \leq C \|R_1 - R_2\|_{L^\infty[0, \tau_1]}$$

by maximum principle. Therefore

$$\|u_1 - u_2\|_{L^\infty(G)} \leq C \|R_1 - R_2\|_{L^\infty[0, \tau_1]} .$$

The function $w = u_1 - u_2$ in the region

$$G = \{x^2 + z^2 > 1, \ x > -L, \ 0 < t < \tau_1\}$$

satisfies:

$$w_t = w_{xx} + w_{zz} + \frac{1}{x + R_1} w + \dot{R}_1 w_x + \Phi$$

where

$$\Phi = \left(\frac{1}{x + R_1} - \frac{1}{x + R_2} \right) u_{2,x} + (\dot{R}_1 - \dot{R}_2) u_{2,x} .$$

Clearly

$$|\nabla_X u_1|, \ |\nabla_X u_2| \leq C .$$

On the boundary $\{x^2 + z^2 = 1, \ 0 \leq t \leq \tau_1\}$,

$$w = \left(e^{\frac{\log R_1}{R_1}} - e^{\frac{\log R_2}{R_2}} \right) h(\theta) \equiv \tilde{h}$$

where

$$\|\tilde{h}\|_{C^{2,1}} \leq C \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]} .$$

We can now proceed as before, applying L^p interior-boundary estimates to w and the imbedding theorem to obtain

$$\left\| \frac{d}{dt} (R_1^2 - R_2^2) \right\|_{C^{\alpha/2}[0,\tau_1]} \leq C \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]}$$

where the constant C is independent of τ_1 . From this estimate we deduce that

$$\|\dot{R}_1 - \dot{R}_2\|_{C^{\alpha/2}[0,\tau_1]} \leq C \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]} ,$$

and hence

$$\|\dot{R}_1 - \dot{R}_2\|_{L^\infty[0,\tau_1]} \leq C \tau_1^{\alpha/2} \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]}$$

and

$$\|R_1 - R_2\|_{L^\infty[0,\tau_1]} \leq C \tau_1^{\alpha/2} \tau_1 \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]} .$$

The last two inequalities imply that

$$\|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]} \leq C \tau_1^{\alpha/2} (1 + \tau_1) \|R_1 - R_2\|_{W^{1,\infty}[0,\tau_1]}$$

and, consequently, $R_1 \equiv R_2$ provided $C \tau_1^{\alpha/2} (1 + \tau_1) < 1$. We can similarly proceed step-by-step to prove that $R_1(t) = R_2(t)$ for all $0 \leq t \leq \tau$.

Proof of Theorem 2.2. Under the assumptions (2.8), (2.9) we can apply the L^p estimates to obtain

$$\|u\|_{W_p^{2,1}(Q)} \leq C$$

where $Q = \left\{ 1 \leq x^2 + z^2 \leq 1 + \frac{\delta}{2}, 0 \leq t \leq T \right\}$. Therefore we can proceed as in the proof of Theorem 2.1, from (2.24) to (2.25), to get

$$\|\dot{R}\|_{C^{\alpha/2}[0,T]} \leq C$$

provided $1 - \frac{4}{p} > \alpha$. Next, by the Schauder estimates,

$$\|u\|_{C^{2+\alpha,1+\alpha/2}(W)} \leq C$$

where $W = \{x^2 + z^2 \geq 1, x \geq -R(t), 0 \leq t \leq T\}$, and, in particular,

$$\|u(\cdot, \cdot, T')\|_{C^{2+\alpha}(W)} \leq C \quad \forall T' < T.$$

Therefore we can use Theorem 2.1 to extend the solution to $0 \leq t \leq T + \tau$ for some $\tau > 0$, and as before, $R \in C^\infty(0, T + \tau]$. The uniqueness follows, of course, from Theorem 1.1.

§3. Nonexistence for $n = 1$. The 1-dimensional case is formulated as follows:

$$\begin{aligned} (3.1) \quad & c_t = c_{xx} \quad \text{if } x < R(t) \quad \text{or if } x > R(t) + 1, \\ & c(x, 0) = c_0(x) \geq 0 \quad \text{if } x < R(0) \quad \text{or if } x > R(0) + 1, \\ & c(R(t), t) = M, \quad c(R(t) + 1, t) = m \quad \text{if } t > 0 \quad (M > m), \\ & \dot{R}(t) = -c_x(R(t) + 1, t) + c_x(R(t), t) \quad \text{if } t > 0. \end{aligned}$$

For simplicity we do not include the factor $\exp(\log R(t)/R(t))$ in the boundary conditions at $x = R(t)$ and at $x = R(t) + 1$.

The constants M, m and $R(0)$ are given.

As in Section 2 one can prove local existence and uniqueness. Further, if

$$(3.2) \quad 0 < \eta \leq R(t) \leq C < \infty, \quad |\dot{R}(t)| \leq C \quad \text{for } 0 < t < T$$

then the solution c, R can be continued to $0 < t < T + \tau$ for some $\tau > 0$.

In this section we assume that

$$(3.3) \quad \|c_0\|_{C^{2+\alpha}(\mathbb{R}_1)} < \infty,$$

that

$$\begin{aligned}
(3.4) \quad & c_0(x) < M \quad \text{if } x < R(0) , \\
& c_0(x) < m \quad \text{if } x > R(0) + 1 \\
& c_0(x) \rightarrow 0 \quad \text{if } x \rightarrow \infty , \quad c'_0(x) \rightarrow 0 \quad \text{if } x \rightarrow \infty , \\
& \int_{R(0)}^{\infty} c_0(x) dx < \infty .
\end{aligned}$$

and that

$$(3.5) \quad m > 1 .$$

From the first two inequalities in (3.4) it follows, by the maximum principle, that

$$(3.6) \quad c_x(R(t), t) > 0 , \quad c_x(R(t) + 1, t) < 0 ;$$

consequently

$$(3.7) \quad \dot{R}(t) > 0 .$$

THEOREM 3.1. *Under the assumptions (3.3)–(3.5), if (c, R) is a solution for $0 < t < t_0$ then*

$$(3.8) \quad R(t) \leq C$$

where C is a constant independent of t_0 .

Proof. Consider the function

$$(3.9) \quad u(x, t) = c(x + R(t), t) .$$

It satisfies

$$(3.10) \quad u_t = u_{xx} + \dot{R}(t)u_x \quad \text{if } t > 0 \quad \text{and} \quad x < 0 \quad \text{or} \quad x > 1 ,$$

$$(3.11) \quad \begin{cases} u(1, t) = m , & u(0, t) = M , \\ u(x, 0) = u_0(x) = c_0(R(0) + x) \end{cases}$$

and

$$(3.12) \quad \dot{R}(t) = -u_x(1, t) + u_x(0, t) .$$

The function u_x satisfies the same differential equation (3.10) as u , and $u_x(x, 0) \rightarrow 0$ if $x \rightarrow \infty$. Hence (by [5]) $u_x(x, t) \rightarrow 0$ if $x \rightarrow \infty$. Integrating (3.10) over $\{x > 1, 0 < t < \tau\}$, we get

$$\int_1^\infty u(x, \tau) dx = - \int_0^\tau u_x(1, t) dt - m \int_0^\tau \dot{R}(t) dt + C$$

where

$$C = \int_1^\infty u_0(x, 0) dx < \infty \quad (\text{by (3.4)}) .$$

Since

$$-u_x(1, t) = \dot{R}(t) - u_x(0, t) \leq \dot{R}(t) , \quad \text{by (3.6),}$$

we obtain

$$\int_1^\infty u(x, t) dx + (m-1)R(t) \leq C + (m-1)R(0) ,$$

and (3.8) follows.

THEOREM 3.2. *Let (3.5) hold and let $0 < \delta < m-1$. Then there exists a small $T > 0$ and a C^3 function $\tilde{c}(x)$ defined for $x \geq R(0) + 1$ and satisfying*

$$(3.13) \quad \tilde{c}(R(0) + 1) = m , \quad \tilde{c}(x) > 0 , \quad \lim_{x \rightarrow \infty} \tilde{c}(x) = \delta$$

such that if $c_0(x)$ is any function satisfying (3.3), (3.4) and

$$(3.14) \quad c_0(x) < \tilde{c}(x) \quad \text{for} \quad R(0) + 1 < x < \infty .$$

then there does not exist a solution of (3.1) for all $0 \leq t \leq T$ (with $\dot{R}(t)$ bounded).

Proof. Suppose such a solution exists and consider the function u defined by (3.9). We wish to compare it with a supersolution $(w, \tilde{R})$ which blows up in time T . More precisely, w and $\tilde{R}$ will satisfy

$$(3.15) \quad w_t \geq w_{xx} + \dot{\tilde{R}}(t)w_x \quad \text{if} \quad x > 1, \quad 0 < t < T ,$$

$$(3.16) \quad \dot{\tilde{R}}(t) = -w_x(1, t) + u_x(0, t) , \quad 0 < t < T ,$$

$$(3.17) \quad w(1, t) = m , \quad 0 < t < T ,$$

and

$$(3.18) \quad \left. \begin{aligned} w(x, 0) &> u(x, 0) \\ w_x(x, t) &< 0 \end{aligned} \right\} \quad \text{if } x > 1$$

$$(3.19) \quad w_x(1, 0) > u_x(1, 0) .$$

In view of (3.19), $\dot{R}(0) > \ddot{R}(0)$ and therefore, by continuity,

$$(3.20) \quad \dot{R}(t) \geq \ddot{R}(t) \quad \text{if } 0 \leq t \leq \tau$$

for some $\tau > 0$.

From the second part of (3.18) it follows that

$$(3.21) \quad \ddot{R}(t) > 0 .$$

Also, by (3.20), (3.15), that

$$w_t \geq w_{xx} + \dot{R}(t)w_x .$$

We conclude that w is a supersolution to the same equation as u and, consequently,

$$\begin{aligned} w(x, t) &> u(x, t) \quad \text{if } x > 0, \quad 0 \leq t \leq \tau , \\ w_x(1, t) &> u_x(1, t) \quad \text{if } 0 \leq t \leq \tau . \end{aligned}$$

This implies that

$$(3.22) \quad \dot{R}(t) > \ddot{R}(t) \quad \text{if } 0 \leq t \leq \tau .$$

Having proved that (3.20) implies (3.22), it follows that (3.20) extends to $0 \leq t \leq \tau'$ for some $\tau' > \tau$ and, similarly, to all $0 < t < T$.

We shall now construct $w, \tilde{R}$ satisfying (3.15) – (3.19) with

$$(3.23) \quad \ddot{R}(t) \rightarrow \infty \quad \text{if } t \rightarrow T .$$

Consider the function

$$\psi(x, t) = (m - \varepsilon)e^{-\beta \frac{x-1}{T-t}} + \varepsilon \quad (1 < x < 1 + \theta, \quad 0 < t < T)$$

for any $0 < \varepsilon < m - 1$, $\theta > 0$, where β is a positive constant to be determined. The inequality (3.15) means that

$$\psi_t - \psi_{xx} - (-\psi_x(1, t) + u_x(0, t))\psi_x \geq 0$$

or

$$-(m - \varepsilon) \frac{\beta(x - 1)}{(T - t)^2} - \frac{(m - \varepsilon)\beta^2}{(T - t)^2} + \left[\frac{(m - \varepsilon)\beta}{T - t} + u_x(1, t) \right] \frac{(m - \varepsilon)\beta}{T - t} \geq 0 .$$

This reduces to

$$-(x - 1) - \beta + [(m - \varepsilon)\beta + u_x(0, t)(T - t)] \geq 0 .$$

Since $u_x(0, t) > 0$, the last inequality is satisfied if $-\theta + (m - 1 - \varepsilon)\beta \geq 0$, or if

$$(3.24) \quad \beta = \frac{\theta}{m - 1 - \varepsilon} .$$

Next we construct a supersolution for $1 + \frac{\theta}{4} < x < \infty$ in the form

$$(3.25) \quad g(x) = Ae^{-x} + \delta .$$

The inequality (3.15) becomes

$$-g'' - \tilde{R}g' > 0$$

or

$$(3.26) \quad -1 + \tilde{R} > 0 .$$

Since, as computed above,

$$(3.27) \quad \tilde{R} = \frac{(m - \varepsilon)\beta}{T - t} + u_x(0, t)$$

and $u_x(0, t) \geq 0$, (3.26) is satisfied if T is sufficiently small.

We wish to patch together the functions ψ and g along a curve $x = \xi(t)$, that is,

$$(3.28) \quad (m - \varepsilon)e^{-\beta \frac{x-1}{T-t}} + \varepsilon = Ae^{-x} + \delta \quad \text{if } x = \xi(t) .$$

Notice that, for $x = 1 + \frac{\theta}{2}$, (3.28) becomes

$$(m - \varepsilon)e^{-\frac{\beta\theta}{2(T-t)}} + \varepsilon = Ae^{-1-\theta/2} + \delta .$$

We choose $0 < \delta < \varepsilon$ and impose the condition $\xi(0) = 1 + \frac{\theta}{2}$, which means that A is determined by

$$A = e^{1+\theta/2} \left[\varepsilon - \delta + (m - \varepsilon)e^{-\frac{\beta\theta}{2T}} \right] ; \quad A > 0 .$$

If T is sufficiently small then there exists a unique solution $\xi(t)$ of (3.28) for $0 \leq t \leq T$ with $\xi(0) = 1 + \theta/2$, and $\xi(t)$ is continuously differentiable. Indeed, if we rewrite (3.28) in the form

$$F(x, t) \equiv (m - \varepsilon)e^{-\beta \frac{x-1}{T-t}} - Ae^{-x} + \varepsilon - \delta = 0$$

then

$$F_t \sim 0 , \quad F_x \sim -Ae^{-x}$$

if $x \sim \theta/2$ and T is small enough. Therefore the assertions concerning $\xi(t)$ follows from the implicit function theorem.

We now define

$$w(x, t) = \begin{cases} \psi(x, t) & \text{if } 1 < x < \xi(t) \\ g(x) & \text{if } \xi(t) < x < \infty . \end{cases}$$

Observe that

$$\begin{aligned} w_x(\xi(t) + 0, t) &= -Ae^{-\xi(t)} \sim -Ae^{-1+\theta/2} , \\ w_x(\xi(t) - 0, t) &= -\frac{(m - \varepsilon)\beta}{T - t} e^{-\beta \frac{\xi(t)-1}{T-t}} \sim -\frac{(m - \varepsilon)\beta}{T - t} e^{-\frac{\beta\theta}{2(T-t)}} ; \end{aligned}$$

hence

$$[w_x(x, t)]_{\xi(t)-0}^{\xi(t)+0} < 0 .$$

This implies that $-w_{xx}$ has a positive Dirac density at $x = \xi(t)$. Since w is also continuous across $x = \xi(t)$, we conclude that w satisfies the differential inequality

$$w_t - w_{xx} - \hat{R}(t)w_x \geq 0$$

in the weak (distribution) sense in $\{1 < x < \infty, \quad 0 < t < T\}$; this (weak sense) suffices in order to carry out the comparison argument which led to (3.20) for all $0 < \tau < T$. Since, by (3.27),

$$\hat{R}(t) \rightarrow \infty \quad \text{if } t \rightarrow T ,$$

the solution (u, R) cannot exist for all $0 \leq t \leq T$.

If we define

$$\tilde{c}(x) = w(x - R(0), 0), \quad R(0) + 1 < x < \infty$$

then the assertion of the theorem clearly follows.

The solution (c, R) , as in Theorem 3.2, exists for some time $0 \leq t < \widehat{T}$, $\widehat{T} \leq T$; denote by T^* the maximal such $\widehat{T}$. By Theorem 3.1,

$$R(t) \leq C \quad \text{if} \quad 0 < t < T^*$$

and, by the proof of Theorem 3.2,

$$\dot{R}(t) > 0 \quad \text{if} \quad 0 < t < T^* .$$

By Theorem 2.2 (modified to $n = 1$) we also know that

$$(3.29) \quad \limsup_{t \rightarrow T^*} \dot{R}(t) = \infty .$$

THEOREM 3.3. *If we further assume that*

$$(3.30) \quad c_0''(x) \geq 0 \quad \text{for} \quad x \geq R(0) + 1 ,$$

then

$$(3.31) \quad \lim_{t \rightarrow T^*} \dot{R}(t) = \infty .$$

Proof. Since $u_t(1, t) = 0$,

$$u_{xx}(1, t) = -\dot{R}(t)u_x(1, t) \geq 0 .$$

Differentiating (3.10) twice in x and applying the maximum principle to u_{xx} , we get

$$(3.32) \quad u_{xx} \geq 0 \quad \text{for} \quad 1 \leq x < \infty , \quad 0 \leq t < T^*$$

To prove (3.31) we assume that

$$(3.33) \quad \dot{R}(t_n) \leq C^* < \infty$$

for a sequence $t_n \uparrow T^*$, and derive a contradiction. Observe that

$$u_x(1, t_n) = u_x(0, t_n) - \dot{R}(t_n) \geq -C^*$$

and therefore, by (3.32),

$$u(x, t_n) \geq m - C^*(x - 1) \quad \text{for} \quad x \geq 1 .$$

Let

$$\theta(x) = \frac{C^{\star 3}}{m^2} \left[\left(\frac{m}{C^{\star}} + 1 - x \right)^+ \right]^3 \quad \text{for } x \geq 1 .$$

Then $\theta \in C^{2,1}[1, \infty)$ and

$$(3.34) \quad u(x, t_n) \geq \theta(x) \quad \text{for } x \geq 1 .$$

Let

$$w_t = w_{xx} \quad \text{for } -\infty < x < 0, \quad t > 0 ,$$

$$w(0, t) = M \quad \text{for } t > 0$$

$$w(x, 0) = w_0(x) \quad \text{for } -\infty < x < 0$$

where $w_0(x) \leq u_0(x)$, $w'_0(x) \geq 0$. Then $w_x \geq 0$, so that

$$w_t - w_{xx} - \dot{R}w_x \leq 0 \quad \text{for } -\infty < x < 0, \quad 0 < t < T^* ,$$

and, by comparison with u ,

$$w(x, t) \leq u(x, t)$$

and

$$(3.35) \quad u_x(0, t) \leq w_x(0, t) < \tilde{C} \quad \text{for } 0 < t < T^* \quad (\tilde{C} > 0) .$$

Consider the problem

$$\psi_t = \psi_{xx} + \dot{\rho}(t)\psi_x \quad \text{for } x \geq 1, \quad t \geq 0$$

$$\psi(1, t) = m ,$$

$$\psi(x, 0) = \theta(x) \quad \text{for } x \geq 1 ,$$

$$\dot{\rho}(t) = -\psi_x(1, t) + \tilde{C} .$$

As in Section 2 one can prove local existence and uniqueness of (ψ, ρ) , say for $0 \leq t \leq \eta$. Since

$$\dot{R}(t_n) \leq C^{\star} < -\frac{d\theta(1)}{dx} < \dot{\rho}(0) ,$$

we have, by continuity,

$$(3.36) \quad \dot{R}(t) \leq \dot{\rho}(t - t_n) \quad \text{for } t_n \leq t \leq t_n + \tau ,$$

for some $\tau > 0$.

But then, since also $\psi_x \leq 0$,

$$\psi_t - \psi_{xx} - \dot{R}(t)\psi_x \leq 0 \quad \text{for } t_n \leq t \leq t_n + \tau .$$

By comparison we conclude that

$$u(x, t) \geq \psi(x, t - t_n) \quad \text{for } x \geq 1 , \quad 0 \leq t - t_n \leq \tau$$

and

$$u_x(1, t) \geq \psi_x(1, t - t_n) = \tilde{C} - \dot{\rho}(t - t_n) .$$

It follows (recalling (3.35)) that

$$\dot{R}(t) = -u_x(1, t) + u_x(0, t) < -(\tilde{C} - \dot{\rho}(t - t_n)) + \tilde{C} ,$$

that is

$$(3.37) \quad \dot{R}(t) < \dot{\rho}(t - t_n) \quad \text{for } 0 \leq t - t_n \leq \tau .$$

Having proved that (3.36) implies (3.37), we conclude that (3.36) can be extended to larger values of τ , until we arrive at $\tau = \eta$. Hence

$$\sup_{t_n \leq t < t_n + \eta} \dot{R}(t) \leq \sup_{0 \leq t \leq \eta} \dot{\rho}(t) < \infty .$$

Taking $t_n \rightarrow T^*$ we get a contradiction to (3.29).

§4. Non-existence for $n = 3$. In this section we extend ideas from Section 3 to prove a non-existence result for the dislocation problem (2.1) – (2.5).

Set

$$\begin{aligned} x &= r - R(t) , \quad X = (x, z) , \\ x &= |X| \cos \theta , \quad z = |X| \sin \theta , \end{aligned}$$

and

$$(4.2) \quad u(x, z, t) = u(X, t) = c(r, z, t) .$$

Then

$$(4.3) \quad \begin{aligned} u_t &= u_{xx} + u_{zz} + \frac{1}{x + R(t)} u_x + \dot{R}(t)u_x \equiv \mathcal{L}u \\ &\text{if } x^2 + z^2 > 1 , \quad x > -R(t) , \end{aligned}$$

$$(4.4) \quad u_x = 0 \quad \text{on} \quad x = -R(t) ,$$

and

$$(4.5) \quad u = e^{\frac{\log R(t)}{R(t)}} h(\theta) \quad \text{on} \quad |X| = 1 .$$

By the maximum principle

$$(4.6) \quad 0 \leq u \leq C ,$$

as long as the solution exists.

The differential equation in (2.5) can be written in the form

$$(4.7) \quad \dot{R}(t) = \int_0^{2\pi} \frac{\partial u}{\partial N} \mu(\theta, R) d\theta$$

where

$$(4.8) \quad \mu(\theta, R) = 2\pi \left(1 + \frac{\cos \theta}{R} \right)$$

and $\partial u / \partial N$ is computed on the boundary $|X| = 1$.

For any $a = (\cos \tilde{\theta}, \sin \tilde{\theta})$ with $|\tilde{\theta}| < \theta_0 < \frac{\pi}{2}$ we shall construct a supersolution similar to the one for dimension $n = 1$. Let

$$(4.9) \quad \begin{aligned} \psi_a &= \left(e^{\frac{\log R}{R}} h(x, z) - \varepsilon \right) e^{-\frac{(X-a) \cdot a}{T-t}} + \varepsilon \\ \text{if} \quad & -\eta_0 \leq (X-a) \cdot a \leq \eta_1 \quad (\eta_0 > 0, \eta_1 > 0) \end{aligned}$$

where $h(x, z)$ is a smooth extension of $h(\theta)$ for $|\theta| \leq \theta_0$ such that $\partial h / \partial x = 0$, i.e., $h(x, z) \equiv h_0(z) = h(\arcsin z)$ for $|z| \leq \sin \theta_0$ and $h_0(z) \geq c_0 > 0$. Then

$$\begin{aligned} \frac{\partial \psi_a}{\partial t} &= h e^{\frac{\log R}{R}} e^{-\frac{(X-a) \cdot a}{T-t}} \left(-\frac{\log R}{R^2} + \frac{1}{R^2} \right) \dot{R} \\ &\quad - \left(h e^{\frac{\log R}{R}} - \varepsilon \right) \frac{(X-a) \cdot a}{(T-t)^2} e^{-\frac{(X-a) \cdot a}{T-t}} , \\ \mathcal{L} \psi_a &= \left\{ (\mathcal{L} h) e^{\frac{\log R}{R}} + \left(h e^{\frac{\log R}{R}} - \varepsilon \right) \left[\frac{1}{(T-t)^2} - \frac{1}{x+R} \frac{\cos \tilde{\theta}}{T-t} - \dot{R} \frac{\cos \tilde{\theta}}{T-t} \right] \right. \\ &\quad \left. - 2 e^{\frac{\log R}{R}} \frac{\nabla h \cdot a}{T-t} \right\} e^{-\frac{(X-a) \cdot a}{T-t}} . \end{aligned}$$

We shall prove later on that

$$(4.10) \quad \dot{R} \geq \frac{\Lambda}{T-t} \quad \text{for some } \Lambda > 0 .$$

Assuming this for the moment, we deduce that if ε, η_1 and T are sufficiently small and

$$(4.11) \quad \Lambda \cos \theta_0 > 1 ,$$

then

$$(4.12) \quad \frac{\partial \psi_a}{\partial t} - \mathcal{L} \psi_a > 0 .$$

Next we introduce a function

$$(4.13) \quad g_a = A e^{-(X-a) \cdot a} + \delta \quad \text{for } (X-a) \cdot a > \frac{1}{4} \eta_1, \quad 0 < \delta < \varepsilon .$$

Using (4.10) we deduce that, for $0 \leq t \leq T$,

$$(4.14) \quad \frac{\partial g_a}{\partial t} - \mathcal{L} g_a = A \left(-1 + \frac{\cos \tilde{\theta}}{R+x} + \dot{R} \cos \tilde{\theta} \right) > 0$$

if T is small enough.

Set

$$F(\xi, \zeta, t) = \left(h e^{\frac{\log R(t)}{R(t)}} - \varepsilon \right) e^{-\frac{\xi}{T-t}} - A e^{-\xi} + \varepsilon - \delta$$

where h is viewed as a function of $\xi = (X-a) \cdot a$ and a transversal variable ζ . We wish to find a C^1 curve $\xi = \xi(t, \zeta)$ such that

$$(4.15) \quad F(\xi(t, \zeta), \zeta, t) = 0 \quad (0 < t < T), \quad \xi(0, 0) = \frac{1}{2} \eta_1 .$$

The second condition determines A ,

$$A = (\varepsilon - \delta) e^{\frac{1}{2} \eta_1} + o(T) \quad (T \rightarrow 0) .$$

Since $\dot{R}(t) > 0$ (by (4.10), $\exp(\log R(t)/R(t))$ is uniformly bounded and continuously differentiable for $0 < t < T$. Noting also that

$$F_\xi = A e^{-\xi} + o(T), \quad F_\zeta = o(T) \quad \text{if } \frac{1}{4} \eta_1 < \xi < \frac{3}{4} \eta_1 ,$$

we deduce that there exists a unique C^1 solution $\xi(t, \zeta)$ of (4.15) and

$$\xi(t, \zeta) = \frac{1}{2} \eta_1 + o(T), \quad \xi_\zeta(t, \zeta) = o(T)$$

uniformly for all $-\infty < \zeta < \infty$. We now we define

$$(4.16) \quad w_a = \begin{cases} \psi_a & \text{if } -\eta_0 < (X-a) \cdot a < \xi(t, \zeta) \\ g_a & \text{if } (X-a) \cdot a > \xi(t, \zeta) . \end{cases}$$

On $(X-a) \cdot a = \xi(t, \zeta)$,

$$\frac{\partial \psi_a}{\partial \xi} = - \left(h e^{\frac{\log R}{R}} - \varepsilon \right) e^{-\frac{\xi}{T-t}} \left(\frac{1}{T-t} + O(1) \right) , \quad \frac{\partial g_a}{\partial \xi} = -A e^{-\xi}$$

and

$$\frac{\partial \psi_a}{\partial \zeta} = O \left(e^{-\frac{\xi}{T-t}} \right) , \quad \frac{\partial g_a}{\partial \zeta} = 0 .$$

It follows that

$$(4.17) \quad \left(\frac{\partial}{\partial t} - \mathcal{L} \right) w_a \geq 0 \quad \text{if } (X-a) \cdot a \geq -\eta_0 ,$$

in the distribution sense.

In view of (1.3), we can construct a function $G_0(X)$ such that

$$(4.18) \quad \begin{aligned} G_0 &= h \quad \text{if } |X| = 1 , \\ G_0 &= M e^{-\varepsilon_0 x} \quad \text{if } x < -R_0 \quad \text{or} \quad x > N , \\ \frac{\partial}{\partial x} G_0 + \varepsilon' G_0 &< 0 , \quad G_0 > 0 \quad \text{if } -R_0 < x < N, \quad x^2 + z^2 > 1 , \end{aligned}$$

where M, N_0, ε_0 are any positive numbers such that

$$(4.19) \quad M e^{\varepsilon_0 R_0} > C + 1 , \quad C \quad \text{as in (4.6)} , \quad M e^{-\varepsilon_0 N} < \min h , \quad N > 1 ,$$

and ε' is a sufficiently small positive number, $0 < \varepsilon' < \varepsilon_0$. Let $G(X, t)$ be the solution of

$$\begin{aligned} G_t &= G_{xx} + G_{zz} \quad \text{if } |X| > 1, \quad t > 0 , \\ G &= h \quad \text{if } |X| = 1 , \quad t > 0 , \\ G(X, 0) &= G_0(X) \quad \text{if } |X| > 1 . \end{aligned}$$

By the maximum principle (for unbounded spatial domains; see [5])

$$(4.20) \quad G(X, t) > 0 .$$

By continuity, $G_x + \varepsilon' G < 0$ on $|X| = 1$, if $0 \leq t \leq T$ and T is sufficiently small. We can apply the maximum principle to $G_x + \varepsilon' G$ and conclude that

$$(4.21) \quad G_x + \varepsilon' G < 0 \quad \text{if} \quad |X| \geq 1, \quad 0 \leq t \leq T .$$

By (4.19) and the continuity of G ,

$$(4.22) \quad G(-R(t), z, t) > C + \frac{1}{2} > u(-R(t), z, t) + \frac{1}{2}$$

if $0 \leq t \leq T$, $z \in \mathbb{R}$ provided T is sufficiently small.

Consider the function

$$\widehat{G}(X, t) = e^{\frac{\log R}{R}} G(X, t) .$$

It satisfies

$$(4.23) \quad \left(\frac{\partial}{\partial t} - \mathcal{L} \right) \widehat{G} = \left[-\frac{1}{R^2} (\log R - 1) G \dot{R} - \frac{G_x}{x + R} - \dot{R} G_x \right] e^{\frac{\log R}{R}} > 0$$

if

$$(4.24) \quad \frac{\log R(0) - 1}{R^2(0)} < \varepsilon' ;$$

here we used (4.20), (4.21).

Take $x_0 = N + 1$ and introduce a function

$$g(x) = K e^{-x} + \delta \quad \text{for} \quad x > \frac{1}{2} x_0$$

where $K > 0, \delta > 0$. By (4.10),

$$\left(\frac{\partial}{\partial t} - \mathcal{L} \right) g > 0 .$$

We want to patch together $\widehat{G}$ and g along a curve $x = x_0(z, t) \sim x_0$. This means that

$$(4.25) \quad \delta \sim M e^{-\varepsilon_0 x_0} - K e^{-x_0} .$$

Along this curve

$$\widehat{G}_x > g_x \quad (0 \leq t \leq T)$$

provided

$$(4.26) \quad \varepsilon_0 M e^{-\varepsilon_0 x_0} < K e^{-x_0}$$

and T is small enough. Clearly there is a $\delta > 0$ such that both (4.26) and (4.25) are satisfied.

Let

$$G^* = \begin{cases} \widehat{G} & \text{if } x < x_0(z, t) \\ g & \text{if } x > x_0(z, t) . \end{cases}$$

Then G^* is a supersolution, i.e.,

$$\left(\frac{\partial}{\partial t} - \mathcal{L} \right) G^* \geq 0$$

in the distribution sense. Furthermore, $G^* = u$ on $|X| = 1$ and, by (4.22), G^* majorizes u on $x = -R(t)$.

So far we have assumed that (4.10) holds where Λ satisfies (4.11). More generally, we proved that if

$$(4.27) \quad \dot{R}(t) \geq \frac{\Lambda}{T-t} \quad \text{for } 0 \leq t \leq \tau$$

where $0 < \tau < T$, then (i) w_a is a supersolution in $\{(X-a) \cdot a \geq -\eta_0, 0 < t < \tau\}$, and (ii) G^* is a supersolution in $\{|X| > 1, 0 < t < \tau\}$ provided $R(0)$ is large enough (so that (4.24) holds). Observe that if T is small enough (depending on η_0) then, for $0 < t < T$,

$$\psi_a > u \quad \text{on} \quad \{(X-a) \cdot a = -\eta_0\} \cap \{|X| > 1\} .$$

Also $\psi_a \geq u$ on $\{(X-a) \cdot a > -\eta_0\} \cap \{|X| = 1\}$ and $\psi_a(a, t) = u(a, t)$. Hence, by the maximum principle, $\psi_a > u$ in $\{-\eta_0 < (X-a) \cdot a, |X| > 1, 0 < t < \tau\}$ and

$$\frac{\partial w_a}{\partial N} < \frac{\partial u}{\partial N} \quad \text{at } X = a, 0 < t < \tau .$$

It follows that, for $|\tilde{\theta}| < \tilde{\theta}_0$,

$$\frac{\partial u}{\partial N} > \left(h(\tilde{\theta}) e^{\frac{\log R}{R}} - \varepsilon \right) \frac{\cos \tilde{\theta}}{T-t} \quad \text{at } (a, t) .$$

For $|\tilde{\theta}| > \theta_0$ we compare u in G^* and conclude that

$$\frac{\partial u}{\partial N} > \frac{\partial G^*}{\partial N} \geq -C^*$$

where C^* is a constant independent of T . It follows that

$$\dot{R}(t) = \int_0^{2\pi} \frac{\partial u}{\partial N} \mu d\theta \geq -3\pi C^* + \frac{4\pi}{T-t} \int_0^{\theta_0} \left(h(\theta) e^{\frac{\log R}{R}} - \varepsilon \right) \cos \theta d\theta .$$

If we define

$$(4.28) \quad \Lambda = 4\pi \int_0^{\theta_0} h(\theta) \cos \theta d\theta - \tilde{\varepsilon}$$

where $\tilde{\varepsilon}$ is any small positive number, then what we have proved is that (4.27) implies

$$\dot{R}(t) > \frac{\Lambda}{T-t} \quad \text{for } 0 \leq t \leq \tau$$

provided $R(0)$ is sufficiently large and (4.11) is satisfied.

We can clearly choose initial data $\tilde{c}(x, y)$ such that, whenever $c_0(x, y) < \tilde{c}(x, y)$, $\dot{R}(0) > \Lambda/T$. But then, a step-by-step argument shows that (4.11) must hold for all $0 < t < T$; hence $\dot{R}(t)$ must “blow up” as $t \uparrow T$, if the solution (c, R) exists for all $0 \leq t < T$.

We summarize:

THEOREM 4.1. *Assume that*

$$(4.29) \quad \left[4\pi \int_0^{\theta_0} h(\theta) \cos \theta d\theta \right] \cos \theta_0 > 1$$

for some $0 < \theta_0 < \frac{\pi}{2}$. Then, if $R(0)$ is sufficiently large, there exist $\delta > 0$, $T > 0$ and a function $\tilde{c}(x, z)$ in $C^3\{(x - R(0))^2 + z^2 \geq 1\}$ satisfying

$$\tilde{c}(x, z) > 0 ,$$

$$\tilde{c} = e^{\frac{\log R(0)}{R(0)}} h \quad \text{on } (x - R(0))^2 + z^2 = 1 ,$$

$$\tilde{c}(x, z) \rightarrow \delta \quad \text{if } x^2 + z^2 \rightarrow \infty ,$$

such that if c_0 satisfies the assumptions in Theorem 2.1 and

$$c_0(x, z) < \tilde{c}(x, z) \quad \text{for } (x - R(0))^2 + z^2 > 1 ,$$

then the solution (c, R) corresponding to c_0 exists only for the time-interval $0 < t < T^*$, where $T^* \leq T$, and

$$\dot{R}(t) \geq 0 , \quad \limsup_{t \rightarrow T^*} \dot{R}(t) = \infty .$$

Remark 4.1. If in (4.9) we take $e^{-\beta \frac{(X-a) \cdot a}{T-t}}$ with $\beta \neq 1$, we actually do not gain any improvement on the size of the quantity Λ , i.e., we end up with the same condition (4.29).

§5. Existence for $n=1$. In Section 3 we proved non-existence in case $n = 1$ when $m > 1$ (Problem (3.1)), and, in Section 4, we established an analogous result for the (3-dimensional) dislocation problem (2.1) – (2.5) under the condition (4.29) on h . In this section we assume that $n = 1$ and $m < 1$ and prove, for a large class of initial data, that there exists a global solution to the problem (3.1). Global existence for the 3-dimensional dislocation problem will be considered in the following sections.

Throughout this section we assume that

$$(5.1) \quad \|c_0\|_{C^{2+\alpha}(\mathbf{R})} < \infty .$$

We begin with the case $0 < m < 1$ and assume:

$$(5.2) \quad \begin{aligned} 0 \leq c_0(x) < M & \quad \text{if } x < R(0) , \\ 0 \leq c_0(x) < m & \quad \text{if } x > R(0) + 1 \quad (M > m) . \end{aligned}$$

THEOREM 5.1. *If (5.1), (5.2) hold and $0 < m < 1$ then there exists a unique bounded solution of (3.1), for all $t > 0$.*

Proof. We may assume that

$$(5.3) \quad c_0(x) \geq \varepsilon > 0 ,$$

for otherwise we can work with $c(x, t) + \varepsilon$ where $0 < \varepsilon < 1 - m$. By (5.2) and the maximum principle,

$$u_x(1, t) < 0 , \quad u_x(0, t) > 0$$

where $u(x, t) = c(x - R(t), t)$; hence $\dot{R}(t) > 0$. In view of Theorem 2.2, it remains to show that

$$(5.4) \quad \dot{R}(t) \leq C .$$

Let

$$(5.5) \quad w(x, t) = \begin{cases} me^{-a(x-1)} & \text{if } x > 1 \\ u(x, t) & \text{if } x < 0 \end{cases} \quad (a > 0)$$

and define $\tilde{R}(t)$ by $\tilde{R}(0) = R(0)$ and

$$(5.6) \quad \dot{\tilde{R}}(t) = -w_x(1, t) + u_x(0, t) .$$

Let Φ be the solution to

$$\Phi_t - \Phi_{xx} = 0 \quad \text{if } x < 0, t > 0,$$

with

$$\Phi(0, t) = M, \quad \Phi(x, 0) < u(x, 0), \quad \Phi_x(x, 0) > 0.$$

Then $\Phi_x > 0$, by the maximum principle, and therefore

$$\Phi_t - \Phi_{xx} - \dot{R}\Phi_x < \Phi_t - \Phi_{xx} = 0.$$

It follows that $\Phi(x, t) < u(x, t)$ if $x < 0$, and

$$u_x(0, t) < \Phi_x(0, t) \leq C.$$

Using (5.6) we conclude that

$$(5.7) \quad \dot{\tilde{R}} \leq am + C.$$

We now deduce that, for $x > 1$,

$$(5.8) \quad w_t - w_{xx} - \dot{\tilde{R}}w_x \leq \{-ma^2 + (am + C)ma\}e^{-a(x-1)} < 0$$

if $a \gg 1$ (recall that $0 < m < 1$).

If a is very large then, because of (5.3), $u(x, 0) > w(x, 0)$ if $x > 1$ and $u_x(1, 0) < w_x(1, 0)$. By continuity we deduce that, for some $\tau > 0$,

$$(5.9) \quad \begin{aligned} w(x, t) &\leq u(x, t) \quad \text{if } x > 0, 0 \leq t \leq \tau, \\ \dot{R}(t) &\leq \dot{\tilde{R}}(t) \quad \text{if } 0 \leq t \leq \tau. \end{aligned}$$

If the solution (u, R) exists for all $t < T$, then (5.9) must hold for all $\tau < T$. Indeed, otherwise there is a maximal τ for which (5.9) holds, and $\tau < T$. Since $w_x < 0$ if $x > 1$,

$$w_t - w_{xx} - \dot{R}w_x \leq w_t - w_{xx} - \dot{\tilde{R}}w_x < 0$$

by the second inequality in (5.9) and by (5.8). Hence, by the maximum principle,

$$w(x, t) < u(x, t) \quad \text{and} \quad w_x(1, t) < u_x(1, t)$$

for $0 \leq t \leq \tau$. It follows that $\dot{R}(t) < \dot{\tilde{R}}(t)$ if $0 \leq t \leq \tau$ and then, by continuity, (5.9) holds for larger values of τ ; a contradiction.

The assertion (5.4) now follows from (5.7) and the second inequality in (5.9).

We next consider the case $m \leq 0$. This case arises if

$$\lim_{x \rightarrow \infty} c_0(x) = \gamma > m$$

provided we work with the function $c - \gamma$ instead of c .

We introduce the functions

$$\begin{aligned} u_1(x, t) &= M - c(x, t) & \text{for } x < R(t) , \\ u_2(x, t) &= m - c(x + 1, t) & \text{for } x > R(t) . \end{aligned}$$

Then (u_1, u_2, R) forms a solution to the 2-phase Stefan problem

$$\begin{aligned} (5.10) \quad & u_{1,t} - u_{1,xx} = 0 & \text{if } x < R(t), t > 0 , \\ & u_{2,t} - u_{2,xx} = 0 & \text{if } x > R(t), t > 0 , \\ & u_1 = u_2 = 0 & \text{on } x = R(t) , \\ & \dot{R}(t) = -u_{1,x} + u_{2,x} & \text{on } x = R(t) \end{aligned}$$

with initial data

$$\begin{aligned} (5.11) \quad & u_1(x, 0) = u_{10}(x) , \quad x < R(0) , \\ & u_2(x, 0) = u_{20}(x) , \quad x > R(0) . \end{aligned}$$

We assume:

$$\begin{aligned} (5.12) \quad & u_{10}(x) \geq 0, \quad \int_{-\infty}^0 |u_{10} - \gamma_1|^2 < \infty \quad (\gamma_1 > 0) , \\ & u'_{10}(x) \rightarrow 0 \quad \text{if } x \rightarrow -\infty , \end{aligned}$$

$$\begin{aligned} (5.13) \quad & u_{20}(x) \leq 0 , \quad \int_0^{\infty} |u_{20} + \gamma_2|^2 < \infty \quad (\gamma_2 > 0) , \\ & u'_{20}(x) \rightarrow 0 \quad \text{if } x \rightarrow +\infty , \end{aligned}$$

THEOREM 5.2. *If (5.12), (5.13) hold then the Stefan problems (5.10), (5.11) has a unique solution for all $t > 0$, and*

$$(5.14) \quad \frac{R(t)}{\sqrt{t}} \rightarrow \alpha, \quad 2\sqrt{t} \dot{R}(t) \rightarrow \alpha \quad \text{if } t \rightarrow \infty,$$

where $\alpha = \gamma_2 - \gamma_1$.

Proof. Set

$$A_i = \sup |u'_{i0}(x)|.$$

It is well known (cf. [6]) that there exists a unique solution to the Stefan problem (5.10), (5.11) and

$$|\dot{R}(t)| \leq \max\{A_1, A_2\}.$$

To prove (5.14) we introduce, for any $\lambda > 1$, the functions

$$w_{i\lambda}(x, t) = u_i(\lambda x, \lambda^2 t)$$

and the corresponding free boundaries $R_\lambda(t) = \frac{1}{\lambda} R(\lambda^2 t)$. By compactness results for the Stefan problem one can show that any sequence $\lambda \rightarrow \infty$ has a subsequence for which

$$w_{i\lambda}(x, t) \rightarrow W_i(x, t), \quad R_\lambda^{(j)}(t) \rightarrow R_\infty^{(j)}(t) \quad (j = 0, 1)$$

uniformly in compact subsets, where (W_1, W_2, R_∞) is the unique solution of the Stefan problem (3.10) with

$$W_1(x, 0) = \gamma_1 \quad \text{if } x < 0,$$

$$W_2(x, 0) = -\gamma_2 \quad \text{if } x > 0$$

Since (cf. [5]) $R_\infty(t) = \alpha\sqrt{t}$, it follows that

$$\frac{1}{\lambda} R(\lambda^2 t) \rightarrow \alpha\sqrt{t} \quad \text{in } C^1 \left[\frac{1}{2} \leq t \leq \frac{3}{2} \right]$$

as $\lambda \rightarrow \infty$. Taking $t = 1$, the assertion (5.14) follows.

§6. Existence for $n=3$ and small c_0, h . In this section we consider (2.1) – (2.5) with

$$(6.1) \quad \begin{cases} h & \text{replaced by } \varepsilon h, \\ c_0 & \text{replaced by } \varepsilon c_0 \end{cases} \quad (\varepsilon > 0),$$

and prove that if ε is sufficiently small then there exists a global solution, and $R(t) \rightarrow \infty$ if $t \rightarrow \infty$.

Introducing the notation of Section 4 (cf. (4.1) – (4.2)), we can write the system (2.1) – (2.5), (6.1) in the form

$$(6.2) \quad u_t = u_{xx} + u_{zz} + \frac{1}{x+R} u_x + \dot{R}(t)u_x \quad \text{if } x^2 + z^2 > 1, \ x > -R(t),$$

$$(6.3) \quad u(x, z, t) = \varepsilon e^{\frac{\log R}{R}} h(\theta) \quad \text{on } \{x = \cos \theta, \ z = \sin \theta\},$$

$$(6.4) \quad \frac{\partial u}{\partial x} = 0 \quad \text{on } x = -R(t),$$

$$(6.5) \quad u(x, z, 0) = \varepsilon u_0(x, z) \geq 0,$$

and

$$(6.6) \quad R(0) = R_0 > 1,$$

$$(6.7) \quad R\dot{R} = 2\pi \int_{|X|=1} (x+R) \frac{\partial u}{\partial N} d\theta,$$

where $x = \cos \theta$ in the integrand.

We shall assume:

$$(6.8) \quad \|u_0\|_{C^{2,\alpha}(\mathbf{R}^2)} \leq 1, \quad u_0 \geq 0,$$

$$(6.9) \quad \begin{cases} \|h\|_{C^{2,\alpha}} \leq 1, \\ h(\theta) > 0, \ h(-\theta) = h(\theta) \quad \text{for } 0 \leq \theta \leq 2\pi, \\ h'(\theta) > 0 \quad \text{for } 0 < \theta < \pi \end{cases}$$

and the additional conditions:

$$(6.10) \quad \begin{cases} \int_{\{x > -R_0\}} (x+R_0)u_0(x, z) dx dz < 1, \\ (1+|x|)|\nabla u_0(x, z)| \rightarrow 0 \quad \text{if } x^2 + z^2 \rightarrow \infty. \end{cases}$$

THEOREM 6.1. *If (6.8) – (6.10) hold and $R_0 - 1 \geq \mu > 0$, then there exists an $\varepsilon_0 > 0$ depending only on μ such that if $0 < \varepsilon \leq \varepsilon_0$ then the system (6.2) – (6.7) has a unique bounded solution for all $0 \leq t < \infty$; furthermore,*

$$(6.11) \quad \liminf_{t \rightarrow \infty} \frac{R(t)}{t^{1/2}} > 0.$$

Proof. We assume that the solution exists for all $0 \leq t < T$, and proceed to establish a priori bounds, independently of T .

Multiplying (6.2) by $x + R$ and integrating over the 2-dimensional domain $\Omega_t \equiv \{x > -R(t)\} \cap \{|X| > 1\}$, we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} (x + R)u &= \int_{\Omega_t} [(x + R)u_t + \dot{R}u] \\ &= \int_{\Omega_t} ((x + R)u_x)_x + ((x + R)u_z)_z + \dot{R}((x + R)u)_x \\ &= \int_{|X|=1} (x + R) \frac{\partial u}{\partial N} d\theta + \dot{R} \int_{|X|=1} (x + R)u \cos(x, N) d\theta ; \end{aligned}$$

the boundary integrals at infinity are equal to zero, by (6.10). Since the right-hand side is equal to

$$\frac{1}{2\pi} R\dot{R} + \varepsilon(c_1 R\dot{R} - c_2 \dot{R})e^{\frac{\log R}{R}} ,$$

where c_1, c_2 are constants, we conclude that

$$\begin{aligned} (6.12) \quad & \int_{\Omega_t} (x + R)u - \int_{\Omega_0} \varepsilon(x + R_0)u_0 \\ &= \frac{1}{2\pi} \left(\frac{1}{2} R^2(t) - \frac{1}{2} R_0^2 \right) + \varepsilon(g(R(t)) - g(R_0)) \end{aligned}$$

where

$$g(R) = \int_1^R (c_1 \xi - c_2) e^{\frac{\log \xi}{\xi}} d\xi .$$

Recalling that $u \geq 0$ we deduce that

$$(6.13) \quad R^2(t) \geq R_0^2 - 4\pi\varepsilon - 4\pi\varepsilon(g(R(t)) - g(R_0)) .$$

Take ε_0 small enough so that

$$4\pi\varepsilon_0 + 4\pi\varepsilon_0 |g(t) - g(R_0)| \leq \frac{1}{2} \left[R_0^2 - \left(\frac{1 + R_0}{2} \right)^2 \right] + \frac{1}{2} [R^2(t) - R_0^2] \quad \text{if } R_0 \leq t \leq \frac{1 + R_0}{2} .$$

Then (6.13) implies that

$$(6.14) \quad R(t) > \frac{1 + R_0}{2} \quad \text{if } 0 \leq t < T .$$

Next, by the maximum principle,

$$(6.15) \quad 0 \leq u(x, z, t) \leq \varepsilon \sup_{\xi \geq 1} e^{\frac{\log \xi}{\xi}} < 2\varepsilon$$

By Theorem 2.1, $T \geq \tau$ where τ is independent of ε . Set

$$K = \max_{0 \leq t \leq \tau} |\dot{R}(t)|.$$

We claim that

$$(6.16) \quad |\dot{R}(t)| < K + 1 \quad \text{for } 0 \leq t < T.$$

Indeed, by continuity

$$(6.17) \quad |\dot{R}(t)| \leq K + 1 \quad \text{for } 0 \leq t \leq \tau + \tau^*$$

for some $\tau^* > 0$. By L^p interior-boundary estimates we then have (since $0 \leq u \leq 2\varepsilon$)

$$\|u\|_{W_p^{2,1}(D_*)} \leq C_p \varepsilon \quad \text{where } D_* = \left\{ 1 \leq x^2 + z^2 \leq \frac{3}{4} + \frac{R_0}{4}, \tau + \frac{\tau^*}{2} < t < \tau + \tau^* \right\},$$

where C_p is a constant independent of τ, τ^* . By imbedding we conclude that

$$\|\nabla u\|_{L^\infty(D_*)} \leq \tilde{C} \varepsilon$$

provided p is taken large enough. Restricting ε_0 to satisfy

$$8\pi^2 \tilde{C} \varepsilon_0 < K + 1$$

we get

$$|\dot{R}(t)| = 2\pi \int_{|X|=1} \frac{\partial u}{\partial N} \left(1 + \frac{\cos \theta}{R} \right) d\theta < K + 1$$

for all $\tau \leq t \leq \tau + \frac{\tau^*}{2}$. Hence, by continuity, (6.17) holds for $0 \leq t \leq \tau + \bar{\tau}$ where $\bar{\tau} > \tau^*$. This argument shows that (6.17) must hold for all $t < T$, so that (6.16) follows.

Having established (6.14), (6.16), we can now appeal to Theorem 2.2 to deduce that there exists a global solution for (6.2) – (6.7) provided $0 < \varepsilon \leq \varepsilon_0$. It remains to prove (6.11).

From (6.12) and the estimate $|g(R)| \leq C(R^2 + 1)$ we see that

$$(6.18) \quad R^2(t) = 4\pi \int_{\Omega_t} (x + R)u(x, z, t) dx dz + R_0^2 + \varepsilon I,$$

$$|I| \leq C(R^2(t) + 1),$$

We shall compare u with the bounded solution w of

$$\begin{aligned}
(6.19) \quad & w_t = w_{xx} + w_{zz} + \frac{1}{x+R} w_x + \dot{R}w_x \quad \text{in} \quad \{x > -R(t)\} \setminus P, \\
& w = \varepsilon \lambda \quad \text{on} \quad \partial P, \\
& w_x = 0 \quad \text{on} \quad x = -R(t), \\
& w(x, y, 0) = 0,
\end{aligned}$$

where

$$P = \left\{ |x| < \frac{\sqrt{2}}{2}, |z| < \frac{\sqrt{2}}{2} \right\}, \quad \lambda = \min h(\theta) > 0.$$

Clearly

$$(6.20) \quad w \leq \varepsilon \lambda$$

and

$$(6.21) \quad w \leq u.$$

It is also clear that

$$(6.22) \quad \frac{\partial w}{\partial N} > 0 \quad \text{on} \quad \partial P$$

where N is the inner normal.

Analogously to the derivation of (6.12),

$$\begin{aligned}
(6.23) \quad & \int_{\{x > -R(t)\} \setminus P} (x + R(t)) w dx dy = \int_0^t \int_{\partial P} (x + R(\tau)) \frac{\partial w}{\partial N} ds d\tau \\
& + \varepsilon \lambda (R(t) - R_0) \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} x \Big|_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} dz \\
& \geq \int_0^t \int_{\partial P} R(\tau) \frac{\partial w}{\partial N} - 2\varepsilon \lambda (R(t) - R_0),
\end{aligned}$$

by (6.22).

To estimate $\int_{\partial P} \partial w / \partial N$, we compare w with the solution $\tilde{w}$ of

$$\begin{aligned}\tilde{w}_t &= \tilde{w}_{zz} \quad \text{if} \quad z > \frac{\sqrt{2}}{2}, \quad t > 0, \\ \tilde{w} &= \varepsilon \lambda \quad \text{on} \quad z = \frac{\sqrt{2}}{2}, \\ \tilde{w} &= 0 \quad \text{for} \quad t = 0.\end{aligned}$$

Clearly $w(x, z, t) < \tilde{w}(z, t)$ if $z > \frac{\sqrt{2}}{2}$ and

$$-\frac{\partial w}{\partial z} \geq -\frac{\partial \tilde{w}}{\partial z} \quad \text{on} \quad \left\{ z = \frac{\sqrt{2}}{2}, \quad |x| < \frac{\sqrt{2}}{2} \right\}.$$

On the other hand, we can construct $\tilde{w}$ explicitly:

$$\tilde{w}(z, t) = -\gamma \int_0^{\frac{z - \sqrt{2}/2}{\sqrt{t}}} e^{-\frac{\xi^2}{4}} d\xi + \varepsilon \lambda \quad \left(\frac{1}{\gamma} = \varepsilon \lambda \int_0^\infty e^{-\frac{\xi^2}{4}} d\xi \right).$$

Hence

$$-\tilde{w}_z \left(\frac{\sqrt{2}}{2}, t \right) = \frac{\gamma}{\sqrt{t}},$$

and, by (6.23), the integral on the right-hand side of (6.23) is

$$\geq c \int_0^t \frac{R(\tau)}{\sqrt{\tau}} d\tau - 2\varepsilon \lambda R(t) \quad \text{for some } c > 0.$$

Using this and (6.21) in (6.18) we get

$$(6.24) \quad R^2(t) \geq c \int_0^t \frac{R(\tau)}{\sqrt{\tau}} d\tau \quad \text{if } t > 1,$$

with another constant $c > 0$.

Solving this Gronwall's inequality, we obtain

$$(6.25) \quad R(t) \geq c\sqrt{t} \quad \text{for all } t \geq 0.$$

Remark 6.1. A careful review of the proof of Theorem 6.1 shows that $\dot{R}(t) \leq C\varepsilon$, where C is a positive constant independent of ε , so that

$$(6.26) \quad \limsup_{t \rightarrow \infty} \frac{R(t)}{t} \leq C .$$

Remark 6.2. Theorem 6.1 can be extended to the case where

$$(6.27) \quad \begin{aligned} \|h - h(0)\|_{C^{2+\alpha}} \ll 1, \quad \|c_0 - c^*\|_{C^{2+\alpha}} \ll 1, \quad \int_{\mathbb{R}^3 \setminus D(R(0))} |c_0 - c^*| \ll 1, \\ (1+r)|\nabla c_0(r, z)| \rightarrow 0 \quad \text{if} \quad r^2 + z^2 \rightarrow \infty \end{aligned}$$

for some constant c^* , provided

$$(6.28) \quad R(0) \gg 1 .$$

Indeed, if we work with $c - c^*$, instead of c , this case is reduced to a slightly different version of Theorem 6.1 (with $R(0) \gg 1$).

§7. Travelling waves solutions (for $n=2$). In this section we consider the case $n = 2$, that is, we drop the term $\frac{1}{r} c_r$ in the diffusion equation for c . Physically this means that instead of a loop dislocation we are considering a cylindrical dislocation.

The function c satisfies

$$(7.1) \quad c_t = c_{rr} + c_{zz} \quad \text{in} \quad \mathbb{R}^2 \setminus D(R(t)) ,$$

$$(7.2) \quad c = e^{A \frac{\log R}{R}} h(\theta) \quad \text{on} \quad \partial D(R(t)) ,$$

$$(7.3) \quad \dot{R} = \int_{\partial D(R(t))} \frac{\partial c}{\partial N} ds \quad (ds = 2\pi d\theta)$$

where $D(R) = \{(r - R)^2 + z^2 \leq 1\}$, and A is a positive parameter. (For the applications in Sections 8,9 it is convenient to have the factor 2π in (7.3).)

We shall consider the case $A = 0$, and look for travelling wave solutions, namely, solutions with $\dot{R} = \lambda$ where λ is a positive constant.

Let

$$c(r, z, t) = G(x, z) \quad \text{where} \quad x = r - \lambda t - R(0) .$$

Then

$$(7.4) \quad G_{xx} + G_{zz} + \lambda G_x = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus D$$

where $D = \{x^2 + z^2 \leq 1\}$, and

$$(7.5) \quad G = h(\theta) \quad \text{on} \quad \{x = \cos \theta, z = \sin \theta\} .$$

Set

$$(7.6) \quad G = e^{-\frac{\lambda}{2}x} \Psi + K .$$

Then

$$(7.7) \quad \Psi_{xx} + \Psi_{zz} = \frac{\lambda^2}{4} \Psi \quad \text{in} \quad \mathbb{R}^2 \setminus D ,$$

$$(7.8) \quad \Psi(\cos \theta, \sin \theta) = e^{\frac{\lambda}{2} \cos \theta} (h(\theta) - K) .$$

We shall make the further transformation

$$(7.9) \quad \Psi = \varphi \exp \left(-\frac{\lambda}{2} \sqrt{x^2 + z^2} \right) .$$

Then

$$(7.10) \quad -(\varphi_{xx} + \varphi_{zz}) + \lambda \frac{x\varphi_x + z\varphi_z}{\sqrt{x^2 + z^2}} + \frac{\lambda}{2} \frac{\varphi}{\sqrt{x^2 + z^2}} = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus D ,$$

$$(7.11) \quad \varphi(\cos \theta, \sin \theta) = e^{\frac{\lambda}{2} \cos \theta + \frac{\lambda}{2}} (h(\theta) - K) \quad \text{for} \quad 0 \leq \theta < 2\pi .$$

LEMMA 7.1. *For any $\lambda > 0$ and $K \in \mathbb{R}^1$ there exists a unique solution of (7.10), (7.11) with*

$$\|\varphi\|_{L^\infty(\mathbb{R}^2 \setminus D)} < \infty .$$

Proof. Consider the truncated problem in $\{|X| \leq \rho\}$ with $\varphi = 0$ on $|X| = \rho$. The corresponding solution φ^ρ is bounded by

$$e^\lambda \sup |h(\theta) - K|$$

and, by compactness, a subsequence is convergent to a bounded solution φ .

To prove uniqueness observe that, since $\lambda > 0$, $\log(x^2 + z^2)$ is a supersolution for (7.10). Hence, if $\tilde{\varphi}$ is another solution, then

$$|\varphi(x, z) - \tilde{\varphi}(x, z)| \leq \left\{ \sup_{\xi^2 + \zeta^2 = \sigma^2} |\varphi(\xi, \zeta) - \tilde{\varphi}(\xi, \zeta)| \right\} \frac{|\log(x^2 + z^2)|}{\log \sigma^2}$$

for any $1 \leq x^2 + z^2 < \sigma^2$. Letting $\sigma \rightarrow \infty$ we get $\varphi \equiv \tilde{\varphi}$.

We can rewrite Lemma 7.1 in terms of G :

LEMMA 7.2. For any $\lambda > 0$ and $K \in \mathbb{R}^1$ there exists a unique solution of (7.4), (7.5) such that

$$(7.12) \quad \sup_{(x,z) \in \mathbb{R}^2 \setminus D} \left| \exp \left(\frac{\lambda}{2} \sqrt{x^2 + z^2} \right) e^{\frac{\lambda}{2} x} (G(x, z) - K) \right| < \infty .$$

Next we want to find a K such that

$$(7.13) \quad \int_{\partial D} \frac{\partial G}{\partial N} ds = \lambda .$$

We can write the corresponding φ in the form $\varphi = \varphi_1 - K\varphi_2$ where φ_1 is the solution corresponding to h and $K = 0$ and φ_2 is the solution corresponding to $h \equiv 1$ and $K = 0$. Therefore

$$G = G_1 - KG_2 + K$$

where

$$G_i = \left\{ e^{-\frac{\lambda}{2} x} \left(\exp \left(-\frac{\lambda}{2} \sqrt{x^2 + z^2} \right) \right) \right\} \varphi_i \quad (i = 1, 2) .$$

Clearly

$$(7.14) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + \lambda \frac{\partial}{\partial x} \right) G_i = 0 \quad \text{in } \mathbb{R}^2 \setminus D ,$$

$$(7.15) \quad \sup \left| \exp \left(\frac{\lambda}{2} \sqrt{x^2 + z^2} \right) e^{\frac{\lambda}{2} x} G_i(x, z) \right| < \infty$$

and

$$(7.16) \quad G_1(\cos \theta, \sin \theta) = h(\theta) ,$$

$$(7.17) \quad G_2(\cos \theta, \sin \theta) = 1 .$$

Condition (7.13) becomes

$$(7.18) \quad \int_{\partial D} \frac{\partial G_1}{\partial N} - K \int_{\partial D} \frac{\partial G_2}{\partial N} = \lambda .$$

The function G_2 is a limit of function G_2^ρ corresponding to the solutions φ_2^ρ of truncated problems with $\varphi_2^\rho = 0$ or $x^2 + z^2 = \rho^2$. By the maximum principle, $G_2^\rho \leq 1$ and, consequently, $G_2 \leq 1$. Since, by (7.15), $G_2 \not\equiv 1$, we also have

$$(7.19) \quad \gamma \equiv \int_{\partial D} \frac{\partial G_2}{\partial N} > 0 .$$

Observing that both G_1 and G_2 are independent of K , it follows from (7.19) that equation (7.18) has precisely one solution K .

We have proved:

THEOREM 7.3. *For any $\lambda > 0$ there exists a unique $K \in \mathbb{R}$ such that there exists a bounded travelling wave solution with velocity λ for the two-dimensional problem (7.1)–(7.3) which satisfies (7.12).*

Remark 7.1. Clearly, if

$$(7.20) \quad \int_{\partial D} \frac{\partial G_1}{\partial N} > \lambda \quad \text{then} \quad K > 0 .$$

If we assume that $h(\theta) = B_0 \hat{h}(\theta)$ where $\|1 - \hat{h}\|_{C^{1,\alpha}} \ll 1$, then, by continuity and (7.19),

$$\int_{\partial D} \frac{\partial G_1}{\partial N} > \frac{1}{2} \gamma B_0$$

so that, by (7.20), $K > 0$ if $\lambda < \frac{1}{2} \gamma B_0$.

In the next two sections we shall consider the 3-dimensional problem with data “near” the travelling wave solution established in Theorem 7.3 for small $\lambda > 0$. We shall need a very careful estimate on the dependence of the constant K on λ , as $\lambda \rightarrow 0$. The remainder of this section is devoted to deriving this estimate.

We begin with a more careful study of the solution of

$$(7.21) \quad \begin{aligned} \Delta u &= \frac{1}{4} u \quad \text{in} \quad \mathbb{R}^2 \setminus B(\lambda) , \\ u &= g(\theta) \quad \text{on} \quad \partial B(\lambda) , \\ u &\text{ bounded} , \end{aligned}$$

where $g \in C^2$ and $B(\lambda) = \{X \in \mathbb{R}^2, |X| < \lambda\}$.

We shall derive an integral representation

$$(7.22) \quad u(r, \theta) = \int_0^{2\pi} T(r, \theta - \bar{\theta}) g(\bar{\theta}) d\bar{\theta}$$

where

$$(7.23) \quad T(r, \theta) = \frac{1}{2\pi} \sum_{n=0}^{\infty} e^{in\theta} \frac{K_n\left(\frac{r}{2}\right)}{K_n\left(\frac{\lambda}{2}\right)}$$

and the K_n are the modified Bessel functions [1; p.5]. They satisfy

$$(7.24) \quad z^2 K_n''(z) + z K_n'(z) - (z^2 + n^2) K_n(z) = 0 ,$$

and [1; p.86]

$$(7.25) \quad K_n(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + O\left(\frac{1}{|z|}\right) \right) \quad \text{if } z > 0, z \rightarrow \infty.$$

We shall need the following inequalities:

$$(7.26) \quad 0 < \frac{K_n(r/2)}{K_n(\lambda/2)} < 1, \quad \frac{K'_n(r/2)}{K'_n(\lambda/2)} < 0.$$

To prove (7.26) note that the function $\xi(r) = K_n(r/2)/K_n(\lambda/2)$ is a solution of (7.24) and $\xi(\lambda) = 1$, $\xi(\infty) = 0$. By the maximum principle, $\xi(r)$ cannot have a local maximum or minimum in $\lambda < r < \infty$; hence $\xi'(r) < 0$ and (7.26) follows.

Using

$$\frac{1}{2\pi} \sum_{n=0}^{\infty} e^{in\theta} = \delta(\theta),$$

$$\left| \int_0^{2\pi} e^{in\bar{\theta}} g(\bar{\theta}) d\bar{\theta} \right| \leq \frac{C}{n^2} \quad \text{if } g \in C^2,$$

and (7.26) it follows that the series obtained by substituting (7.23) into (7.22) is uniformly convergent and is, in fact, the solution to (7.21).

Since $g(\theta) \geq 0$ implies $u(r, \theta) \geq 0$ (by the maximum principle), we deduce that

$$(7.27) \quad T(r, \theta) \geq 0.$$

As a special case of the representation (7.22),

$$(7.28) \quad \begin{aligned} \left| \frac{K_n(r/2)}{K_n(\lambda/2)} \right| &= \left| \int_0^{2\pi} T(r, \theta - \bar{\theta}) e^{in\bar{\theta}} d\bar{\theta} \right| \\ &\leq \int_0^{2\pi} T(r, \theta - \bar{\theta}) d\bar{\theta} = \frac{K_0(r/2)}{K_0(\lambda/2)}. \end{aligned}$$

Using this estimate in the integral representation for u , and using also (7.26) with $n = 0$, we get

$$(7.29) \quad |u(r, \theta)| \leq C \frac{K_0(r/2)}{K_0(\lambda/2)} \|g\|_{C^2} \leq C \|g\|_{C^2} \quad (C > 0).$$

Recalling (7.25) we also get

$$(7.30) \quad |u| \leq \frac{C e^{-r/2}}{\sqrt{r} K_0(\lambda/2)} \quad (r \rightarrow \infty)$$

with another constant C .

To estimate the first derivatives of u near $r = \lambda$ we consider $v(X) = u(\lambda X)$, for $1 < |X| < 2$. Since

$$\begin{aligned} \Delta v &= \frac{\lambda^2}{4} v, \\ v|_{\partial B(1)} &= g(\theta), \\ |v| &\leq C \|g\|_{C^2}, \end{aligned}$$

we get, by elliptic estimates,

$$\|\nabla v\|_{L^\infty(B(2) \setminus B(1))} \leq C \|g\|_{C^2}$$

with another constant C . We conclude that

$$\|\nabla u\|_{L^\infty(\partial B(\lambda))} \leq \frac{C}{\lambda} \|g\|_{C^2}.$$

By the representation (7.22) for ∇u we then get

$$(7.31) \quad |\nabla u(r, \theta)| \leq \int_0^{2\pi} T(r, \theta - \bar{\theta}) \frac{C}{\lambda} \|g\|_{C^2} d\bar{\theta} \leq \frac{C}{\lambda} \|g\|_{C^2} \frac{K_0(r/2)}{K_0(\lambda/2)},$$

where the second inequality in (7.28) was used.

We summarize:

LEMMA 7.4. *The solution of (7.21) satisfies the bounds (7.29), (7.31).*

Consider the function Ψ defined in (7.6) when $K = K_\lambda$ (λ small) is chosen as in Theorem 7.3. Set

$$(7.32) \quad M(x, z) = \Psi\left(\frac{x}{\lambda}, \frac{z}{\lambda}\right).$$

Then

$$(7.33) \quad \Delta M = \frac{1}{4} M \quad \text{in } \mathbf{R}^2 \setminus B(\lambda),$$

$$(7.34) \quad M|_{\partial B(\lambda)} = (h - K_\lambda) e^{\frac{\lambda}{2} \cos \theta}$$

and by (7.13),

$$(7.35) \quad \int_0^{2\pi} e^{-\frac{x}{2}} \frac{\partial}{\partial N} \left(M + K_\lambda e^{\frac{1}{2}x} \right) \Big|_{|X|=\lambda} d\theta = \frac{1}{2\pi} - \frac{1}{2} \int_0^{2\pi} h(\theta) \cos \theta d\theta .$$

Let us assume that

$$(7.36) \quad h(\theta) = B_0 \widehat{h}(\theta), \quad \|\widehat{h}(\theta) - 1\|_{C^2} \leq B \leq o(\lambda) ,$$

Then we can write, on $\partial B(x)$,

$$(7.37) \quad M = (B_0 - K_\lambda) + (B_0 - K_\lambda)g_2(\theta) + g_3(\theta)$$

where

$$\begin{aligned} g_2(\theta) &= e^{\frac{\lambda}{2} \cos \theta} - 1 , \\ g_3(\theta) &= B_0 e^{\frac{\lambda}{2} \cos \theta} (\widehat{h} - 1) . \end{aligned}$$

Clearly

$$\|g_2\|_{C^2} \leq C\lambda , \quad \|g_3\|_{C^2} \leq CB .$$

We can decompose M as follows:

$$(7.38) \quad M = (B_0 - K_\lambda) \frac{K_0(r/2)}{K_0(\lambda/2)} + (B_0 - K_\lambda)M_2(r, \theta) + M_3(r, \theta)$$

where M_j is the bounded solution of (7.21) corresponding to $g = g_j$

By Lemma 7.4

$$(7.39) \quad \begin{aligned} |M_2(r, \theta)| &\leq C\lambda \frac{K_0(r/2)}{K_0(\lambda/2)} , \quad |M_3(r, \theta)| \leq CB \frac{K_0(r/2)}{K_0(\lambda/2)} , \\ |\nabla M_2(r, \theta)| &\leq C \frac{K_0(r/2)}{K_0(\lambda/2)} , \quad |\nabla M_3(r, \theta)| \leq C \frac{B}{\lambda} \frac{K_0(r/2)}{K_0(\lambda/2)} . \end{aligned}$$

Note that the integral with K_λ in (7.35) is equal to zero. Substituting M from (7.38) into (7.35) we obtain

$$\begin{aligned} \frac{1}{2\pi} - \frac{B_0}{2} \int_0^{2\pi} \widehat{h}(\theta) \cos \theta d\theta &= (B_0 - K_\lambda) \left(-\frac{1}{2} \int_0^{2\pi} e^{\frac{\lambda}{2} \cos \theta} d\theta \right) \frac{K'_0(\lambda/2)}{K_0(\lambda/2)} \\ &\quad - \int_0^{2\pi} e^{-\frac{1}{2}x} (B_0 - K_\lambda) \frac{\partial M_2}{\partial r} \Big|_{r=\lambda} d\theta - \int_0^{2\pi} e^{-\frac{1}{2}x} \frac{\partial M_3}{\partial r} \Big|_{r=\lambda} d\theta . \end{aligned}$$

Using the asymptotic formula

$$(7.40) \quad K_0(r) \sim c^* \log r , \quad K'_0(r) \sim c^* \frac{1}{r} \quad (c^* \text{ constant } \neq 0)$$

and recalling that $B \leq o(\lambda)$, we easily conclude that

$$(7.41) \quad B_0 - K_\lambda \sim C \frac{K_0(\lambda/2)}{K'_0(\lambda/2)} \quad \text{as } \lambda \rightarrow 0 .$$

We summarize:

LEMMA 7.5. For λ small and h satisfying (7.36), the constant $K = K_\lambda$ which occurs in Theorem 7.3, satisfies the estimate (7.41).

8. The linearized problem near the travelling wave solution. As before we denote by $X = (x, z)$ a variable point in $\mathbb{R}^2$, and set

$$B(\rho) = \{|X| < \rho\} , \quad B = B(1) .$$

In this section we consider the linearization of the 2-dimensional dislocation problem near the travelling wave solution constructed in Section 7. That means that if

$$\begin{aligned} u &= G(x - R(t), z) + \varepsilon \varphi(x - R(t), z, t) , \\ R(t) &= R_0 + \lambda t + \varepsilon \tilde{R}(t) \end{aligned}$$

then, dropping $O(\varepsilon^2)$ terms, φ satisfies

$$(8.1) \quad \varphi_t = \Delta \varphi + \lambda \varphi_x + \left(2\pi \int_0^{2\pi} \frac{\partial \varphi}{\partial N} d\theta \right) G_x \quad \text{in } \mathbb{R}^2 \setminus B$$

and

$$(8.2) \quad \varphi|_{\partial B} = 0 ;$$

the integral in (8.1) is evaluated on ∂B (The factor 2π in (8.1) should actually be dropped out; however it is convenient to include it, for the application in Section 9 to the 3-dimensional dislocation problem.). By the transformation

$$\varphi(x, z, t) = e^{-\frac{\lambda}{2} x} \psi(\lambda x, \lambda z, \lambda^2 t) ,$$

(8.1) and (8.2) become (for $\psi = \psi(x, z, t)$)

$$(8.3) \quad \psi_t = \Delta \psi - \frac{1}{4} \psi + \left[2\pi \int_0^{2\pi} e^{-\frac{x}{2}} \frac{\partial \psi}{\partial N} d\theta \right]_{\partial B(\lambda)} \left(M_x - \frac{1}{2} M \right) \quad \text{in } \mathbb{R}^2 \setminus B(\lambda) ,$$

$$(8.4) \quad \psi = 0 \quad \text{on } \partial B(\lambda)$$

where M is the function defined in (7.32).

Consider the operator

$$\mathcal{A}\psi = \Delta \psi - \frac{1}{4} \psi + 2\pi \int_0^{2\pi} \left[e^{-\frac{x}{2}} \frac{\partial \psi}{\partial N} d\theta \right]_{\partial B(\lambda)} \left(M_x - \frac{1}{2} M \right)$$

in $L^2(\mathbf{R}^2 \setminus B(\lambda))$, with domain $D(\mathcal{A}) = H^2(\mathbf{R}^2 \setminus B(\lambda)) \cap H_0^1(\mathbf{R}^2 \setminus B(\lambda))$, and denote its spectrum by $\sigma(\mathcal{A})$ and its resolvent set by $\rho(\mathcal{A})$.

In what follows we shall need a stronger assumption than (7.36), namely,

$$(8.5) \quad h(\theta) = B_0 \widehat{h}(\theta), \quad \|\widehat{h}(\theta) - 1\|_{C^2} = O(\lambda^2 |\log \lambda|) \quad \text{for } \lambda \text{ small.}$$

LEMMA 8.1. *If λ is sufficiently small and (8.5) holds, then $\sigma(\mathcal{A}) \subset \{\operatorname{Re} z < 0\}$.*

Proof. We shall prove that any z with $\operatorname{Re} z \geq 0$ belongs to $\rho(\mathcal{A})$. This means that for any $g \in L^2(\mathbf{R}^2 \setminus B(\lambda))$ we need to solve uniquely the equation $(\mathcal{A} - z)\psi = g$ in $D(\mathcal{A})$. Set

$$(8.6) \quad H(z) = \int_0^{2\pi} e^{-\frac{\pi}{2}} \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(M_x - \frac{1}{2} M \right) \Big|_{\partial B(\lambda)} d\theta.$$

Since the resolvent $(\Delta - \frac{1}{4} - z)^{-1}$ does exist, we can apply it to the equation $(\mathcal{A} - z)\psi = g$ to obtain

$$(8.7) \quad \psi = -c \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(M_x - \frac{1}{2} M \right) + \left(\Delta - \frac{1}{4} - z \right)^{-1} g$$

where

$$c = 2\pi \left[\int_0^{2\pi} e^{-\frac{\pi}{2}} \frac{\partial \psi}{\partial N} d\theta \right]_{\partial B(\lambda)}.$$

Substituting ψ from (8.7) into the expression for c , we get

$$c[1 + 2\pi H(z)] = 2\pi \left[\int_0^{2\pi} e^{-\frac{\pi}{2}} \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} g d\theta \right]_{\partial B(\lambda)}.$$

If

$$(8.8) \quad H(z) \neq -\frac{1}{2\pi}$$

then c is uniquely determined (by g) from the last equation, and (8.7) determines the solution ψ uniquely. We shall prove (8.8) by showing that $|H(z)| < \frac{1}{2\pi}$.

From (8.6) we get

$$(8.9) \quad |H(z)| \leq \left| \int_0^{2\pi} \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(M_x - \frac{1}{2} M \right) \Big|_{\partial B(\lambda)} d\theta \right| + O(\lambda) \int_0^{2\pi} \left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(M_x - \frac{1}{2} M \right) \Big|_{\partial B(\lambda)} d\theta.$$

Later on we shall need the fact that

$$(8.10) \quad \int \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right)_x \Big|_{\partial B(\lambda)} d\theta = 0 .$$

To prove it we simply consider the solution w of

$$\begin{aligned} \Delta w - \frac{1}{4} w - zw &= \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right)_x \quad \text{in } L^2(\mathbb{R}^2 \setminus B(\lambda)), \\ w &= 0 \quad \text{or } \partial B(\lambda) . \end{aligned}$$

Its average, $\bar{w}$, with respect to θ satisfies

$$\begin{aligned} \Delta \bar{w} - \frac{1}{4} \bar{w} - z\bar{w} &= 0, \quad \bar{w} \in L^2(\mathbb{R}^2 \setminus B(\lambda)) , \\ \bar{w}|_{\partial B(\lambda)} &= 0 \end{aligned}$$

and by uniqueness must vanish identically. Hence $\partial \bar{w} / \partial r = 0$, which gives (8.10).

We shall also need the following comparison lemma.

LEMMA 8.2. *If $\operatorname{Re} z \geq 0$ then*

$$(8.11) \quad \left| \left(\Delta - z - \frac{1}{4} \right)^{-1} f \right| \leq -\sqrt{2} \left(\Delta - \frac{1}{4} \right)^{-1} |f| \quad \text{in } \mathbb{R}^2 \setminus B(\lambda) ,$$

and

$$(8.12) \quad \left| \frac{\partial}{\partial N} \left(\Delta - z - \frac{1}{4} \right)^{-1} f \right| \leq \sqrt{2} \left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} \right)^{-1} |f| \right| \quad \text{on } \partial B(\lambda) .$$

Proof. By the semigroup representation

$$\left(\Delta - z - \frac{1}{4} \right)^{-1} f = - \int_0^\infty e^{-zt} e^{-\frac{t}{4}} e^{\Delta t} f dt, \quad \operatorname{Re} z \geq 0 .$$

Taking the absolute value on both sides and noting that $|f \cdots| \leq |f| \cdots$ and that $|e^{-\frac{t}{4}} e^{\Delta t} f| \leq \sqrt{2} e^{-\frac{t}{4}} e^{\Delta t} |f|$ (f complex valued) by the maximum principle for parabolic equations, we get

$$\left| \left(\Delta - z - \frac{1}{4} \right)^{-1} f \right| \leq \sqrt{2} \int_0^\infty e^{-\frac{t}{4}} e^{\Delta t} |f| dt = -\sqrt{2} \left(\Delta - \frac{1}{4} \right)^{-1} |f| ;$$

this proves (8.11). The inequality (8.12) follows immediately from (8.11) since both functions $\left(\Delta - z - \frac{1}{4}\right)^{-1} f$ and $\left(\Delta - \frac{1}{4}\right)^{-1} |f|$ vanish on $\partial B(\lambda)$.

We now continue to estimate the right-hand side of (8.9). We use (7.38), (7.39) and (8.10) in order to replace $M_x - \frac{1}{2} M$ in (8.9) by

$$((B_0 - K_\lambda)M_2 + M_3)_x + \frac{1}{2} |M| ,$$

and estimate the resulting expression by

$$C(B_0 - K_\lambda) \frac{K_0(r/2)}{K_0(\lambda/2)} \quad \text{provided} \quad \frac{|B|}{\lambda} < C|B_0 - K_\lambda|,$$

which follows by (8.5) and (7.41). Using also Lemma 8.2 we get

$$(8.13) \quad \begin{aligned} |H(z)| &\leq C \int_0^{2\pi} \left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} \right)^{-1} \left(|B_0 - K_\lambda| \frac{K_0(r/2)}{K_0(\lambda/2)} \right) \right|_{\partial B(\lambda)} d\theta \\ &+ C\lambda \int_0^{2\pi} \left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} \right)^{-1} |B_0 - K_\lambda| \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right)_x \right|_{\partial B(\lambda)} d\theta \equiv I_1 + I_2 \end{aligned}$$

where C is a constant independent of z, λ .

To estimate the first term on right-hand side, set

$$\varphi = - \left(\Delta - \frac{1}{4} \right)^{-1} \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right) .$$

Then

$$(8.14) \quad \begin{aligned} - \left(\Delta \varphi - \frac{1}{4} \varphi \right) &= \frac{K_0(r/2)}{K_0(\lambda/2)} \quad \text{in} \quad \mathbb{R}^2 \setminus B(\lambda), \\ \varphi &= 0 \quad \text{as} \quad \partial B(\lambda), \\ \varphi &\rightarrow 0 \quad \text{as} \quad r \rightarrow \infty . \end{aligned}$$

We shall construct a supersolution w :

$$(8.15) \quad \begin{aligned} - \left(\Delta w - \frac{1}{4} w \right) &\geq \frac{K_0(r/2)}{K_0(\lambda/2)} \quad \text{in} \quad \mathbb{R}^2 \setminus B(\lambda), \\ w &= 0 \quad \text{on} \quad \partial B(\lambda) , \\ w &\rightarrow w_0 > 0 \quad \text{as} \quad r \rightarrow \infty . \end{aligned}$$

Fix $\varepsilon > \lambda$, and let

$$(8.16) \quad w_2 = \left[1 - \frac{K_0(r/2)}{K_0(\lambda/2)} \right] \frac{4K_0(\varepsilon/2)}{K_0(\lambda/2)} .$$

Then, for $r > \lambda$

$$(8.17) \quad - \left(\Delta w_2 - \frac{1}{4} w_2 \right) = \frac{K_0(\varepsilon/2)}{K_0(\lambda/2)} \geq \frac{K_0(r/2)}{K_0(\lambda/2)} \chi_{[\varepsilon, \infty)}$$

by the second part of (7.26). Also

$$(8.18) \quad w_2 = 0 \quad \text{on} \quad r = \lambda , \quad \lim_{r \rightarrow \infty} w_2(r) = \frac{4K_0(\varepsilon/2)}{K_0(\lambda/2)} > 0 .$$

Next we construct a function w_1 satisfying

$$(8.19) \quad - \left(\Delta w_1 - \frac{1}{4} w_1 \right) \geq \frac{K_0(r/2)}{K_0(\lambda/2)} \chi_{[\lambda, \varepsilon]} \quad \text{in} \quad \mathbb{R}^2 \setminus B(\lambda) .$$

with the same boundary conditions as φ .

Let $\bar{\varphi}(r)$ be the solution of

$$\begin{aligned} -\Delta \bar{\varphi} &= 1 \quad \text{in} \quad \mathbb{R}^2 \setminus B , \\ \bar{\varphi} &= \bar{\varphi}_r = 0 \quad \text{at} \quad r = 1 . \end{aligned}$$

Then

$$\bar{\varphi}(r) \sim -\frac{r^2}{4} \quad \text{as} \quad r \rightarrow \infty .$$

Take

$$\bar{w}_1 = \lambda^2 \bar{\varphi} \left(\frac{r}{\lambda} \right) + \frac{\varepsilon^{1/2}}{|\log \lambda|} \log \frac{r}{\lambda} .$$

Then, for $r > \lambda$,

$$(8.20) \quad - \left(\Delta \bar{w}_1 - \frac{1}{4} \bar{w}_1 \right) = 1 + \frac{1}{4} \bar{w}_1 \geq \chi_{[\lambda, \varepsilon]} \geq \frac{K_0(r/2)}{K_0(\lambda/2)} \chi_{[\lambda, \varepsilon]}$$

provided $\bar{w}_1 > 0$. If $r/\lambda = O(1)$ then

$$\bar{w}_1 \sim \lambda^2 O \left(\left(\frac{r}{\lambda} - 1 \right)^2 \right) + c \frac{\varepsilon^{1/2}}{|\log \lambda|} \frac{r - \lambda}{\lambda} > 0 \quad (c > 0)$$

If $r/\lambda \gg 1$ and $|\log r| < \theta |\log \lambda|$ for some $0 < \theta < 1$ then

$$\bar{w}_1 \sim -\frac{1}{4} r^2 + \frac{\varepsilon^{1/2}}{|\log \lambda|} (\log r - \log \lambda) > -\frac{1}{4} r^2 + (1 - \theta)\varepsilon^{1/2} > 0 \quad \text{if } r^2 < 4(1 - \theta)\varepsilon^{1/2} .$$

Finally, if $r/\lambda \gg 1$ and $|\log r| \geq \theta |\log \lambda|$ then $r < \lambda^\theta$ and

$$\bar{w}_1 > -\frac{\lambda^{2\theta}}{4} + \frac{\varepsilon^{1/2}}{|\log \lambda|} > 0$$

if λ is small enough. Hence $\bar{w}_1 > 0$ and (8.20) holds if $r < \varepsilon^{1/4}$.

We shall match $\bar{w}_1$ with another supersolution $\tilde{w}_1$ defined for all $r > \varepsilon$:

$$\tilde{w}_1(r) = (\varepsilon^{1/2} + 2\varepsilon^{3/2})e^{-\frac{1}{2} r} .$$

Clearly

$$-\left(\Delta \tilde{w}_1 + \frac{1}{4} \tilde{w}_1\right) = \left(\varepsilon^{1/2} + 2\varepsilon^{3/2}\right) \frac{1}{2r} e^{-\frac{1}{2} r} > \chi_{[\lambda, \varepsilon]} .$$

It is easy to check that $\bar{\varphi}_{rr} < 0$ for all $r > 0$, so that also $\bar{w}_{1,rr} < 0$ for all $r > 0$. Also $\tilde{w}_{1,rr} > 0$ for all $r > 0$, and $\tilde{w}_1 > \bar{w}_1$ at $r = \lambda$ and for all large r . It follows that $\bar{w}_1$ and $\tilde{w}_1$ may coincide in at most two points $r = r_1, r = r_2$. That these points do exist and lie in the interval $\varepsilon < r < 2\varepsilon^{1/2}$ can be proved by using the expansions

$$\begin{aligned} \bar{w}_1 &\sim -\frac{1}{4} r^2 + \varepsilon^{1/2} , \\ \tilde{w}_1 &\sim (\varepsilon^{1/2} + 2\varepsilon^{3/2}) \left(1 - \frac{1}{2} r\right) \sim \varepsilon^{1/2} + 2\varepsilon^{3/2} - \frac{\varepsilon^{1/2}}{2} r , \end{aligned}$$

which are valid in that interval. We find that

$$r_1 \sim 4\varepsilon , \quad r_2 \sim 2\varepsilon^{1/2} - 4\varepsilon .$$

As expected,

$$\tilde{w}_{1,r} \sim -\frac{\varepsilon^{1/2}}{2} < -\frac{1}{2} 4\varepsilon \sim \bar{w}_{1,r} \quad \text{at } r = r_1 ,$$

provided λ is small enough. It follows that the function

$$w_1 = \begin{cases} \bar{w}_1 & \text{if } r < r_1 \\ \tilde{w}_1 & \text{if } r > r_1 \end{cases}$$

satisfies (8.19).

We can now take $w = w_1 + w_2$ as the supersolution satisfying (8.15) and conclude, by comparing with the solution φ of (8.14), that, at $|X| = \lambda$,

$$\left| \frac{\partial \varphi}{\partial N} \right| \leq \frac{2K_0(\varepsilon/2)}{K_0(\lambda/2)} \frac{K'_0(\lambda/2)}{K_0(\lambda/2)} + \frac{\varepsilon^{1/2}}{|\log \lambda| \lambda}.$$

Using this estimate and recalling (7.41), we get for the first term I_1 in (8.13) the bound

$$(8.21) \quad I_1 \leq C\varepsilon^{1/2}$$

as $\lambda \rightarrow 0$; here we used the estimate

$$\left| \frac{K_0(\lambda/2)}{K'_0(\lambda/2)} \right| \leq \lambda |\log \lambda|.$$

The function

$$u = \frac{K_0(r/2)}{K_0(\lambda/2)}$$

satisfies (7.21) with $g \equiv 1$. Hence by (7.31),

$$|\nabla u| \leq \frac{C}{\lambda} \frac{K_0(r/2)}{K_0(\lambda/2)}.$$

It follows that

$$\left| \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right)_x \right| \leq \frac{C}{\lambda} \frac{K_0(r/2)}{K_0(\lambda/2)}$$

for $r \geq \lambda$, and by comparison

$$\left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} \right)^{-1} \left| \left(\frac{K_0(r/2)}{K_0(\lambda/2)} \right)_x \right| \right| \leq \frac{C}{\lambda} \left| \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} \right)^{-1} \frac{K_0(r/2)}{K_0(\lambda/2)} \right|$$

on $\partial B(\lambda)$. Hence the second term I_2 in (8.13) can be estimated by a constant times I_1 . It then follows, from (8.21), that

$$|H(z)| \leq C\varepsilon^{1/2},$$

and consequently (8.8) holds if ε is chosen small enough (independently of z , for $\operatorname{Re} z \geq 0$).

The above proof shows that, for $\operatorname{Re} z \geq 0$, the equation $(\mathcal{A} - z)\psi = g$ is solved uniquely by the formula

$$\begin{aligned} \psi = & \left(\Delta - \frac{1}{4} - z \right)^{-1} g \\ & - \frac{2\pi}{1 + 2\pi H(z)} \left[\int_0^{2\pi} e^{-\frac{\theta}{2}} \frac{\partial}{\partial N} \left(\Delta - \frac{1}{4} - z \right)^{-1} g d\theta \right]_{\partial B(\lambda)} \left(\Delta - \frac{1}{4} - z \right)^{-1} \left(M_x - \frac{1}{2} M \right), \end{aligned}$$

and hence

$$(8.22) \quad \|(\mathcal{A} - z)^{-1} g\|_{H^2(\mathbb{R}^2 \setminus B(\lambda))} \leq C \|g\|_{L^2(\mathbb{R}^2 \setminus B(\lambda))};$$

this completes the proof of the lemma.

§9. Global solution for n=3 near the travelling wave solution. We introduce the space

$$Y = L^2(\mathbb{R}^2 \setminus B; e^{\lambda x}), \quad B = B(1),$$

i.e., $L^2(\mathbb{R}^2 \setminus B)$ provided with the norm

$$\|f\|_Y = \left[\iint_{\mathbb{R}^2 \setminus B} |f(x, z)|^2 e^{\lambda x} dx dz \right]^{1/2}.$$

Similarly we define the spaces

$$H^j(\mathbb{R}^2 \setminus B; e^{\lambda x})$$

and the subspace $H_0^1(\mathbb{R}^2 \setminus B; e^{\lambda x})$ of $H^1(\mathbb{R}^2 \setminus B; e^{\lambda x})$ which consists of functions vanishing on ∂B . Set

$$(9.1) \quad Y_0 = H_0^1(\mathbb{R}^2 \setminus B; e^{\lambda x}) \cap H^2(\mathbb{R}^2 \setminus B; e^{\lambda x}).$$

In this section we consider the 3-dimensional problem (2.1) – (2.5) with initial data “near” the travelling wave solution G corresponding to λ small and $R(0)$ large. We shall prove that there exists a global solution which, roughly speaking, converges asymptotically to the travelling wave solution as $t \rightarrow \infty$.

More precisely, we assume that $h(\theta)$ satisfies (8.5) and fix λ to be positive and small enough as in Lemma 8.1. We take initial data

$$(9.2) \quad \begin{aligned} c_0(r, z) &= G(r - R(0), t) + \psi_0(r - R(0), z), \\ c_0 &\text{ satisfies (2.6),} \end{aligned}$$

where G is the travelling wave solution with velocity λ constructed in Lemma 8.1, and

$$(9.3) \quad \begin{aligned} e^{\frac{\lambda}{2}x} |\nabla^\ell \psi_0(x, z)| &\leq \frac{C}{(1 + |x| + |z|)^2} \quad (\ell = 0, 1, 2) \\ \|\psi_0\|_{H^1(\mathbb{R}^2 \setminus B)} &\leq \varepsilon, \quad \|\psi_0\|_{C^{2,\alpha}(\mathbb{R}^2 \setminus B)} \leq \varepsilon \end{aligned}$$

where ε is a positive constant independent of $R(0)$. We shall prove:

THEOREM 9.1. *Assume that (8.5), (9.2) and (9.3) hold. If ε is sufficiently small and $R(0)$ is sufficiently large, then there exists a unique global solution (c, R) of (2.1) – (2.5) and*

$$(9.4) \quad R(t) = (R(0) + \lambda t) + O(\log(R(0) + \lambda t)),$$

$$(9.5) \quad c(r, z, t) = e^{\frac{\log R(t)}{R(t)}} G(r - R(t), z) + v(r - R(t), z, t)$$

for all $t > 0$, where

$$(9.6) \quad \|v\|_{H^1(\mathbf{R}^3 \setminus D(R(t)); e^{\lambda x})} = O\left(\frac{\log R(t)}{R(t)}\right).$$

To prove the theorem it will be more convenient to take $R(0) = 0$ and the initial time $t = t_0$ sufficiently large.

We write c and R in the form

$$(9.7) \quad c(r, z, t) = G(r - R(t), z) + \tilde{\psi}(r - R(t), z, t),$$

$$(9.8) \quad R(t) = \lambda t + 2\pi\mu(t).$$

Set

$$x = r - R(t), \quad X = (x, z), \quad u(x, z, t) = c(r, z, t), \quad \psi(x, z, t) = \tilde{\psi}(r - R(t), z, t).$$

Then (2.1), (2.2) and (2.5) reduce to

$$(9.9) \quad \begin{aligned} \psi_t &= \psi_{xx} + \psi_{zz} + \lambda\psi_x + 2\pi\dot{\mu}\psi_x + 2\pi\dot{\mu}G_x + \frac{1}{R(t) + x} (G_x + \psi_x) \\ \text{if } -R(t) &< x < \infty, -\infty < z < \infty, x^2 + z^2 > 1, \end{aligned}$$

$$(9.10) \quad \psi|_{\partial B} = \left(e^{\frac{\log R}{R}} - 1\right) h,$$

$$(9.11) \quad \dot{\mu} = \int_0^{2\pi} \frac{\partial\psi}{\partial N} d\theta + \frac{1}{R(t)} \int_0^{2\pi} \frac{\partial G}{\partial N} \cos\theta d\theta + \frac{1}{R(t)} \int_0^{2\pi} \frac{\partial\psi}{\partial N} \cos\theta d\theta$$

where the integrals are taken over $|X| = 1$.

The initial conditions for c is

$$(9.12) \quad c_0(r, z) = G(r - \lambda t_0, z) + \psi_0(r - \lambda t_0, z) \quad \text{at } t = t_0,$$

i.e., $R(0)$ is then replaced by $R(t_0) = \lambda t_0$ in (9.2).

By Theorem 2.1 there exists a unique solution (c, R) for $t_0 \leq t \leq t_0 + \eta$ (η small enough) such that,

$$(9.13) \quad 2\pi|\dot{\mu}(t)| < M, \quad M < \frac{\lambda}{2},$$

$$(9.14) \quad \|\nabla^\ell \psi\|_{L^\infty(\partial B)} < 1 \quad (\ell = 0, 1, 2), \quad \int_0^{2\pi} \left| \frac{\partial\psi}{\partial N} \right| d\theta < N$$

for some constants M, N ; furthermore, using the first part of (9.3) one can establish the corresponding inequalities for ψ :

$$(9.15) \quad e^{\frac{\lambda}{2}x} |\nabla^\ell \psi(x, z, t)| \leq \frac{C}{(1 + |x| + |z|)^2} \quad (\ell = 0, 1, 2)$$

where C is a constant depending on t . In the derivation of (9.15) we make use of the fact that such inequalities hold for the function G (in the region $x > -R(t)$, $t_0 \leq t \leq t_0 + \eta$). In the derivation of (9.13) we need to use the assumption that $R(0) \gg 1$ or $t_0 \gg 1$.

LEMMA 9.2. *If (9.13), (9.14) and (9.15) hold for $t_0 \leq t \leq t_0 + \eta$ with some $\eta > 0$, then these inequalities also hold, with the same M, N but not necessarily the same constant C , for $t_0 \leq t \leq t_0 + \eta'$ for some $\eta' > \eta$.*

In the sequel we denote by P a generic constant which is independent of N, M and C .

To prove the lemma it will be convenient to extend $c(r, z, t)$ to $r < 0$. Set

$$c(-r, z, t) = c(r, z, t) \quad \text{for } r > 0, (r - R(t))^2 + z^2 \geq 1 ;$$

since $c_r = 0$ at $r = 0$, the extended c satisfies the same differential equation as c . Next we extend c smoothly into the disc $(r + 2R(t))^2 + z^2 < 1$ so that its C^2 norm in the disc is bounded by an absolute constant times its C^2 norm on the boundary of the disc.

We find that ψ satisfies (9.9) in $\{x < -R(t), (x + 2R(t))^2 + z^2 > 2\}$, and

$$(9.16) \quad \begin{aligned} \psi_t &= \psi_{xx} + \psi_{zz} + \lambda\psi_x + 2\pi\dot{\mu}\psi_x + 2\pi\dot{\mu}(t)G_x + \tilde{f}(x, z, t) , \quad |\tilde{f}| \leq P \\ &\text{if } x < -R(t), (x + 2R(t))^2 + z^2 < 1 ; \end{aligned}$$

notice that G is already defined in the entire plane except on $\{(r - R(t))^2 + z^2 < 1\}$. Set

$$(9.17) \quad \psi = \left(e^{\frac{\log R}{R}} - 1 \right) G + v .$$

Then v satisfies:

$$(9.18) \quad \begin{aligned} v_t &= v_{xx} + v_{zz} + \lambda v_x + 2\pi\dot{\mu}v_x + 2\pi\dot{\mu}G_x + \dot{R}e^{\frac{\log R}{R}} \frac{1 - \log R}{R^2} G \\ &\quad + \frac{1}{R(t) + x} \left(e^{\frac{\log R}{R}} G_x + v_x \right) \end{aligned}$$

if

$$x^2 + z^2 \geq 1 \quad \text{and} \quad (x + 2R(t))^2 + z^2 \geq 1 ,$$

$$(9.19) \quad v = 0 \quad \text{on} \quad \partial B ,$$

and

$$(9.20) \quad \begin{aligned} \dot{\mu}(t) &= \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \lambda \left(e^{\frac{\log R}{R}} - 1 \right) + \frac{1}{R(t)} \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta \\ &\quad + \frac{1}{R(t)} \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \frac{1}{R(t)} \left(e^{\frac{\log R}{R}} - 1 \right) \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta . \end{aligned}$$

In

$$\{(x + 2R(t))^2 + z^2 \leq 1\}$$

v satisfies

$$(9.21) \quad v_t = v_{xx} + v_{zz} + \lambda v_x + 2\pi \dot{\mu} v_x + 2\pi \dot{\mu} G_x + \tilde{f}(x, z, t), \quad |\tilde{f}| \leq P.$$

If we substitute $\dot{\mu}$ from (9.20) into (9.18) and (9.21), we get

$$(9.22) \quad v_t = \mathcal{L}v + 2\pi e^{\frac{\log R}{R}} \frac{G_x}{R(t)} \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \frac{1}{R(t) + x} \left(e^{\frac{\log R}{R}} G_x + v_x \right) + 2\pi \Gamma$$

where

$$(9.23) \quad \mathcal{L}v = v_{xx} + v_{zz} + \lambda v_x + 2\pi G_x \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta,$$

and

$$(9.24) \quad \begin{aligned} \Gamma = \Gamma(x, z, t) &= \lambda e^{\frac{\log R}{R}} G_x \left(e^{\frac{\log R}{R}} - 1 \right) + \frac{G_x}{R(t)} e^{\frac{\log R}{R}} \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta \\ &- \dot{R} \frac{1 - \log R}{R^2} G + \frac{G_x}{R(t)} e^{\frac{\log R}{R}} \left(e^{\frac{\log R}{R}} - 1 \right) \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta \\ &+ v_x \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \lambda \left(e^{\frac{\log R}{R}} - 1 \right) v_x + \frac{v_x}{R(t)} \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta \\ &+ \frac{v_x}{R(t)} \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + v_x \frac{1}{R(t)} \left(e^{\frac{\log R}{R}} - 1 \right) \int_0^{2\pi} \frac{\partial G}{\partial N} \cos \theta d\theta. \end{aligned}$$

We shall be working with Sobolev norms with weight $e^{\lambda x}$. From the asymptotic form of G it easily follows (recall that λ is now fixed, and t_0 is sufficiently large) that

$$(9.25) \quad \Gamma = \left(1 + \frac{1}{R(t)} \right) v_x \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + v_x \tilde{g}(t) \frac{\log t}{t} + g(x, z, t) \frac{\log t}{t}, \text{ and}$$

$$\|g\|_{H^1(\mathbf{R}^2 \setminus B; e^{\lambda x})} \leq P, \quad |\tilde{g}(t)| \leq P.$$

In the region $(x + 2R(t))^2 + z^2 \leq 1$

$$(9.26) \quad v_t = \mathcal{L}v + \left(1 + \frac{1}{R(t)}\right) v_x \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \widehat{f}(x, z, t), \quad |\widehat{f}| \leq P.$$

Here we used the fact that, by (9.14),

$$(9.27) \quad \left| \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta \right| \leq P.$$

Set

$$(9.28) \quad H(x, z, t) = \begin{cases} 2\pi \frac{G_x}{R(t)} \left(\int_0^{2\pi} \frac{\partial v}{\partial N} d\theta \right) + 2\pi \Gamma \\ \quad + \frac{1}{R(t) + x} \left(e^{\frac{\log R}{R}} G_x + v_x \right) & \text{if } (x + 2R(t))^2 + z^2 \geq 1 \\ 2\pi \left(1 + \frac{1}{R(t)}\right) v_x \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta + \widehat{f}(x, z, t) & \text{if } (x + 2R(t))^2 + z^2 < 1. \end{cases}$$

We can write, at least formally,

$$(9.29) \quad v(\cdot, t) = e^{(t-t_0)\mathcal{L}} v_0 + \int_{t_0}^t e^{(t-s)\mathcal{L}} H(\cdot, s) ds \quad (v_0 = v(\cdot, t_0))$$

where $e^{\tau\mathcal{L}}$ is the semigroup associated to $\mathcal{L}$. In order to justify the last formula, set

$$Av = v_{xx} + v_{zz} + \lambda v_x = e^{-\lambda x} \nabla (e^{\lambda x} \nabla v).$$

Then A is a sectorial operator in Y ; in fact, it is self-adjoint closed operator with domain $D(A) = Y_0$ (Y_0 as in (9.1)), and its spectrum $\sigma(A)$ lies in $\operatorname{Re} z \leq -\delta_0 < 0$. Set

$$Bv = \left(2\pi \int_0^{2\pi} \frac{\partial v}{\partial N} d\theta \right) G_x.$$

Then, for any $\delta > 0$,

$$\|Bv\|_Y \leq \delta \|Av\|_Y + C_\delta \|v\|_Y, \quad (C_\delta = \frac{1}{\delta^k}, \quad \text{for some } k \geq 1).$$

It follows that $\mathcal{L} \equiv A+B$ is also sectorial. Under the transformation $v(x, z, t) = e^{\frac{\lambda}{2}x} w(\lambda x, \lambda z, \lambda^2 t)$, the operator $\mathcal{L}$ in the weighted space $\tilde{Y}$ corresponds to $\mathcal{A}$ in section 8. Therefore, by Lemma 8.1

$$\gamma \equiv \sup \operatorname{Re} \sigma(\mathcal{L}) < 0 .$$

In particular, the semigroup $e^{\tau \mathcal{L}}$ is analytic.

Since $(\mathcal{L} - A)A^{-\alpha}$ is bounded in Y , for some $\alpha \in (0, 1)$, it follows by [7; Th. 1.4.6] that

$$(9.31) \quad \mathcal{L}^\beta A^{-\beta} \quad \text{and} \quad A^\beta \mathcal{L}^{-\beta} \quad \text{are bounded operator in } Y, \quad \text{for any } \beta \in (0, 1) .$$

This fact will be needed later on.

Before proceeding to estimate v from the representation (9.29), we need to derive some bounds on c .

LEMMA 9.3. *The following estimates hold for $t_0 \leq t \leq t_0 + \eta$:*

$$(9.32) \quad \int_{\mathbb{R}^3 \setminus D(R(t))} (c^2 + |\nabla c|^2) \leq P(1+t)^2 ,$$

$$(9.33) \quad \int_{-\infty}^{\infty} \left[\sum_{l=0}^2 \sup_{0 \leq r \leq \frac{1}{2}} |\nabla^l c(r, z, t)| \right]^2 dz \leq P(1+t)^2 .$$

The last estimate will be needed to get rid of possible difficulties around $r = 0$.

Proof. Multiplying (2.1) by c and integrating over $\mathbb{R}^3 \setminus D(R(t))$, and using (9.15), we get

$$\int_{\mathbb{R}^3 \setminus D(R(t))} \partial_t \left(\frac{c^2}{2} \right) = \int_{\partial D(R(t))} c \frac{\partial c}{\partial N} - \int_{\mathbb{R}^3 \setminus D(R(t))} |\nabla c|^2 \leq \int_{\partial D(R(t))} c \frac{\partial c}{\partial N} .$$

Hence

$$\partial_t \int_{\mathbb{R}^3 \setminus D(R(t))} \frac{c^2}{2} \leq \int_{\partial D(R(t))} |c| \left| \frac{\partial c}{\partial N} \right| + \frac{1}{2} \dot{R}(t) \int_{\partial D(R(t))} |c|^2 .$$

Using (9.13), (9.14) we conclude that

$$\partial_t \int_{\mathbb{R}^3 \setminus D(R(t))} \frac{c^2}{2} dx \leq PR(t) .$$

In a similar way, applying the above argument to ∇c we get

$$\partial_t \int_{\mathbb{R}^3 \setminus D(R(t))} (c^2 + |\nabla c|^2) \leq PR(t) ,$$

and (9.32) follows.

Introduce the (one-dimensional) r -intervals $I_\rho = \{0 < r < \rho\}$.

By (9.32) and parabolic estimates,

$$\|c(\cdot, t)\|_{W^{2,\infty}(I_1 \times (n, n+1))} \leq P \sup_{t-\delta \leq s \leq t} \left\{ \int_{I_2 \times [n-1, n+2]} |c(r, z, s)|^2 dr dz \right\}^{1/2}$$

for any fixed $\delta > 0$. Using this estimate for $t \geq t_0 + \delta$ and recalling that (9.15) holds if $t_0 < t < t_0 + \delta$ (δ small enough), we conclude (after using (9.32)) that

$$\int_{-\infty}^{\infty} \left(\sup_{r \leq \frac{1}{2}} |\nabla^l c(r, z, t)| \right)^2 dz \leq P(1+t)^2 \quad (l = 0, 1, 2)$$

for $t_0 \leq t \leq t_0 + \eta$.

Returning to (9.29), we shall use the theory of fractional powers of operators [7]; in particular, by [7, Th. 1.5.4]

$$(9.34) \quad \|\mathcal{L}^{1/2} e^{t\mathcal{L}} f\|_Y = \|e^{t\mathcal{L}} f\|_{Y^{1/2}} \leq P \min \left(\frac{e^{-\gamma t}}{t^{1/2}} \|f\|_Y, \quad e^{-\gamma t} \|f\|_{Y^{1/2}} \right)$$

and, by (9.31),

$$\|\cdot\|_{Y^{1/2}} \cong \|\cdot\|_{H^1(\mathbb{R}^2 \setminus B; e^{\lambda x})}.$$

Using (9.34) we derive from (9.27), (9.29)

$$(9.35) \quad \|v(\cdot, t)\|_{Y^{1/2}} \leq K e^{-\gamma(t-t_0)} \|v_0\|_{Y^{1/2}} + K \int_{t_0}^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} \|H(\cdot, s)\|_Y ds.$$

We proceed to estimate the various terms in $\|H(\cdot, s)\|_Y$. First

$$(9.36) \quad \|\widehat{f} \chi_{[(x+2R(t))^2 + z^2 \leq 1]}\|_Y \leq P e^{-R(t)\lambda}.$$

Next we estimate

$$(9.37) \quad \zeta \equiv \frac{1}{R(t) + x} \left(e^{\frac{\log R}{R}} G_x + v_x \right) = \frac{c_r}{r},$$

by splitting:

$$\begin{aligned} \|\zeta\|_Y &= \|\zeta \chi_{[-R(t)-\frac{1}{2}, -R(t)+\frac{1}{2}]} \|_Y + \|\zeta \chi_{[-\infty, -R(t)-\frac{1}{2}]} \|_Y \\ &\quad + \|\zeta \chi_{[-R(t)+\frac{1}{2}, -\frac{R(t)}{2}]} \|_Y + \|\zeta \chi_{[-\frac{R(t)}{2}, \infty]} \|_Y = T_1 + T_2 + T_3 + T_4. \end{aligned}$$

The term T_1 involves the region near $r = 0$.

Since $\frac{1}{r} c_r$ can be estimated by c_{rr} , we get from (9.33),

$$T_1 \leq P(1+t)e^{-\frac{\lambda}{2}R(t)} .$$

In T_3 , $1/(R(t)+x) \leq \frac{1}{2}$, and therefore

$$T_3 \leq Pe^{-\frac{\lambda}{2}R(t)} \left(\int |\nabla c|^2 \right)^{1/2} \leq P(1+t)e^{-\frac{\lambda}{2}R(t)} ,$$

by (9.32). Similarly

$$T_2 \leq P(1+t)e^{-\frac{\lambda}{2}R(t)} .$$

Finally,

$$\begin{aligned} T_4 &\leq \frac{P}{R(t)} \|G_x\|_Y + \frac{P}{R(t)} \|\nabla v\|_Y \\ &\leq \frac{P}{R(t)} + \frac{P}{R(t)} \|v\|_{Y^{1/2}} . \end{aligned}$$

Using (9.28), the bound on $\|\zeta\|_Y$ and (9.25), (9.27), (9.36), we obtain a bound on $\|H\|_Y$, which we put in (9.35) to get

$$\begin{aligned} (9.38) \quad \|v(\cdot, t)\|_{Y^{1/2}} &\leq Pe^{-\gamma(t-t_0)} \|v_0\|_{Y^{1/2}} + P \int_{t_0}^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} \left(\frac{1}{s} + \frac{\log s}{s} \right) ds \\ &\quad + P \int_{t_0}^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} \left(N + \frac{\log s}{s} \right) \|v(\cdot, s)\|_{Y^{1/2}} ds ; \end{aligned}$$

here we used the estimate $R(t) \geq \frac{\lambda}{2} t$, which follows from (9.13).

One can easily establish that

$$\begin{aligned} (9.39) \quad &\int_1^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} \frac{\log s}{s} ds \leq C_1^* \frac{|\log t|}{t} , \\ &\int_{t_0}^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} e^{-\frac{\gamma}{2}(s-t_0)} ds = e^{-\frac{\gamma}{2}(t-t_0)} \int_{t_0}^t \frac{e^{-\frac{\gamma}{2}(t-s)}}{(t-s)^{1/2}} ds \leq C_2^* e^{-\frac{\gamma}{2}(t-t_0)} , \end{aligned}$$

where the first inequality is obtained by splitting the integral into $\int_1^{t-2|\log t|/\gamma} + \int_{t-2|\log t|/\gamma}^t$, with the constants C_1^* and C_2^* independent of t_0 . Hence

$$\begin{aligned} \|v(\cdot, t)\|_{Y^{1/2}} &\leq P e^{-\gamma(t-t_0)} \|v_0\|_{Y^{1/2}} + P \frac{\log t}{t} \\ &\quad + P \int_{t_0}^t \frac{e^{-\gamma(t-s)}}{(t-s)^{1/2}} \left(N + \frac{\log s}{s} \right) \|v(\cdot, s)\|_{Y^{1/2}} ds, \end{aligned}$$

and using (9.39) we conclude, by continuation in t , that

$$\|v(\cdot, t)\|_{Y^{1/2}} < 2P e^{-\frac{\gamma}{2}(t-t_0)} \|v_0\|_{Y^{1/2}} + 2P \frac{\log t}{t}$$

provided

$$(9.40) \quad 2P \max(C_1^*, C_2^*) \left(N + \frac{\log t_0}{t_0} \right) < 1.$$

Assuming that $\left| \frac{\log R(0)}{R(0)} \right| \leq \varepsilon$ and that ε is small enough, N can be chosen so that (9.40) is satisfied. Using (9.3) we obtain

$$\|v(\cdot, t)\|_{Y^{1/2}} \leq 2P \varepsilon e^{-\frac{\gamma}{2}(t-t_0)} + 2P \frac{\log t}{t}.$$

This implies that

$$(9.41) \quad \|v(\cdot, t)\|_{H^1(\Omega_3)} \leq P \frac{\log t}{t} + P \varepsilon e^{-\frac{\gamma}{2}(t-t_0)}$$

where

$$\Omega_\rho = \{1 < x^2 + z^2 < \rho\}.$$

By parabolic estimates it follows that

$$\int_{\partial D} \left| \frac{\partial v}{\partial N} \right| d\theta \leq P \frac{\log t}{t} + P \varepsilon e^{-\frac{\gamma}{2}(t-t_0)}$$

and then, by (9.11),

$$2\pi |\dot{\mu}(t)| \leq P \frac{\log t}{t} + P \varepsilon e^{-\frac{\gamma}{2}(t-t_0)}.$$

Similarly, by parabolic estimates,

$$\|\nabla^\ell \psi\|_{L^\infty(\partial D(R(t)))} \leq P \frac{\log t}{t} + P \varepsilon e^{-\frac{\gamma}{2}(t-t_0)} \quad (\ell = 0, 1, 2).$$

Choosing ε small and t_0 sufficiently large, we conclude that

$$\begin{aligned} 2\pi|\dot{\mu}(t)| &< \frac{M}{2} , \\ \|\nabla^\ell \psi\|_{L^\infty(\partial D(R(t)))} &< 1/2 \quad (\ell = 0, 1, 2) , \\ \int_0^{2\pi} \left| \frac{\partial \psi}{\partial N} \right| d\theta &< N/2 . \end{aligned}$$

Consequently (9.13) and (9.14) extend to $t_0 \leq t \leq t_0 + \eta'$ for some $\eta' > \eta$. The estimate (9.15) also extends to this interval with a constant another C , depending on η' (by parabolic estimates).

We have now completed the proof of (9.13), (9.14) for all $t > t_0$, and of (9.15) with C which is a constant depending on $t_0 + \eta'$. The above proof also shows that

$$(9.42) \quad |\dot{\mu}(t)| \leq \frac{C \log t}{t} ,$$

$$(9.43) \quad \|\nabla^\ell \psi\|_{L^\infty(\partial D(R(t)))} \leq \frac{C \log t}{t} ,$$

$$(9.44) \quad \|v(\cdot, t)\|_{Y^{1/2}} \leq \frac{C \log t}{t} .$$

Remark 9.1. The data in Theorem 9.1 do not satisfy, in general, the third condition in (6.27). Therefore the existence part of Theorem 9.1 does not follow from the existence results of Section 6.

§10. Linearized problem near steady-state solution for $n=3$.

LEMMA 10.1. *For any $\overline{R} > 1$ there exists a unique $\xi > 0$ such that the solution Φ to*

$$\begin{aligned} (10.1) \quad \Delta \Phi &= 0 \quad \text{in} \quad R^3 \setminus D(\overline{R}) , \\ \Phi|_{\partial D(\overline{R})} &= h(\theta) \exp\left(\frac{\log \overline{R}}{\overline{R}}\right) , \\ \Phi(X) &\rightarrow \xi \quad \text{as} \quad |X| \rightarrow \infty \end{aligned}$$

satisfies

$$(10.2) \quad \iint_{\partial D(\overline{R})} \frac{\partial \Phi}{\partial N} dS = 0 .$$

Here $D(R)$ is the torus generated by rotating $\{(x - R)^2 + z^2 \leq 1, y = 0\}$ about the z -axis, and $X = (r, z)$ where $r = (x^2 + y^2)^{1/2}$.

The pair $(\Phi, \overline{R})$ forms a steady state solution to the dislocation problem (in 3-dimensions).

Proof. Denote by Φ_ξ the solution to (10.1). Then Φ_ξ is monotone increasing in ξ and, consequently, the function

$$g(\xi) = \iint_{\partial D(\overline{R})} \frac{\partial \Phi_\xi}{\partial N} dS$$

is monotone decreasing in ξ . Clearly $\partial \Phi_\xi / \partial N > 0$ if $\xi = 0$ and therefore, by continuity, $g(\xi) > 0$ if ξ is positive and small. On the other hand if ξ is large then Φ_ξ tends to its supremum at ∞ , so that, by Green's formula,

$$g(\xi) = - \lim_{M \rightarrow \infty} \iint_{x^2+y^2+z^2=M} \frac{\partial \Phi_\xi}{\partial N} < 0 .$$

Hence there is a unique ξ such that $g(\xi) = 0$.

We want to find a solution to the 3-dimensional dislocation problem near $(\Phi, \overline{R})$, in the form

$$(10.3) \quad c = \Phi + \varepsilon \Psi , \quad R = \overline{R} + \varepsilon \lambda(t)$$

where $c = c(X, t)$ and

$$(10.4) \quad \begin{aligned} c(X, 0) &= \Phi(X) + \varepsilon \Psi_0(X) , \\ \Psi_0(X) &\text{ has compact support .} \end{aligned}$$

We assume that

$$(10.5) \quad h(\theta) \text{ is analytic .}$$

We can then extend the function $\Phi(X)$ as a harmonic function into a neighborhood W_0 of $\partial D(\overline{R})$; we shall later establish a priori bound on $\lambda(t)$ which ensures that $\mathbb{R}^3 \setminus D(R(t))$ will be contained in $(\mathbb{R}^3 \setminus D(\overline{R})) \cup W_0$ for all sufficiently small ε (and all $t > 0$).

The pair (c, R) satisfies:

$$(10.6) \quad c_t - \Delta c = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus D(R(t)) ,$$

$$(10.7) \quad c = h(\theta) \exp \left(\frac{\log R}{R} \right) \quad \text{on} \quad \partial D(R(t)) ,$$

$$(10.8) \quad R \dot{R} = \iint_{\partial D(R(t))} \frac{\partial c}{\partial N} dS .$$

Equations (10.7), (10.8) can be written in the form

$$(10.9) \quad \Psi = \frac{1}{\varepsilon} \left[h(\theta) \exp \frac{\log R}{R} - \Phi \right] \quad \text{on} \quad \partial D(R(t)) ,$$

$$\dot{R} = \iint_{\partial D(R(t))} \frac{\partial \Phi}{\partial N} dS + \varepsilon \iint_{\partial D(R(t))} \frac{\partial \Psi}{\partial N} dS .$$

Since, by Green's formula,

$$\iint_{\partial D(R(t))} \frac{\partial \Phi}{\partial N} = \lim_{\substack{A \rightarrow \infty \\ |X|=A}} \iint \frac{\partial \Phi}{\partial N} = \iint_{\partial D(\bar{R})} \frac{\partial \Phi}{\partial N} = 0 ,$$

the equation for $\dot{R}$ can be written in the form

$$(10.10) \quad \dot{\lambda}(t) = \iint_{\partial D(\bar{R} + \varepsilon \lambda(t))} \frac{\partial \Psi}{\partial N} dS .$$

To evaluate (10.9) we write

$$\exp \left(\frac{\log R}{R} \right) = \exp \left(\frac{\log(\bar{R} + \varepsilon \lambda)}{\bar{R} + \varepsilon \lambda} \right) = \exp \left(\frac{\log \bar{R}}{\bar{R}} \right) \left\{ 1 + \frac{1 - \log \bar{R}}{\bar{R}^2} \lambda \varepsilon \right\} + O(\lambda^2 \varepsilon^2) ,$$

and

$$\begin{aligned} \Phi(X) &= \Phi(R(t) + \cos \theta, \sin \theta) \\ &= h(\theta) \exp \left(\frac{\log \bar{R}}{\bar{R}} \right) + \frac{\partial \Phi}{\partial r} \lambda \varepsilon + O(\lambda^2 \varepsilon^2) . \end{aligned}$$

We then obtain from (10.9)

$$(10.11) \quad \Psi = Q(\theta) \lambda + \varepsilon O(\lambda^2) \quad \text{on} \quad \partial D(\bar{R} + \varepsilon \lambda)$$

where $\lambda^{-2} O(\lambda^2)$ is a smooth function of (λ, θ) and

$$(10.12) \quad Q(\theta) = \exp \left(\frac{\log \bar{R}}{\bar{R}} \right) \frac{1 - \log \bar{R}}{\bar{R}^2} h(\theta) - \frac{\partial \Phi}{\partial r} \quad \text{on} \quad \partial D(\bar{R}) .$$

In summary, (Ψ, λ) should satisfy

$$(10.13) \quad \Psi_t = \Delta \Psi \quad \text{in} \quad \mathbf{R}^3 \setminus D(\bar{R} + \varepsilon \lambda(t)) ,$$

(10.11), (10.10), and an initial condition.

In Section 11 we shall prove that this system has a unique global solution with $\lambda(t) \rightarrow 0$ if $t \rightarrow \infty$, provided

$$(10.14) \quad Q(\theta) < 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi .$$

Condition (10.14) is a “stability” condition; it is satisfied if, for instance,

$$(10.15) \quad |h'(\theta)| \ll h(\theta) \quad \text{and} \quad \overline{R} > e .$$

Before tackling the system for (Ψ, λ) , it will be convenient to study the linearized problem

$$(10.16) \quad \begin{aligned} \Psi_t &= \Delta \Psi \quad \text{in} \quad \mathbb{R}^3 \setminus D(\overline{R}) , \\ \Psi &= Q(\theta)\lambda \quad \text{on} \quad \partial D(\overline{R}) , \\ \dot{\lambda} &= \iint_{\partial D(\overline{R})} \frac{\partial \Psi}{\partial N} dS , \quad \lambda(0) = \lambda_0 , \\ \Psi(X, 0) &= \Psi_0(X) , \end{aligned}$$

This system has a unique local solution if, $\Psi_0 \in C^{2+\alpha}$. Our main goal here is to derive a priori estimates which will be needed in Section 11.

LEMMA 10.2. *If*

$$(10.17) \quad \begin{aligned} \tilde{\Psi}_t &\geq \Delta \tilde{\Psi} \quad \text{in} \quad [\mathbb{R}^3 \setminus D(\overline{R})] \times (0 < t < T) , \\ \tilde{\Psi} &= \tilde{\lambda} Q \quad \text{on} \quad [\partial D(\overline{R})] \times (0 < t < T) , \\ \dot{\tilde{\lambda}} &\leq \iint_{\partial D(\overline{R})} \frac{\partial \tilde{\Psi}}{\partial N} dS \quad \text{for} \quad 0 < t < T , \\ \tilde{\Psi}(X, 0) &\geq \Psi_0(X) \quad \text{in} \quad \mathbb{R}^3 \setminus D(\overline{R}) , \\ \tilde{\lambda}(0) &\leq \lambda(0) , \end{aligned}$$

then $\tilde{\lambda}(t) \leq \lambda(t)$ and, consequently, $\tilde{\Psi}(X, t) \geq \Psi(X, t)$ for $0 \leq t < T$.

Proof. We shall assume that $\tilde{\lambda}(0) < \lambda(0)$; the general case can be obtained by considering $\lambda(0) + \delta$ as an initial data in (10.16), with $\delta > 0$, $\delta \rightarrow 0$.

We claim that $\tilde{\lambda}(t) < \lambda(t)$ for all $0 < t < T$. Indeed, if this is not true then there exists a $t_0 \in (0, T)$ such that

$$(10.18) \quad \begin{aligned} \tilde{\lambda}(t) &< \lambda(t) \quad \text{if } 0 \leq t < t_0, \\ \tilde{\lambda}(t_0) &= \lambda(t_0). \end{aligned}$$

By the maximum principle, $\tilde{\Psi}(X, t) > \Psi(X, t)$ if $0 \leq t < t_0$ and, since $\tilde{\Psi}(X, t_0) = \Psi(X, t_0)$ on $\partial D(\bar{R})$,

$$\frac{\partial \tilde{\Psi}}{\partial N} < \frac{\partial \Psi}{\partial N} \quad \text{at } t = t_0.$$

This implies that $\dot{\tilde{\lambda}}(t_0) < \dot{\lambda}(t_0)$, a contradiction to (10.18).

We now construct explicitly a supersolution $(\tilde{\Psi}, \tilde{\lambda})$ as in (10.17).

Let G, H be functions defined by

$$(10.19) \quad \begin{aligned} \Delta G &= 0 \quad \text{in } \mathbb{R}^3 \setminus D(\bar{R}), \\ G &= Q(\theta) \quad \text{on } \partial D(\bar{R}), \\ G &\rightarrow 0 \quad \text{as } |X| \rightarrow \infty, \end{aligned}$$

and

$$(10.20) \quad \begin{aligned} \Delta H &= 0 \quad \text{in } \mathbb{R}^3 \setminus D(\bar{R}), \\ H &= 0 \quad \text{on } \partial D(\bar{R}), \\ H &\rightarrow 1 \quad \text{as } |X| \rightarrow \infty. \end{aligned}$$

Clearly

$$G < 0, \quad H > 0.$$

Let $\tilde{\Gamma} \gg \Gamma$ be positive constants, and let $\tilde{\varepsilon}$ be a small positive constant, $\beta \in (0, 1)$ and

$$(10.21) \quad \tilde{\tilde{\varepsilon}} = \tilde{\varepsilon}\beta.$$

Choose a function $g_0(X) \geq 0$ with compact support such that

$$(10.22) \quad \tilde{\varepsilon}(g_0(X) + 1) > \tilde{\Gamma}|G(X)| \quad \text{for all } X,$$

and define a function Z by

$$\begin{aligned}\Delta Z &= -g_0(X) \quad \text{in } \mathbb{R}^3 \setminus D(\bar{R}) , \\ Z &= \frac{1}{6} |X|^2 \quad \text{on } \partial D(\bar{R}) , \\ Z &\rightarrow 0 \quad \text{as } |X| \rightarrow \infty .\end{aligned}$$

The function

$$W = Z - \frac{1}{6} |X|^2 - \tilde{C}H$$

satisfies

$$\begin{aligned}(10.23) \quad \Delta W &= -g_0(X) - 1 \quad \text{in } \mathbb{R}^3 \setminus D(\bar{R}) , \\ W &= 0 \quad \text{on } \partial D(\bar{R}) , \\ W &\sim -\frac{1}{6} |X|^2 \quad \text{as } |X| \rightarrow \infty .\end{aligned}$$

Furthermore,

$$(10.24) \quad W < 0 \quad \text{in } \mathbb{R}^3 \setminus D(\bar{R})$$

if we take $\tilde{C}$ to be large enough.

We shall construct supersolutions

$$\begin{aligned}\tilde{\Psi}_i &\quad \text{in } |X| \leq 2\eta\sqrt{t+C^*} , \\ \tilde{\Psi}_e &\quad \text{in } |X| > \eta\sqrt{t+C^*} .\end{aligned}$$

The following lemma will be used to patch together the supersolutions.

LEMMA 10.3. Suppose $\tilde{\Psi}_i, \tilde{\Psi}_e$ are smooth functions satisfying

$$(10.25) \quad (\partial_t - \Delta)\tilde{\Psi}_i \geq 0 \quad \text{in } \left\{ |X| < 2\eta\sqrt{t+C^*} \right\} ,$$

$$(10.26) \quad (\partial_t - \Delta)\tilde{\Psi}_e \geq 0 \quad \text{in } \left\{ |X| > \eta\sqrt{t+C^*} \right\} ,$$

$$(10.27) \quad \tilde{\Psi}_i < \tilde{\Psi}_e \quad \text{on } \left\{ |X| = \eta\sqrt{t+C^*} \right\} ,$$

$$(10.28) \quad \tilde{\Psi}_i > \tilde{\Psi}_e \quad \text{on } \left\{ |X| = 2\eta\sqrt{t+C^*} \right\} .$$

Then the function

$$\tilde{\Psi} = \begin{cases} \tilde{\Psi}_i & \text{if } |X| < \eta\sqrt{t+C^*} \\ \min \left\{ \tilde{\Psi}_i, \tilde{\Psi}_e \right\} & \text{if } \eta \leq \frac{X}{\sqrt{t+C^*}} \leq 2\eta \\ \tilde{\Psi}_e & \text{if } |X| > 2\eta\sqrt{t+C^*} \end{cases}$$

is a supersolution.

This means that if P is a smooth function such that

$$P_t - \Delta P \leq 0 \quad \text{in a domain } \Omega ,$$

$$P \leq \tilde{\Psi} \quad \text{on the parabolic boundary } \partial_p \Omega ,$$

then $P \leq \tilde{\Psi}$ in Ω .

To prove this we consider the function $P_\delta = P - \delta t$ for $\delta > 0$. Then

$$\left(\frac{\partial}{\partial t} - \Delta \right) (P_\delta - \tilde{\Psi}) < 0 \quad \text{in } \Omega ,$$

$$P_\delta - \tilde{\Psi} < 0 \quad \text{on } \Omega .$$

One can easily show (by the maximum principle) that $P_\delta - \tilde{\Psi}$ cannot attain a positive maximum at any point (x_0, t_0) in Ω , and therefore $P_\delta - \tilde{\Psi} < 0$ in Ω , and the lemma then follows by taking $\delta \rightarrow 0$.

Set $\tau = t + C^*$ with $C^* \gg 1$ to be determined, and define

$$(10.29) \quad \begin{aligned} \tilde{\Psi}_i &= -\frac{\tilde{\Gamma}}{\tau^{3\beta/2}} G(X) + \frac{\Gamma}{\tau^{3\beta/2}} H(X) + \frac{3\beta}{2}(\Gamma + \tilde{\varepsilon}) \frac{W(X)}{\tau^{1+3\beta/2}} , \\ \tilde{\Psi}_e &= \frac{\Gamma + \tilde{\varepsilon}}{\tau^{3\beta/2}} \exp \left\{ -\frac{|X|^2}{4\tau} \right\} , \end{aligned}$$

and

$$(10.30) \quad \tilde{\lambda}(t) = -\frac{\tilde{\Gamma}}{\tau^{3\beta/2}} .$$

Then

$$(10.31) \quad \tilde{\Psi}_i = \tilde{\lambda}Q \quad \text{on } \partial D(\bar{R}) .$$

For $|X|$ large but $|X|/\sqrt{\tau}$ small,

$$\begin{aligned}\tilde{\Psi}_i &\sim \frac{\Gamma - \frac{\beta}{4} (\Gamma + \tilde{\varepsilon}) \frac{|X|^2}{\tau}}{\tau^{3\beta/2}}, \\ \tilde{\Psi}_e &\sim \frac{(\Gamma + \tilde{\varepsilon}) \left(1 - \frac{|X|^2}{4\tau}\right)}{\tau^{3\beta/2}}.\end{aligned}$$

If we choose η such that

$$\begin{aligned}[(\Gamma + \tilde{\varepsilon}) - (\Gamma + \tilde{\varepsilon})\beta]\eta^2 &< 4\tilde{\varepsilon}, \\ [(\Gamma + \tilde{\varepsilon}) - (\Gamma + \tilde{\varepsilon})\beta]4\eta^2 &> 4\tilde{\varepsilon},\end{aligned}$$

then (10.27), (10.28) are satisfied. By (10.21), the last two inequalities are satisfied if

$$(10.32) \quad \eta^2 = \frac{2\tilde{\varepsilon}}{\Gamma(1-\beta)}.$$

Next

$$(10.33) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{\Psi}_e = \left(-\frac{3\beta}{2} + \frac{3}{2}\right) \frac{\Gamma + \tilde{\varepsilon}}{\tau^{3\beta/2+1}} \exp\left\{-\frac{|X|^2}{4\tau}\right\} > 0.$$

As for $\tilde{\Psi}_i$, we have

$$\begin{aligned}(10.34) \quad \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{\Psi}_i &= \frac{3\beta}{2} \frac{\Gamma}{\tau^{3\beta/2+1}} \{g_0(X) + 1 - H(X)\} \\ &+ \frac{3\beta}{2} \frac{1}{\tau^{3\beta/2+1}} \left\{ \tilde{\varepsilon}(g_0(X) + 1) + \tilde{\Gamma}G(X) \right\} \\ &+ \left(-\frac{3\beta}{2}\right) \left(\frac{3\beta}{2} + 1\right) \frac{\Gamma + \tilde{\varepsilon}}{\tau^{3\beta/2+2}} W(X).\end{aligned}$$

The first term on the right-hand side is positive since $H(X) \leq 1$; the second term is positive by (10.22), and the third term is positive since $W < 0$. We conclude that $\tilde{\Psi}_i$ satisfies (10.26).

We have shown so far that $\tilde{\Psi}$ satisfies the first two conditions in (10.17). We next show that also

$$(10.35) \quad \dot{\lambda} \leq \iint_{\partial D(\bar{R})} \frac{\partial \tilde{\Psi}_i}{\partial N},$$

so that we shall be in a position to apply Lemma 10.2.

Observe first that since

$$\begin{aligned} G &= Q < 0 \quad \text{on} \quad \partial D(\bar{R}) , \\ G &\rightarrow 0 \quad \text{as} \quad |X| \rightarrow \infty , \end{aligned}$$

we have

$$-\frac{\gamma_1}{|X|} \leq G \leq -\frac{\gamma_2}{|X|}$$

where γ_1, γ_2 are positive constants. Hence

$$-\iint_{\partial D(\bar{R})} \frac{\partial G}{\partial N} = \lim_{\substack{\rho \rightarrow \infty \\ |X|=\rho}} \iint \frac{\partial G}{\partial N} \geq \gamma_2 > 0 .$$

Since $\tilde{\Gamma} \gg \Gamma$ and $\tau \geq C^* \gg 1$, we have

$$\begin{aligned} \iint_{\partial D(\bar{R})} \frac{\partial \tilde{\Psi}_i}{\partial N} &> \frac{1}{2} \frac{\tilde{\Gamma}}{\tau^{3\beta/2}} \left(- \iint_{\partial D(\bar{R})} \frac{\partial G}{\partial N} \right) \\ &> \frac{1}{2} \frac{\tilde{\Gamma} \gamma_2}{\tau^{3\beta/2}} > \frac{3\beta}{2} \frac{\tilde{\Gamma}}{\tau^{3\beta/2+1}} = \dot{\lambda} . \end{aligned}$$

Applying Lemma 10.2, we deduce that

$$|\lambda(t)| \leq \frac{\tilde{\Gamma}}{\tau^{3\beta/2}}$$

provided

$$|\lambda(0)| < \frac{\tilde{\Gamma}}{(C^*)^{3\beta/2}} , \quad |\Psi_0(X)| \leq \tilde{\Psi}(X, 0) .$$

If we differentiate (10.16) in t , we find that Ψ_t satisfies the same linear differential system with the only change being in the initial condition:

$$\Psi_t|_{t=0} = \Delta \Psi_0(X) .$$

Hence if

$$|\dot{\lambda}(0)| < \frac{\tilde{\Gamma}}{(C^*)^{3\beta/2}} , \quad |\Delta \Psi_0(X)| < \tilde{\Psi}(X, 0)$$

then we can estimate $|\Psi_t|$ and $|\dot{\lambda}(t)|$ as before.

Since $\tilde{\Psi}(X, 0) > 0$, we can establish the a priori estimates on λ and $\dot{\lambda}$ provided

$$(10.36) \quad |\lambda(0)|, \|\Psi_0\|_{L^\infty} \quad \text{and} \quad \|\Delta \Psi_0\|_{L^\infty} \quad \text{are sufficiently small}$$

Using these a priori estimates we can extend the solution of the linearized problem (10.16) to all t :

THEOREM 10.4. suppose (10.5) holds, $\Psi_0 \in C_0^{2+\alpha}(\mathbf{R}^3)$, $\lambda(0)Q(\theta) = \Psi_0(X)$ and (10.36) is satisfied. Then there exists a unique solution to the linearized problem (10.16) for all $t > 0$.

§11. Global solution near steady-state solution for $n=3$. We assume that $\lambda(t) = (R(t) - \bar{R})/\varepsilon$ satisfies

$$(11.1) \quad |\lambda(t)| + |\dot{\lambda}(t)| < \frac{C_1}{(t + C^*)^{3\beta/2}}, \quad 0 \leq t < T$$

and prove that (11.1) remains valid for $t = T$; this will establish (by continuity) the a priori bound (11.1) for all t provided C_1 is chosen so that (11.1) is satisfied at $t = 0$. In view of Theorems 2.1, 2.2, there is then a unique global solution near the stationary solution $(\Phi, \bar{R})$, and $R(t) \rightarrow \bar{R}$ as $t \rightarrow \infty$.

In (11.1), C^* is chosen as in Section 10 and C_1 is a positive constant still to be determined; it will be independent of ε , thus ensuring that $\partial D(R(t))$ remains in the open set W_0 where Φ is analytic, provided ε is small enough.

We begin by mapping the tori $D(R(t))$ onto the fixed torus $D(\bar{R})$ without changing the variable for large $|X|$:

$$r \rightarrow r + \varepsilon \lambda(t) \xi(r), \quad z = z$$

where $\xi \in C^2$, $0 \leq \xi \leq 1$ and

$$\xi(r) = \begin{cases} 1 & \text{if } |r - \bar{R}| < 1 \\ 0 & \text{if } |r - \bar{R}| > 2; \end{cases}$$

for simplicity we assume here that $\bar{R} > 3$.

Let

$$w(r, z, t) = c(r + \varepsilon \lambda(t) \xi(r), z, t).$$

Set $D = D(\bar{R})$. Then

$$(11.2) \quad \dot{\lambda} = \frac{1}{\varepsilon} \iint_{\partial D} \frac{\partial w}{\partial N} dS.$$

The differential equation for c becomes

$$w_t = Lw$$

where

$$(11.3) \quad \begin{aligned} Lu = & \frac{u_{rr}}{(1 + \varepsilon \xi' \lambda)^2} - \frac{\varepsilon \xi'' \lambda}{(1 + \varepsilon \xi' \lambda)^3} u_r + \frac{1}{(r + \varepsilon \xi \lambda)(1 + \varepsilon \xi' \lambda)} u_r \\ & + \frac{\varepsilon \xi \dot{\lambda}}{1 + \varepsilon \xi' \lambda} u_r + u_{zz} . \end{aligned}$$

Writing analogously to (10.3),

$$(11.4) \quad w = \varphi(r, z, t) + \varepsilon \psi(r, z, t)$$

where

$$\varphi(r, z, t) = \Phi(r + \varepsilon \lambda(t) \xi(r), z) ,$$

we get

$$(11.5) \quad \psi_t = L\psi \quad \text{in} \quad \mathbb{R}^3 \setminus D .$$

By (11.2)

$$(11.6) \quad \dot{\lambda} = \int_{\partial D} \frac{\partial \psi}{\partial N} dS$$

and, by (10.11),

$$(11.7) \quad \psi = Q(\theta) \lambda + \varepsilon O(\lambda^2) = Q^*(\theta, t) \lambda$$

where, $\lambda^{-2} O(\lambda^2)$ is a smooth function of (λ, θ) and, by (11.1),

$$(11.8) \quad 2Q(\theta) < Q^*(\theta, t) < \frac{1}{2} Q(\theta) ,$$

$$(11.9) \quad \|Q_t^*\|_{L^\infty} \leq \frac{\sqrt{\varepsilon}}{(t + C^*)^{3\beta/2}}$$

if ε is small enough.

LEMMA 11.1. *If $\psi_0 \equiv \psi(r, z, 0)$ has compact support and $|\lambda(0)|$, $\|\psi_0\|_{C^{2,\alpha}}$ are small enough, then there exists a constant C_2 depending only on C_1 such that*

$$(11.10) \quad |\ddot{\lambda}(t)| < \frac{C_2}{(t + C^*)^{3\beta/2}} \quad \text{if} \quad 0 \leq t < T ,$$

and, for some $\alpha \in (0, 1)$,

$$(11.11) \quad \|Q^*\|_{C^{2+\alpha, 1+\alpha/2}} \leq C, \quad \|Q_t^*\|_{C_x^{2+\alpha}} \leq \frac{\sqrt{\varepsilon}}{(t + C^*)^{3\beta/2}},$$

$$(11.12) \quad \|Q_{tt}^*\|_{L^\infty} \leq \frac{\sqrt{\varepsilon}}{(t + C^*)^{3\beta/2}}$$

provided ε is small enough; the constant C is independent of C_1, C_2 .

Proof. By (10.33), for any $K > 0$,

$$(\partial_t - L) \left\{ \frac{K}{(t + C^*)^{3\beta/2}} \exp \left[-\frac{(r + \varepsilon \lambda(t) \xi(r))^2 + z^2}{4(t + C^*)} \right] \right\} > 0.$$

Denote the function in braces by Ψ^* . Then $\Psi^* \pm \psi$ is a supersolution and, by (11.1), it is nonnegative on the parabolic boundary if

$$C^* > 1 \quad \text{and} \quad K \geq 2C_1 \exp \left(\frac{(\bar{R} + 2)^2 + 1}{4} \right) \max |Q(\theta)|.$$

Therefore, by the maximum principle,

$$(11.14) \quad |\psi(r, z, t)| \leq \frac{K}{(t + C^*)^{3\beta/2}}.$$

Let

$$(11.15) \quad \chi(r, z, t) = (t + C^*)^{3\beta/2} \psi(r, z, t).$$

Then

$$(\partial_t - L)\chi = \frac{3\beta}{2} (t + C^*)^{\frac{3\beta}{2}-1} \psi = \frac{3\beta}{2} \frac{\chi}{t + C^*}.$$

By (11.14)

$$\|\chi(r, z, t)\|_{L^\infty} \leq K.$$

On ∂D

$$\chi(r, z, t) = (t + C^*)^{3\beta/2} Q^*(\theta, t) \lambda$$

and therefore, by (11.1), (11.9),

$$\|\chi\|_{L^\infty(\partial D)} + \|\chi_t\|_{L^\infty(\partial D)} \leq C$$

where C depends on C_1 . Using L^p interior estimates (for χ) and the imbedding theorem, as well as (11.6), (11.15), we deduce that

$$(11.16) \quad \|(t + C^*)^{3\beta/2} \dot{\lambda}\|_{C^{\alpha/2}[0,T]} \leq C .$$

Further, by (11.7), (11.16),

$$\|(t + C^*)^{3\beta/2} Q_t^*\|_{C^{\alpha,\alpha/2}(\partial D \times [0,T])} \leq C .$$

It follows that

$$\|\chi_t\|_{C^{\alpha,\alpha/2}(\partial D \times [0,T])} \leq C$$

and then, by Schauder's boundary estimates,

$$(11.17) \quad \|\chi\|_{C^{2+\alpha,1+\alpha/2}((\mathbb{R}^3 \setminus D) \times [0,T])} \leq C .$$

All the constants C above depend only on C_1 .

From (11.17) and (11.15), (11.6) we conclude immediately that (11.10) holds and then (11.11), (11.12) follow immediately from the definition of $Q^*(\theta, t)$ in (11.7).

We now construct a supersolution $\tilde{\Psi}$ and the corresponding $\tilde{\lambda}$ as in Section 10 with two minor differences: (i) We replace $G(X)$ by a function $G(X, t)$ satisfying

$$\begin{aligned} \Delta G &= 0 \quad \text{in } \mathbb{R}^3 \setminus D , \\ G &= Q^*(\theta, t) \quad \text{on } \partial D , \\ G &\rightarrow 0 \quad \text{as } |X| \rightarrow \infty . \end{aligned}$$

(ii) We take $g_0(X)$ with compact support such that (cf. (10.22))

$$\frac{1}{2} \tilde{\varepsilon}(g_0(X) + 1) > \tilde{\Gamma}|G(X, t)| \quad \text{for all } X ;$$

this can be done since Q^* is bounded independently of t .

Then

$$\begin{aligned} \left(\frac{\partial}{\partial t} - L\right) \tilde{\Psi}_i &= \left(\frac{\partial}{\partial t} - \Delta\right) \tilde{\Psi}_i + (\Delta - L) \tilde{\Psi}_i \\ &\geq \frac{\tilde{\varepsilon}}{2} \frac{3\beta}{2} \frac{1}{\tau^{3\beta/2+1}} + (\Delta - L) \tilde{\Psi}_i . \end{aligned}$$

By (11.1), the coefficients of $\Delta - L$ are bounded by $\varepsilon C_1 / \tau^{3\beta/2}$. Therefore, if ε is small enough,

$$\left(\frac{\partial}{\partial t} - L\right) \tilde{\Psi}_i > 0 \quad \text{for } |X| \leq 2\eta\sqrt{\tau} ;$$

the constants $\tilde{\Gamma}, \Gamma, \tilde{\varepsilon}, \tilde{\varepsilon}$ are independent of C_1 and C_2 .

Since η is defined by (10.32), if C^* is large enough then $|X| \geq \eta\sqrt{\tau}$ implies $|X| > 3$, so that $L = \Delta$. Hence $\tilde{\Psi}_e$ is again a supersolution. Applying Lemmas 10.3 and 10.2 (with Δ replaced by L), we obtain the estimate

$$(11.18) \quad |\lambda(t)| \leq \frac{\overline{C}}{(t + C^*)^{3\beta/2}} \quad \text{if } t < T ,$$

provided $|\lambda(0)|$ and $\|\psi_0\|_{C^{2+\alpha}}$ are small enough, where the constant $\overline{C}$ is independent of C_1, C_2 .

Next we want to estimate $\dot{\lambda}$. We differentiate (11.5) in t and obtain

$$(11.19) \quad \psi_{tt} - L\psi_t = f$$

where

$$\|f\|_{L^\infty} \leq C\varepsilon \|D^2\psi\|_{L^\infty} (|\lambda(t)| + |\dot{\lambda}(t)|), \quad f = 0 \quad \text{for } |X| > 3 .$$

From (11.17), (11.1) it follows that

$$(11.20) \quad \|f\|_{L^\infty} \leq \frac{\sqrt{\varepsilon}}{(t + C^*)^{3\beta}}$$

provided ε is small enough.

We next differentiate (11.7) and (11.6) in t ,

$$(11.21) \quad \psi_t = Q_t^*(\theta, \lambda)\lambda + Q^*(\theta, t)\dot{\lambda} \quad \text{on } \partial D ,$$

$$(11.22) \quad \ddot{\lambda} = \iint_{\partial D} \frac{\partial \psi_t}{\partial N} dS .$$

We shall need the following version of Lemma 10.2 whose proof is the same as that of Lemma 10.2:

If

$$(11.23) \quad \tilde{\Psi}_t \geq L\tilde{\Psi} + f \quad \text{in } [\mathbb{R}^3 \setminus D] \times (0 < t < T) ,$$

$$(11.24) \quad \tilde{\Psi} = \eta Q^* + Q_t^*(\theta, \lambda)\lambda \quad \text{on } \partial D \times (0, T) ,$$

$$(11.25) \quad \dot{\eta} \leq \iint_{\partial D} \frac{\partial \tilde{\Psi}}{\partial N} dS \quad \text{for } 0 < t < T ,$$

$$(11.26) \quad \tilde{\Psi}(X, 0) > |\psi_t(X, 0)| \quad \text{in } \mathbb{R}^3 \setminus D ,$$

$$(11.27) \quad \eta(0) < \dot{\lambda}(0) ,$$

then $\eta(t) < \dot{\lambda}(t)$ for $0 < t < T$.

Let

$$\begin{aligned}\Delta \tilde{G} &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\ \tilde{G} &= Q_t^*(\theta, \lambda) \quad \text{on } \partial D, \\ \tilde{G} &\rightarrow 0 \quad \text{as } |X| \rightarrow \infty,\end{aligned}$$

and modify the $\tilde{\Psi}_i$ used above in this section by adding the term $\lambda \tilde{G}(X, t)$. By (11.11), (11.12),

$$\|\tilde{G}\|_{C_X^{2+\alpha}} + \|\tilde{G}_t\|_{L_X^\infty} \leq \frac{C\sqrt{\varepsilon}}{(t + C^*)^{3\beta/2}}$$

and, therefore,

$$\|\lambda \tilde{G}\|_{C_X^{2+\alpha}} + \|(\lambda \tilde{G})_t\|_{L_X^\infty} \leq \frac{C\sqrt{\varepsilon}}{(t + C^*)^{3\beta}} \leq \frac{\varepsilon^{1/4}}{(t + C^*)^{3\beta}}.$$

Using this estimate and (11.20), we find that (11.23) is satisfied if ε is small enough. Also (11.24), (11.25) hold if $\eta(t)$ is defined as $\tilde{\lambda}$ in (10.30) (cf. the proof of (10.35)). Hence, under the assumptions of Lemma 11.1,

$$|\dot{\lambda}(t)| \leq \frac{\overline{C}_0}{(t + C^*)^{3\beta/2}}$$

where $\overline{C}_0$ is a constant independent of C_1, C_2 . From this and (11.8) it follows that (11.1) can be continued beyond $t = T$ if $|\lambda(0)|, |\dot{\lambda}(0)|$ and $\|\psi\|_{C^{2+\alpha}}$ are sufficiently small, if C_1 is a priori chosen larger than $\overline{C}$ and $\overline{C}_0$, and if ε is small enough.

We have thus proved the following stability theorem:

THEOREM 11.2. *Let $(\Phi, \overline{R})$ be the steady-state solution constructed in Lemma 10.1, and assume that (10.5) and (10.14) are satisfied. Suppose also that $|R(0) - \overline{R}|$ is sufficiently small, that $c(X, 0) - \Phi(X)$ has compact support, and that*

$$\|c(X, 0) - \Phi(X)\|_{C^{2,\alpha}}$$

is sufficiently small. Then the solution (c, R) to the dislocation problem exists and is unique for all $t > 0$, and, for any $\beta \in (0, 1)$ there is a constant C such that

$$\begin{aligned}|R(t) - \overline{R}| &\leq C(1+t)^{-3\beta/2}, \\ |c(r - R(t), z, t) - \Phi(X)| &\leq C(1+t)^{-3\beta/2}\end{aligned}$$

in $[\mathbb{R}^3 \setminus D(t)] \times [0 < t < \infty)$.

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