INSTABILITY IN THE IMPLEMENTATION OF WALRASIAN ALLOCATIONS

By

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Abstract

The existence of game forms which implement Walrasian allocations as Cournot (Nash) equilibrium outcomes is well known. However, if the equilibria are also required to be locally dynamically stable, at least for environments with unique Walrasian allocations, this paper shows that the requisite game forms do not exist. Our definition of a game form entails certain regularity conditions, and requires the Cournot equilibrium to be unique when the Walrasian equilibrium is unique. The main result is that for such a game form, there does not exist a continuous-time strategy adjustment process which ensures the local stability of Cournot equilibria throughout a certain class of environments having unique Walrasian equilibria. Each trader adjusts his strategy in response to his own characteristics and the observed current strategies of others; but the direction and magnitude of adjustments are not constrained by any behavioral assumptions. The definitions permit the inclusion of an artificial player, such as an auctioneer, so the well known tatonnement instability emerges as a special case.

1. Introduction

An allocation rule associates an allocation with each economic environment. Depending on the allocation rule, verifying the correct allocation requires a certain amount of information about the environment, information which is initially dispersed among the individual economic agents. The goal of implementation theory is the design of mechanisms, usually termed "game forms", which make the necessary communication compatible with individual incentives. There are various definitions of implementation, depending on the definition of incentive-compatibility, but the most successful in terms of positive results has been Cournot (Nash) implementation (see Dasgupta, Hammond, and Maskin (1979) for a survey of concepts and results in implementation theory).^{1/} In particular, game forms whose Cournot equilibrium outcomes are Walrasian allocations for exchange environments are now well known.

Unfortunately, the study of implementation has thus far been confined to comparing equilibrium outcomes and desired allocations, ignoring the question of stability. If an equilibrium is not at least locally stable, then even if the initial position is an equilibrium, a slight change in the environment could prevent equilibrium from ever being reestablished. Thus, in the absence of stability, the equilibrium conditions may be irrelevant to the actual behavior of the mechanism.

Just as implementation theory has ignored the stability question, stability research has proceeded independently of implementation theory. The tatonnement price adjustment process was the initial focus of research on the stability of Walrasian equilibrium, but Scarf (1960) and Gale (1963) constructed examples of exchange environments with unique Walrasian equilibria which were not even locally tatonnement stable. The subsequent character-

ization of market excess demand functions (see especially Mas-Colell (1977, Corollary 2)) showed that such examples constitute a large subset of the traditional domain of demand theory. More recently, Smale (1976) developed the "Global Newton Method" which generically converges to Walrasian equilibrium. Price adjustments in the Newton method depend on the derivative of the aggregate excess demand function, and subsequent research by Saari and Simon (1976) and the present author, Jordan (1982 and 1983) showed that any price adjustment process which makes Walrasian equilibria locally stable requires information about the derivative as well as the value of aggregate excess demand. The Walrasian equilibrium condition merely requires that aggregate excess demand equal zero. Thus, at least within the realm of price adjustment processes, stabilizing a Walrasian allocation requires more environmental information than is needed to verify a Walrasian allocation.

This informational finding gives additional urgency to the stability issue in implementation theory. If the communication needed for stability exceeds the communication needed merely to verify the desired allocation, then the incentives for communication which have been developed thus far may be seriously incomplete. The main result of the present paper suggests that this is indeed the case, and that a complete system of incentives is impossible. More specifically, we show that any game form which is well-behaved in a sense explained below, and which implements Walrasian equilibria as Cournot equilibrium outcomes, admits an environment, with a unique Walrasian allocation, such that the corresponding Cournot equilibrium is not locally stable. We generalize the concept of Cournot implementation by permitting the game form to include an artificial player, such as an auctioneer, whose behavior is specified as part of the game form, and by dropping the "balance" condition, so that disequilibrium

outcomes are not required to be feasible. This generalization includes the tatonnement process, as well as game forms constructed in the implementation literature. Thus our result shows that the examples of tatonnement instability constructed by Scarf (1960) and Gale (1963) are merely symptomatic of a general instability which is intrinsic to the Walrasian allocation rule.

The Instability Theorem (6.1 below) is a consequence of two fundamental principles in economic theory: the Correspondence Principle and the monotonicity property of Cournot-Nash implementation. First, if a slight change in endowments, for example, results in disequilibrium, a locally stable price adjustment process will find the new equilibrium. Thus the price dynamics must, to a certain minimal extent, correspond to the comparative statics. More generally, if a game form implements Walrasian allocations as Cournot equilibria, then a locally stable strategy adjustment process must also reproduce the Walrasian comparative statics. This means that the dynamics must be sensitive to the same environmental information that determines the derivative of Walrasian demand, including the second derivatives of individual preferences. Moreover, each player's strategy adjustment can be sensitive to the characteristics of other players only to the extent that their characteristics are revealed by their strategies, so the information must be revealed by the strategies.

Second, the monotonicity property of Cournot equilibrium prevents the required information from being revealed. Suppose an agent has chosen an action, or strategy, s^i , yielding an allocation x^i which maximizes his preferences given the available alternatives. If his preferences change in such a way that every allocation which previously was not preferred to x^i is still not preferred to x^i , then the individual has no incentive to change s^i . If x^i is in the interior of the consumption set and preferences are

smooth, such a change in preferences preserves the tangent to the indifference surface at x^i but may increase the curvature of the indifference surface, and change the Engle curve through x^i . Thus, with no change in observed strategies, the substitution effect may be reduced and the income effect changed, significantly changing the Walrasian comparative statics. The failure of such information to be revealed by observed strategies does not depend on the particular outcome function from strategies to allocations. The nonelicitation of second order preference information is a feature of any incentive system with any number of agents.

The various sufficient conditions for tatonnement stability restrict the range of second order preference behavior so that the adjustment process can ignore variations within this range without compromising stability. The Instability Theorem implies that such restrictions are essential to local stability in any mechanism satisfying certain assumptions, described below, which implements Walrasian allocations through individual maximizing behavior.

Section 2 below describes a class of exchange environments with three commodities and four traders. The stability of the tatonnement process with two commodities indicates that our result requires at least three commodities. With some sacrifice of convenience, the result could probably be obtained with only three traders. The number of traders and the number of commodities could be increased (independently) with only minor modifications. In the environments considered here, individual preferences are represented by quadratic utility functions, but are nonsatiated and satisfy the usual regularity properties. Within the general class, E , of such environments, there is a subclass, E^* , with two distinctive properties. First, individual preferences can be modified outside a fixed neighborhood of a commodity bundle x^0 so that the Walrasian allocation, which lies in this neighborhood for each trader, is unique. The modified

preferences are C^1 , strictly increasing, and convex. This result is stated as Proposition 2.5. Second, for any (σ_1, σ_2) in the unit circle, there is an environment in E^* having the matrix

$$\begin{pmatrix} \sigma_1 & -\sigma_2 \\ \sigma_2 & \sigma_1 \end{pmatrix}$$

as the derivative of aggregate excess demand (for commodities 1 and 2 with respect to prices 1 and 2) at the unique equilibrium. This diversity of comparative statics prevents any game form from having a locally stable Cournot equilibrium for every environment in E^* . To study local stability, it suffices to consider only allocations within the fixed neighborhood of x^0 , so we can maintain the quadratic representation of preferences throughout the remainder of the paper.

Section 3 studies the Cournot equilibria of game forms which implement Walrasian allocations. A game form consists of a strategy set S^i for each trader $1 \leq i \leq 4$, as well as a strategy set S^0 for the artificial player, or auctioneer. For each trader there is an outcome function $h^i: S \rightarrow R^3$, which associates a net trade $h^i(s)$ with each strategy profile $s = (s^i)_{i=0}^4 \in S = \prod_i S^i$. The auctioneer has, as part of the game form, a strategy adjustment function $s^0 = f^0(s)$. Each S^i is assumed to be an open subset of a finite-dimensional Euclidean space. Of course, we can dispense with the auctioneer by setting $S^0 = \{0\}$ and $f^0 \equiv 0$. A Cournot equilibrium is a strategy profile which is stationary for the auctioneer, and utility maximizing for each of the traders. Each trader, in determining his maximizing strategy, regards the strategies of the other traders and the auctioneer as fixed. The Implementation assumption (3.3) states that Cournot equilibrium outcomes correspond to Walrasian trades, although Cournot equilibria are not required to exist for all

environments in $E \setminus E^*$. The Uniqueness assumption (3.5) states that environments in E^* , which are local restrictions of environments with unique Walrasian equilibria, also have unique Cournot equilibria. This assumption allows equilibrium strategy profiles to reveal only as much environmental information as can be elicited in a determinate fashion. We do not impose any qualitative restrictions on the disequilibrium behavior of traders, so in the absence of Uniqueness, an example in Section 6.2 shows that strategy adjustment processes can be designed to reveal complete information in disequilibrium, even when the outcome functions provide no incentives for such revelation. The Regularity assumption (3.7) imposes continuity on the relation between environments and equilibria, and requires the set of potential equilibria to be a C^0 submanifold of S . Section 4 discusses some examples of game forms satisfying these assumptions. In particular, the game form constructed by Schmeidler (1980), modified slightly to ensure uniqueness, satisfies the above assumptions, despite having discontinuous outcome functions.

The Equilibrium Theorem (3.10) establishes two key properties of Cournot equilibria. First, because of the decentralized nature of Cournot implementation, an equilibrium strategy profile must identify the Walrasian equilibrium price as well as the allocation. Second, monotonicity and Uniqueness prevent the revelation of any further information.

Section 5 defines strategy adjustment processes and the local stability of equilibrium. Each trader, in adjusting his own strategy, is presumed not to possess any information about the characteristics of other traders except the information revealed by the observed strategy profile. The auctioneer, if any, has no direct environmental information. The adjustment functions are assumed to be sufficiently regular to yield, for each environment, a well-defined ordinary differential equation on the strategy space. The adjustment process

is related to the game form by the assumption that strategy profiles are stationary if and only if they are Cournot equilibria. At disequilibrium strategy profiles, the direction and magnitude of individual strategy adjustments are not restricted by any behavioral assumptions.

The Instability Theorem states that under the above assumptions, it is not possible for every environment in E^* to have a locally stable Cournot equilibrium. The proof shows that if the adjustment process is locally stable, the adjustment function must be a continuous deformation of the function from environments to Cournot equilibria. The two conclusions of the Equilibrium Theorem are then used to show that this "homotopy correspondence principle" requires more information than the game form can elicit.

2. Exchange Environments

This section describes the set of exchange environments studied in this paper. As in the Scarf and Gale examples, there are three commodities. Individual preferences are represented by nonsatiated quadratic utility functions. The utility functions have a simple geometrical interpretation, which we give after the following definitions.

2.1. Definitions: There are three commodities, indexed by the subscript j , $1 \leq j \leq 3$. Commodity 3 is the numeraire, so the price space is R_{++}^2 with generic element $p = (p_1, p_2)$. For each $p \in R_{++}^2$, let $\bar{p} = (p_1, p_2, 1) \in R_{++}^3$. Let $p^0 = (1, 1)$, so that $\bar{p}^0 = (1, 1, 1)$.

There are four traders, indexed by the superscript i , $1 \leq i \leq 4$. For each i , let $x^0 = (10, 10, 10)$, let $y^{01} = (-8, 0, 8)$, $y^{02} = (8, 0, -8)$, $y^{03} = (0, -8, 8)$, $y^{04} = (0, 8, -8)$, and for each i , let $\omega^{0i} = x^0 - y^{0i}$. Then $\sum_{i=1}^4 y^{0i} = 0$ and $4x^0 = \sum_i \omega^{0i}$. For each $b > 0$, and $a \in R_{++}^3$ with $\sum_{j=1}^3 a_j = 1$, define the function $u(a, b; \cdot): R_+^3 \rightarrow R$ by

$$u(a, b; x) = \bar{p}^0(x - x^0) + (x - x^0)' A^{-1} B A^{-1} (x - x^0),$$

where

$$A = \begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ -1 & -1 & a_3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -b^{-1} & 0 & 0 \\ 0 & -b^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The utility function $u(a, b; \cdot)$ is concave, nonsatiated, and for x near x^0 , strictly increasing. For each i let Ω^i be an open neighborhood of ω^{0i} with $\Omega^i \subset \{\omega^i \in R_{++}^3: \omega^i + y^{0i} \in R_{++}^3\}$. Also, for each $p \in R_{++}^2$, if p is

near enough to p° and ω^i is near enough to ω^i to prevent a boundary solution, the excess demand for commodities 1 and 2 at p determined by the endowment ω^i and the utility function $u(a,b;\cdot)$ is given by

$$\begin{aligned} z_j^i(p) &= (x_j^\circ - \omega_j^i) - b(\bar{p}a)^{-1}(p_j - 1) - a_j(\bar{p}a)^{-1}\bar{p}(x^\circ - \omega^i) \\ &\quad + b(\bar{p}a)^{-2}[(p_1 - 1)^2 + (p_2 - 1)^2], \quad j = 1, 2. \end{aligned}$$

In particular, $z_j^i(p^\circ) = (x_j^\circ - \omega_j^i) - a_j\bar{p}^\circ(x^\circ - \omega^i)$, $j = 1, 2$; and

$$D_p z^i(p^\circ) = \begin{pmatrix} -b & -a_1 y_1^{\circ i} & -a_1 y_2^{\circ i} \\ -a_2 y_1^{\circ i} & -b & -a_2 y_2^{\circ i} \end{pmatrix}$$

where $D_p z^i$ denotes the derivative with respect to p of the excess demand for commodities 1 and 2. Summing over i , and using the particular values of $y^{\circ i}$ chosen above, we obtain

$$\sum_i z_j^i(p^\circ) = \sum_i (x_j^\circ - \omega_j^i) - \sum_i a_j \bar{p}^\circ (x^\circ - \omega^i) \quad j = 1, 2;$$

and

$$(1) \quad \sum_i D_p z^i(p^\circ) = \begin{pmatrix} 8(a_1^1 - a_1^2) - \sum_i b^i & 8(a_1^3 - a_1^4) \\ 8(a_2^1 - a_2^2) & 8(a_2^3 - a_2^4) - \sum_i b^i \end{pmatrix}.$$

2.2. Remarks: The utility function $u(a,b;\cdot)$ induces an income expansion path through a commodity bundle x of the form $x + \lambda a$, $\lambda \in \mathbb{R}$. The vector of marginal utilities is constant along this path. Since $\bar{p}^\circ a = \sum_j a_j = 1$, a is

the derivative of demand with respect to income. Since $a \in R_{++}^3$, all goods are normal. Although the function u_a is not strictly concave, the preference relation it represents is strictly convex. All indifference surfaces are parallel.

The utility $u(a,b;x)$ can be determined as follows. Represent x as $x = (x^\circ + \lambda a) + z$, where $\lambda \in R$ and $\bar{p}^\circ z = 0$. Then $u(a,b;x) = \lambda - b^{-1}(z_1^2 + z_2^2)$.

2.3. A Special Class of Environments: Fix $0 < b^\circ$ and $a^2 = a^4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Let $\Sigma = \{\sigma \in R^2: \sigma_1^2 + \sigma_2^2 = 1\}$, and for each $\sigma \in \Sigma$ let $a^1(\sigma) = (\sigma_1/8 + b^\circ/2 + \frac{1}{4}, \sigma_2/8 + \frac{1}{4}, a_3^1(\sigma))$, $a^3(\sigma) = (\frac{1}{4} - \sigma_2/8, \sigma_1/8 + b^\circ/2 + \frac{1}{4}, a_3^3(\sigma))$, where $a_3^i(\sigma)$ is determined by the requirement that $\sum_j a_j^i(\sigma) = 1$, $i = 1, 3$. Then each σ on the unit circle determines an exchange environment $e(\sigma) = (u(a^i(\sigma), b^\circ; \cdot), \omega^\circ)^4_{i=1}$. At p° , the derivative of aggregate excess demand, given by (1), is

$$D(\sigma) = \begin{pmatrix} \sigma_1 & -\sigma_2 \\ \sigma_2 & \sigma_1 \end{pmatrix},$$

an orthonormal matrix with determinant +1. Let $E(\Sigma)$ denote the set of environments $\{e(\sigma): \sigma \in \Sigma\}$.

2.4. Remarks: Our interest in $E(\Sigma)$ is motivated by the behavior of the tatonnement process on this class of environments. The local behavior of the tatonnement process

$$\dot{p}_j = \sum_i \xi_j^i(p) \quad j = 1, 2,$$

near the equilibrium p° is determined by the matrix $D(\sigma)$. As σ varies in the unit circle Σ , the tatonnement process is variously stable, unstable, and cyclic. Suppose we attempt to stabilize the tatonnement process by multiplying

the aggregate excess demand vector for commodities 1 and 2 by a 2×2 non-singular matrix C , yielding the system

$$\dot{p}_j = \sum_{k=1}^2 c_{jk} (\sum_i x_k^i(p)),$$

where

$$C = (c_{jk}) \quad 1 \leq j \leq 2 \\ 1 \leq k \leq 2.$$

Then the local dynamics are determined by the matrix $CD(\sigma)$. Since individual traders only communicate net trades, σ is not observed by the Walrasian auctioneer so C must be fixed independently of σ . Since $\det D(\sigma) > 0$, stability requires $\det C > 0$. Let C^* be the 2×2 matrix whose first column is the first row of C normalized to Euclidean norm one, and whose second column is the second row of C projected to the orthogonal complement of the first row, and normalized. Then there is some $\sigma^* \in \Sigma$ such that $D(\sigma^*) = C^*$, so that $CD(\sigma^*)$ is a lower triangular matrix with positive diagonal entries. Hence both characteristic roots of $CD(\sigma^*)$ are real and positive, so the modified tatonnement process is unstable at the equilibrium p° for the environment $e(\sigma^*)$. Hence the class of environments $E(\Sigma)$ is rich enough to contradict the stability of all such modifications of the tatonnement process.

The main result of this paper is that $E(\Sigma)$ is also essentially rich enough to contradict the stability of any adjustment process, with or without an auctioneer, which implements Walrasian allocations as Cournot equilibria. Of course, if it is known that p° is the Walrasian equilibrium price, as is the case for all environments in $E(\Sigma)$, then one can replace the tatonnement process by the price adjustment process

$$\dot{p}_j = p_j^0 - p_j \quad j = 1, 2,$$

which is obviously stable at p^0 . To avoid this trivialization, we will enrich $E(\Sigma)$ by permitting small perturbations of the endowment profile $(\omega^i)_i$, which of course give rise to small shifts in the equilibrium price. The magnitude of admissible endowment perturbations is specified in Proposition 2.5 below.

If p^0 is only one of several equilibria, then stability or instability at p^0 is not the whole story. For this reason we need to show that the equilibrium is unique. Unfortunately, p^0 is not the unique equilibrium for all environments in $E(\Sigma)$. However, we can modify the utility functions on consumption bundles bounded away from x^0 , the allocation corresponding to p^0 , so that p^0 is the unique equilibrium for the modified preferences. In this way we can ensure the uniqueness of equilibrium while retaining the local behavior determined by the utility functions $u(a^i(\sigma), b; \cdot)$. This result is stated in Proposition 2.5 below. The proof is rather long and technically unrelated to the analysis in the rest of this paper, so we have relegated the proof to Appendix A.

2.5. Proposition: There exist $\epsilon_x > 0$ and $\epsilon_\omega^* > 0$ such that for each $(\omega^i)_i \in (R_+^3)^4$ with $\|\omega^i - \omega^0\| < \epsilon_\omega^*$ for each i , and each $\sigma \in \Sigma$, there is a utility function $v^i: R_+^3 \rightarrow R$, for each i , which

- i) is C^1 , quasi-concave, strictly increasing; and
- ii) agrees with $u(a^i(\sigma), b^0; \cdot)$ on $\{x \in R_+^3: \|x - x^0\| < \epsilon_x\}$;

and the exchange economy $(v^i, \omega^i)_i$ has a unique Walrasian equilibrium $(p, (y^i)_i)$. Moreover $\|x^0 - (\omega^i + y^i)\| < \epsilon_x$ for each i .

2.6 Remarks: As a result in demand theory, Proposition 2.5 is hardly surprising. A well known result of Mas-Colell (1977, Corollary 2, p. 119) implies that each matrix $D(\sigma)$ can be the derivative of an aggregate excess demand function at a unique equilibrium. However, the individual preference relations constructed by Mas-Colell are nondifferentiable and do not have a convenient parameterization, because Mas-Colell's paper is devoted to rationalizing a very general class of excess demand functions and equilibrium price sets. We will rely heavily on the differentiability and transparent parameterization of individual preferences near the unique equilibrium, so Proposition 2.5 is essential.

The preferences constructed in Proposition 2.5 satisfy the conventional assumptions of general equilibrium theory, but are not easily described away from x^0 . However, since each trader's Walrasian equilibrium allocation is near x^0 , a locally stable adjustment process will keep each trader's allocation within the convenient range. The following definitions will be used to make this localization precise.

2.7. Definitions: Let $0 < \epsilon_\omega < \epsilon_\omega^*$ and let $\epsilon_y > 0$ such that

- i) $\epsilon_\omega + \epsilon_y < \epsilon_x$;
- ii) for each $\sigma \in \Sigma$ and $(\omega^i)_i$ with $\|\omega^i - \omega^{0i}\| < \epsilon_\omega$ for each i , the environment $(u(a^i(\sigma), b^0; \cdot), \omega^i)_i$ has a Walrasian equilibrium $(p, (y^i)_i)$ with $\|y^{0i} - y^i\| < \epsilon_y$ for each i .^{3/}

For each i , let $\Omega^i = \{\omega^i: \|\omega^i - \omega^{0i}\| < \epsilon_\omega\}$, let $\Omega^{*i} = \{\omega^i: \|\omega^i - \omega^{0i}\| < (\frac{1}{2})\epsilon_\omega\}$ and let $Y^i = \{y^i: \|y^i - y^{0i}\| < \epsilon_y\}$.

Let E^* denote the set of environments $\{(u(a^i(\sigma), b^0; \cdot), \omega^i)_i: \sigma \in \Sigma \text{ and } \omega^i \in \Omega^{*i} \text{ for each } i\}$. The space E^* can be parameterized as $E = \Sigma \times \prod_i \Omega^{*i}$. For each i , let $E^i = \{(u(a^i, b^i; \cdot), \omega^i)_i: \omega^i \in \Omega^i \text{ for each } i\}$, and let $E = \prod_i E^i$. The space E^i can be parameterized as $E^i = \Delta \times R_{++} \times \Omega^i$, where $\Delta = \{a \in R_{++}^3: \sum_j a_j = 1\}$, and an element $e^i \in E^i$ can be denoted

$e^i = (a^i, b^i, \omega^i)$. The topology on E^i is given by the Euclidean topology on the parameter space $\Delta \times R_{++} \times \Omega^i$, and E is given the product topology. Note that the relative topology on E^* makes the parameterization $E^* = \Sigma \times \Pi_i \Omega^{*i}$ a homeomorphism, and that $clE^* = \Sigma \times \Pi_i cl\Omega^{*i}$ is a compact subset of E , where "cl" denotes closure.

A local Walrasian equilibrium for an environment $e \in E$ is a Walrasian equilibrium $(p, (y^i)_i)$ with $y^i \in Y^i$ for each i . The trade profile $(y^i)_i$ is a local Walrasian trade profile for e . By (i) above, a local Walrasian trade profile satisfies $\|\omega^i + y^i - x^0\| < \epsilon_x$ for each i . By (ii) above, each $e \in clE^*$ has a local Walrasian equilibrium, which is unique by Proposition 2.5.

3. Game Forms Implementing Walrasian Allocations

This section describes game forms which implement Walrasian allocations as Cournot equilibrium outcomes. Our definitions allow for the possibility of an artificial player or auctioneer. Moreover, the definition given below even permits the auctioneer to adjust the strategies of the real players. The convenience of this additional generality is discussed below.

3.1. Definitions: For each $0 \leq i \leq 4$, let S^i be an open subset of R^{n^i} , where n^i is a nonnegative integer. For each i , S^i is the strategy space for player i . If $i \geq 1$, player i is trader i ; player 0 is the auctioneer. Let $S = \prod_{i=0}^4 S^i$ and let $f^0: S \rightarrow R^n$, where $n = \sum_i n_i$, denote the auctioneer's adjustment function. That is, the auctioneer adjusts s according to the equation $\dot{s} = f^0(s)$. For each $i \geq 1$, let $h^i: S \rightarrow Y^i$ be an outcome function, and let $h = (h^i)_i: S \rightarrow \prod_i Y^i$. The outcome $h^i(s)$ is interpreted as the net trade for trader i determined by the strategy profile $s = (s^i)_{i=0}^4$. A Cournot equilibrium for an environment $(u(a^i, b^i; \cdot), \omega^i)_i \in E$ is a strategy profile s satisfying

- i) $f^0(s) = 0$; and
- ii) s^i maximizes $u(a^i, b^i; \omega^i + h^i(s^0, \dots, s^{i-1}, \cdot, s^{i+1}, \dots, s^4))$.

The triple (s, h, f^0) is a game form.

3.2. Remarks: The auctioneer is an artificial participant whose characteristics are part of the game form rather than part of the environment. For this reason the auctioneer's strategy adjustment function is specified as part of the game form, and used in place of a utility function in the definition of equilibrium. Allowing the auctioneer to adjust the strategies of the real players permits the imposition of equilibrium conditions across strategy profiles which the real players are not assumed to impose on themselves. In some cases, it also permits equilibrium to be defined as a single-valued selection from what would otherwise

be an equilibrium set. In particular, certain "balanced" game forms, which we would like to include in the scope of our definitions, have a linear subspace of equilibria for each environment. All equilibria yield the same outcomes and same attainable sets for each player. By including an adjustment of strategies within this set, we can artificially remove this multiplicity without affecting the equilibrium outcomes or the strategic possibilities of the original game form. Since the auctioneer's adjustment is a function of the current strategy profile, which is assumed to be public knowledge, each player can negate the auctioneer's adjustment of his own strategy by making a corresponding adjustment in the opposite direction. Also, this device permits the model of strategy adjustment processes, described in Section 5, below, to include higher order differential equations. Of course, a game form without an auctioneer can be placed in this framework by setting $S^0 = \{0\}$ and $f^0 \equiv 0$.

The game form is local in the sense that $h^i(S) \subset Y^i$, and since all environments in E have $\omega^i \in \Omega^i$, each trader's allocation satisfies $\|\omega^i + h^i(s) - x^0\| < \epsilon_x$. Also, we only require the game form to implement local Walrasian equilibria. In fact, we only require the full implementation of local Walrasian equilibria for environments in clE^* and those environments in E which have $(p^0, (y^{0i})_i)$ as a Walrasian equilibrium.

3.3. Implementation: Each environment $e \in clE^*$ has a Cournot equilibrium s such that $(h^i(s))_i$ is a local Walrasian trade profile for e . Also, for each environment $e \in E$ such that $(p^0, (y^{0i})_i)$ is a Walrasian equilibrium, there is a Cournot equilibrium s with $h^i(s) = y^{0i}$ for each i . Finally, for every environment $e \in E$, if s is a Cournot equilibrium for e then $(h^i(s))_i$ is a local Walrasian trade profile for e .

3.4. Remarks: Environments in $E \setminus clE^*$ for which $(p^0, (y^{0i})_i)$ is not a

Walrasian equilibrium are not assumed to possess Cournot equilibria, but if such environments have Cournot equilibria, the equilibrium outcomes are required to be Walrasian.

Environments in cE^* have unique local Walrasian equilibria, but even under the Implementation assumption, this does not imply the uniqueness of Cournot equilibria. Unfortunately, when there are multiple equilibria, stability is an ambiguous property. The example of the two-commodity tatonnement process suggests that it would be most natural to expect the local stability of at least one equilibrium. However, we will show in Section 6.2 below that if multiple Cournot equilibria are permitted, we can trivially build in enough Cournot equilibria so that at least one equilibrium is locally stable for each environment. To avoid such artificial examples, we assume that Cournot equilibria are unique for environments in cE^* .

3.5. Uniqueness: Each environment in cE^* has a unique Cournot equilibrium. Define the function $\eta: cE^* \rightarrow S$ by setting $\eta(e)$ equal to the unique Cournot equilibrium for each $e \in cE^*$.

3.6. Remarks: Thus far we have not directly imposed any mathematical regularity conditions on the game form (S, h, f^0) . In order to study the stability of strategy adjustment processes, we will need to analyze the comparative statics embodied in the equilibrium function η . The continuity of the equilibrium function η is implied by the local stability of a strategy adjustment process (Proposition 5.9 below), but it will be convenient to have this property for use in the present section. It should be emphasized that we will not impose the continuity of the individual outcome functions, so that the game forms constructed by Maskin (1977) (see also Williams (1984a) and (1984b)) and Schmeidler (1980) are not excluded on the basis of regularity conditions. These game

forms do violate the Uniqueness assumption, although Schmeidler's can be modified to satisfy Uniqueness (see Section 4, below). We will also need to assume that $\eta(E^*)$, the set of strategy profiles which are equilibria for some environment in E^* , is well-behaved as a subset of S . More precisely, we assume that $\eta(E^*)$ is a C^0 submanifold of S , so that each $s^0 \in \eta(E^*)$ has an open neighborhood in S which is homeomorphic to $R^{n_1} \times R^{n_2}$, where $n_1 = \dim \eta(E^*)$ and $n_2 = \dim S - n_1$. Each strategy profile s in this neighborhood has an image $(s_1, s_2) \in R^{n_1} \times R^{n_2}$, and $s \in \eta(E^*)$ if and only if $s_2 = 0$. Such a technical assumption does not have a direct economic interpretation, but may be motivated as follows. It is natural to expect that the Cournot equilibrium conditions will cause the set of strategy profiles which are equilibria for some environment in E to be submanifold of S . Since E^* is a submanifold of E ($\dim E^* = \dim \Sigma + \dim \Pi_i \Omega^{*i} = 13$), it is then reasonable to expect $\eta(E^*)$ to be a, perhaps smaller, submanifold of S . In the examples given in Section 4 below, the two submanifolds are identical.

3.7. Regularity:

- i) the equilibrium function $\eta: cE^* \rightarrow S$ is continuous; and
- ii) $\eta(E^*)$ is a C^0 submanifold of S .

3.8. Remarks: We can now take the first major step toward the Instability Theorem. The decentralized nature of Cournot equilibrium implies the existence of a function g which maps each Cournot equilibrium $\eta(e)$ to the associated Walrasian equilibrium $(p, (y^i)_i)$. The trade coordinates of g are simply the individual outcome functions h^i , and a fundamental result of Informational Decentralization Theory (e.g., Hurwicz (1972)) is that we can also recover the supporting price from $\eta(e)$. However, the Equilibrium Theorem, below, states that $g^{-1}(p^0, (y^0)_i)$ is a singleton, so that no further information is conveyed

by $\eta(e)$ when $(p^0, (y^0)^i)_i$ is a Walrasian equilibrium for e .

3.9. Definition: Let $W = \{(p, (y^i)_i) \in R_{++}^2 \times (R^3)^4 : \bar{p}y^i = 0 \text{ for each } i \text{ and } \sum_i y^i = 0\}$. Because of the equations $\bar{p}y^i = 0$ for each i , and $\sum_i y^i = 0$, the map $(p, (y^i)_i) \mapsto (p, (y_1^i, y_2^i)_{i \geq 2})$ is a homeomorphism on W to $R_{++}^2 \times (R^2)^3$.

3.10. The Equilibrium Theorem: Suppose that (S, h, f^0) satisfies Implementation (3.3), Uniqueness (3.5) and Regularity (3.7), and let $S^* = \eta(E^*)$. Then there is a continuous function $g: S^* \rightarrow W$ such that for each $e \in E^*$, $g(\eta(e))$ is a local Walrasian equilibrium for e . Moreover, $g^{-1}(p^0, (y^0)^i)_i$ is a singleton.

Proof: To construct the function g , let $s \in S^*$, let $e \in E^*$ with $s = \eta(e)$, and let $(p, (y^i)_i)$ be the local Walrasian equilibrium for e . Then $h^i(s) = y^i$ for each i . For any $e' \in E^*$ with $\eta(e') = s$, $(y^i)_i$ must be a Walrasian allocation for e' , and we need to show that if p' is the Walrasian equilibrium price for e' , then $p' = p$. Let $(u(a^i(\sigma), b^0; \cdot), \omega^i)_i = e$ and $(u(a^i(\sigma'), b^0; \cdot), \omega^i)_i = e'$. It follows from the definition of Cournot equilibrium that s is also a Cournot equilibrium for the mixed environment $((u(a^1(\sigma'), b^0; \cdot), \omega^1), (u(a^i(\sigma), b^0; \cdot), \omega^i)_{i=2}^4)$. But $D_x u(a^1(\sigma'), b^0; \omega^1) + h^1(s)$ is proportional to $\bar{p}'$, and $D_x u(a^2(\sigma), b^0; \omega^2) + h^2(s)$ is proportional to $\bar{p}$, so we must have $p' = p$. Hence for each $s \in S^*$, all environments in $\eta^{-1}(s)$ have the same local Walrasian equilibrium, which defines $g(s)$.

The continuity of g is established as follows. Let $\{s_n\}_{n=1}^\infty \subset S^*$ be a sequence converging to some $s^0 \in S^*$, and for each n , let $e_n \in \eta^{-1}(s_n)$. Since clE^* is compact and η is continuous, there is a subsequence of $\{e_n\}$ converging to some $e^0 \in clE^*$, and $s^0 = \eta(e^0)$. Also, the local Walrasian equilibria for the subsequence of environments converge to the local Walrasian equilibrium

of e^0 , so $g(s_n) \rightarrow g(s^0)$.

To prove the final assertion, let $s, s' \in g^{-1}(p^0, (y^0)^i)_i$, and let $e = (u(a^i(\sigma), b^0; \cdot), \omega^i)_i$, $e' = (u(a^i(\sigma'), b^0; \cdot), \omega^i)_i \in E^*$ with $\eta(e) = s$, $\eta(e') = s'$. Note that for any $(a^i, b^i)_i$, $(p^0, (y^0)^i)_i$ is a local Walrasian equilibrium for the environment $(u(a^i, b^i; \cdot); \omega^i)_i \in E$. Pick a^i arbitrarily for each i , and choose b sufficiently large so that for each i , $\{y^i \in Y^i: u(a^i(\sigma), b^0; \omega^i + y^i) \geq u(a^i(\sigma), b^0; \omega^i + y^0)^i)\} \cup \{y^i \in Y^i: u(a^i(\sigma'), b^0; \omega^i + y^i) \geq u(a^i(\sigma'), b^0; \omega^i + y^0)^i)\} \subset \{y^i \in Y^i: u(a^i, b; \omega^i + y^i) \geq u(a^i, b; \omega^i + y^0)^i)\}$. By the Implementation assumption, the environment thus constructed has a Cournot equilibrium s'' with $h^i(s'') = y^0^i$ for each i . By the above inclusion of preferred trade sets, s'' is also a Cournot equilibrium for e and e' . The Uniqueness assumption then implies that $s = s' = s''$, which completes the proof.

4. Examples

This section describes some specific game forms which satisfy the assumptions of the previous section. The examples we describe are defined globally, but are easily localized by replacing the strategy sets with small open subsets.

The most obvious example is the classical Walrasian auctioneer mechanism. Let $S^0 = R_{++}^2$, the price space, and for each $1 \leq i \leq 4$, let $S^i = R^2$, with generic element $s^i = (y_1^i, y_2^i)$. For each $i \geq 1$, the outcome function $h^i: S \rightarrow R^3$ is defined by $h^i(p, (y_1^i, y_2^i)_i) = (y_1^i, y_2^i, -p_1 y_1^i - p_2 y_2^i)$. The auctioneer's adjustment function is given by

$$p_j = \sum_{i=1}^4 y_j^i \quad j = 1, 2.$$

The Implementation and Uniqueness assumptions are obviously satisfied. Also, $S^* = \eta(E^*)$ is an open subset of the set $\{(p, (y_1^i, y_2^i)_i) \in S: \sum_i y_j^i = 0, j = 1, 2\}$, a submanifold of S , so the Regularity assumption is also satisfied.

Several game forms without an auctioneer have also been constructed to implement Walrasian allocations. Perhaps the simplest of these is due to Schmeidler (1980). In this game form, applied to the present 4-trader, 3-commodity case, each trader announces a price $p^i = (p_1^i, p_2^i)$ and a proposed trade (z_1^i, z_2^i) for commodities 1 and 2. Let $T^i = \{1 \leq i' \leq 4: p^{i'} = p^i\}$. The outcome $y^i = h^i((p^k, z^k)_{k=1}^4) \in R^3$ is defined by

$$y_j^i = z_j^i - [\sum_{i' \in T^i} z_j^{i'}] / \#T^i \quad j = 1, 2; \text{ and}$$

$$y_3^i = -p_1^i y_1^i - p_2^i y_2^i.$$

That is, trader i receives his proposed trade minus the average proposed trade of all traders, including himself, whose price announcements agree with p^i .

The agreement set T^i varies discontinuously with the price announcements, so

the outcome functions are discontinuous. A Cournot equilibrium for an environment e is easily shown to be a strategy profile $s = (p^i, z^i)_i$ such that $p^1 = \dots = p^4$ and $(p, (h^i(s))_i)$ is a Walrasian equilibrium for e . Hence the Implementation assumption is satisfied. Also S^* is an open subset of the set $\{s: p^1 = \dots = p^4\}$ so condition (ii) of the Regularity assumption is satisfied.

The game form is "balanced" in the sense that, even out of equilibrium, $\sum_i h^i(s) = 0$, so the outcome is feasible in the aggregate. Since each h^i is a linear function of the proposed trades, this causes the Uniqueness assumption to be violated, since, for a given trade profile $(y^i)_i$ with $\sum_i y_i = 0$, the set of proposed trade profiles $(z^i)_i$ with $(h^i(s))_i = (y^i)_i$ is a linear subspace. In particular, if each trader i increases z_j^i by one unit, neither the existing outcome, nor the set of outcomes which a trader could obtain by varying his own strategy, are affected. Therefore we can remove this multiplicity of equilibria without affecting the strategic nature of the mechanism. We introduce an auctioneer, with no strategies of his own, who adjusts the proposed trades to achieve the current outcome with the minimal proposed volume of trade. That is, the auctioneer sets $\dot{z}_j^i = -\sum_k z_j^k$ for each i and each $j = 1, 2$. With this modification, a Cournot equilibrium $s = (p, z^i)_i$ now satisfies $z^i = h^i(s)$ for each i , so the proposed trades are the Walrasian trades. The Uniqueness assumption and condition (i) of the Regularity assumption are now satisfied since the Cournot and Walrasian equilibria exactly coincide.

The game form constructed by Hurwicz (1979a) is quite different, and has continuous outcome functions, but it satisfies Implementation and Regularity. Also, it has the same multiplicity of equilibria as Schmeidler's game form, which can be removed in exactly the same way.

The technique of Maskin (1977) (see also S. Williams (1984a and 1984b)) can also be applied to the present context to construct game forms implementing

Walrasian equilibria. However, a Maskin game form will have a multiplicity of equilibria which cannot be removed by an artificial auctioneer without affecting the players' strategic possibilities. The difficulty can be explained without too much detail. In a Maskin game form, each trader's strategy is a complete description of the environment $e_i \in E$. If the announced descriptions agree, the outcome is the Walrasian outcome for the common announcement $e = e_1 = \dots = e_4$. When, for example, $e_1 = e_2 = e_3 = e$, trader 4 can, by varying e_4 , attain for himself any trade which, according to the other traders' common announcement of his preferences, is less preferred than his Walrasian trade in the environment e . In this game form, any environment will have a continuum of equilibria along the diagonal $e^1 = \dots = e^4$, with all of the commonly announced environments having the same Walrasian equilibrium. However, this multiplicity cannot be removed without adjusting the commonly announced preferences, and thus affecting each trader's attainable outcome set.

5. Strategy Adjustment Processes

The strategy adjustment function of the auctioneer, if any, is part of the game form, and stationarity for the auctioneer is a condition of Cournot Equilibrium. The equilibrium condition for the real players is simply individual optimization, so their strategy adjustment functions influence only the disequilibrium behavior of the game. We define, in Sections 5.1 and 5.3 below, two classes of strategy adjustment processes.

5.1. Continuous Adjustment Processes: Given a game form (S, h, f^0) , for each $i \geq 1$, let $f^i: E^i \times S \rightarrow R^{n^i}$. We assume that f^i is continuous and for each $e^i \in E^i$, $f^i(e^i, \cdot)$ is C^1 . The function f^i determines player i 's strategy adjustment according to the equation $\dot{s}^i = f^i(e^i, s)$. For notational convenience, we interpret $f^i(e^i, s)$ as the sum of the adjustment in s^i made by trader i and the adjustment, if any, made in s^i by the auctioneer. Under this notational convention, the auctioneer's adjustment function f^0 refers only to the adjustment in s^0 , the auctioneer's strategy, so $f^0: S \rightarrow R^{n^0}$. At this point, we also assume that f^0 is C^1 . Given an environment e and an initial strategy profile s^0 , the strategy adjustment functions determine an ordinary differential equation

$$(*) \quad \begin{aligned} \dot{s} &= F(e, s) \\ s(0) &= s^0 \end{aligned}$$

on S , where $F(e, s) = (f^0(s), f^1(e^1, s), \dots, f^4(e^4, s))$. The function F is the strategy adjustment process.

5.2 Remarks: The regularity conditions on $(f^i)_{i \geq 0}$ ensure that the ODE $(*)$ has a unique local solution near every initial condition.

The formal device of an artificial player who can adjust the strategies of

the real players enables higher order differential equations to be included in the above definition. Given a game form (S, h, f^0) , we can define a new game form as follows. For each i , let $S^i = S^i \times R^{n^i}$ and define h' by $h'((s_1^i, s_2^i)) = h((s_1^i)_i)$. The auctioneer adjusts each strategy, including his own, if any, according to the rule $\dot{s}_1^i = s_2^i$. Then a Cournot equilibrium requires $s_2^i = 0$ for all i , so the equilibria of the new game form correspond to the original equilibria. Thus the new strategy coordinates do not violate Uniqueness even though they do not enter the outcome function. In disequilibrium, each player i chooses $\dot{s}_2^i$, which, through the auctioneer's adjustment rule, corresponds to $\ddot{s}_1^i$.

Unfortunately, the classical tatonnement process does not formally satisfy the above definition. The auctioneer rule, $\dot{p} = \sum_i y^i$, determines the price adjustment as a function of announced net trades, but if traders adjust their announcements continuously, say $\dot{y}^i = \xi^i(p) - y^i$, where $\xi^i(p)$ is trader i 's desired trade at the announced price p , then announced trades will generally differ from the desired trades, so $\dot{p} \neq \sum_i \xi^i(p)$. The stability of the adjustment process will depend on the relative speeds of price and quantity adjustments. If the announced trades are adjusted too slowly relative to the speed of price adjustment, the process may oscillate even if the environment satisfies the sufficient conditions for tatonnement stability. The following, more general definition includes the tatonnement process by allowing traders to adjust some or all of their strategy coordinates instantaneously.

5.3. Mixed Adjustment Processes: For each $i \geq 0$, let $S^i = S_C^i \times S_I^i$, and let $S_C = \prod_i S_C^i$. For each i , let $g^i: S_C \rightarrow S_I^i$, and let $f^i: E^i \times X \rightarrow R^{n^i}$, where $n_C^i = \dim S_C^i$. Let $n = \dim S$ and for each $e \in E$, $s \in S$, define $\dot{s} = F(e, s)$ by

$$\dot{s}_C^i = f^i(e^i, (s_C^j, g^j(e^j, s_C))_{j=0}^4)$$

$$\dot{s}_I^i = g^i(e^i, s_C) - s_I^i,$$

for each i , where $s_C = s_C^0, \dots, s_C^4$. We assume that for each i , f^i and g^i are continuous, and that for each $e^i \in E^i$, $f^i(e^i, \cdot)$ and $g^i(e^i, \cdot)$ are C^1 . Again, the function F is termed the strategy adjustment process, and defines an ODE on the strategy space S for each $e \in E$:

$$(*) \quad \dot{s} = F(e, s)$$

$$s(0) = s^0.$$

5.4. Remarks: For each i , s_C^i is adjusted continuously, and s_I^i is, in substance, adjusted instantaneously to $g^i(e^i, s_C)$. The adjustment $\dot{s}_I^i = g^i(e^i, s_C) - s_I^i$ is purely formal, and has no influence on the essential dynamics of the process. That is, $\dot{s}_C$ depends on $(g^i(e^i, s_C))_i$, not $(s_I^i)_i$; and if, along a trajectory of $(*)$, $s_C(t)$ converges to a stationary point s_C^* , then each $s_I^i(t)$ converges to $g^i(e^i, s_C^*)$. A strategy profile $s = (s_C, s_I)$ is stationary if $f^i(e^i, s) = 0$ and $s_I^i = g^i(e^i, s_C)$ for each i . Of course, we will assume that stationary points and Cournot equilibria coincide. This restricts the mixed adjustment processes which could be applied to a given game form. In particular, it must be the case that trader i 's equilibrium s_I^i is independent of s_I^j for each $j \neq i$.

5.5. Stationarity: Given a game form (S, h, f^0) , we assume that the strategy adjustment process F has the property that for each $e \in E$ and each $s \in S$, $F(e, s) = 0$ if and only if s is a Cournot equilibrium for e .

5.6. Definitions: Let (S, h, f^0) be a game form satisfying Implementation (3.3) and Uniqueness (3.5), and let F be a strategy adjustment process satis-

fying Stationarity (5.5).

Let $s(e, s^0, t)$ denote the solution of (*) at time t . Let $\mathcal{D} = \{(e, s^0) \in E^* \times S : \text{the solution } s(e, s^0, t) \text{ of (*) exists for all } 0 \leq t \leq \infty\}$. The strategy adjustment process is locally stable at an environment $e \in E^*$ if for each neighborhood U of the unique Cournot equilibrium $\eta(e)$, there is a neighborhood N_e of e in E^* , and a smaller neighborhood of $\eta(e)$, $N_s \subset U$, such that $N_e \times N_s \subset \mathcal{D}$ and for each environment $e' \in N_e$ and each initial strategy profile $s^0 \in N_s$,

- i) $s(e', s^0, t) \in U$ for all t ; and
- ii) $\lim_{t \rightarrow \infty} s(e', s^0, t) = \eta(e')$.

5.7. Remarks: If we hold the environment e fixed, the above definition is a standard mathematical definition of local stability for the system (*). Condition (i) is usually termed Lyapunov stability, and condition (ii) is usually termed asymptotic stability. We have, in addition, required stability in response to small changes in the environment. Given an environment $e \in E^*$, suppose the strategy profile is in the equilibrium position $\eta(e)$. Then a small shift of e to e' should result in a path of strategy profiles $s(e', \eta(e), \cdot)$ which stays near $\eta(e)$ (5.6 (i)) and converges to the new equilibrium $\eta(e')$ (5.6 (ii)). This requirement is entirely natural in the present context, since disequilibrium is more likely to result from a shift in the environment than an arbitrary perturbation of the strategy profile. Also, the standard sufficient condition for the narrower stability property usually implies our definition as well. More explicitly, let $D_s F(e, \eta(e))$ denote the derivative of the right hand side of (*) at an equilibrium, and suppose that for each $e \in E^*$, all of the characteristic roots of $D_s F(e, \eta(e))$ have negative real parts. If, in addition, the function $(e, s) \mapsto D_s F(e, s)$ is continuous, then it is straightforward to

show that the strategy adjustment process is locally stable on E^* .

Stability in response to small changes in the environment requires the solution $s(e, s^0, t)$ to be continuous in e as well as s^0 and t , and to converge continuously to the equilibrium $\eta(e)$. This result is recorded as Proposition 5.9 below. Continuous convergence involves the continuity of $s(\cdot)$ at $t = \infty$, which we model by adding ∞ to the time domain in the following definitions.

5.8. Definitions: Let $[0, \infty]$ denote the one-point compactification of R_+ (e.g., Kelley (1955), p. 150). The sets of the form $(a, \infty) \cup \{\infty\}$ form a neighborhood base at ∞ . In particular, $[0, \infty]$ is homeomorphic to $[0, 1]$.
 $\mathcal{D} = \{(e, s^0) \in E^* \times S : \text{the solution } s(e, s^0, t) \text{ of } (*) \text{ exists for all } 0 \leq t < \infty\}$.
 Define $s^*: \mathcal{D} \times [0, \infty] \rightarrow S$ by

$$s^*(e, s^0, t) = \begin{cases} s(e, s^0, t) & \text{if } t < \infty \\ \eta(e) & \text{if } t = \infty. \end{cases}$$

5.9. Proposition: Suppose that the strategy adjustment process is locally stable at each environment in E^* . Let $G = \{(e, s) \in E^* \times S : s = \eta(e)\}$. Then there is an open set $U \subset E^* \times S$ with $G \subset U$ such that s^* is continuous on $U \times [0, \infty]$.

Proof: Let $e \in E^*$ and let $s = \eta(e)$. Since $(f^i)_i$ is locally stable at e , there is a neighborhood N_e of e in E^* and a neighborhood N_s of s such that $N_e \times N_s \subset \mathcal{D}$, and for each $(\bar{e}, s^0) \in N_e \times N_s$, $\lim_{t \rightarrow \infty} s(\bar{e}, s^0, t) = \eta(\bar{e})$. We need to show that s^* is continuous on $N_e \times N_s \times [0, \infty]$. A standard result in ordinary differential equations theory (e.g., Coddington and Levinson (1955), Theorem 7.1, p. 22) implies that s^* is continuous on $N_e \times N_s \times [0, \infty)$, since s^* agrees with s on this domain. Now let $e_n \rightarrow \bar{e} \in N_e$, let $s_n \rightarrow s^0 \in N_s$, and let $t_n \rightarrow \infty$. Let $\bar{s} = \eta(\bar{e})$ and let $\epsilon > 0$. Since $(f^i)_i$ is locally stable at $\bar{e}$, there is a neighborhood $N_{\bar{s}}^\epsilon$ of $\bar{s}$ and a neighborhood $N_{\bar{e}}^\epsilon$ such that if

$(e', s') \in N_e^c \times N_s^c$ then $\|s(e', s', t) - \bar{s}\| < \epsilon$ for all t . Given n sufficiently large, $s(\bar{e}, s^o, t_n) \in N_s^c$ since $\lim_{t \rightarrow \infty} s(\bar{e}, s^o, t) = \bar{s}$, and also $e_n \in N_e^c$. Given n , $s(\cdot, t_n)$ is continuous as noted above. Hence for n sufficiently large, and m sufficiently large depending on n , with $m > n$, we have $e_m \in N_e^c$ and $s(e_m, s_m, t_n) \in N_s^c$. Since $m > n$, $s(e_m, s_m, t_m) = s(e_m, [s(e_m, s_m, t_n)], t_m - t_n)$. Hence $\|s(e_m, s_m, t_m) - \bar{s}\| < \epsilon$ since $e_m \in N_e^c$ and $s(e_m, s_m, t_n) \in N_s^c$. This proves that s^* is continuous on $N_e \times N_s \times [0, \infty]$. Finally, let $U = \cup \{N_e \times N_s : (e, s) \in G\}$.

5.10. Remarks: Proposition 5.9 relies on Implementation and Uniqueness through the definition of Local Stability, but does not assume Regularity (3.7). However, the continuity of η , which is part of the Regularity assumption, is implied by the continuity of s^* at ∞ , since $s^*(e, s^o, \infty) = \eta(e)$ for each $(e, s^o) \in U$. This implication simply expresses the obvious fact that the Lyapunov and asymptotic stability properties are incompatible if slight changes in the environment result in large changes in the equilibrium.

Our definition of strategy adjustment processes involves no behavioral assumptions on individual strategy adjustments. Trader i may adjust s^i in utility increasing, decreasing, or constant directions. The only restriction is that $\dot{s}^i$ is sensitive to the characteristics of other traders only to the extent that those characteristics are revealed in the observed strategy profile. It might seem natural to assume that strategies are adjusted only in utility increasing directions, that is, to assume that if $\dot{s}^i \neq 0$, then the inner product $[D_{s^i} u(a^i, b^i; \omega^i + h^i(s))] \dot{s}^i > 0$. However, in conjunction with the stationarity of Cournot equilibria, this means that the first order condition, $D_x u(a^i, b^i; \omega^i + h^i(s)) D_{s^i} h^i(s) = 0$ for each i , is sufficient for Cournot equilibrium. This directly implies that equilibrium strategy profiles cannot convey information about the second derivatives of traders' utilities. In contrast, the Equilibrium Theorem (3.10) derives this informational limitation as

a more general consequence of individual optimization.

6. The Instability Theorem

This section contains the main result of the paper, establishing that no strategy adjustment process can stabilize the equilibrium of every environment in E^* . The key step in the proof is a "homotopy correspondence principle" which relates the direction of motion of the strategy profile to the actual direction of the Cournot equilibrium. Recall from Proposition 5.9 the open set $U \subset E^* \times S$ consisting of pairs (e, s°) with s° close enough to the Cournot equilibrium $\eta(e)$ that $s(e, s^\circ, t) \rightarrow \eta(e)$. Let $U^* = \{(e, s^\circ) \in U: s^\circ \neq \eta(e)\}$, the set of disequilibrium situations in U . Then

- a) $F: U^* \rightarrow R^n - 0$; and
- b) $\eta^*: U^* \rightarrow R^n - 0$,

where $R^n - 0 = \{r \in R^n: r \neq 0\}$, $n = \dim S$, and η^* is defined by $\eta^*(e, s^\circ) = \eta(e) - s^\circ$. The function F describes the initial adjustment of the strategy profile, and η^* describes the total change in the strategy profile needed to achieve equilibrium. If the strategy adjustment process is locally stable on E^* , then the paths $s(e, s^\circ, \cdot)$ constitute a continuous deformation of the function F to the function η^* , so that, in technical language, F and η^* are homotopic as maps on U^* to $R^n - 0$. In other words, modulo homotopy equivalence, the strategy adjustment process must move the strategy profile directly toward the Cournot equilibrium. We cannot analyze the function η^* itself without more concrete knowledge of the game form. However, if we apply the function g given by the Equilibrium Theorem (3.10), which maps Cournot equilibria to Walrasian equilibria, we can obtain sufficient knowledge of the homotopy class of η^* to prove that η^* cannot be homotopic to F .

The reasons for this are the two assertions of the Equilibrium Theorem. First, the equilibrium function η , through g , must duplicate the classical local comparative statics of Walrasian equilibrium. The Walrasian comparative statics are determined by the inverse of the derivative of aggregate excess demand at equilibrium. This involves the second derivatives of the traders' utility functions. Second, $g^{-1}(p^0, (y^0)^i)_i$ is a singleton, so the different comparative statics of the various environments having $(p^0, (y^0)^i)_i$ as a Walrasian equilibrium are not distinguished by different equilibrium strategy profiles. Trader i , in adjusting his own strategy, can refer to the second derivatives of his own utility function, but the Walrasian comparative statics cannot be decomposed in this fashion. The proof of the Instability Theorem, below, shows that for these two reasons, the dynamics cannot correspond to the comparative statics.

Some results from homotopy theory which are used in the proof are collected in Appendix B for reference.

6.1. Theorem: Suppose that (S, h, f^0) is a game form satisfying Implementation (3.3), Uniqueness (3.5), and Regularity (3.7), and that F is a strategy adjustment process satisfying Stationarity (5.5). Then F fails to be locally stable at some environment in E^* .

Proof of the Theorem: Recall that for each $\sigma \in \Sigma$, $(p^0, (y^0)^i)_i$ is the unique Walrasian equilibrium for the environment $(\sigma, (\omega^0)^i)_i \in E(\Sigma)$. Let $W = \{(p, (y^i)_i) \in \mathbb{R}_{++}^2 \times (\mathbb{R}^3)^4 : \bar{p}y^i = 0 \text{ for all } i \text{ and } \sum_i y^i = 0\}$, and let $g: S_N \rightarrow W$ be the map given by the Equilibrium Theorem, and let $s^0 = g^{-1}(p^0, (y^0)^i)_i$. Then $\eta(\sigma, (\omega^0)^i)_i = s^0$ for all $\sigma \in \Sigma$.

Suppose by way of contradiction that F is locally stable at each environment in E^* . Let U_{s^0} be a neighborhood of s^0 and let U^* be an

open subset of E^* with $E(\Sigma) \subset U^*$ such that $\eta(U^*) \subset U_{S^0}$ and $U^* \times U_{S^0} \subset U$, where U is given by Proposition 5.9. Since S^* is a submanifold of S , we can choose U_{S^0} smaller, if necessary, so that U_{S^0} is homeomorphic to $(U_{S^0} \cap S^*) \times \mathbb{R}^{n_2}$, where $n_2 = \dim S - \dim S^*$. Thus each $s \in U_{S^0}$ can be written $s = (s_1, s_2) \in (U_{S^0} \cap S^*) \times \mathbb{R}^{n_2}$, where $(s_1, s_2) \in S^*$ if and only if $s_2 = 0$. Let V be a convex neighborhood of 0 in $(\mathbb{R}^2)^4 \times \mathbb{R}^{n_2}$, with generic element $((\delta^i)_i, s_2)$, such that for each $((\delta^i)_i, s_2) \in V$ and each $\sigma \in \Sigma$, $(\sigma, (\omega^{\circ i} + (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i)))_i \in E^*$ and $(s_1^{\circ}, s_2) \in U_{S^0}$. For each i , $\xi^i(p^{\circ}; a^i(\sigma), b^{\circ}, \omega^{\circ i} + (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i)) = y^{\circ i} - (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i)$, where $\xi^i(p^{\circ}; e^i)$ is the excess demand at p° determined by the characteristic e^i . Hence if $(\delta^i)_i \neq 0$, $(p^{\circ}, (y^{\circ i})_i)$ is not a Walrasian equilibrium for $(\sigma, \omega^{\circ i} + (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i))_i$. Define $\eta^*: \Sigma \times V \rightarrow \mathbb{R}^n$, where $n = \dim S$, by $\eta^*(\sigma, (\delta^i)_i, s_2) = \eta(\sigma, (\omega^{\circ i} + (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i))_i) - (s_1^{\circ}, s_2)$. Then $\eta^*(\sigma; (\delta^i)_i, s_2) = 0$ if and only if $((\delta^i)_i, s_2) = 0$, so $\eta^*: \Sigma \times (V-0) \rightarrow \mathbb{R}^n - 0$, where $\mathbb{R}^n - 0 = \{z \in \mathbb{R}^n: z \neq 0\}$ and $V-0$ is defined analogously. Define $F^*: \Sigma \times (V-0) \rightarrow \mathbb{R}^n - 0$ by $F^*(\sigma; (\delta^i)_i, s_2) = F((\sigma, (\omega^{\circ i} + (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i))_i), s_1^{\circ}, s_2)$.

We now show that $F^* \simeq \eta^*$. Define $H: \Sigma \times (V-0) \times [0, \infty] \rightarrow \mathbb{R}^n - 0$ by

$$H(\sigma; (\delta^i)_i, s_2; t) = \begin{cases} F^*(\sigma; (\delta^i)_i, s_2) & \text{if } t = 0; \\ [s^*((\sigma, (\omega^{\circ i} - (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i))_i); s_1^{\circ}, s_2; t) \\ \quad - (s_1^{\circ}, s_2)]/t & \text{if } 0 < t \leq 1; \text{ and} \\ [s^*(\sigma, (\omega^{\circ i} - (\delta_1^i, \delta_2^i, -p^{\circ} \delta^i))_i); s_1^{\circ}, s_2; t) \\ \quad - (s_1^{\circ}, s_2)] & \text{if } 1 < t \leq \infty. \end{cases}$$

The continuity of H follows from Proposition 5.9 and the fact that F^* is continuous. Thus η^* and F^* are homotopic as maps on $\Sigma \times (V-0)$ to $\mathbb{R}^n - 0$.

Let $\sigma^{\circ} \in \Sigma$ and define $F^*_{\sigma^{\circ}}: \Sigma \times (V-0) \rightarrow \mathbb{R}^n - 0$ by $F^*_{\sigma^{\circ}}(\sigma; (\delta^i)_i, s_2) = F^*(\sigma^{\circ}; (\delta^i)_i, s_2)$. Define $H': \Sigma \times (V-0) \times [0, 1] \rightarrow \mathbb{R}^n - 0$ by $H'(\sigma; (\delta^i)_i, s_2; \lambda) =$

$F((\lambda a^i(\sigma^0) + (1-\lambda)a^i(\sigma), b^0, \omega^0 + (\delta_1^i, \delta_2^i, -p^0 \delta^i))_i; s_1^0, s_2)$. Then $H^*(\cdot, 0) = F^*$ and $H^*(\cdot, 1) = F_{\sigma^0}^*$, so F^* and $F_{\sigma^0}^*$ are homotopic as maps on $\Sigma \times (V-0)$ to $R^n - 0$. Applying the homotopy H defined above yields $F_{\sigma^0}^* \approx \eta_{\sigma^0}^*$, where $\eta_{\sigma^0}^*$ is defined by $\eta_{\sigma^0}^*(\sigma; (\delta^i)_i, s_2) = \eta^*(\sigma^0; (\delta^i)_i, s_2)$. Therefore $\eta^* \approx \eta_{\sigma^0}^*$. We now investigate the homotopy classes of these two maps to prove that they cannot be homotopic.

Let M be a neighborhood of zero in R^n such that $\eta^*(\Sigma \times V) \subset M$ and $\{m_1 \in R^{n_1}: (m_1 - s_1^0, m_2) \in M \text{ for some } m_2 \in R^{n_2}\} \subset U_{\sigma^0} \cap S^*$, where $n_1 = \dim S_1^*$ and $n_2 = n - n_1$. Let $W = \{(p, (y^i)_i) \in R_{++}^2 \times (R^3)^4: py^i = 0 \text{ for all } i \text{ and } \sum_i y^i = 0\}$, and recall the map $g: S^* \rightarrow W$ given by the Equilibrium Theorem, and recall the homeomorphism $(p, (y^i)_i) \leftrightarrow (p, (y_1^i, y_2^i)_{i \geq 2})$ on W to $R_{++}^2 \times R^6$. Let $g: S^* \rightarrow R^{8+n_2}$ denote the composition of this homeomorphism with g . Then define $g^*: M \rightarrow R^{8+n_2}$ by $g^*(m_1, m_2) = (\hat{g}(m_1 + s^0) - \hat{g}(s^0), m_2) = (\hat{g}(m_1 + s^0) - (p^0, (y_1^0, y_2^0)_{i \geq 2}), m_2)$. By the final assertion of the Equilibrium Theorem, $g^*: M-0 \rightarrow R^{8+n_2} - 0$. We are interested in the composition

$$g^* \circ \eta^*: \Sigma \times (V-0) \rightarrow R^{8+n_2} - 0,$$

which takes the form $g^* \circ \eta^*(\sigma; (\delta^i)_i, s_2) = ((p - p^0, (y_1^i - y_1^0, y_2^i - y_2^0)_{i \geq 2}), -s_2)$, where $(p, (y^i)_i)$ is the Walrasian equilibrium for the environment $(\sigma, (\omega^0 + (\delta_1^i, \delta_2^i, -p^0 \delta^i))_i)$. By Proposition B.2 of the Appendix B, $g^* \circ \eta^* \approx L$, where $L: \Sigma \times (V-0) \rightarrow R^{8+n_2} - 0$ is defined by $L(\sigma; (\delta^i)_i, s_2) =$

$$\begin{pmatrix} D(\sigma)^{-1} & D(\sigma)^{-1} & D(\sigma)^{-1} & D(\sigma)^{-1} & 0 \\ D(\sigma)^{-1} D^2(\sigma) & D(\sigma)^{-1} D^2(\sigma) - I_2 & D(\sigma)^{-1} D^2(\sigma) & D(\sigma)^{-1} D^2(\sigma) & 0 \\ D(\sigma)^{-1} D^3(\sigma) & D(\sigma)^{-1} D^3(\sigma) & D(\sigma)^{-1} D^3(\sigma) - I_2 & D(\sigma)^{-1} D^3(\sigma) & 0 \\ D(\sigma)^{-1} D^4(\sigma) & D(\sigma)^{-1} D^4(\sigma) & D(\sigma)^{-1} D^4(\sigma) & D(\sigma)^{-1} D^4(\sigma) - I_2 & 0 \\ 0 & 0 & 0 & 0 & -I_{n_2} \end{pmatrix} \begin{pmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \\ \delta^4 \\ s_2 \end{pmatrix}$$

where $D(\sigma)^{-} = \begin{pmatrix} \sigma_1 & \sigma_2 \\ -\sigma_2 & \sigma_1 \end{pmatrix}$, the transpose of $D(\sigma)$; $D^i(\sigma) = D_p f^i(p^0; a^i(\sigma), b^0, \omega^0)$;

and I_2 and I_{n_2} denote the 2×2 and $n_2 \times n_2$ identity matrices respectively. For each $0 \leq \lambda \leq 1$, the following matrix

$$\begin{pmatrix} D(\sigma)^{-} & \lambda D(\sigma)^{-} & \lambda D(\sigma)^{-} & \lambda D(\sigma)^{-} \\ D(\sigma)^{-} D^2(\sigma) & \lambda D(\sigma)^{-} D^2(\sigma) - I_2 & \lambda D(\sigma)^{-} D^2(\sigma) & \lambda D(\sigma)^{-} D^2(\sigma) \\ D(\sigma)^{-} D^3(\sigma) & \lambda D(\sigma)^{-} D^3(\sigma) & \lambda D(\sigma)^{-} D^3(\sigma) - I_2 & \lambda D(\sigma)^{-} D^3(\sigma) \\ D(\sigma)^{-} D^4(\sigma) & \lambda D(\sigma)^{-} D^4(\sigma) & \lambda D(\sigma)^{-} D^4(\sigma) & \lambda D(\sigma)^{-} D^4(\sigma) - I_2 \end{pmatrix}$$

is easily seen to have $\{0\}$ as its null space, and thus is nonsingular, so

$L \approx L'$ where L' is defined by $L'(\sigma; (\delta^i)_i, s_2) =$

$$\begin{pmatrix} D(\sigma)^{-} & 0 & 0 & 0 & 0 \\ D(\sigma)^{-} D^2(\sigma) & -I_2 & 0 & 0 & 0 \\ D(\sigma)^{-} D^3(\sigma) & 0 & -I_2 & 0 & 0 \\ D(\sigma)^{-} D^4(\sigma) & 0 & 0 & -I_2 & 0 \\ 0 & 0 & 0 & 0 & -I_{n_2} \end{pmatrix} \begin{pmatrix} \delta^1 \\ \delta^2 \\ \delta^3 \\ \delta^4 \\ s_2 \end{pmatrix}.$$

Similarly, multiplying $D(\sigma)^{-} D^i(\sigma)$ by $0 \leq \lambda \leq 1$ for each $i \geq 2$ in the above matrix shows that $L' \approx L^*$, where $L^*(\sigma; (\delta^i)_i, s_2) = (D(\sigma)^{-} \delta^1, -\delta^2, -\delta^3, -\delta^4, -s_2)$.

Hence $g^*_{\sigma} \eta^* \approx L^*$. By the same argument, $g^*_{\sigma} \eta^*_{\sigma} \approx L^*_{\sigma}$,

$L^*_{\sigma}(\sigma; (\delta^i)_i, s_2) = (D(\sigma^0)^{-} \delta^1, -\delta^2, -\delta^3, -\delta^4, -s_2)$. By Proposition B.3 of Appendix B, L^*_{σ} is not homotopic to L^* , so by Proposition B.1 of the Appendix B, η^* is not homotopic to η^*_{σ} , which proves the Theorem.

6.2. The Role of Uniqueness: The Implementation and Stationarity assumptions are definitionally related to the stability question, and the Regularity assumption

seems innocuous. However, the Uniqueness assumption restricts game forms in a way that is not directly related to the intuition behind the Instability Theorem, so its role should be carefully considered. We will now describe, somewhat informally, an example showing that stability of at least one equilibrium for each environment in E^* is easily obtained if a multiplicity of equilibria is permitted.

The game form is a variant of the Walrasian mechanism, in which the auctioneer's strategy is a price p , and each trader's strategy is a trade $y^i = (y_1^i, y_2^i)$. Trader i 's outcome is the trade $(y_1^i, y_2^i, -p_1 y_1^i - p_2 y_2^i)$. In addition, trader i 's strategy includes an announced characteristic, $e_a^i \in E^i$, which is not an argument of the outcome functions. For each $e \in E^*$, let $\pi(e)$ denote the unique local Walrasian equilibrium price for e , and extend π arbitrarily (but smoothly) to all of E . The auctioneer's adjustment function is $\dot{p} = \|\sum_i y^i\|^2 (\pi(e_a) - p)$, where $e_a \in E$ is the announced profile of characteristics. Thus, if $\sum_i y^i \neq 0$, the auctioneer moves the price toward $\pi(e_a)$. To prevent equilibria with $p = \pi(e_a)$ and $\sum_i y^i \neq 0$, we will suppose that the auctioneer also adjusts each y^i according to the equation $\dot{y}^i = -\sum_j y^j$. Since the announced characteristics do not enter any trader's outcome function, a Cournot equilibrium for an environment e is a Walrasian equilibrium for e together with any profile of announced characteristics. However, we can design the traders' strategy adjustment functions so that, in response to disequilibrium, they will resolve this indeterminacy by announcing their true characteristics. Specifically, if trader i is in disequilibrium, so that y^i is not his most preferred trade at the current price p , he adjusts his announced characteristic e_a^i toward his true e^i , and adjusts y^i towards his preferred trade. Then for each environment in E^* , the Walrasian equilibrium together with the true profile of characteristics is a locally stable

equilibrium.

This example is implausible because we have built into the strategy adjustment process the revelation of information which is not elicited by the outcome functions of the game form. The Uniqueness assumption precludes such examples by preventing strategy profiles from containing information which is not elicited in a determinate fashion by the game form.

6.3. Alternative Dynamics: This paper has concentrated solely on continuous-time dynamics, but it would also be interesting to study the problem in a discrete time framework. That is, suppose that player i chooses s_{t+1}^i as a utility maximizing reply to the strategy profile s_t . This would eliminate the need for some of the regularity conditions we have had to impose to obtain well-behaved differential equations, although one would have to impose sufficient regularity to ensure that best reply strategies always exist. However, stability is usually more difficult to obtain in discrete time than in continuous time, due to the possibility of "over-shooting". For example, the tatonnement process in discrete time may be unstable even in the two commodity case. Also, the standard Cournot oligopoly equilibrium with linear demand and constant marginal cost is stable in continuous time under the gradient adjustment process (Arrow and Hurwicz (1977)), but unstable in discrete time, with more than two firms, when producers use best reply output levels (Theocharis (1960)). Loosely speaking, in continuous time, only the direction of adjustment matters, whereas in discrete time the norm of the adjustment vector is also important. In a discrete time framework, it may even be possible to obtain a version of the Instability Theorem for a class of two commodity exchange environments. Saari (1983) has shown that, in the two commodity case, global, or "system", stability cannot be ensured by any mechanism within a broad class of price adjustment mechanisms. Muench and Walker, in a series of papers, have investigated the required

sensitivity of the magnitude of strategy adjustments to the characteristics of the environment for certain game forms on a class of two commodity environments (see Muench and Walker (1979), especially Section 8, for a survey of this work). Although Muench and Walker are primarily interested in asymptotic incentive and stability properties as the size of the economy increases, their work appears to offer a useful point for departure.

7. Conclusion

We have shown that if Walrasian allocations are implemented as the Cournot equilibria of a well-behaved game form, with or without an auctioneer, then dynamic instability in some exchange environments is inevitable. Thus the instability demonstrated by Scarf (1960) and Gale (1963) is not special to the tatonnement process but is intrinsic to the Walrasian allocation rule.

Our proof of instability relies heavily on the properties of Walrasian allocations, so it is natural to ask whether other interesting allocation rules could be implemented as stable Cournot equilibrium outcomes. However, quite aside from the stability question, the Cournot implementation of individually rational, Pareto efficient, nonWalrasian allocations is problematic. Hurwicz (1979a, Theorem 2) has shown that any game form which implements individually rational Pareto efficient allocations as Cournot equilibrium outcomes for a reasonably large class of exchange environments, and which has outcome functions that are concave in own strategies, can only implement Walrasian outcomes.

The reason for instability is that individual optimizing behavior does not generate the minimal information needed to guide a stable adjustment process. Since the required communication cannot be elicited by a system of individual incentives, stability can only be ensured by imposing significant restrictions, such as tatonnement stability conditions, on the class of environments permitted by the theory. The attainment of socially optimal allocations through individual optimization may be confined to much smaller classes of environments than the presently accepted domain of general equilibrium theory. The characterization of "maximal" domains remains open.

Appendix A

This appendix provides the proof of Proposition 2.5. The proof is by construction, and is somewhat tedious, but can be sketched as follows. We first modify the environments in $E(\Sigma)$. This construction is divided into two cases: Case 1, in which σ is bounded away from $(1,0)$; and Case 2, in which σ is contained in an interval about $(1,0)$. In Case 1, let $a^i = a^i(\sigma)$ and consider the indifference surface through x° . Let $\epsilon > 0$, and imagine drawing an ellipse on this surface about x° consisting of commodity bundles supported by a price p with $\|p - p^\circ\| = \epsilon$. Inside this ellipse, the indifference surface is left as is, but outside the ellipse the surface is flattened, so that the commodity bundles supported by a price p with $\|p - p^\circ\| = \epsilon$ now constitute a line segment between a point on the ellipse and a point on the boundary of the consumption set R_+^3 . Now modify all

indifference surfaces in the same way, or equivalently, construct the new preference field by translating the modified surface through x° parallel to itself along lines of the form $x + \lambda a^i$, $\lambda \in \mathbb{R}$. This construction coincides with the original preferences on an elipsoidal tube around the income expansion line through x° . We have to verify that p° is the unique equilibrium for the environment with endowments $(\omega^\circ)^i_i$ and preferences modified in this way. If $\|p - p^\circ\| < \epsilon$, the aggregate excess demand is that of the original environment, so the nonsingularity of $D(\sigma)$ implies that we can choose ϵ small enough so there is no equilibrium $p \neq p^\circ$ with $\|p - p^\circ\| < \epsilon$. If $\|p - p^\circ\| > \epsilon$, all individual demands are on boundary of R_+^3 , and it is straightforward to verify that there is no boundary equilibrium. This leaves the possibility of an equilibrium p with $\|p - p^\circ\| = \epsilon$. In this case demand is set-valued since p supports a line segment on each indifference surface for each trader. However, movements along these segments, for ϵ small enough, are approximately in the direction

$p^\circ - p$. Hence if σ is bounded away from $(1,0)$ so that $D(\sigma)(p-p^\circ)$ is bounded away from $p-p^\circ$, then ϵ can be chosen small enough so that movements along these line segments cannot cancel the excess demand $\approx D(\sigma)(p-p^\circ)$ created by moving the price from p° to p . This concludes Case 1.

For Case 2, where σ lies on an interval containing $(1,0)$, indifference surfaces are flattened in the same way. However, outside a small interval containing the flattened indifference surface through x° , we need to bend the income expansion paths so that a price movement from p° to p with $\|p-p^\circ\| = \epsilon$ results in an aggregate excess demand z for the first two commodities with $\|z\|^{-1}z$ bounded away from $\epsilon^{-1}(p-p^\circ)$. Except for this complication, Case 2 is similar to Case 1. Finally, the fact that p° is a regular equilibrium for each such modified environment easily implies that the uniqueness of equilibrium is preserved under small perturbations of the endowment profile.

Proof: We will first set $\omega^i = \omega^{\circ i}$ for each i . For each environment in $E(\Sigma)$, p° is an equilibrium price, so we need to show that it is the unique equilibrium. We will consider two cases.

Case 1: $\sigma_1 \leq (.96)^{\frac{1}{2}}$. Let $\sigma \in \Sigma$ with $\sigma_1 \leq (.96)^{\frac{1}{2}}$ and for each i , let $a^i = a^i(\sigma)$. We will work with a given i for the moment, and suppress the superscript i . Also, since b is fixed, let u_a denote the utility function $u(a,b;\cdot)$. For each $x \in R^3$, let z denote the vector $A^{-1}(x-x^\circ)$, which expresses commodity bundles in the coordinates $(1,0,-1)$ and $(0,1,-1)$, which span

the null space of $\bar{p}^\circ$, and a ; with the origin at x° . Define $u_a^z: R^3 \rightarrow R$ by $u_a^z(z) = u_a(x^\circ + Az) = \bar{p}^\circ \cdot Az + (\frac{1}{2})z \cdot Bz$. For each p , the budget constraint becomes $p^z z + \bar{p}y^\circ = 0$, where $p^z = \bar{p} \cdot A = (p_1-1, p_2-1, \bar{p}a)$. Ignoring nonnegativity constraints, the demand function is given by

$$(1) \quad \begin{aligned} z_1(p) &= -b(\bar{p}a)^{-1}(p_1-1) \\ z_2(p) &= -b(\bar{p}a)^{-1}(p_2-1) \\ z_3(p) &= -(\bar{p}a)^{-1}(\bar{p}y^\circ) + b(\bar{p}a)^{-2}[(p_1-1)^2 + (p_2-1)^2]. \end{aligned}$$

Let $0 < \epsilon_p < 1$, and define

$$Z(\epsilon_p) = \{z \in R^3: (z_1, z_2) = (z_1(p), z_2(p)) \text{ for some } p \text{ with } [(p_1-1)^2 + (p_2-1)^2]^{\frac{1}{2}} = \|p-p^\circ\| < \epsilon_p\}.$$

Using (1) above, it is straightforward to show that $Z(\epsilon_p)$ is a closed convex set. Then $X(\epsilon_p)$, defined as $X(\epsilon_p) = \{x \in R^3: x = x^\circ + Az \text{ for some } z \in Z(\epsilon_p)\}$ is also closed and convex. Also, for each $x \in X(\epsilon_p)$, $Du_a(x)$ is proportional to $\bar{p}$ for some p with $\|p-p^\circ\| < \epsilon_p$, so $Du_a(x) \in R_{++}^3$. Now define $v_a: R^3 \rightarrow R$ by $v_a(x) = \min \{d(x-x^\circ): d = Du_a(x') \text{ for some } x' \in X(\epsilon_p)\}$. Then $v_a(x) = u_a(x)$ for all $x \in X(\epsilon_p)$. Also, v_a is C^1 (though not C^2), concave (though not strictly quasi-concave), and strictly increasing. Reducing ϵ_p , if necessary, to avoid boundary solutions, the demand function determined by v_a and the endowment ω° (again ignoring the superscript i) for $\|p-p^\circ\| < \epsilon_p$ is given by

$$(2) \quad \begin{aligned} x_j(p) &= x_j^\circ + z_j(p) + a_j z_3(p), \quad j = 1, 2 \\ x_3(p) &= x_3^\circ + a_3 z_3(p) - z_1(p) - z_2(p). \end{aligned}$$

If $\|p-p^\circ\| = \epsilon_p$, the demand set is

$$\{x(p) + \lambda[(1-p_1), (1-p_2), 0] + \epsilon_p^2 (\bar{p}a)^{-1} a: 0 \leq \lambda \leq \bar{\lambda}(p)\},$$

where $x(p)$ is defined in (2) above and $\bar{\lambda}(p)$ is determined by the nonnegativity constraints.

We now reintroduce the superscript i to study aggregate excess demand. Suppose that each trader i has the characteristics (v_a^i, ω^i) where $a^i = a^i(\sigma)$ as above. For each p with $\|p - p^0\| \leq \epsilon_p$, define $\xi_j(p) = \sum_{i=1}^I (z_j^i(p) + a_j z_3^i(p))$, $j = 1, 2$. Then if $\|p - p^0\| < \epsilon_p$, the aggregate excess demand for commodities 1 and 2 is given by $(\xi_1(p), \xi_2(p))$, and if $\|p - p^0\| = \epsilon_p$, the aggregate excess demand set for commodities 1 and 2 is $\{(\xi_1(p), \xi_2(p)) + (\sum_i \lambda^i)[(1-p_1), (1-p_2)] + \epsilon_p^2 \sum_i \lambda^i (\bar{p} a^i)^{-1} a^i : \text{for each } i, 0 \leq \lambda^i \leq \bar{\lambda}^i(p)\}$. We need to verify that $p^0 = (1, 1)$ is the unique equilibrium. Decomposing $\xi(p)$ into the terms which are linear in $\|p - p^0\|$ and the higher order terms, we obtain

$$(3) \quad (\xi_1(p), \xi_2(p)) = D(\sigma)(p - p^0) + r(p),$$

where

$$D(\sigma) = \begin{pmatrix} \sigma_1 & -\sigma_2 \\ \sigma_2 & \sigma_1 \end{pmatrix},$$

and there is some $K > 0$ with $\|r(p)\| < K\|p - p^0\|^2$. Since $D(\sigma)$ is orthonormal, if $\epsilon_p < K^{-1}$ there is no equilibrium p with $\|p - p^0\| < \epsilon_p$. If $\|p - p^0\| = \epsilon_p$, equilibrium requires that

$$(4) \quad [D(\sigma) - (\sum_i \lambda^i)I](p - p^0) = -r(p) - \epsilon_p^2 \sum_i \lambda^i (\bar{p} a^i)^{-1} a^i$$

for some $(\lambda^i)_i$ with $0 \leq \lambda \leq \bar{\lambda}^i(p)$ for each i , where I is the 2×2 identity matrix. Let $\lambda^0 = \sum_i \lambda^i$. Then there is some $K' > 0$ with $\|\sum_i \lambda^i (\bar{p} a^i)^{-1} a^i\| \leq \lambda^0 K'$ for all $\lambda^0 \geq 0$. Also, $((\sigma_1 - \lambda^0)^2 + \sigma_2^2)^{-1/2} [D(\sigma) - \lambda^0 I]$ is orthonormal, so (4) requires that

$((\sigma_1 - \lambda^\circ)^2 + \sigma_2^2)^{\frac{1}{2}} \epsilon_p \leq (K + \lambda^\circ K') \epsilon_p^2$, which implies

$$(5) \quad ((\sigma_1 - \lambda^\circ)^2 + \sigma_2^2)^{\frac{1}{2}} (K + \lambda^\circ K')^{-1} \leq \epsilon_p.$$

As $\lambda^\circ \rightarrow \infty$, the LHS of (5) converges to K'^{-1} , so there is some $\bar{\lambda} > 0$ such that if $\epsilon_p < K'^{-1}$ and λ° minimizes the LHS of (5), $0 \leq \lambda^\circ \leq \bar{\lambda}$. If $\sigma_1 \leq 0$, then $(\sigma_1 - \lambda^\circ)^2 \geq \sigma_1^2$ so the LHS of (5) is bounded below by $(K + \bar{\lambda} K')^{-1}$. If $\sigma_1 > 0$, then since $\sigma_1 \leq (.96)^{\frac{1}{2}}$, $\sigma_2^2 \geq .04$, so the LHS of (5) is bounded below by $.04(K + \bar{\lambda} K')^{-1}$. Hence if $\epsilon_p < .04(K + \bar{\lambda} K')^{-1}$, (5) cannot be satisfied, so there is no equilibrium with $\|p - p^\circ\| = \epsilon_p$. It is straightforward to verify that the constants K , K' , and $\bar{\lambda}$, which determine the admissible ϵ_p , can be chosen uniformly for all σ satisfying Case 1. Hence ϵ_p can be chosen uniformly for Case 1. Finally, we need to show that there is no equilibrium p with $\|p - p^\circ\| > \epsilon_p$. For such p , each trader's demand is on the boundary of R_+^3 . In particular, the demand of trader i takes the form

$$x^i = x^\circ + \begin{pmatrix} 1 & 0 & a_1^i \\ 0 & 1 & a_2^i \\ -1 & -1 & a_3^i \end{pmatrix} \begin{pmatrix} a^i(1-p_1) \\ a^i(1-p_2) \\ \beta^i \end{pmatrix},$$

where $a^i, \beta^i > 0$. Hence $\bar{p}^\circ \Sigma_i (x^i - x^\circ) = \Sigma_i \beta^i > 0$, so $\Sigma_i (x^i - x^\circ) \neq 0$, and thus p cannot be an equilibrium.

Case 2: $\sigma_1 \geq (.96)^{\frac{1}{2}}$. Let $\sigma \in \Sigma$ with $\sigma_1 \geq (.96)^{\frac{1}{2}}$, and for each i , let $a^i = a^i(\sigma)$. Suppress the superscript i and, given $0 < \epsilon_p < 1$, define v_a as in Case 1 above. Let $h: R \rightarrow [0,1]$ be a smooth function with the properties

$$\begin{aligned} h(t) &= 0 & \text{if } |t| &\leq 8\epsilon_p^2 \\ h(t) &= 1 & \text{if } |t| &\geq 8(\epsilon_p^2 + \epsilon_p^4) \end{aligned}$$

$$\begin{aligned} Dh(t) &\geq 0 \quad \text{if } 8\epsilon_p^2 \leq t \leq 8(\epsilon_p^2 + \epsilon_p^4) \\ Dh(t) &\leq 0 \quad \text{if } -8(\epsilon_p^2 + \epsilon_p^4) \leq t \leq -8\epsilon_p^2. \end{aligned}$$

Let $\sigma^* = (.9, (.19)^{1/2})$, let $a^* = a^1(\sigma^*)$, and define $g: \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$g(0) = 0$$

$$Dg(t) = h(t)a^* + (1-h(t))a \quad \text{for each } t \in \mathbb{R}.$$

Now define $w_a: \mathbb{R}^3 \rightarrow \mathbb{R}$ implicitly by the equation

$$v_a(x + g(w_a(x))) = 0 = v_a(x^0).$$

Since v_a and g are both strictly increasing, w_a is well-defined. Since v_a is C^1 and g is smooth, w_a is C^1 . Geometrically, the utility function w_a is defined by translating the indifference surface $v_a^{-1}(0)$ along the curve $x^0 + g(\mathbb{R})$, and setting $w_a(x)$ equal to the unique value of t with $x \in v_a^{-1}(0) + g(t)$. Hence it is clear that w_a is quasi-concave. Since v_a itself is defined by setting $v_a(x)$ equal to the unique value of t with $x \in v_a^{-1}(0) + ta$, and $g(t) = ta$ for $|t| \leq 8\epsilon_p^2$, we can conclude that w_a and v_a agree on the set $\{x + ta: v_a(x) = 0 \text{ and } |t| \leq 8\epsilon_p^2\}$, which contains an open neighborhood of x^0 . Hence w_a agrees with u_a on a neighborhood of x^0 whose radius is determined by ϵ_p .

The demand function generated by w_a and ω^1 for $\|p - p^0\| < \epsilon_p$ can be derived by finding the point $x \in v_a^{-1}(0)$ with $Dv_a(x)$ proportional to $\bar{p}$, and adding $g(t)$ according to the budget constraint. To this end, for each p with $\|p - p^0\| \leq \epsilon_p$, define $\hat{x}(p)$ by

$$\begin{aligned} \hat{x}_j(p) &= x_j^0 + b(\bar{p}a)^{-1}(1-p_j) + (a_j/2)b(\bar{p}a)^{-2}[(p_1-1)^2 + (p_2-1)^2], \\ &\quad j=1,2; \end{aligned}$$

$$\hat{x}_3(p) = x_3^o + (a_3/2)b(\bar{p}a)^{-2}[(p_1-1)^2 + (p_2-1)^2] + b(\bar{p}a)^{-1}[(p_1-1) + (p_2-1)].$$

The demand is given by $\hat{x}(p) + g(t(p))$ where $t(p)$ is determined by

$$\begin{aligned}\bar{p}[(\hat{x}(p) - x^o + g(t(p))) + (x^o - w^o)] &= 0, \text{ or} \\ \bar{p}g(t(p)) &= -\bar{p}y^o + (b/2)(\bar{p}a)^{-1}\|p-p^o\|^2.\end{aligned}$$

Let $x(p) = \hat{x}(p) + g(t(p))$ for $\|p-p^o\| \leq \epsilon_p$. Define $\delta_1: R \rightarrow R$ by

$$\begin{aligned}\delta_1(s) &= \min \{s, \epsilon_p^2\} \quad \text{if } s \geq 0; \text{ and} \\ \delta_1(s) &= \max \{s, -\epsilon_p^2\} \quad \text{if } s \leq 0; \text{ and}\end{aligned}$$

define $\delta_2: R \rightarrow R$ by $\delta_2(s) = s - \delta_1(s)$. Using the definition of g we decompose $x(p) - x^o$ into first and second order terms in $\|p-p^o\|$ as

$$x_j(p) - x_j^o = b(\bar{p}a)^{-1}(1-p_j) - a_j\delta_1\left(\frac{\bar{p}y^o}{8\bar{p}a}\right) - a_j^*\delta_2\left(\frac{\bar{p}y^o}{8\bar{p}a}\right) + r_j(p), \quad j=1,2.$$

Using $\bar{p}^o y^o = 0$, $\bar{p}^o a = \bar{p}^o a^* = 1$, and the fact that δ_1 and δ_2 are Lipschitzian, a further decomposition yields

$$(6) \quad x_j(p) - x_j^o = -b(p_j-1) - a_j\delta_1(\bar{p}y^o/8) - a_j^*\delta_2(\bar{p}y^o/8) + r_j(p), \quad j=1,2$$

where there is some $K > 0$ such that $\|r(p)\| < K\|p-p^o\|^2$ for all p with $\|p-p^o\| \leq \epsilon_p$.

To study aggregate excess demand, we now reintroduce the superscripts i .

If $\|p-p^o\| < \epsilon_p$, equilibrium for the exchange economy $(w_a^i, \omega^o{}^i)_i$ requires

$$\sum_{i=1}^4 (x_j^i(p) - x_j^o) = 0, \quad j=1,2.$$

For each p with $\|p-p^o\| \leq \epsilon_p$, let $\xi(p)$ denote the first two coordinates of

of $\sum_i (x^i(p) - x^{o,i})$ and let $\delta_k(p-p^o) = (\delta_k(p_1-p^o_1), \delta_k(p_2-p^o_2))$, $k=1,2$. Summing (6) over i yields

$$\xi(p) = D(\sigma)\delta_1(p-p^o) + D(\sigma^*)\delta_2(p-p^o) + \sum_i r^i(p).$$

Since $\sigma_1 \geq (.96)^{\frac{1}{2}}$, geometrical considerations indicate that the vectors $D(\sigma)\delta_1(p-p^o)$ and $D(\sigma^*)\delta_2(p-p^o)$ have a nonnegative inner product, so the norm of their sum is at least $(\|\delta_1(p-p^o)\|^2 + \|\delta_2(p-p^o)\|^2)^{\frac{1}{2}}$, which in turn is at least $\|p-p^o\|/\sqrt{2}$, since $\delta_1(p-p^o) + \delta_2(p-p^o) = p-p^o$. Let $K^o > 0$ such that $\|\sum_i r^i(p)\| < K^o\|p-p^o\|^2$ for all p with $\|p-p^o\| < \epsilon_p$. Then if $\epsilon_p < (K^o\sqrt{2})^{-1}$, $\|\xi(p)\| > 0$ for all $p \neq p^o$ with $\|p-p^o\| \leq \epsilon_p$, so p^o is the unique equilibrium p satisfying $\|p-p^o\| < \epsilon_p$. If $\|p-p^o\| = \epsilon_p$, then since $\|\delta_1(p-p^o)\| < \epsilon_p^2\sqrt{2}$, there is some $K' > 0$ such that

$$\xi(p) = D(\sigma^*)(p-p^o) + r'(p)$$

where $\|r'(p)\| \leq K'\epsilon_p^2$ for all p with $\|p-p^o\| = \epsilon_p$. Since σ^* satisfies the hypothesis of Case 1 above, it can be shown exactly as in Case 1 that for ϵ_p sufficiently small, there is no equilibrium p with $\|p-p^o\| = \epsilon_p$. As in Case 1, there is also no equilibrium p with $\|p-p^o\| > \epsilon_p$. Also, ϵ_p can be chosen uniformly for Case 2, and thus for Cases 1 and 2 together.

In Case 1, for each i , v_a^i agrees with u_a^i on the set $X^i(\epsilon_p)$. In Case 2, w_a^i agrees with u_a^i on $X^i(\epsilon_p) \cap \{x + ta^i : u_a^i(x) = 0 \text{ and } |t| \leq 8\epsilon_p^2\}$. Thus there is some $\epsilon_x > 0$ such that the above intersection contains the ϵ_x ball centered at x^o for all i and all $(a^i)_i$. For each $(a^i, \omega^i)_i \in E(\Sigma)$, the extended preferences constructed above cause p^o to be the unique equilibrium. Also, p^o is a regular equilibrium and the derivative of excess demand at p^o has determinant +1 for all $(a^i)_i$. Since the deriva-

tive of aggregate excess demand at p^0 is continuous in $(\omega^i)_i$, and since the aggregate excess demand correspondence is upper semi-continuous in p and $(\omega^i)_i$, it is straightforward to show the existence of $\epsilon_\omega > 0$ such that if $\|\omega^{0i} - \omega^i\| < \epsilon_\omega$ for each i , then for each $\sigma \in \Sigma$, the environment with the endowments $(\omega^i)_i$ and preferences extended as above has a unique equilibrium $(p, (y^i)_i)$. Moreover, $\|x^0 - (\omega^i + y^i)\| < \epsilon_x$ for each i . This completes the proof.

The modified preferences constructed in the proof are C^1 but not C^2 , and convex but not strictly convex. However, it seems likely that a more intricate construction could prove the Proposition with smooth, strictly convex preferences. It does appear to be essential to split the proof into at least two cases. That is, the modified preferences cannot be forced to vary in a globally continuous fashion with respect to σ . This creates no difficulties for our purposes since we are only interested in behavior near x^0 , which is determined by the original characteristics (u_a^i, ω^i) .

Appendix

This section collects some definitions and results in homotopy theory for reference in the proof of Theorem 6.1.

Let X and Y be topological spaces, and let $f_0, f_1: X \rightarrow Y$ be continuous functions. The functions f_0 and f_1 are homotopic, written $f_1 \approx f_0$ if there is a continuous function $H: X \times [0,1] \rightarrow Y$ with $H(\cdot, 0) = f_0$ and $H(\cdot, 1) = f_1$. The function H is called a homotopy. It is straightforward to verify that the relation $\approx$ is reflexive, symmetric, and transitive, so that $\approx$ is an equivalence relation.

B.1. Proposition: Suppose that $f_0 \approx f_1$ and that $g: Y \rightarrow Z$ is a continuous function on Y to a topological space Z . Then $g \circ f_0 \approx g \circ f_1$.

Proof: Let $H: X \times [0,1] \rightarrow Y$ be a homotopy with $H(\cdot, 0) = f_0$ and $H(\cdot, 1) = f_1$. Define $H': X \times [0,1] \rightarrow Z$ by $H' = g \circ H$. Then $g \circ H(\cdot, 0) = g \circ f_0$ and $g \circ H(\cdot, 1) = g \circ f_1$, so $g \circ f_0 \approx g \circ f_1$.

B.2. Proposition: Let V be a convex neighborhood of 0 in $\mathbb{R}^n$ and let $f: X \times V \rightarrow \mathbb{R}^n$ be a continuous function such that for each x , $f(x, \cdot)$ is differentiable, and the function $(x, v) \mapsto D_v f(x, v)$ is continuous. Suppose further that for each $x \in X$,

- i) $f(x, v) = 0$ if and only if $v=0$; and
- ii) $D_v f(x, 0)$ is nonsingular.

Then there is a homotopy $H: X \times (V-0) \times [0,1] \rightarrow \mathbb{R}^n - 0$ with $H(\cdot, 1) = f$, and $H(x, v; 0) = D_v f(x, 0)v$ for each $(x, v) \in X \times (V-0)$.

Proof: Define H by

$$H(x, v; \lambda) = \begin{cases} \lambda^{-1} f(x, \lambda v) & \text{if } \lambda > 0; \text{ and} \\ D_v f(x, 0) v & \text{if } \lambda = 0. \end{cases}$$

B.3. Proposition: Let V be a convex neighborhood of 0 in $\mathbb{R}^n$ and define the functions $L, L_\sigma: \Sigma \times (V-0) \rightarrow \mathbb{R}^n - 0$ by $L(\sigma; r_1, r_2, r_3, \dots, r_n) =$

$$\begin{pmatrix} \sigma_1 & -\sigma_2 & & \\ \sigma_2 & \sigma_1 & & \\ & & 0 & \\ & & & -I_{n-2} \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{pmatrix},$$

where I_{n-2} is the $(n-2) \times (n-2)$ identity matrix, and $L_\sigma(\sigma; r) = -r$. Then L and L_σ are not homotopic.

Proposition B.3 follows directly from Whitehead (1942, Theorem 4, p. 638).^{4/} We cannot provide a self-contained proof here, but we can give some motivation for readers somewhat familiar with homotopy theory. For this purpose we will modify the map L , and thus L_σ , by replacing $-I_{n-2}$ with I_{n-2} , which does not affect the result. For each σ , let $M(\sigma)$ denote the matrix which defines $L(\sigma, \cdot)$. By projecting $V-0$ radially onto the unit sphere Σ^{n-1} in $\mathbb{R}^n$, we can show that L is homotopy equivalent to the map $(\sigma, r) \mapsto M(\sigma)r$ on $\Sigma \times \Sigma^{n-1}$ to Σ^{n-1} . Whitehead (1942) describes an operation which associates with each map on $\Sigma \times \Sigma^{n-1}$ to Σ^{n-1} a map on Σ^{n+1} to Σ^n . This operation preserves homotopy equivalence, that is, two maps on $\Sigma \times \Sigma^{n-1}$ to Σ^{n-1} which are homotopic are associated with homotopic maps on Σ^{n+1} to Σ^n . The maps associated with L and L_σ are not homotopic for the following reason. The function $\sigma \mapsto M(\sigma)$ generates the fundamental group of $SO(n)$, the space of orthogonal $n \times n$ matrices with positive determinant. The constant map $\sigma \mapsto M(\sigma^0)$

of course represents the zero element of $\Pi_1(SO(n))$. Each element $[f] \in \Pi_1(SO(n))$ is represented by a map $f: \Sigma \rightarrow SO(n)$, which induces the map $\tilde{f}: \Sigma \times \Sigma^{n-1} \rightarrow \Sigma^{n-1}$ defined by $\tilde{f}(\sigma, r) = f(\sigma)r$. Theorem 4 of Whitehead (1942) states that the operation mentioned above induces an isomorphism on $\Pi_1(SO(n))$ to $\Pi_{n+1}(\Sigma^n)$ in the sense that f is a generator of $\Pi_1(SO(n))$ if and only if the map on Σ^{n+1} to Σ^n associated with $\tilde{f}$ is a generator of $\Pi_{n+1}(\Sigma^n)$ (both groups are isomorphic to the integers mod 2). Hence L is associated with a generator of $\Pi_{n+1}(\Sigma^n)$, and $L_{\sigma} \circ$ is associated with a representative of 0 in $\Pi_{n+1}(\Sigma^n)$, so L and $L_{\sigma} \circ$ are not homotopic.

Notes

1. Unfortunately, the term "Nash equilibrium", for reasons which are not apparent from its definition, is now widely interpreted as requiring every agent to have complete knowledge of the characteristics of all. For example, Sonnenschein (1983, p. 16) writes "The weakness of the Nash equilibrium concept is that it requires each agent to know the preferences of all agents". The term "Cournot equilibrium", which has the same definition, has not been burdened with this interpretation (e.g., Walker (1981)), so we use this term instead.
2. $R_+^n = \{x \in R^n: x_j \geq 0 \ \forall j\}$ and $R_{++}^n = \{x \in R^n: x_j > 0 \text{ for all } j\}$.
3. This is possible since $(p^0, (y^0)^i)_i$ is a regular equilibrium for $(u^i(a^i(\sigma), b^0; \cdot), \omega^i)_i$ for each $\sigma \in \Sigma$ and each Σ is compact.
4. I would like to thank Professor H. Gershenson for this reference.

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