# LEVEL SET METHODS FOR OPTIMIZATION PROBLEMS INVOLVING GEOMETRY AND CONSTRAINTS II. OPTIMIZATION OVER A FIXED SURFACE

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# Level set methods for optimization problems involving geometry and constraints II. Optimization over a fixed surface

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**Abstract**

In this work, we consider an optimization problem described on a surface. The approach is illustrated on the problem of finding a closed curve whose arclength is as small as possible while the area enclosed by the curve is fixed. This problem exemplifies a class of optimization and inverse problems that arise in diverse applications. In our approach, we assume that the surface is given parametrically. A level set formulation for the curve is developed in the surface parameter space. We show how to obtain a formal gradient for the optimization objective, and derive a gradient-type algorithm which minimizes the objective. The algorithm is a projection method which has a PDE interpretation. We demonstrate and verify the method in numerical examples.

## 1 Introduction

This work represents a continuation of our investigation into optimization problems involving geometry and constraints [8]. In the present study, we are motivated by the need to solve optimization and inverse problems which are described on a surface. The problems are geometric in nature; i.e., we wish to find a set (possibly multiply connected) on the surface which extremizes certain cost functionals. The approach we will present is quite general but we will focus on a specific problem arising in differential geometry.

Consider a smooth fixed surface $S$ included in some bounded open set $\Omega \subset \mathbb{R}^3$. On this surface, we denote a closed curve by $\Gamma$. The arclength of

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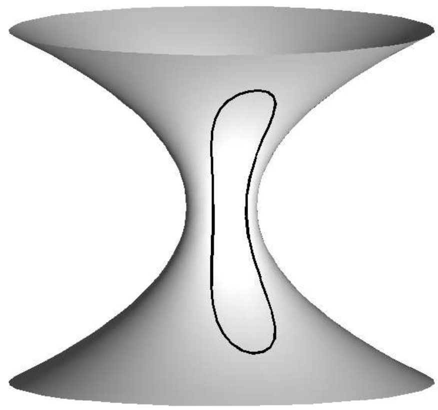

FIGURE 1: The optimization problem is to find the curve with the shortest arclength while keeping the area contained by the curve on the surface fixed. This is an instance of an optimization problem involving geometry and constraints.

the curve, denoted by $\ell(\Gamma)$, is to be minimized while the area enclosed by the curve is $A(\Gamma)$ is fixed. The optimization problem then is

$$\min_{\text{Area}(\Gamma)=C} \ell(\Gamma).$$

In the planar case, this is a classical problem whose solution is given by the isoperimetric theorem (the unique solution is a circle). On general surfaces the problem is harder and recent advances left some open questions (see [5] and references therein). The goal of this work is to develop an effective numerical method for solving problems of this type.

Motion of curve on surfaces has been studied by [3, 2]. Let $S$ be the surface and assume that it is included in $\Omega \subset \mathbb{R}^3$. The curve $\Gamma_t$ which moves on $S$ as a function of time $t$. In the work cited, the authors introduce a classical level-set representation of $S$ and $\Gamma_t$ in $\mathbb{R}^3$ (see [7] for an introduction to level-set methods). $S$ is the zero level-set of a fixed function $\Psi : \Omega \to \mathbb{R}$, while $\Gamma_t$ is the intersection of $S$ and another time-dependent level-set function $\Phi : \Omega \times (0, T) \to \mathbb{R}$. By dealing with the level set functions $\Phi(x)$ and $\Psi(x)$, the authors show how one can move curves on surfaces while satisfying constraints. The computation involves solving a time-dependent PDE in 3-D using finite differences. It is worth noting that this method can also treat the case of a moving supporting surface $S(t)$.

In our work, the surface $S$ is kept fixed. We further assume that we have a parametric representation of $S$. Given that in $\mathbb{R}^3$ the number of parameters is two, we wish to exploit this fact in our approach. We will still use a level set representation for the curve $\Gamma$, but it will be given by a function mapping a two dimensional domain in parameter space to the reals. This means that our computational method is two dimensional, and can be expected to be efficient. Our approach still has the benefits of a level set method. Singularities which could develop, such as merging and splitting, as the curve $\Gamma$ evolves, are easily handled.

In order to do the optimization, we need to develop some formulas to calculate such quantities as arclength, area, and their variations with respect to the level set function. They will be used to derive an iterative method whereby we start with an initial guess for the curve and proceed to take steps towards minimization by moving the curve.

The paper is organized as follows. Section 2 details the general framework for the computational method. We also provide geometrical formulas for arclength and area, and describe curve evolutions which preserve the area. A descent algorithm for curve shortening is presented in Section 3. In Section 4, we derive the equation for the geodesic curvature in terms of the level set function. Additionally, we show that the geodesic curvature is constant on the curve when the velocity for the flow of the curve is zero. Numerical examples are presented in Section 5, where we also validate our computational results. A summarizing discussion is contained in Section 6. For the convenience of the reader, we provide a list of our notation below.

## Notation

The following notation is used throughout the paper.

- $\gamma(r,s) \ : \ J^2 \to \mathbb{R}^3$ is the parameterization of the fixed surface $S$. In component form $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$.
- $\nabla = (\partial_r, \partial_s)^T$. The 3-D cartesian gradient is denoted by $\nabla_x$.
- $\varphi(r,s) = 0$ is the level set function for the curve on $S$ described in the parameter space.
- $$\nabla\gamma = \begin{pmatrix} \gamma_{1,r} & \gamma_{1,s} \\ \gamma_{2,r} & \gamma_{2,s} \\ \gamma_{3,r} & \gamma_{3,s} \end{pmatrix}, \qquad \nabla\gamma^T = \begin{pmatrix} \gamma_{1,r} & \gamma_{2,r} & \gamma_{3,r} \\ \gamma_{1,s} & \gamma_{2,s} & \gamma_{3,s} \end{pmatrix}.$$

- $$\nabla \times \gamma = \begin{pmatrix} -\gamma_{1,s} & \gamma_{1,r} \\ -\gamma_{2,s} & \gamma_{2,r} \\ -\gamma_{3,s} & \gamma_{3,r} \end{pmatrix}, \quad \nabla \times \gamma^T = \begin{pmatrix} -\gamma_{1,s} & -\gamma_{2,s} & -\gamma_{3,s} \\ \gamma_{1,r} & \gamma_{2,r} & \gamma_{3,r} \end{pmatrix}.$$

- $\nabla\varphi = (\varphi_r, \varphi_s)^T$, and $\nabla \times \varphi = (-\varphi_s, \varphi_r)^T$.

## 2 Motion of curves on a fixed surface

Since the surface $S$ is fixed, we can choose the following parametrization. Let $J$ be an interval, and $\gamma : J^2 \to \mathbb{R}^3$ be such that

$$S = \{x \,|\, x = \gamma(r, s), \quad (r, s) \in J^2\}.$$

We will view the iterative optimization method as a discretization of a 'flow'. Therefore, it will be most convenient to consider the problem in the continuous setting. To this end, the curve on the surface is denoted by $\Gamma_t$, where the subscript $t$ denotes its dependence on time $t$. The curve $\Gamma_t$ is given a level-set representation in the parameter domain $J^2$. Let $\varphi : J^2 \times (0, T) \to \mathbb{R}$ such that

$$\Gamma_t = \{x \,|\, x = \gamma(r, s), \quad \varphi(r, s, t) = 0\}.$$

We will consider two cases:

(i) $S$ has a boundary but the curve $\Gamma_t$ does not touch this boundary. We assume $\varphi > \alpha > 0$ on $\partial J^2$.

(ii) $S$ has no boundary. In that case $\gamma$ is taken periodic in $r$ and $s$.

An obvious generalization is the case where $S$ is a truncated cylinder, then $\gamma$ will be periodic in one direction and $\varphi$ will be constrainted to be positive on the boundary of the parameter space of the other direction. All that follows applies to that case as well.

To move the curve $\Gamma_t$, we will evolve the level-set function $\varphi(r, s, t)$ according to a transport equation with a given velocity field. To constrain the area enclosed by the curve $\Gamma_t$ on $S$, we will need to find a projection for the velocity field. These ideas are discussed in more detail below.

## 2.1 Arclength and surface area

The computation of arclength of $\Gamma_t$ on the surface $S$ takes a few steps. We introduce a parametric representation of the zero-level set $\{(s,r)\,|\,\varphi(r,s,t)=0\}$. Let $K$ be an interval and $\tau \in K$ be a parameter. The map $\beta : \tau \in K \to J^2$ is such that

$$\varphi(\beta(\tau,t),t) = 0.$$

The curve $\Gamma_t$ is then $\{x\,|\,x = \gamma(\beta(\tau,t))\}$, and it is easy to calculate arclength from this. The length of $\Gamma_t$ is

$$\ell(\Gamma_t) = \int_K \left| \frac{d}{d\tau}\gamma(\beta(\tau,t)) \right| d\tau = \int_K |\nabla\gamma\beta_{,\tau}| d\tau = \int_K \left| \nabla\gamma \frac{\beta_{,\tau}}{|\beta_{,\tau}|} \right| |\beta_{,\tau}| d\tau.$$

The vector $\beta_{,\tau}/|\beta_{,\tau}|$ is simply the unit tangent on the curve in the parameter domain $J^2$. The component $|\beta_{,\tau}|d\tau$ is the infinitesimal arclength on $J^2$. We replace both these with their level set function counterparts

$$\ell(\Gamma_t) = \int_{J^2} \left| \nabla\gamma \frac{\nabla\times\varphi}{|\nabla\varphi|} \right| |\nabla\varphi|\delta(\varphi)\, drds.$$

Here, we have introduced the notation $\nabla\times\varphi = [-\varphi_{,s}, \varphi_{,r}]^T$, and $\delta(\cdot)$ is the Dirac delta function. We will approximate this integral as a limit of an approximate delta function. Letting $\zeta(\cdot)/\varepsilon$ be the approximate delta function, we obtain from above

$$\begin{aligned}\ell(\Gamma_t) &= \lim_{\varepsilon\to 0} \int_{J^2} \left| \nabla\gamma \frac{\nabla\times\varphi}{|\nabla\varphi|} \right| |\nabla\varphi| \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\, drds \\ &= \lim_{\varepsilon\to 0} \int_{J^2} |\nabla\gamma\nabla\times\varphi| \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\, drds.\end{aligned}$$

Next, letting

$$\nabla\times\gamma = \begin{pmatrix} -\gamma_{1,s} & \gamma_{1,r} \\ -\gamma_{2,s} & \gamma_{2,r} \\ -\gamma_{3,s} & \gamma_{3,r} \end{pmatrix},$$

we denote

$$\ell_\varepsilon(\Gamma_t) = \int_{J^2} |\nabla\gamma\nabla\times\varphi| \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\, drds = \int_{J^2} |\nabla\times\gamma\nabla\varphi| \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\, drds, \quad (1)$$

which represents the approximate arclength of $\Gamma_t$.

The area enclosed by $\Gamma_t$ onto $S$ is a little simpler to calculate. Let $\mathcal{H}(\cdot)$ be the Heaviside function, and $H(\cdot)$ be its approximation, then

$$\begin{aligned}\text{Area}(\Gamma_t) &= \int_{J^2} (1 - \mathcal{H}(\varphi(r,s,t))|\gamma_{,r} \times \gamma_{,s}|\, drds \\ &= \lim_{\varepsilon \to 0} \int_{J^2} \left(1 - H\left(\frac{\varphi}{\varepsilon}\right)\right) |\gamma_{,r} \times \gamma_{,s}|\, drds.\end{aligned}$$

## 2.2 Area preserving velocity field

Recall that our goal is to solve the problem

$$\min_{\Gamma} \ell(\Gamma) \quad \text{subject to } \text{Area}(\Gamma) = C. \tag{2}$$

As we mentioned, our approach will be to obtain a 'flow' that reduces the objective while respecting the constraint. The flow deforms the curve by transporting the level-set function $\varphi(r,s,t)$. This is done through the equation

$$\varphi_t + w \cdot \nabla \varphi = 0. \tag{3}$$

The velocity field $w(r,s,t)$ will be such that the objective is decreased, but we also need to make certain that the area is preserved. Therefore, we need to determine the condition satisfied by $w$ such that area is preserved during the flow.

We use the approximate area as the surrogate for the area, therefore

$$\text{Area}_\varepsilon(\Gamma_t) := \int_{J^2} \left(1 - H\left(\frac{\varphi}{\varepsilon}\right)\right) |\gamma_{,r} \times \gamma_{,s}|\, drds = C.$$

Then differentiating leads to

$$\int_{J^2} -\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right) \varphi_t |\gamma_{,r} \times \gamma_{,s}|\, drds = 0.$$

Since $\varphi_t + w \cdot \nabla\varphi = 0$, and $-\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi = \nabla(1 - H(\frac{\varphi}{\varepsilon}))$, we have, from above,

$$\int_{J^2} [w(r,s,t)|\gamma_{,r} \times \gamma_{,s}|] \cdot \nabla \left(1 - H\left(\frac{\varphi}{\varepsilon}\right)\right) drds = 0.$$

Thus, for a velocity field to preserve the area, it must satisfy

$$\text{div}(|\gamma_{,r} \times \gamma_{,s}| w) = 0. \tag{4}$$

The next step is to find a velocity field that not only preserves the area, but also reduces the arclength.

## 3 Descent algorithm

The algorithm we propose is a projected gradient approach. We will describe it in terms of a flow in which the objective function, viewed as energy, is decreased in time. The flow is characterized by a velocity field $w$ for the level set function $\varphi$ which preserves the area.

Recall that the objective we wish to minimize is the arclength of the curve $\ell_\epsilon(\Gamma_t)$, given in (1). We do this by evolving the curve $\Gamma_t$ by prescribing velocity $w$ to its level-set representation. We posit that the $t$-derivative of arclength takes the form

$$\frac{d\ell_\varepsilon(\Gamma_t)}{dt} = -\int_{J^2} F(\varphi)\cdot w\, drds. \tag{5}$$

This can be interpreted 'physically' as follows. Viewing the arclength $\ell_\epsilon(\Gamma_t)$ as 'energy', then its time rate of change is 'power', which must take the form of the dot product of 'force' $F(\varphi)$ and velocity $w$. We will show below that this is true, at least formally, by directly calculating the derivative.

We choose a velocity of the form

$$w = \frac{F}{|\gamma_{,r}\times\gamma_{,s}|^2} - \frac{\nabla p}{|\gamma_{,r}\times\gamma_{,s}|}. \tag{6}$$

The choice of the first term is to make a negative contribution to $d\ell_\varepsilon(\Gamma_t)/dt$. The second term is for projection to constrain $w$ so that the area inside $\Gamma_t$ is preserved. The normalizations are taken for convenience, but will be important later when we give an interpretation for the 'force' $F$. In order to determine the term $p$, we require $w$ to satisfy the constraint (4), which means that

$$\Delta p = \operatorname{div}\left(\frac{F}{|\gamma_{,r}\times\gamma_{,s}|}\right). \tag{7}$$

To see that $w$ so determined leads to a flow that reduces arclength, we substitute $F$ in (6) in (5). We obtain

$$\frac{d\ell_\varepsilon(\Gamma_t)}{dt} = -\int_{J^2} |\gamma_{,r}\times\gamma_{,s}|^2|w|^2\, drds - \int_{J^2} |\gamma_{,r}\times\gamma_{,s}|w\cdot\nabla p\, drds.$$

Next, we use the identity

$$\operatorname{div}\left[(|\gamma_{,r}\times\gamma_{,s}|w)p\right] = (|\gamma_{,r}\times\gamma_{,s}|w)\cdot\nabla p + \operatorname{div}(|\gamma_{,r}\times\gamma_{,s}|w)\, p,$$

and integrate by parts to obtain

$$\frac{d\ell_\varepsilon(\Gamma_t)}{dt} = -\int_{J^2} |\gamma_{,r}\times\gamma_{,s}|^2|w|^2 drds + \int_{J^2} p\ \operatorname{div}(|\gamma_{,r}\times\gamma_{,s}|w)\, drds,$$

using the boundary condition $w|_{\partial J^2} = 0$ (or periodic boundary conditions). By (4) we see that the second term on the right-hand side is zero. Therefore we have established that

$$\frac{d\ell_\varepsilon(\Gamma_t)}{dt} \leq 0.$$

and that the length of the curve stops decreasing if, and only if, $w$ vanishes.

### 3.1 Computation of "the force"

Computing forcing term $F$ in terms of $\varphi$ and $\gamma$ is a matter of differential calculus. We start by formally differentiating (1)

$$\frac{d\ell_\varepsilon(\Gamma_t)}{dt} = \int_{J^2} \left[ |\nabla\times\gamma\nabla\varphi|_t \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right) + |\nabla\times\gamma\nabla\varphi| \frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\varphi_t \right] drds \qquad (8)$$

From (3) we have $\frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\varphi_t = -\frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi\cdot w = -\nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\cdot w$ so that the second term of the integrand, which we denote by $I_2$ reads

$$I_2 = -|\nabla\times\gamma\nabla\varphi|\nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\cdot w. \qquad (9)$$

We will see later that this term is cancelled by a component of the first term.

We need to get an expression for the first term in the integrand. We start by taking the gradient of the transport equation (3)

$$\nabla\varphi_t + D^2\varphi\, w + \nabla w^T\nabla\varphi = 0.$$

Since $\nabla\times\gamma$ is time independent, we can premultiply the equation above by it to get

$$(\nabla\times\gamma\nabla\varphi)_t + \nabla\times\gamma D^2\varphi\, w + \nabla\times\gamma\nabla w^T\nabla\varphi = 0.$$

Now, taking the scalar product of this equation with $\nabla\times\gamma\nabla\varphi$ gives

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\times\gamma\nabla\varphi|^2 + \nabla\varphi^T\nabla\times\gamma^T\nabla\times\gamma D^2\varphi\, w + \nabla\varphi^T\nabla\times\gamma^T\nabla\times\gamma\nabla w^T\nabla\varphi = 0.$$

By transposing, and using the identity $(u\otimes v) : A = u^T A v = v^T A^T u$, we arrive at

$$\frac{1}{2}\frac{\partial}{\partial t}|\nabla\times\gamma\nabla\varphi|^2 + (D^2\varphi\nabla\times\gamma^T\nabla\times\gamma\nabla\varphi)\cdot w + \left[\nabla\varphi\otimes(\nabla\times\gamma^T\nabla\times\gamma\nabla\varphi)\right] : \nabla w = 0,$$

which upon division by $|\nabla\times\gamma\nabla\varphi|$ gives

$$|\nabla\times\gamma\nabla\varphi|_t + D^2\varphi\nabla\times\gamma^T\frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}w + \left[\nabla\varphi\otimes(\nabla\times\gamma^T\frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|})\right] : \nabla w = 0. \qquad (10)$$

Next we recall two tensor identities involving a matrix $A$, and vectors $a, b$, namely

$$\operatorname{div}(a \otimes b) = (\operatorname{div} b)a + (\nabla a)b \quad \text{and} \quad A : \nabla b = \operatorname{div}(A^T b) - (\operatorname{div} A) \cdot b.$$

Applying the second identity to the third term in (10) gives

$$\begin{aligned}
&\left(\nabla\varphi \otimes (\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|})\right) : \nabla w \\
&= \operatorname{div}\left([\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi] w\right) - \operatorname{div}\left(\nabla\varphi \otimes (\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|})\right) \cdot w \\
&= \operatorname{div}\left([\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi] w\right) - \operatorname{div}\left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\right)(\nabla\varphi \cdot w) \\
&\qquad - D^2\varphi \nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} w,
\end{aligned}$$

after applying the first identity. After collecting terms (10) becomes

$$\begin{aligned}
|\nabla\times\gamma\nabla\varphi|_t = \operatorname{div}\left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\right)(\nabla\varphi \cdot w) \\
- \operatorname{div}\left(\left[\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi\right] w\right).
\end{aligned}$$

We now want to use this expression in (8), thus it will be multiplied by $\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)$, and then integrated over $J^2$.

Let us first look at what happens to the second term. We multiply it by $\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)$ and use the product rule on the divergence to get

$$\begin{aligned}
&\operatorname{div}\left(\left[\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi\right] w\right) \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right) \\
&\qquad = \operatorname{div}\left(\left[\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi\right] w \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right) \\
&\qquad\quad - \left(\left[\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \otimes \nabla\varphi\right] w\right) \cdot \nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right].
\end{aligned} \tag{11}$$

When we integrate the expression on the right-hand side over $J^2$, the first term gives zero as long as $\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)$ vanishes on $\partial J^2$. This was assumed in the case where $S$ has a boundary (case (i)). In that case cancellation of this terms occurs as long as the curve is not too close to the boundary of the parameter space. In the case of a closed surface (case (ii)), this term vanishes by periodicity of $\gamma, \varphi$ and $w$. The second term in the right-hand side of (11) becomes

$$\begin{aligned}
&\left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \cdot \nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\right)(\nabla\varphi\cdot w)\\
&\quad= \left\{\left[\nabla\varphi\otimes\left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\right)\right]\nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\right\}\cdot w\\
&\quad= \left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \cdot \nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\right)(\nabla\varphi\cdot w)\\
&\quad= \left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|} \cdot \frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi\right)(\nabla\varphi\cdot w)\\
&\quad= \left(\nabla\varphi^T\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\right)(\nabla\varphi\cdot w)\\
&\quad= |\nabla\times\gamma\nabla\varphi|\frac{1}{\varepsilon^2}\zeta'\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi\cdot w\\
&\quad= |\nabla\times\gamma\nabla\varphi|\nabla\left[\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\right]\cdot w.
\end{aligned}$$

This cancels out $I_2$ in (9). We finally arrive at the expression for the force

$$F(\varphi) = -\operatorname{div}\left(\nabla\times\gamma^T \frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\right)\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi. \tag{12}$$

**Remark 1** *Considering the special case where $S$ is a plane, with $\gamma(r,s) = (r,s,0)^T$ then $\nabla\times\gamma^T\nabla\times\gamma = \mathbb{I}_2$ and $|\nabla\times\gamma\nabla\varphi| = |\nabla\varphi|$ thus we recover the classic formula.*

*Note however that this holds because the parameter space has a the trivial first fundamental form. If we consider another parametrization of the plane, say $\gamma(r,s) = (r^3,s^3,0)^T$, then we do not recover the classical formula for the curvature. Similarly, the force $F$ is not invariant under a change of parameter space, since it represents an object onto that space.*

**Remark 2** *For a matrix $A$ and a vector $v$, we have $\operatorname{div}(A^Tv) = \operatorname{div} A\cdot v + A : \nabla v$, where $\operatorname{div} A$ is as usual the (size 3) column vector made of divergence of (size 2) row vectors of $A$. But $\operatorname{div}\nabla\times\gamma = 0$, thus the formula for the force can also be written as*

$$F(\varphi) = -\nabla\times\gamma : \nabla\left(\frac{\nabla\times\gamma\nabla\varphi}{|\nabla\times\gamma\nabla\varphi|}\right)\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi. \tag{13}$$

### 3.2 Curve moving algorithm

To sum up, the minimization process is done by solving the following system of PDEs

$$\varphi_t + w \cdot \nabla\varphi = 0, \tag{14}$$

$$w + \frac{1}{|\gamma_{,r} \times \gamma_{,s}|}\nabla p = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2}F(\varphi), \tag{15}$$

$$\mathrm{div}(|\gamma_{,r} \times \gamma_{,s}|w) = 0. \tag{16}$$

The evolution terminates when the velocity field $w$ becomes zero.

The divergence-free condition may be implemented by a slightly modified projection method. For example for the classical Chorin-type projection [1] we perform these steps

$$\begin{aligned}
\frac{\varphi^{n+1} - \varphi^n}{\delta t} + w^n \cdot \nabla\varphi^n &= 0, \\
\tilde{w}^{n+1} &= \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2}F(\varphi^{n+1}), \\
\Delta p^{n+1} &= \mathrm{div}(|\gamma_{,r} \times \gamma_{,s}|\tilde{w}^{n+1}), \\
w^{n+1} &= \tilde{w}^{n+1} - \frac{1}{|\gamma_{,r} \times \gamma_{,s}|}\nabla p^{n+1}.
\end{aligned}$$

We may of course use some more advanced time-stepping scheme but this algorithm is presented here for the sake of simplicity. For example we can use $F(\frac{3}{2}\varphi^{n+1} - \frac{1}{2}\varphi^n)$ rather than $F(\varphi^{n+1})$, so that $w^{n+1}$ will approximate the velocity at time $n + \frac{3}{2}$ and the next step in the transport of $\varphi$ will be more accurate.

## 4 Geodesic curvature

We will next provide a geometric interpretation of the force $F$ in (13). When the miminization (2) is solved using the algorithm in (14)-(16), the process terminates when the velocity $w$ is zero. Recall from differential geometry that curves which minimize their length under a fixed enclosed area constraint are linked to constant geodesic curvature curves [5]. We will show that the geodesic curvature of the curves becomes constant when the velocity is zero. Further, the geodesic curvature provides a method for verifying numerical calculations.

An intrinsic and simple way to define the geodesic curvature [4] is to use the classical representation of the curve as a level-set $\Phi$ on $S$ [3]. Then this curvature is defined by

$$\kappa_g = \mathrm{div}_S \frac{\nabla_S \Phi}{|\nabla_S \Phi|}.$$

Note that in this formula, $\Phi$ needs only to be defined on $S$, since the surface operators do not depend on its values outside $S$. From our formulation, we can easily define a function on $S$ whose level-set is the curve, by setting

$$\Phi(\gamma(r,s)) = \varphi(r,s) \qquad \forall (r,s) \in J^2. \tag{17}$$

It is therefore possible to express $\kappa_g$ in terms of $\gamma$ and $\varphi$. We start by taking the gradient (with respect to $(r,s)$) of (17)

$$\nabla\gamma^T(r,s)\nabla_x\Phi(\gamma(r,s)) = \nabla\varphi(r,s),$$

where $\nabla_x$ denotes the usual gradient in $\mathbb{R}^3$. We implicitly extended $\Phi$ outside $S$ in a smooth but arbitrary way. Now a definition of the surface gradient is

$$\nabla_S\Phi = \nabla\Phi - (\nabla\Phi \cdot n)\, n,$$

where $n$ is a unit normal to $S$. Thus on $S$,

$$\nabla\gamma^T(r,s)\nabla_S\Phi(\gamma(r,s)) = \nabla\gamma^T(r,s)\nabla_x\Phi(\gamma(r,s)) - (\nabla\Phi\cdot n)\,\nabla\gamma^T n = \nabla\varphi(r,s),$$

since $\nabla\gamma^T n = 0$. Hence $\nabla_S\Phi$ is the vector in $\mathbb{R}^3$ such that

$$\nabla\gamma^T(r,s)\nabla_S\Phi(\gamma) = \nabla\varphi \quad \text{and} \quad n\cdot\nabla_S\Phi(\gamma) = 0,$$

where $n = \frac{\gamma_{,r}\times\gamma_{,s}}{|\gamma_{,r}\times\gamma_{,s}|}$. This can be written as

$$A\nabla_S\Phi(\gamma) = \begin{pmatrix}\nabla\varphi \\ 0\end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix}\gamma_{,r}^T \\ \gamma_{,s}^T \\ (\gamma_{,r}\times\gamma_{,s})^T\end{pmatrix}.$$

The following holds for vectors $a$ and $b$

$$\begin{pmatrix} a^T \\ b^T \\ (a\times b)^T \end{pmatrix}^{-1} = \frac{1}{|a\times b|^2}\begin{pmatrix} b\times(a\times b) & -a\times(a\times b) & a\times b\end{pmatrix}.$$

Applying this to calculate the inverse of $A$, we obtain

$$\begin{aligned}\nabla_S\Phi(\gamma) &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|}\begin{pmatrix}\gamma_{,s}\times n & -\gamma_{,r}\times n & n\end{pmatrix}\begin{pmatrix}\nabla\varphi\\ 0\end{pmatrix}\\ &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|}\left[\varphi_{,r}(\gamma_{,s}\times n)-\varphi_{,s}(\gamma_{,r}\times n)\right].\end{aligned}$$

Using the triple cross-product formula we have

$$\begin{aligned}\gamma_{,s}\times n &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|}\left[|\gamma_{,s}|^2\gamma_{,r}-(\gamma_{,r}\cdot\gamma_{,s})\gamma_{,s}\right],\\ \gamma_{,r}\times n &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|}\left[(\gamma_{,r}\cdot\gamma_{,s})\gamma_{,r}-|\gamma_{,s}|^2\gamma_{,s}\right].\end{aligned}$$

Thus

$$\begin{aligned}\nabla_S\Phi(\gamma) &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|^2}\left[(|\gamma_{,s}|^2\varphi_{,r}-(\gamma_{,r}\cdot\gamma_{,s})\varphi_{,s})\gamma_{,r}\right.\\ &\qquad\left.+(|\gamma_{,r}|^2\varphi_{,s}-(\gamma_{,r}\cdot\gamma_{,s})\varphi_{,r})\gamma_{,s}\right]\\ &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|^2}\nabla\gamma\begin{pmatrix}(|\gamma_{,s}|^2\varphi_{,r}-(\gamma_{,r}\cdot\gamma_{,s})\varphi_{,s})\\ -(\gamma_{,r}\cdot\gamma_{,s})\varphi_{,r}+|\gamma_{,r}|^2\varphi_{,s})\gamma_{,s}\end{pmatrix}\\ &= \frac{1}{|\gamma_{,r}\times\gamma_{,s}|^2}\nabla\gamma\begin{pmatrix}|\gamma_{,s}|^2 & -\gamma_{,r}\cdot\gamma_{,s}\\ -\gamma_{,r}\cdot\gamma_{,s} & |\gamma_{,r}|^2\end{pmatrix}\nabla\varphi\\ &= \nabla\gamma\begin{pmatrix}|\gamma_{,r}|^2 & \gamma_{,r}\cdot\gamma_{,s}\\ \gamma_{,r}\cdot\gamma_{,s} & |\gamma_{,s}|^2\end{pmatrix}^{-1}\nabla\varphi.\end{aligned}$$

Hence the surface gradient is expressed in term of $\varphi$ and $\gamma$ by

$$\nabla_S\Phi(\gamma) = \nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1}\nabla\varphi. \tag{18}$$

Note that this formula should not depend on the parametrization, since the surface gradient is intrinsic. Let us check that this is indeed the case by considering a diffeomorphism $\theta : J^2 \to J^2$ which defines a new paramatrization and level-set such that $\gamma = \tilde{\gamma}(\theta)$ and $\varphi = \tilde{\varphi}(\theta)$. Then plugging these relation in the expression for suface gradient leads to

$$\nabla_S\Phi(\gamma) = \nabla\tilde{\gamma}\nabla\theta(\nabla\theta^T\nabla\tilde{\gamma}^T\nabla\tilde{\gamma}\nabla\theta)^{-1}\nabla\theta^T\nabla\tilde{\varphi} = \nabla\tilde{\gamma}(\nabla\tilde{\gamma}^T\nabla\tilde{\gamma})^{-1}\nabla\tilde{\varphi}.$$

In order to compute the geodesic curvature we need now to write the divergence operator. The expression for surface gradient reads componentwise

$$\partial_S^i\Phi = \mathcal{A}_{i\alpha}\varphi_{u_\alpha}, \qquad \mathcal{A} = \nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1} \in \mathbb{R}^{3\times 2}.$$

where $(u_1, u_2) = (r, s)$ and the summation over repeated indices has been used. Thus for a velocity field $V$ defined on $S$ by $V(\gamma(r,s)) = v(r,s)$ with $v$ defined from $J^2$ to $\mathbb{R}^3$, we get

$$\operatorname{div}_S V(\gamma) = \partial_S^i V_i(\gamma) = \mathcal{A}_{i\alpha} v_{i,u_\alpha} = \mathcal{A} : \nabla v,$$

which leads to the following formula for geodesic curvature

$$\kappa_g = \mathcal{A} : \nabla \left( \frac{\mathcal{A}\nabla\varphi}{|\mathcal{A}\nabla\varphi|} \right), \qquad \mathcal{A} = \nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1}. \tag{19}$$

We would like to connect this expression to the force in (12). We know that the minimizer of our optimization problem is somehow related to curves such that $\kappa_g =$ constant [5]. If such a relation is available, we would be able to state what the force satisfies at termination of the evolution. Before doing this let us make a few observations.

**Remark 3** *The following identities holds.*

$$(\nabla\gamma^T\nabla\gamma)^{-1} = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2} \nabla\times\gamma^T\nabla\times\gamma,$$

$$\mathcal{A}^T\mathcal{A} = (\nabla\gamma^T\nabla\gamma)^{-T}\nabla\gamma^T\nabla\gamma(\nabla\gamma^T\nabla\gamma)^{-1} = (\nabla\gamma^T\nabla\gamma)^{-1}.$$

**Remark 4** *Using the above identities, we have*

$$\begin{aligned} |\mathcal{A}\nabla\varphi|^2 &= \langle \mathcal{A}\nabla\varphi, \mathcal{A}\nabla\varphi \rangle = \left\langle \nabla\varphi, \mathcal{A}^T\mathcal{A}\nabla\varphi \right\rangle \\ &= \left\langle \nabla\varphi, (\nabla\gamma^T\nabla\gamma)^{-1}\nabla\varphi \right\rangle = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2} \left\langle \nabla\varphi, \nabla\times\gamma^T\nabla\times\gamma\nabla\varphi \right\rangle, \end{aligned}$$

*thus*

$$|\mathcal{A}\nabla\varphi| = \frac{|\nabla\times\gamma\nabla\varphi|}{|\gamma_{,r} \times \gamma_{,s}|}.$$

We next calculate $|\gamma_{,r} \times \gamma_{,s}|\kappa_g$

$$\begin{aligned} |\gamma_{,r} \times \gamma_{,s}|\kappa_g &= (|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) : \nabla \left( \frac{\mathcal{A}\nabla\varphi}{|\mathcal{A}\nabla\varphi|} \right) \\ &= \operatorname{div} \left( \frac{|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}^T\mathcal{A}\nabla\varphi}{|\mathcal{A}\nabla\varphi|} \right) - \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) \cdot \frac{\mathcal{A}\nabla\varphi}{|\mathcal{A}\nabla\varphi|}. \end{aligned}$$

We use the identities in Remark 3 in the first term on the right-hand side, and rewrite the left-hand side to get

$$|\gamma_{,r} \times \gamma_{,s}|\kappa_g = \operatorname{div}\left(\nabla\!\times\!\gamma^T \frac{\nabla\!\times\!\gamma\nabla\varphi}{|\nabla\!\times\!\gamma\nabla\varphi|}\right) - A^T \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|A) \cdot \frac{\nabla\varphi}{|A\nabla\varphi|}.$$

We will show below that $\mathcal{A}^T \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) = 0$ so that we have the following formula for geodesic curvature

$$|\gamma_{,r} \times \gamma_{,s}|\kappa_g = \operatorname{div}\left(\nabla\!\times\!\gamma^T \frac{\nabla\!\times\!\gamma\nabla\varphi}{|\nabla\!\times\!\gamma\nabla\varphi|}\right). \tag{20}$$

We pause to examine (20) and connect it with the formula for the force in (12). It can be seen that

$$F(\varphi) = -|\gamma_{,r} \times \gamma_{,s}|\kappa_g \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi. \tag{21}$$

We showed at the beginning of Section 3 that the curve length stops decreasing when the velocity $w$ is zero in (15). Let us show that we get the expected minimizer. As $w = 0$, from (15) there holds

$$\frac{1}{|\gamma_{,r} \times \gamma_{,s}|}\nabla p = \frac{1}{|\gamma_{,r} \times \gamma_{,s}|^2}F(\varphi).$$

**Remark 5** *One could be surprised that the force does not vanish at equilibrium. This is due to the fact that the curve is still willing to shorten its length, but is prevented from doing so by the area constraint. Thus this generates a gradient-like force (corrected by the metric).*

Then from (21),

$$\nabla p = -\kappa_g \frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right)\nabla\varphi = -\kappa_g \nabla\left[Z(\frac{\varphi}{\varepsilon})\right] \tag{22}$$

where $Z'(r) = \zeta(r)$. Intuitively for this to hold $\kappa_g$ must be constant in the direction orthogonal to the nablas. This shows that $-\kappa_g$ has to be constant along level-sets of $\varphi$ in a neighborhood of $\varphi = 0$. To show this more rigourosly, we can use the curl of a 2D vector field which is the scalar defined by $\operatorname{curl} v := v_{2,x_1} - v_{1,x_2}$ and verifies for a scalar function $f$ and a velocity field $v$, $\operatorname{curl}(fv) = f\operatorname{curl} v + \nabla\times f \cdot u = f\operatorname{curl} v - \nabla f \cdot u^\perp$ where $u^\perp$ is orthogonal to $u$. Thus taking the curl of (22) above gives

$$0 = -\nabla\kappa_g \cdot \nabla\!\times\! Z(\frac{\varphi}{\varepsilon}) = -\nabla\kappa_g \cdot \nabla\!\times\!\varphi\frac{1}{\varepsilon}\zeta\left(\frac{\varphi}{\varepsilon}\right),$$

which just says that $\kappa_g$ is constant along the tangent to the level sets of $\varphi$ when $\zeta > 0$, i.e. in a neighborhood of $\varphi = 0$. This proves that our minimizing curves have, as expected, constant geodesic curvature [5]. We thus proved:

**Proposition 1** *The algorithm (14)-(16) makes the length of curve defined as the image of the zero level-set of $\varphi$ by $\gamma$ diminishing in time. If the length reaches an equilibrium, then the corresponding curve has constant geodesic curvature.*

Turning back to show that $\mathcal{A}^T \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) = 0$, we note that

$$\mathcal{A}^T = (\nabla\gamma^T\nabla\gamma)^{-T}\nabla\gamma^T,$$

so that the requirement is equivalent to

$$\nabla\gamma^T \operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) = 0. \tag{23}$$

Demonstrating (23) is through brute-force calculation. We had hoped to find a clever known fact from geometry to help us but we were unable to do so.

We begin by calculating $(\nabla\gamma^T\nabla\gamma)$

$$(\nabla\gamma^T\nabla\gamma) = \begin{pmatrix} \gamma_{,r}\cdot\gamma_{,r} & \gamma_{,r}\cdot\gamma_{,s} \\ \gamma_{,r}\cdot\gamma_{,s} & \gamma_{,s}\cdot\gamma_{,s} \end{pmatrix}.$$

Computing the inverse of this 2-by-2 matrix and using it in the definition of $\mathcal{A}$, we get

$$\mathcal{A} = \frac{1}{\det(\nabla\gamma^T\nabla\gamma)}\Big((\gamma_{,s}\cdot\gamma_{,s})\gamma_{,r} - (\gamma_{,r}\cdot\gamma_{,s})\gamma_{,s} \quad -(\gamma_{,r}\cdot\gamma_{,s})\gamma_{,r} + (\gamma_{,r}\cdot\gamma_{,r})\gamma_{,s}\Big).$$

Using the fact that $|\gamma_{,r} \times \gamma_{,s}| = \sqrt{\det(\nabla\gamma^T\nabla\gamma)}$, we obtain

$$\begin{aligned}\operatorname{div}(|\gamma_{,r} \times \gamma_{,s}|\mathcal{A}) = \Big(&\frac{(\gamma_{,s}\cdot\gamma_{,s})\gamma_{,r} - (\gamma_{,r}\cdot\gamma_{,s})\gamma_{,s}}{\sqrt{\det(\nabla\gamma^T\nabla\gamma)}}\Big)_{,r} \\ &+ \Big(\frac{-(\gamma_{,r}\cdot\gamma_{,s})\gamma_{,r} + (\gamma_{,r}\cdot\gamma_{,r})\gamma_{,s}}{\sqrt{\det(\nabla\gamma^T\nabla\gamma)}}\Big)_{,s}\end{aligned}$$

Calculations showing that

$$\begin{aligned}\gamma_{,r}\cdot\Big(&\frac{(\gamma_{,s}\cdot\gamma_{,s})\gamma_{,r} - (\gamma_{,r}\cdot\gamma_{,s})\gamma_{,s}}{\sqrt{\det(\nabla\gamma^T\nabla\gamma)}}\Big)_{,r} \\ &+ \gamma_{,r}\cdot\Big(\frac{-(\gamma_{,r}\cdot\gamma_{,s})\gamma_{,r} + (\gamma_{,r}\cdot\gamma_{,r})\gamma_{,s}}{\sqrt{\det(\nabla\gamma^T\nabla\gamma)}}\Big)_{,s} = 0,\end{aligned}$$

needed to prove the first of (23) are omitted as they are too tedious. A similar result involving $\gamma_{,s}$ also holds. Thus we can conclude that the formula for geodesic curvature in (20) is indeed correct.

## 5 Numerical examples

In the context of moving a curve with given enclosed surface, our projection algorithm has a clear advantage over other algorithms that use a penalty term to enforce the fixed area constraint. Our implementation uses a MAC grid which ensures accurate divergence-free condition [1]. Thus the surface area constraint is not penalized but enforced. Surface area loss from initialization to stationary state in the case of an ellipse on a cylinder relaxing to a circle is under one percent (0.66 %) for a $64 \times 64$ grid. Moreover, as the Poisson equation associated to the projection method lies on the rectangular parametric space, fast FFT solvers (e.g. FISHPACK [9]) may be used, leading to very small computational costs.

The boundary conditions are of Dirichlet type in case of non-closed supporting surfaces, and periodic in one direction in the case of surfaces of revolution. Note however that our algorithm as presented above in its native form requires a regular parametrical representation of the supporting surface. This fact rules out, for example, the case where the supporting surface is a closed sphere. But one could easily adapt the algorithm to deal with parametrical patches.

Numerically, we found different ways to compute the force. While each give overall the same evolution, some are more stable than others. In this respect it is worth noticing that our problem has no diffusion in velocity, which could regularize some numerical oscillations. For that reason we use WENO [6] schemes to solve the advection equation (14) and to compute gradients of the level set function. We found that the form (13) leads to a more stable evolution than the divergence form (12). An even more stable form could be found by using the identity

$$A : \nabla\left(\frac{A\nabla\varphi}{|A\nabla\varphi|}\right) = \frac{1}{|A\nabla\varphi|} A : \left(\mathbb{I} - \frac{A\nabla\varphi \otimes A\nabla\varphi}{|A\nabla\varphi|}\right) \nabla(A\nabla\varphi),$$

applied with $A = \nabla \times \gamma$ to equation (13). Note that in the classical computation of curvature on a plane, one uses an expanded form involving second order partial derivatives of $\varphi$. This might be done here too (by expanding the last gradient term), but would lead to a huge formula. This intermediate formula showed good stability behavior while remaining relatively easy to implement.

## 5.1 Paraboloid supporting surface

We start to illustrate the results of our minimization algorithm in the case of a paraboloid supporting surface. The minimizer is known: it consists of a horizontal circle, in accordance with FIGURE 2.

## 5.2 Cylindrical supporting surface

As noted above, our algorithm has built-in volume conservation. To illustrate this, we consider the minimization problem for a cylindrical supporting surface with as initialization an ellipse. The simplest case, without topological changes, is when the cylinder has a radius large enough so that the minimizer is a circle (see below for the other case). With the cylinder oriented vertically, we calculated the geodesic curvature on the curve. The value of the geodesic curvature is sampled at two points, corresponding to the vertical and horizontal (with respect to the orientation of the cylinder) curvatures. In FIGURE 3 we plot these horizontal and vertical curvatures as a function of evolution. Both converge towards a common value which is the curvature of the minimizing circle. The volume conservation property clearly holds, even in the rough grid $64 \times 64$ used, since the asymptotic behavior is horizontal. Volume loss would have induced a negative slope for increasing iterations.

In the next example, the supporting surface is a cylinder of radius $a = 1$. An ellipse in the parametric space is chosen as initialization, which gives the curve drawn on the left-most picture in FIGURE 4. This curve is wrapped around the cylinder: the top and bottom loops are running on the back part of the surface while the thinest part of domain enclosed by the curve is drawn on the front. Computations are made on a $128 \times 128$ grid. Due to the fact that the area enclosed by the curve is greater than $4\pi a^2$, the minimizing curve is known to be made of two circles [5], a fact that our computations recover. Note however that starting from a circle in the parametric space, with a radius greater than $a$, will not give the absolute minimizer since this corresponds to a local minimum.

## 5.3 Hyperboloid supporting surface

In the preceding example the metric was flat. To illustrate the fact that our method works for an arbitrary supporting surface, we consider the hyperboloid shape of FIGURE 5 and perform the same kind of minimization, starting from an ellipse in the parametrical space. This leads also to a minimizer which is made of two closed circles.

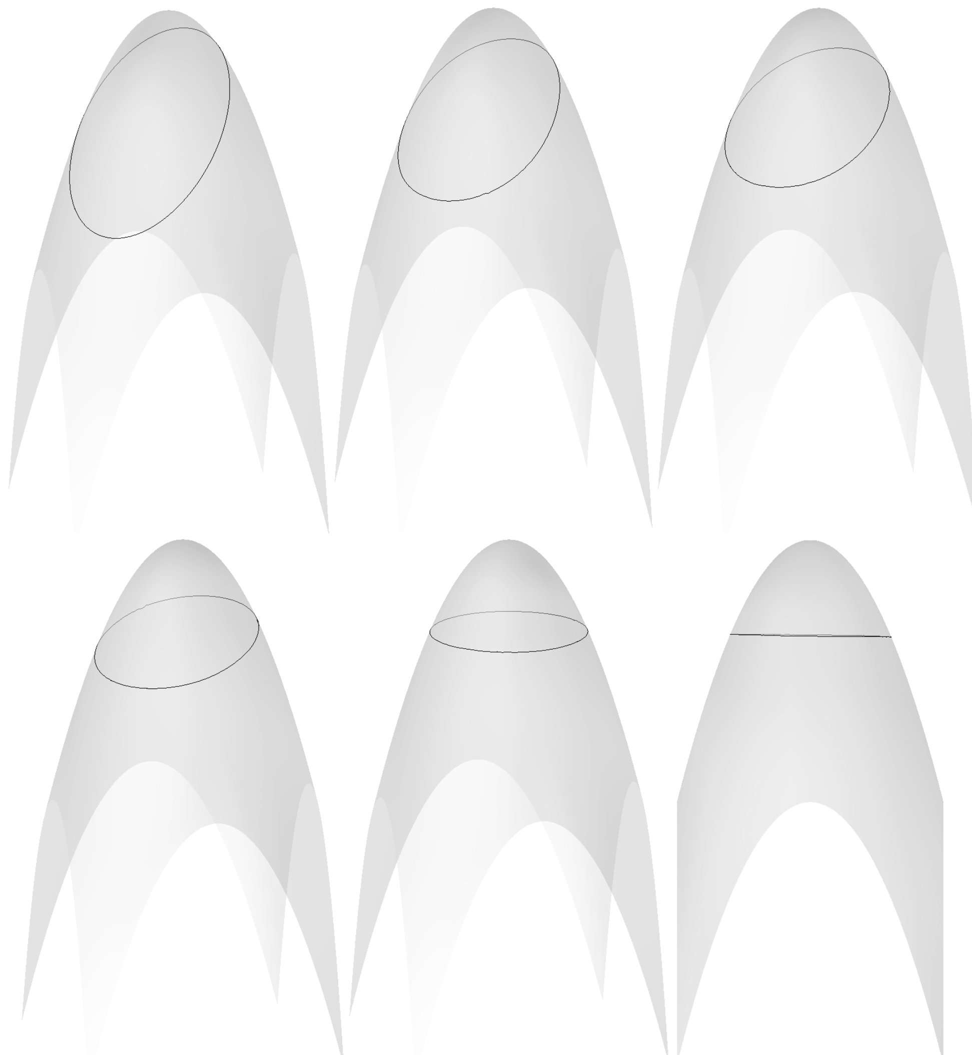

FIGURE 2: Minimization of curve length at prescribed enclosed surface area, on a paraboloid. Convergence toward the horizontal circle. Last picture shows a non perspective plot of the final state.

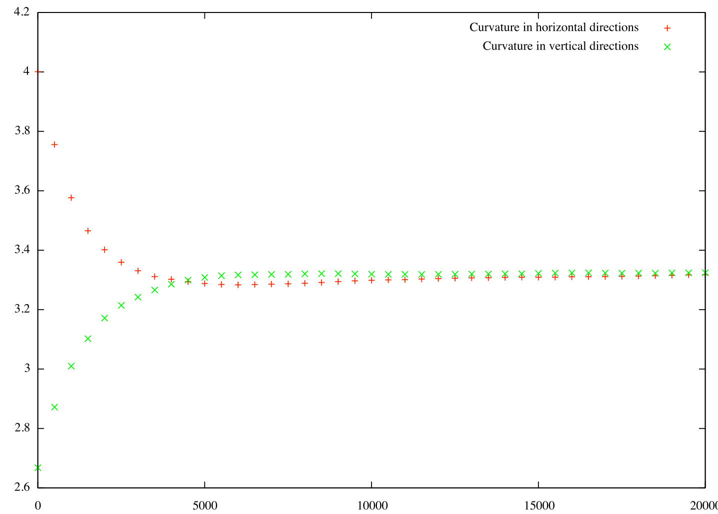


FIGURE 3: Convergence to a constant curvature curve. Horizontal and vertical cur

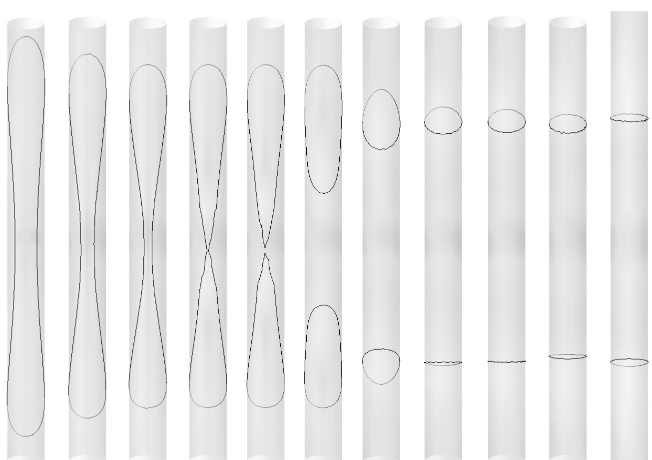

FIGURE 4: Minimization of curve length minimization leading to a break. Last picture on the right is without perspective to demonstrate symmetry property.

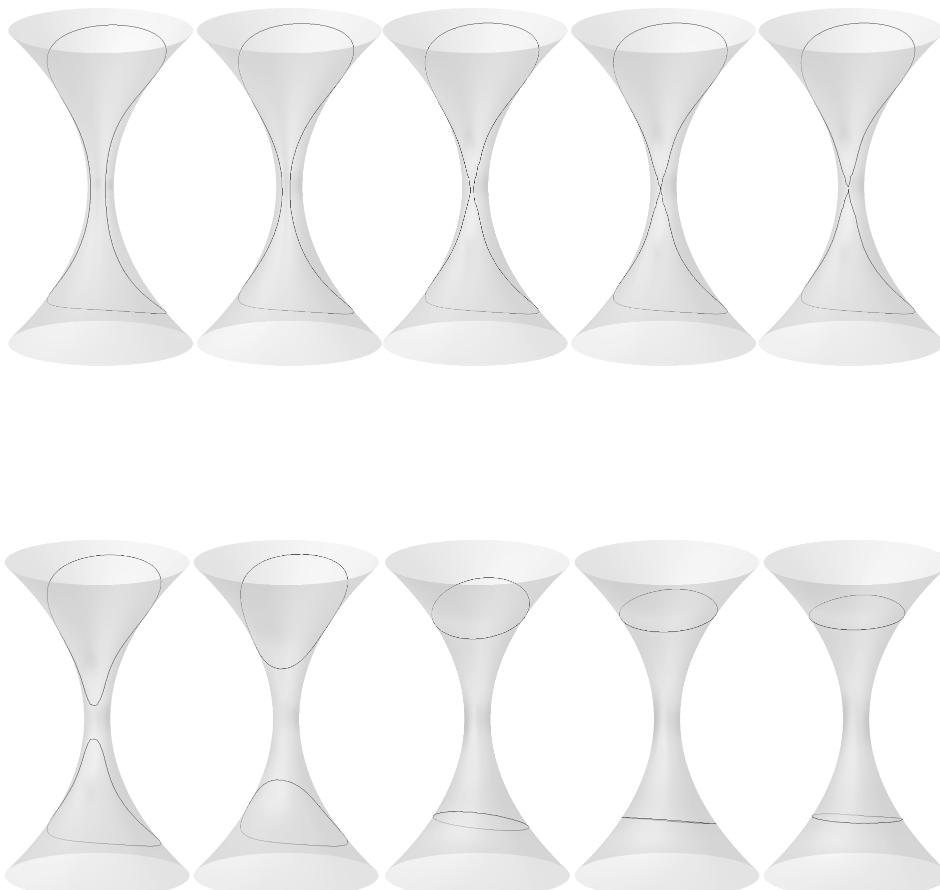

FIGURE 5: Minimization of curve length at prescribed enclosed surface area, in the case of a non flat surface.

## 6 Discussion

In this work, we have considered a geometrical optimization problem on a fixed curved surface. As a model, we considered the isoparametric problem of finding a curve of least length with a given area. The method we propose uses a level set function to represent the unknown geometry. The level set function is defined in the 2-D parameter space. Thus the computation takes place in two dimensions, leading to a very efficient method. The level set function's evolution is governed by a constrained gradient flow which reduces the arclength of the curve. The velocity field is calculated by a projection method. Thus the curve moves in such a way that the enclosed area remains constant. The approach we propose is a framework for inverse and optimization problems on curved surfaces. In a future work, we will apply the strategy on an inverse problem involving geometry on curved surfaces.

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