COMBINATORIAL ORTHOGONALITY

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Abstract

By considering a combinatorial property of orthogonality we determine the minimum number of nonzero entries in an orthogonal matrix of order n which cannot be decomposed into two smaller orthogonal matrices.

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1 Introduction

Let $Q = [q_{ij}]$ be a real orthogonal (or complex unitary) matrix of order $n \geq 2$. At least how many nonzero entries must Q have? The short answer is n , since the identity matrix I_n is an orthogonal matrix. In order to make the question more interesting, let us assume that Q cannot be written as the direct sum of two (orthogonal) matrices of smaller order (no matter how the rows and columns are permuted). M. Fiedler¹ conjectured that Q must have at least $4(n - 1)$ nonzero entries. For each $n \geq 2$ there exists an orthogonal matrix of order n with exactly $4(n - 1)$ nonzero entries which cannot be written as a direct sum. Examples of such matrices are

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \quad (1)$$

and

$$\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \quad (2)$$

We show that Fiedler's conjecture follows from a more general combinatorial theorem.

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¹private communication

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors. We define x and y to be *combinatorially orthogonal* provided

$$|\{i : x_i y_i \neq 0\}| \neq 1.$$

Thus the combinatorial orthogonality of two vectors depends only on the positions of their nonzero coordinates. The vectors $z_1, \dots, z_k$ are *combinatorially orthogonal* provided z_i and z_j are combinatorially orthogonal for all $i \neq j$.

A *combinatorially row-orthogonal matrix* is a square matrix whose rows are combinatorially orthogonal. Combinatorial column-orthogonality is defined analogously. A *combinatorially orthogonal matrix* is a square matrix whose rows are combinatorially orthogonal and whose columns are combinatorially orthogonal. Clearly, an orthogonal matrix is combinatorially orthogonal. The combinatorial orthogonality of a matrix depends only on the zero-nonzero pattern of the matrix. Hence in discussing the combinatorial orthogonality of a square matrix A we may assume that A is a $(0,1)$ -matrix. A $(0,1)$ -matrix A is combinatorially orthogonal if and only if for all $i \neq j$ the inner product of row i with row j and that of column i with column j is not one. A zero matrix as well as the matrix

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (3)$$

are examples of combinatorially orthogonal matrices for which there does not exist an orthogonal matrix with the same zero-nonzero pattern.

Recall that a matrix A of order $n \geq 2$ is *partly decomposable* provided there exist permutation matrices P_1 and P_2 such that

$$P_1 A P_2 = \begin{bmatrix} A_1 & O \\ X & A_2 \end{bmatrix} \quad (4)$$

where A_1 and A_2 are square (nonvacuous) matrices. Equivalently, A is partly decomposable if and only if A has an r by s zero submatrix for some positive integers r and s with $r + s = n$. A matrix of order $n \geq 2$ is *fully indecomposable* provided it is not partly decomposable. If A is an orthogonal matrix which satisfies (4) then X is a zero matrix. Hence a partly decomposable

orthogonal matrix can be written as a direct sum (after row and column permutations). The matrix (3) is a partly decomposable matrix.

There are fully indecomposable, combinatorially orthogonal (0,1)-matrices for which there does not exist an orthogonal matrix with the same zero-nonzero pattern. Let P be the permutation matrix of order 11 with 1's in positions $(1, 2), \dots, (10, 11), (11, 1)$ and let

$$A = I_{11} + P + P^3 + P^6 + P^{10}.$$

Then

$$AA^T = A^T A = 3I_{11} + 2J_{11} \tag{5}$$

where J_{11} denotes the all 1's matrix of order 11 (A is an incidence matrix of the (11,5,2)-design). Thus A is a fully indecomposable, combinatorially orthogonal (0,1)-matrix. Suppose $Q = [q_{ij}]$ is an orthogonal matrix with the same zero-nonzero pattern as A . Let R be the $(0, 1, -1)$ -matrix obtained from Q by replacing the positive entries with 1 and the negative entries with -1 . It follows from (5) that $RR^T = 5I_{11}$ and hence $(\det R)^2 = 5^{11}$. We conclude that there is no orthogonal matrix with the same zero-nonzero pattern as A .

In the next section we prove that a fully indecomposable, combinatorially orthogonal matrix of order $n \geq 2$ has at least $4(n - 1)$ nonzero entries thereby proving Fiedler's conjecture. For $n \geq 5$, we show that up to row and column permutations and transposition, there is a unique fully indecomposable combinatorially orthogonal (0,1)-matrix of order n with exactly $4(n - 1)$ 1's. In the last section we show that a fully indecomposable, combinatorially row-orthogonal (0,1)-matrix of order $n \geq 4$ can have $3n$ but no fewer 1's if n is even and $3n + 1$ if n is odd. Those matrices for which equality holds are characterized.

2 Combinatorially orthogonal matrices

We begin by recursively defining a family of (0,1)-matrices of order $n \geq 2$. Let

$$B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

If n is odd, define

$$B_n = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ & & & & 1 \\ \hline 0 & \dots & 0 & 1 & 1 \end{array} \right].$$

If n is even, define

$$B_n = \left[\begin{array}{cccc|c} & & & & 0 \\ & & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ \hline 0 & \dots & 0 & 1 & 1 \end{array} \right].$$

For example,

$$B_3 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

and

$$B_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

We shall make use of the following facts concerning the matrices B_n . If n is odd, then B_n has a unique row and a unique column with exactly two 1's. If n is even and greater than 2, then B_n has exactly two columns with exactly two 1's and every row has at least three 1's. For $n \geq 3$, there are exactly four 1's in B_n each of which is in a row with exactly two 1's and a column with exactly three 1's or a column with exactly two 1's and a row with exactly three 1's. Two rows (or two columns) of B_n whose inner product is greater than 2 are identical. It is easy to verify that B_n is a fully indecomposable, combinatorially orthogonal matrix for each $n \geq 2$. The matrices B_5 and B_6

have the same zero-nonzero pattern as the orthogonal matrices (1) and (2), respectively. For each n a similar construction yields an orthogonal matrix with the same zero-nonzero pattern as B_n .

The following technical lemma is used in the inductive proof of the main result of this section. In general we use J to denote an all 1's matrix. The r by s matrix of all 1's is denoted by $J_{r,s}$; in case $r = s$ this is shortened to J_r .

Lemma 2.1 *Let*

$$M = \left[\begin{array}{c|c|c} U & V & O \\ \hline W & X & Y \\ \hline O & O & Z \end{array} \right]$$

be a fully indecomposable, combinatorially orthogonal $(0,1)$ -matrix of order n where U is k by k , X is ℓ by p , and $\ell \geq 2$. Assume each row of $[W X]$ contains a 1, and if $p = 1$ then the inner product of each pair of rows of Y is at least 1. Let

$$\overline{M} = \left[\begin{array}{cc} J_{\ell,p} & Y \\ O & Z \end{array} \right]$$

be the matrix of order $n - k$. Then $\overline{M}$ is a fully indecomposable, combinatorially orthogonal matrix.

Proof. A zero submatrix of $\overline{M}$ is either a zero submatrix of

$$\left[\begin{array}{cc} O & Z \end{array} \right] \quad \text{or} \quad \left[\begin{array}{c} Y \\ Z \end{array} \right].$$

Since U is a square matrix, the full indecomposability of M implies the full indecomposability of $\overline{M}$. The assumptions imply that each nonzero column of Y contains at least two 1's. This fact, the combinatorial orthogonality of M , and the assumption in the case $p = 1$ now imply that $\overline{M}$ is combinatorially orthogonal. $\square$

The number of nonzero entries of a matrix A is denoted by $\sigma(A)$. The number of zeros of a matrix A is denoted by $\sigma_0(A)$.

Theorem 2.2 *Let $M = [m_{ij}]$ be a fully indecomposable, combinatorially orthogonal $(0,1)$ -matrix of order $n \geq 2$. Then*

$$\sigma(M) \geq 4(n - 1).$$

Equality holds if and only if there exist permutation matrices P_1 and P_2 such that $P_1 M P_2$ equals B_n or B_n^T , or $n = 4$ and $P_1 M P_2$ equals $J_4 - I_4$.

Proof. Let r_i and s_i denote the number of 1's in row i and column i of M , respectively. It is easy to verify the theorem for $n \leq 4$. We now assume that $n > 4$ and proceed by induction on n .

First suppose that there exist i and j such that $m_{ij} = 1$ and $r_i = 2$. Let the other 1 in row i be in column k . Combinatorial row-orthogonality of M implies that columns j and k are identical. Full indecomposability of M implies that $s_j \geq 3$. We conclude that there do not exist integers i and j such that $m_{ij} = 1$ and $r_i = s_j = 2$. We divide the remainder of the proof into several cases.

Case 1: There exist i and j such that $m_{ij} = 1$, $r_i = 2$ and $s_j = 3$ (or $r_i = 3$ and $s_j = 2$).

Without loss of generality we assume that

$$M = \left[\begin{array}{c|c|c} 1 & 1 & 0 \cdots 0 \\ \hline 1 & X & Y \\ 1 & & \\ \hline 0 & & \\ \vdots & T & Z \\ 0 & & \end{array} \right].$$

The combinatorial row-orthogonality of M implies that $X = J_{2,1}$ and $T = O$. The combinatorial column-orthogonality of M implies that the two rows of Y are identical. Since M is fully indecomposable, the two rows of Y have a nonzero inner product. Hence by Lemma 2.1 the matrix

$$\overline{M} = \begin{bmatrix} J_{2,1} & Y \\ O & Z \end{bmatrix}$$

is a fully indecomposable, combinatorially orthogonal matrix of order $n - 1$. We have $\sigma(M) = \sigma(\overline{M}) + 4$, and hence by induction $\sigma(M) \geq 4(n - 1)$. Suppose that $\sigma(M) = 4(n - 1)$ and $n \geq 6$. Then $\sigma(\overline{M}) = 4(n - 2)$ and by induction $\overline{M}$ can be permuted to B_{n-1} or B_{n-1}^T . It follows that M can be permuted to B_n or B_n^T . It can be verified that the same conclusion holds if $n = 5$.

Case 2: There exist i, j, p, q, u, v such that $m_{ij} = 0$, $m_{iu} = m_{iv} = m_{pj} = m_{qj} = 1$, $m_{pu} = 1$, and $r_i = s_j = 2$.

Combinatorial row- and column-orthogonality imply that $m_{pv} = m_{qu} = m_{qv} = 1$. Without loss of generality we assume that

$$M = \left[\begin{array}{c|cc|c} 0 & 1 & 1 & 0 \cdots 0 \\ \hline 1 & & & \\ 1 & & & \\ 0 & X & & Y \\ \vdots & & & \\ 0 & & & \\ \hline 0 & & & \\ \vdots & O & & Z \\ 0 & & & \end{array} \right]$$

where X has no zero rows. Combinatorial orthogonality implies that rows 2 and 3 of M are identical and columns 2 and 3 are also identical; in particular, we have $X = J$. The matrix $\overline{M}$ obtained by applying Lemma 2.1 is a fully indecomposable, combinatorially orthogonal matrix of order $n - 1$. Since $X = J$ we have $\sigma(M) = \sigma(\overline{M}) + 4$ and hence by induction $\sigma(M) \geq 4(n - 1)$. If $\overline{M}$ can be permuted to B_{n-1} or B_{n-1}^T , then it is easy to see that M is partly decomposable. Hence it follows by induction that when $n \geq 6$ we have $\sigma(M) > 4(n - 1)$. The same conclusion holds when $n = 5$.

Case 3: There exists a row with exactly two 1's and a row with exactly three 1's whose inner product is 2 (or there exist two columns with these properties).

We assume that the rows in Case 3 are rows 1 and 2, respectively, and that $m_{11} = m_{12} = m_{13} = m_{22} = m_{23} = 1$. Row-orthogonality implies that columns 2 and 3 of M are identical. Permuting rows, M has the form

$$\left[\begin{array}{c|cc|c} 1 & 1 & 1 & 0 \cdots 0 \\ \hline W & J & & Y \\ \hline O & O & & Z \end{array} \right].$$

The matrix $\overline{M}$ obtained by applying Lemma 2.1 is a fully indecomposable, combinatorially orthogonal matrix of order $n - 1$. Since M is fully indecomposable, $\sigma(W) \geq 1$ and hence $\sigma(M) = \sigma(\overline{M}) + 3 + \sigma(W)$. By induction, $\sigma(M) \geq 4(n - 1)$ with equality if and only if $\sigma(W) = 1$. If $\sigma(W) = 1$ then Case 1 applies.

Case 4: There exists a row with exactly two 1's and a row with exactly four 1's whose inner product is 2 (or there exist two columns with these properties).

Without loss of generality we assume that

$$M = \left[\begin{array}{c|cc|c} J_2 & 0 & 0 & O \\ & 1 & 1 & \\ \hline W & X & Y & \\ \hline O & O & Z & \end{array} \right]$$

where $[W \ X]$ has no zero rows. Row-orthogonality implies that each row of $[W \ X]$ is either $0 \ 0 \ 1 \ 1$ or is of the form $1 \ 1 \ * \ *$. We again apply Lemma 2.1 and obtain a matrix $\overline{M}$ of order $n - 2$ for which

$$\sigma(M) = \sigma(\overline{M}) + 6 + \sigma(W) - \sigma_0(X).$$

Considering inner products of columns 3 and 4 with column 1 of M , we see that either some row of $[W \ X]$ equals $1 \ 1 \ 1 \ 1$, or both $1 \ 1 \ 1 \ 0$ and $1 \ 1 \ 0 \ 1$ occur as rows of $[W \ X]$. Hence $\sigma(W) - \sigma_0(X) \geq 2$, and by induction $\sigma(M) \geq 4(n-1)$. Suppose that $\sigma(M) = 4(n-1)$. Then $\sigma(\overline{M}) = 4(n-3)$, and by induction $\overline{M}$ can be permuted to B_{n-2} or B_{n-2}^T , or to $J_4 - I_4$ if $n = 6$. It is easy to verify that the latter cannot occur. The definition of B_{n-2} implies that there is a 1 in Z such that the row of $\overline{M}$ which contains it has exactly two 1's and the column which contains it has exactly three 1's, or the other way around. The row and column of M that contains this 1 have the same number of 1's as the corresponding row and column of $\overline{M}$. This reduces the case of equality to that of Case 1.

Case 5: There exist two rows with exactly two 1's and another row whose inner product with each of these two rows is nonzero (or there exist columns with these properties).

The combinatorial row-orthogonality of M allow us to assume that

$$M = \left[\begin{array}{cc|cc|c} 1 & 1 & 0 & 0 & O \\ 0 & 0 & 1 & 1 & \\ \hline W & X & Y & \\ \hline O & O & Z & \end{array} \right]$$

where $[W \ X]$ has no zero rows and has at least one row equal to 1 1 1 1. By Lemma 2.1 the matrix $\overline{M}$ of order $n - 2$ satisfies

$$\sigma(M) = \sigma(\overline{M}) + 4 + \sigma(W) - \sigma_0(X).$$

The combinatorial row-orthogonality of M implies that each row of $[W \ X]$ is one of 0 0 1 1, 1 1 0 0, and 1 1 1 1. The combinatorial orthogonality of columns 1 and 3 implies that at least two rows of $[W \ X]$ equal 1 1 1 1. Hence by induction, $\sigma(M) \geq 4(n - 1)$. As above we reduce the case of equality to that of Case 1.

Case 6: There exist two rows, one of which has exactly three 1's, whose inner product is 3 (or there exist two columns with these properties).

Without loss of generality we assume that

$$M = \left[\begin{array}{c|cc|c} 1 & 1 & 1 & O \\ \hline W & X & & Y \\ \hline O & O & & Z \end{array} \right],$$

where $[W \ X]$ has no zero rows and has at least one row equal to 1 1 1. The combinatorial row-orthogonality of M implies that each row of $[W \ X]$ contains at least two 1's. The matrix $\overline{M}$ obtained by applying Lemma 2.1 satisfies

$$\sigma(M) = \sigma(\overline{M}) + 3 + \sigma(W) - \sigma_0(X).$$

Since $[W \ X]$ has at least one row equal to 1 1 1, it follows by induction that $\sigma(M) \geq 4(n - 1)$. Again the case of equality can be reduced to that of Case 1.

Case 7: There exist i and j such that $m_{ij} = 1$ and $r_i = s_j = 3$.

Without loss of generality we assume that $i = j = 1$ and that

$$M = \left[\begin{array}{c|cc|c} 1 & 1 & 1 & O \\ \hline W & X & & Y \\ \hline O & O & & Z \end{array} \right],$$

where $[W \ X]$ has no zero rows. If $[W \ X]$ has a row equal to 1 1 1, then Case 6 applies. Otherwise, we may assume that $m_{21} = m_{31} = m_{22} = m_{33} = 1$ and that $m_{32} = m_{23} = 0$. By combinatorial row-orthogonality of M each

row of $[W \ X]$ other than rows 1 and 2 equals 0 1 1. We may apply a similar argument to M^T and hence may assume that

$$M = \begin{bmatrix} 1 & J_{1,2} & O & O \\ J_{2,1} & I_2 & J_{2,u} & O \\ O & J_{v,2} & Y_1 & Y_2 \\ O & O & Z_1 & Z_2 \end{bmatrix}.$$

where u and v are positive integers. By applying Lemma 2.1 to M we obtain the fully indecomposable, combinatorially orthogonal matrix

$$\overline{M} = \begin{bmatrix} J_2 & J_{2,u} & O \\ J_{v,2} & Y_1 & Y_2 \\ O & Z_1 & Z_2 \end{bmatrix}.$$

Considering inner products of rows of M with row 2 we see that Y_1 has no zero rows. Similarly, Y_1 has no zero columns. One easily verifies that the matrix $\tilde{M}$ obtained from $\overline{M}$ by changing the 1 in its first row and second column to 0 is also fully indecomposable and combinatorially orthogonal. We have $\sigma(M) = \sigma(\overline{M}) + 3 = \sigma(\tilde{M}) + 4$. Applying induction to $\tilde{M}$ we obtain $\sigma(M) \geq 4(n-1)$.

Suppose $\sigma(M) = 4(n-1)$. Then $\sigma(\tilde{M}) = 4(n-2)$ and by induction $\tilde{M}$ can be permuted to B_{n-1} or B_{n-1}^T , or to $J_4 - I_4$ if $n = 5$. It is easy to verify that the latter cannot occur. By the full indecomposability of M we have $u \geq 1$ and $v \geq 1$. If $u \geq 2$, then rows 1 and 2 of $\tilde{M}$ have inner product 3 or more and are not identical, a property that does not hold for any two rows of B_{n-1} or B_{n-1}^T . We conclude that $u = v = 1$. Considering the inner product of row 2 and 4 of M we have $m_{44} = 1$ and hence the inner product of rows 2 and 3 of $\tilde{M}$ equals 3. Therefore rows 2 and 3 of $\tilde{M}$ are identical. Since $n \geq 5$ M is partly decomposable, contrary to assumption. Hence equality cannot occur. (Note that if in the above analysis we had $n = 4$ then we obtain $M = J_4 - I_4$.)

Case 8: There exist two rows each with exactly three 1's whose inner product is 2 (or there exist two columns with these properties).

We assume that the rows in Case 8 are rows 1 and 2. Without loss of generality we assume that

$$\left[\begin{array}{c|c|c} J_2 & I_2 & O \\ \hline W & X & Y \\ \hline O & O & Z \end{array} \right],$$

where $[W \ X]$ has no zero rows. Assume that Case 6 does not apply. Then considering inner products of rows 1 and 2 with the other rows we see that each row of $[W \ X]$ is one of $1 \ 1 \ 0 \ 0$, $1 \ 0 \ 1 \ 1$, and $0 \ 1 \ 1 \ 1$. By considering inner products of columns 1 and 2 with columns 3 and 4, we conclude that both $1 \ 0 \ 1 \ 1$ and $0 \ 1 \ 1 \ 1$ occur as a row of $[W \ X]$. The matrix $\overline{M}$ of order $n - 2$ obtained by applying Lemma 2.1 satisfies

$$\sigma(M) \geq \sigma(\overline{M}) + 8,$$

and hence by induction $\sigma(M) \geq 4(n - 1)$. The case of equality once again can be reduced to that of Case 1.

We now assume that none of Cases 1 to 8 applies. Then every row with two or three 1's has an inner product of 0 with every other row with two or three 1's. A similar statement holds for the columns of M . We now permute the rows and columns so that all rows with exactly two 1's and the columns containing their 1's come first. Then all the rows with three 1's and the columns containing their 1's come next. The columns containing exactly two 1's and the rows containing their 1's can be taken to be last (since Cases 1 and 7 do not occur) preceded by the columns containing exactly three 1's and the rows containing their 1's. Without loss of generality we may assume that the number of rows of M with exactly three 1's is at least as large as the number of columns with exactly three 1's. We may further permute rows and columns to bring M to the form

$$\left[\begin{array}{c|c|c|c|c|c|c} C & O & O & O & O & O & O \\ \hline O & F & O & O & O & O & O \\ \hline O & O & G & O & O & O & O \\ \hline & & & & & O & O \\ \hline & & & & & O & O \\ \hline & & & & & H & O \\ \hline S & O & T & O & U & O & D \end{array} \right],$$

where each row of C and each column of D has exactly two 1's, each row of F and G and each column of H has exactly three 1's, each column of U has at least one 1, and no row of G has a inner product zero with all the rows of T . Since Case 2 does not apply we have $S = O$. Since Case 4 does not apply each column of M containing a column of U has at least five 1's.

Consider a row α of F or G and the three columns of M containing its 1's. Each of these columns contains at least four 1's. Since Case 6 does not apply, the inner products of the row of M containing α with the other rows of M are 0 or 2. This implies that the three specified columns of M have a total of at least thirteen 1's. If α is a row of G then since Cases 3 and 4 do not apply, the three columns of M have at least fifteen 1's.

Let the number of columns of C, F, G, U, H and D be c, f, g, u, h and d , respectively. Note that the number of rows of M with exactly three 1's is $(f + g)/3$ and hence $h \leq (f + g)/3$. Each of the last d columns of M has a nonzero inner product with at least three other columns of M . Moreover, since Case 5 does not apply, a column of M cannot have a nonzero inner product with two of the last d columns of M . This implies that $g + u \geq 3d$.

Computing the number of 1's in each of the seven groups of columns of M we obtain

$$\begin{aligned}
\sigma(M) &\geq 4c + \frac{13f}{3} + \frac{15g}{3} \\
&\quad + 4(n - c - f - g - u - h - d) + 5u + 3h + 2d \\
&= 4n + g + u + \frac{f}{3} - h - 2d \\
&= 4n + \frac{f + g}{3} - h + \frac{2}{3}(g + u) - 2d + \frac{u}{3} \\
&\geq 4n + \frac{u}{3}.
\end{aligned}$$

Hence if none of Cases 1 to 8 apply the number of 1's of M is at least $4n$, and the proof is complete. $\square$

3 Combinatorially row-orthogonal matrices

A real matrix whose rows are mutually orthogonal, unit vectors is an orthogonal matrix and hence has mutually orthogonal columns. But there exist combinatorially row-orthogonal (0,1)-matrices with the same number of 1's in each row which are not combinatorially column-orthogonal. For example,

the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

is a combinatorially row-orthogonal but not column-orthogonal matrix.

Let n be a positive even integer and let

$$R_n = \begin{bmatrix} J_2 & O & O & \cdots & O & X \\ X & J_2 & O & \cdots & O & O \\ O & X & J_2 & \cdots & O & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \cdots & J_2 & O \\ O & O & O & \cdots & X & J_2 \end{bmatrix},$$

be a matrix of order n where

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Now let $n > 1$ be an odd integer. We define matrices S_n and T_n by

$$S_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & \vdots \\ & & R_{n-1} & & & 0 \\ & & & & & 1 \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 \end{array} \right],$$

and

$$T_n = \left[\begin{array}{ccccc|c} & & & & & 0 \\ & & & & & \vdots \\ & & R_{n-1} & & & 0 \\ & & & & & 1 \\ & & & & & 0 \\ \hline 0 & \cdots & 0 & 1 & 1 & 1 \end{array} \right].$$

Then R_n , S_n and T_n are fully-indecomposable, combinatorially row-orthogonal matrices. If $n \geq 4$ is an even integer, then $\sigma(R_n) = 3n$. If $n \geq 5$ is an odd integer then $\sigma(S_n) = \sigma(T_n) = 3n + 1$.

Theorem 3.1 *Let $M = [m_{ij}]$ be a fully-indecomposable, combinatorially row-orthogonal $(0,1)$ -matrix of order $n \geq 5$. Then*

$$\sigma(M) \geq \begin{cases} 3n & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases} \quad (6)$$

Proof. We prove by induction on n that (6) holds and that if n is even and $\sigma(M) = 3n$, then each column of M has exactly three 1's. This can be verified for $n = 5$ and $n = 6$. If $\sigma(M) \geq 3n + 1$, there is nothing to prove. We now assume that $n \geq 7$ and that $\sigma(M) \leq 3n$.

Let r_i and s_i denote the number of 1's in row i and column i , respectively. First assume that there do not exist integers i and j with $m_{ij} = 1$, $r_i = 2$ and $s_j = 3$. If each column of M has at least three 1's, then since $\sigma(M) \leq 3n$ each row and column of M has exactly three 1's. Since a fully indecomposable, combinatorially row-orthogonal $(0,1)$ -matrix with exactly three 1's in each row and column has order less than 5, some column of M has exactly two 1's. After row and column permutations,

$$M = \begin{bmatrix} 1 & u \\ 1 & v \\ O & Z \end{bmatrix}.$$

Let

$$\overline{M} = \begin{bmatrix} u \oplus v \\ Z \end{bmatrix}$$

where $u \oplus v$ is the $(0,1)$ -vector of the same length as u and v with 1's in those positions in which u or v has a 1. Then $\overline{M}$ is a fully indecomposable, combinatorially row-orthogonal matrix of order $n-1$. The combinatorial row-orthogonality of rows 1 and 2 of M implies that $\sigma(M) \geq \sigma(\overline{M}) + 3$. If n is even, then by induction $\sigma(M) \geq 3n + 1$. Suppose n is odd. Since $\sigma(M) \leq 3n$, it follows from the inductive assumption that $\sigma(M) = 3n$, $\sigma(\overline{M}) = 3(n-1)$, $uv^T = 1$ and each column of $\overline{M}$ has exactly three 1's. This implies that M has a column with exactly two 1's, a column with exactly four 1's and the remaining columns each have exactly three 1's. Since no 1 lies in a row with two 1's and a column with three 1's, each row of M has exactly three 1's. The full indecomposability and combinatorial row-orthogonality of M now lead to the contradiction that $n = 4$.

Now assume that there exist integers i and j with $m_{ij} = 1$, $r_i = 2$ and $s_j = 3$. Without loss of generality we assume that $i = j = 1$. The combinatorial row-orthogonality of M implies that M can be permuted to the form

$$M = \begin{bmatrix} 1 & 1 & 0 \cdots 0 \\ 1 & 1 & w \\ 1 & 1 & x \\ O & & Z \end{bmatrix}.$$

Then

$$\overline{M} = \begin{bmatrix} w \oplus x \\ Z \end{bmatrix},$$

is a fully indecomposable, combinatorially row-orthogonal matrix of order $n - 2$ with $\sigma(M) \geq \sigma(\overline{M}) + 6$. Hence, by induction (6) holds. Suppose n is even and $\sigma(M) = 3n$. Then $\sigma(\overline{M}) = 3n - 6$ and $wx^T = 0$. By induction each column of $\overline{M}$ has exactly three 1's and hence each column of M has exactly three 1's. $\square$

With a more careful inductive argument it is possible to characterize those matrices M for which equality holds in (6).

Theorem 3.2 *Let M be a fully indecomposable, combinatorially row-orthogonal $(0,1)$ -matrix of order $n \geq 6$. If n is even, then $\sigma(M) = 3n$ if and only if M can be permuted to R_n . If n is odd, then $\sigma(M) = 3n + 1$ if and only if M can be permuted to S_n or T_n .*

The inequality (6) and Theorem 3.2 can be proved together by showing inductively that if M is a fully indecomposable, combinatorially row-orthogonal $(0,1)$ -matrix of order $n \geq 6$ then

- (i) (6) holds with equality if and only if M is as given in Theorem 3.2;
- (ii) if n is even and $\sigma(M) = 3n + 1$ then each row of M has at most four 1's.

This involves some tedious checking whose details we omit.

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