EVOLUTION OF NONPARAMETRIC SURFACES WITH SPEED DEPENDING ON CURVATURE, III. SOME REMARKS ON MEAN CURVATURE AND ANISOTROPIC FLOWS

By

Vladimir I. Oliker

and

Nina N. Uraltseva

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Evolution of nonparametric surfaces with speed depending on curvature, III. Some remarks on mean curvature and anisotropic flows

Vladimir I. Oliker[1] and Nina N. Uraltseva[2]

Dedicated to James Serrin on the occasion of his 65th birthday

1. Introduction

This paper is a sequel to our paper [OU] where we investigated questions concerning solvability and asymptotic behavior of solutions to the mean curvature evolution problem

$$u_t = \sqrt{1+|Du|^2}H(u) \text{ in } \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,t) = 0 \text{ on } \partial\Omega \times [0,\infty), \tag{1.2}$$

$$u(x,0) = u_0(x) \text{ in } \overline{\Omega},\ u_0 \in C_0^\infty(\overline{\Omega}), \tag{1.3}$$

where Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with C^∞ boundary $\partial\Omega$, H is the mean curvature operator

$$H(u) := div \frac{Du}{\sqrt{1+|Du|^2}},$$

$Du = \text{grad}\, u$, $|Du|^2 = \langle Du, Du\rangle$, and $u_t = \partial u/\partial t$.

In the first part of this paper, we investigate the same equation (1.1) in case of nonhomogeneous Dirichlet boundary condition and in the second part we study the problem (1.1)-(1.3) with $H(u)$ replaced by its "anisotropic" version (see the equation 1.22) below).

In the nonhomogeneous case the mean curvature evoluton problem is formulated as follows.

Suppose there exists a function $\phi \in C^\infty(\overline{\Omega})$ such that

$$H(\phi) = 0 \text{ in } \Omega, \tag{1.4}$$

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that is, the graph of the function ϕ is a minimal surface. Here the graph of ϕ is considered in $\mathbf{R}^{n+1}$ in which a Cartesian coordinate system $x_1, \ldots, x_n, x_{n+1}$ is chosen so that Ω lies in the (hyper)plane $x_{n+1} = 0$ and the graph of ϕ is given by the set $(x, \phi(x))$, $x \in \overline{\Omega}$.

We are interested in solvability of the problem

$$u_t = \sqrt{1 + |Du|^2}\, H(u) \text{ in } \Omega \times [0, \infty), \tag{1.5}$$

$$u(x,t) = \phi(x) \text{ in } \partial\Omega \times [0, \infty), \tag{1.6}$$

$$u(x,0) = \phi(x) + u_0(x) \in \overline{\Omega},\ u_0 \in C_0^\infty(\overline{\Omega}). \tag{1.7}$$

The equation (1.5) describes a motion of the surface $(x, u(x,t))$ evolving with normal speed equal to the mean curvature H(u). The boundary of the surface remains fixed. At the initial moment $u(x,0)$ is a smooth bounded perturbation of the minimal surface $x_{n+1} = \phi(x)$, $x \in \Omega$.

We show in this paper that (1.5)-(1.7) admits a "solution" u for all time; however, this solution may develop singularities on $\partial\Omega$ in finite time and, in general, will not satisfy (1.6). After a sufficiently long time the singularities will disappear and the solution will be smooth in $\overline{\Omega}$. Furthermore, the solution $u(x,t) \to \phi(x)$ as $t \to \infty$.

It is known that if the domain Ω satisfies the condition of H. Jenkins and J. Serrin (see [JS] and [S]) then a smooth in $\overline{\Omega}$ solution of (1.5)-(1.7) exists for all time. This was shown by G. Huisken [H]. The novelty in our case is that we do not assume that Ω satisfies the Jenkins-Serrin condition.

In order to formulate the results we need the notion of a generalized solution; it is similar to the one in [LT] and [OU].

A function $u(x,t)$ is called a **generalized solution** of (1.5)-(1.7) if it satisfies the following conditions:

$$u \in C^\infty(\Omega \times [o, \infty));\ u \in L_\infty([0,\infty);\ W^{1,1}(\Omega)); \tag{1.8}$$

$$u_t = L_\infty(\Omega \times [0, \infty)); \tag{1.9}$$

$$Lu := u_t - \sqrt{1 + |Du|^2}\, H(u)$$

$$\equiv u_t - \Delta u + \frac{u_i u_j}{1 + |Du|^2} u_{ij} = 0 \text{ in } \Omega \times (0, \infty), \tag{1.10}$$

where $u_i := \partial u / \partial x_i$, $u_{ij} := \partial^2 u / \partial x_i \partial x_j$, Δ is the Laplace operator, and the convention about summation over repeated indices is assumed here and throughout the paper;

$$-\frac{\langle Du,\nu\rangle}{\sqrt{1+|Du|^2}} \in \; sign(u-\phi) \;\text{ a.e. on }\; \partial\Omega\times(0,\infty), \tag{1.11}$$

where ν is the exterior unit normal field on $\partial\Omega$;

$$u(x,0) \;=\; \phi(x). \tag{1.12}$$

Now we can formulate the first result.

Theorem 1. *Problem (1.5)-(1.7) admits a generalized solution.*

In order to construct this generalized solution, we consider a family of regularized problems:

$$L^\varepsilon u^\varepsilon \;:=\; Lu^\varepsilon - \varepsilon\sqrt{1+|Du^\varepsilon|^2}\,\Delta u^\varepsilon \;=\; 0 \;\text{ in }\; \Omega\times(0,\infty), \tag{1.13}$$
$$u^\varepsilon(x,t) \;=\; \phi(x) \;\text{ on }\; \partial\Omega\times[0,\infty), \tag{1.14}$$

$$u^\varepsilon(x,0) = \phi(x) + u_0(x) \;\text{ in }\; \overline{\Omega}. \tag{1.15}$$

The problems (1.13)-(1.15) are uniformly parabolic for each $\varepsilon > 0$ and each of them admits a unique solution u^ε from $C^\infty(\overline{\Omega}\times[0,\infty))$. The generalized solution of (1.5)-(1.7) that we construct is obtained as a limit of a subsequence of $\{u^\varepsilon\}$ converging in $C^k(\overline{\Omega}'\times[0,T])$ for any $k \geq 0$, any $T > 0$, and any $\Omega' \subset\subset \Omega$ to some function u satisfying (1.8)-(1.12).

The next result shows that $u(x,t)$ is smooth in $\overline{\Omega}$, possibly, after a sufficiently long time.

Theorem 2. *Let* $u(x,t) \;=\; \lim_{\varepsilon\to 0} u^\varepsilon(x,t)$ *be the generalized solution of (1.5)-(1.7). There exists* $\bar{t} \geq 0$ *such that*

$$u \in C^\infty(\overline{\Omega}\times[\bar{t},\infty)), \tag{1.16}$$

$$u(x,t) = \phi(x),\;\; (x,t) \in \partial\Omega\times[\bar{t},\infty). \tag{1.17}$$

Furthermore,

$$|u(x,t)-\phi(x)| \leq Ce^{-\mu t},\;\; x\in\overline{\Omega},\;\; t\geq 0, \tag{1.18}$$

where C and μ are positive constants depending on initial data and domain Ω.

The proofs of Theorems 1 and 2 follow the same plans as the proofs of Theorem A and the first part of Theorem D in [OU]. However, some of the barriers needed here for the C^0 estimates are different from those in [OU] and new efforts are needed to obtain these estimates. These proofs are given in sections 2 and 3.

Remark. Everywhere in the paper the capitals C, C_1, ... etc. denote positive constants depending, at most, on the domain Ω, initial data, and function ϕ.

In the second part of this paper, we show how the techniques in [OU] can be extended to allow investigation of the "anisotropic" version of the evolution flow under mean curvature. For closed surfaces, this evolution was considered by Y.-G. Chen, Y. Giga, and S. Goto in [GG] and [CGG]. In this connection see also the paper by S. Angenent and M. Curtin [AC]. We study here the evolution of graphs and establish the following result.

Let $F(p)$, $p \in \mathbf{R}^n$, be a positive C^∞ function satisfying for all $p, \xi \in \mathbf{R}^n$ the following structure conditions:

$$|F_i(p)| \leq \alpha_0, \tag{1.19}$$

$$F_i(p)p_i \geq \alpha_1 \frac{p^2}{\sqrt{1+p^2}} - \alpha_2, \tag{1.20}$$

$$\frac{\alpha_3 |\xi_p|^2}{(1+p^2)^{3/2}} + \frac{\alpha_3 |\xi'|^2}{\sqrt{1+p^2}} \leq F_{ij}(p)\xi_i\xi_j \leq \frac{\alpha_4 |\xi|^2}{\sqrt{1+p^2}}, \tag{1.21}$$

where

$$F_i := \frac{\partial F}{\partial p_i}, \; F_{ij} := \frac{\partial^2 F}{\partial p_i \partial p_j},$$

$$\xi_p = \left\langle \xi, \frac{p}{|p|} \right\rangle \frac{p}{|p|} \text{ for } p \neq 0, \; \xi' = \xi - \xi_p,$$

α_0, α_1, α_3, α_4 are positive constants and $\alpha_2 \geq 0$.

These structure conditions are derived from postulated properties of a function defined in $\mathbf{R}^{n+1}$ which represents physically the interfacial energy; for more details see [AC], [GG], and [CGG]. The special case where $F(p) = \sqrt{1+p^2}$ leads to the mean curvature evolution equation as it can be seen from the equation (1.22) below.

Consider the initial boundary value problem

$$u_t = \sqrt{1+|Du|^2} \frac{d}{dx_i}\left(\frac{\partial F(p)}{\partial p_i}\right) \text{ in } \Omega \times (0,\infty), \tag{1.22}$$

$$u(x,t) = 0 \text{ on } \partial\Omega \times [0,\infty), \tag{1.23}$$

$$u(x,0) = u_0(x) \text{ in } \Omega, \; u_0 \in C_0^\infty(\Omega), \tag{1.24}$$

where d/dx_i denotes the total derivative, and Ω, as before, is a bounded domain in $\mathbf{R}^n$ with smooth boundary.

Again, we define the generalized solution of (1.22)-(1.24) as a function $u(x,t)$ with properties:

$$u \in C^\infty(\Omega \times [0,\infty)); \; u \in L_\infty([0,\infty); W^{1,1}(\Omega)); \tag{1.25}$$

$$u_t \in L_\infty(\Omega \times [0,\infty)); \tag{1.26}$$

$$Lu \; := u_t - \sqrt{1+|Du|^2}\frac{d}{dx_i}\left(\frac{\partial F(p)}{\partial p_i}\right) = 0 \text{ in } \Omega \times (0,\infty); \tag{1.27}$$

$$-\frac{\langle Du, \nu \rangle}{\sqrt{1+|Du|^2}} \in sign(u) \text{ a.e. on } \partial\Omega \times (0,\infty); \tag{1.28}$$

$$u(x,0) = u_0(x) \text{ in } \Omega. \tag{1.29}$$

In order to construct a generalized solution, we consider the regularized problem

$$L^\varepsilon u^\varepsilon \; := Lu^\varepsilon - \varepsilon\sqrt{1+|Du^\varepsilon|^2}\,\Delta u^\varepsilon = 0 \text{ in } \Omega \times [0,\infty), \tag{1.30}$$

$$u^\varepsilon = 0 \text{ on } \partial\Omega \times [0,\infty), \tag{1.31}$$

$$u^\varepsilon = u_0 \text{ on } \Omega \times \{0\}. \tag{1.32}$$

Theorem 3. *The problem (1.22)-(1.24) admits a generalized solution which can be constructed as a limit of solutions of (1.30)-(1.32). For this generalized solution, there exists some $\bar{t} \geq 0$ such that*

$$u \in C^\infty(\overline{\Omega} \times [\bar{t},\infty)) \text{ and } u = 0 \text{ on } \partial\Omega \times [\bar{t},\infty).$$

The proof of Theorem 3 is given in section 4. It follows the same basic steps as the proofs of Theorems 1 and 2. In comparison to [OU] essentially new arguments are needed only for construction of C^0-barriers. We present these estimates in detail. The local C^1-estimates in x and t are obtained by a slight modification of the arguments in [OU].

2. C^0-estimates for solutions of (1.13)-(1.15)

Lemma 2.1. *There exist positive constants C and μ depending on Ω, u_0, and ϕ such that for all $(x,t) \in \overline{\Omega} \times [0,\infty)$*

$$|u^\varepsilon(x,t) - \phi(x)| \le C e^{-\mu t}. \tag{2.1}$$

Proof. The proof is obtained by constructing appropriate barriers and then applying the usual maximum principle.
It will be convenient to use the following notation:

$$\tilde{u}^\varepsilon := u^\varepsilon - \phi,$$

$$a_{ij}(D\tilde{u}^\varepsilon + D\phi) := \left[1 + \varepsilon\sqrt{1 + |D\tilde{u}^\varepsilon + D\phi|^2}\right]\delta_{ij} - \frac{\partial_i(\tilde{u}^\varepsilon + \phi)\partial_j(\tilde{u}^\varepsilon + \phi)}{1 + |D\tilde{u}^\varepsilon + D\phi|^2}, \ \partial_i := \partial/\partial x_i,$$

$$b_{ij}(D\tilde{u}^\varepsilon) := a_{ij}(D\tilde{u}^\varepsilon + D\phi).$$

By (1.13) we have in $\overline{\Omega} \times [0,\infty)$

$$L^\varepsilon \tilde{u}^\varepsilon = \tilde{u}_t - b_{ij}(D\tilde{u}^\varepsilon)\tilde{u}^\varepsilon_{ij} - b_{ij}(D\tilde{u}^\varepsilon)\phi_{ij} = 0, \tag{2.2}$$

where $\tilde{u}^\varepsilon_{ij} = \partial^2 \tilde{u}^\varepsilon/\partial x_i \partial x_j$ and $\phi_{ij} = \partial^2\phi/\partial x_i \partial x_j$.

The equation (2.2) is invariant relative to parallel translations of the origin O of the coordinate system in the plane $x_{n+1} = 0$. Therefore, we may assume that $O \in \Omega$. Denote by B_R a ball of radius R centered at O. Assume that $R > 1$ and $\Omega \subset B_{R-1}$.

Consider the functions

$$v^\pm(x,t) = \pm\frac{m_0}{m} e^{-\mu t}(e^m - e^{\psi(x)}),$$

where $\psi(x) = -m(R^2 - 1/2\,|x|^2 - 1)$ and the constants m_0, m, and μ are positive and to be chosen so that

$$L^\varepsilon(v^+ + \phi) = v^+_t - b_{ij}(Dv^+)v^+_{ij} - b_{ij}(Dv^+)\phi_{ij} > 0 \ \text{ in } \ \overline{\Omega} \times [0,\infty) \tag{2.3}$$

$$L^\varepsilon(v^- + \phi) < 0 \ \text{ in } \ \overline{\Omega} \times [0,\infty), \tag{2.3$'$}$$

$$v^+(x,t) \ge 0 \ \text{ in } \ \partial\Omega \times [0,\infty), \tag{2.4}$$

$$v^+(x,0) \ge |u_0(x)| \ \text{ in } \ \overline{\Omega}. \tag{2.5}$$

Later we show that (2.3), (2.3)′, (2.4), and (2.5) imply (2.1).

The inequality (2.4) is satisfied trivially. It is also clear that there exists $\overline{m} > 0$ such that for all $m \geq \overline{m}$, $(1/m)(e^m - e^{\psi(x)}) \geq 1$. Setting

$$m_0 = \max_{\overline{\Omega}} |u_0(x)|,$$

we obtain (2.5) with any $m \geq \overline{m}$.

Consider now (2.3). We have

$$v_t^+ = -\mu v^+, \; v_i^+ = -m_0 e^{-\mu t} e^{\psi(x)} x_i, \; (v_i = \partial_i v)$$

$$v_{ij}^+ = -m_0 e^{-\mu t} e^{\psi(x)} (m x_i x_j + \delta_{ij}), \; (v_{ij} = \partial^2 v / \partial x_i \partial x_j).$$

Note next that

$$b_{ij}(Dv^+) v_{ij}^+ = -m_0 e^{-\mu t} e^{\psi(x)} [m b_{ij}(Dv^+) x_i x_j + b_{ii}(Dv^+)],$$

$$b_{ij}(Dv^+) x_i x_j \geq \frac{|x|^2}{1 + |Dv^+|^2 + |D\phi|^2}, \tag{2.6}$$

$$b_{ii}(Dv^+) \geq n - 1. \tag{2.7}$$

From the expression for v_i^+ we get

$$|Dv^+| \leq m_0 e^{-\mu t} e^{\psi(x)} |x|. \tag{2.8}$$

Let Φ be a constant such that

$$\|\phi\|_{C^2(\overline{\Omega})} \leq \Phi.$$

From (2.6) and (2.8) we obtain

$$b_{ij}(Dv^+) x_i x_j \geq \frac{|x|^2}{1 + m_0^2 R^2 + \Phi^2}. \tag{2.9}$$

We also have,

$$b_{ij}(Dv^{+})\phi_{ij} \;=\; a_{ij}(D\phi)\phi_{ij} + \int_0^1 \left(\frac{d}{d\tau} a_{ij}(\tau Dv^{+} + D\phi) \right) d\tau \, \phi_{ij} \;=\; \overline{a}_{ij,k} v_k^{+} \phi_{ij},$$

where

$$\overline{a}_{ij,k} = \int_0^1 \frac{\partial}{\partial p_k} a_{ij}(\tau Dv^{+} + D\phi) d\tau, \;\; p_k = v_k^{+}.$$

It is straight-forward to check that

$$\max_{i,j,k} \max_{\overline{\Omega}} |\overline{a}_{ij,k}| \;\le\; 1.$$

Since $a_{ij}(D\phi)\phi_{ij} \;=\; 0$, we obtain

$$|b_{ij}(Dv^{+})\phi_{ij}| \;\le\; \Phi |Dv^{+}| \;\le\; m_0 \Phi e^{-\mu t} e^{\psi(x)} |x|.$$

From this inequality and (2.7), (2.9) we get

$$L^{\varepsilon}(v^{+} + \phi) \;\ge\; \left\{ -\frac{\mu}{m} + e^{-m\left(R^2 - \frac{1}{2}|x|^2\right)} \left(\frac{m|x|^2}{1 + m_0^2 R^2 + \Phi^2} - \Phi|x| + n - 1 \right) \right\} m_0 e^{-\mu t} e^{m}.$$

Let

$$m = \max\left\{ \frac{\phi^2(1 + m_0^2 R^2 + \phi^2)}{4(n - 3/2)}, \overline{m} \right\}, \;\; \mu = \frac{1}{4} m e^{-mR^2}.$$

Then $L^{\varepsilon}(v^{+} + \phi) > 0$ in $\overline{\Omega} \times [0, \infty)$ and (2.3) is satisfied.

Consider now (2.3)′. In this case one shows in the same way as above that

$$L^{\varepsilon}(v^{-} + \phi) \;\le\; \left\{ \frac{\mu}{m} - e^{-m\left(R^2 - \frac{1}{2}|x|^2\right)} \left(\frac{m|x|^2}{1 + m_0^2 R^2 + \phi^2} - \phi|x| + n - 1 \right) \right\} m_0 e^{-\mu t} e^{m}$$

and, consequently, $L^{\varepsilon}(v^{-} + \phi) < 0$ in $\overline{\Omega} \times [0, \infty)$.

It is also clear from (2.4) and (2.5) that

$$v^{+}(x,t) + \phi(x) \;\ge\; \phi(x) \;=\; u^{\varepsilon}(x,t) \;\; \text{on} \;\; \partial\Omega \times [0, \infty),$$

$$\tag{2.10}$$

$$v^{+}(x,0) + \phi(x) \;\ge\; u_0(x) + \phi(x) \;=\; u^{\varepsilon}(x,0) \;\; \text{on} \;\; \overline{\Omega},$$

$$v^-(x,t)+\phi(x) \le \phi(x) = u^\varepsilon(x,t) \text{ on } \partial\Omega\times[0,\infty),$$

$$v^-(x,0)+\phi(x) \le u_0(x)+\phi(x) = u^\varepsilon(x,0) \text{ on } \overline{\Omega}. \tag{2.10$'$}$$

By the maximum principle, it follows from (2.3), (2.10), and (2.3)′, (2.10)′ that in $\overline{\Omega}\times[0,\infty)$

$$|u^\varepsilon(x,t)-\phi(x)| \le v^+(x,t).$$

Letting $C = m_0/m \cdot e^m$, we obtain (2.1). The lemma is proved.

Lemma 2.2. *Let $d(x) = dist(x,\partial\Omega)$, $x\in\overline{\Omega}$. There exist positive constants C_1 and T depending on Ω, the constant C in (2.1), and $\Phi = \|\phi\|_{C^2(\overline{\Omega})}$ such that*

$$|u^\varepsilon(x,t)-\phi(x)| \le C_1 d(x)e^{-\mu t},\ (x,t)\in\overline{\Omega}\times[T,\infty). \tag{2.11}$$

Proof. Let $\Omega_\delta = \{x\in\Omega \mid d(x)<\delta\}$ where δ is so small that $d\in C^\infty(\overline{\Omega}_\delta)$. Put $f(d(x)) = (1/k)(1-e^{-kd(x)})$, where k is a positive constant to be chosen later, and $g(t) = [(T-t)_+]^2$, where $(\ )_+$ denotes the nonnegative part. Consider the functions

$$w(x,t) = v(x,t)+\phi(x),$$

$$v(x,t) = C_2[f(d(x))+g(t)]e^{-\mu t}.$$

We want to show that by choosing appropriately T, C_2, k, and δ we can arrange so that w is an upper barrier for $u^\varepsilon(x,t)$ in $\overline{\Omega}_\delta\times[T-1,\infty)$. For that we need to verify the following inequalities for some $T\ge 1$:

$$v(x,t) \ge 0 \text{ in } \partial\Omega\times[T-1,\infty), \tag{2.12}$$

$$w(x,t) \ge u^\varepsilon(x,t) \text{ for } d(x)=\delta \text{ and } t\ge T-1, \tag{2.13}$$

$$w(x,T-1) \ge u^\varepsilon(x,T-1) \text{ in } \overline{\Omega}_\delta \tag{2.14}$$

$$L^\varepsilon w \ge 0 \text{ in } \Omega_\delta\times(T-1,\infty). \tag{2.15}$$

The inequality (2.12) is obviously true in $\partial\Omega\times[0,\infty)$. Further, if

$$C_2\delta e^{k\delta} \geq C, \tag{2.16}$$

where C is as in (2.1), then because of (2.1) and since $f(\delta) \geq \delta e^{-k\delta}$ we have

$$v(x,t) \geq Ce^{-\mu t} \geq u^\varepsilon(x,t) - \phi(x) \text{ for } d(x) = \delta \text{ and } t \geq 0.$$

This implies (2.13).

Next we note that by (2.1)

$$v(x,T-1) \geq C_2 e^{-\mu(T-1)} \geq Ce^{-\mu(T-1)} \geq u^\varepsilon(x,T-1) - \phi(x) \text{ in } \overline{\Omega}_\delta,$$

provided

$$C_2 \geq C, \tag{2.17}$$

and this implies (2.14).

Thus, if (2.16), (2.17), and (2.15) can be satisfied, then w is indeed an upper barrier. Then by the maximum principle

$$u^\varepsilon(x,t) \leq w(x,t) \text{ in } \overline{\Omega}_\delta \times [T-1,\infty). \tag{2.18}$$

Since $g(t) = 0$ for $t \geq T$, and because one can choose a constant C_1 so that

$$C_2 f(d(x)) \leq C_1 d(x) \text{ in } \overline{\Omega},$$

we obtain

$$u^\varepsilon(x,t) \leq C_1 d(x) e^{-\mu t} + \phi(x) \text{ in } \overline{\Omega} \times [T,\infty).$$

The bound from below is obtained by constructing similarly a lower barrier. Thus, we obtain (2.11).

Consider now (2.15). We have

$$v_i = C_2 e^{-\mu t} f' d_i, \; v_{ij} = C_2 e^{-\mu t}[f'' d_i d_j + f' d_{ij}],$$

where $d_i = \partial d/\partial x_i$, $d_{ij} = \partial^2 d/\partial x_i \partial x_j$ and similarly v_i, v_{ij}. Using the same notation a_{ij}, b_{ij} as in the proof of Lemma 2.1, we get

$$b_{ij}(Dv)v_{ij} = f''\left[q - \frac{\langle D(v+\phi), Dd\rangle^2}{1+|D(v+\phi)|^2}\right] + f'\left[q\Delta d - \frac{2v_i\phi_j + \phi_i\phi_j}{1+|D(v+\phi)|^2}d_{ij}\right], \tag{2.19}$$

where $q = 1 + \varepsilon\sqrt{1 + |D(v + \phi)|^2}$. Here, we used the facts that $|Dd| = 1$ and $d_i d_j d_{ij} = 0$.

In the same way as in the proof of Lemma 2.1 we get

$$|b_{ij}(Dv)\phi_{ij}| \le \Phi|Dv| \le \Phi C_2 f' e^{-\mu t}. \tag{2.20}$$

Now using (2.19), (2.20) and noting that $f'' < 0$, we obtain

$$L^\varepsilon w \ge C_2 e^{-\mu t}\{-\mu(f + g) + g_t + [1 + |D(v + \phi)|^2]^{-1} \times$$
$$[-f'' - f'[(1 + |D(v + \phi)|^2)(q|\Delta d| + \Phi) + |2v_i\phi_j + \phi_i\phi_j)d_{ij}|]\}$$

Recall that $d(x) \in C^2(\overline{\Omega}_\delta)$ for any $\delta < \delta_0 := \inf_{\partial\Omega}(1/\kappa_v(x))$ where $\kappa_v(x)$ is the maximum of the absolute value of the normal curvature of $\partial\Omega$ at $x \in \partial\Omega$ (see [S], p. 421). Put $\rho := \|d\|_{C^2(\overline{\Omega}_\delta)}$.

On the other hand, $|Dv| \le C_2 e^{-\mu t}$. Thus, if

$$C_2 e^{-\mu(T-1)} \le 1 \tag{2.21}$$

then

$$L^\varepsilon w \ge C_2 e^{-\mu t}\{-\mu(1 + \delta) - 2 + (2 + \Phi^2)^{-1}[k - M]d^{-kd}\},$$

where

$$M = (2 + \Phi^2)[(3 + \Phi^2)\rho + \Phi] + \rho\Phi(2 + \Phi)$$

and it is assumed that $\varepsilon \le 1$.

Set

$$k = M + (2 + \Phi^2)[2 + \mu(1 + \delta_0)]e, \ \delta = \min(\delta_0/2, 1/k, 1), \ C_2 = C\delta^{-1}e^{k\delta},$$

and choose T so that (2.21) is satisfied.

Then (2.16), (2.17) are satisfied and

$$L^\varepsilon w \ge C_2\mu\frac{\delta_0}{2}e^{-\mu t} > 0 \text{ in } \Omega_\delta \times [T - 1, \infty).$$

The lemma is proved.

3. Proofs of Theorems 1 and 2

Before we can proceed with the proofs of Theorems 1 and 2, we need to record some preliminary estimates which allow applications of the estimates in x and t from [OU].

Proposition 3.1. *Let*

$$M_0 = \max\{\max_\Omega |\phi + u_0|, \max_{\partial\Omega}\phi\}$$

and $u^\varepsilon(x,t)$ a solution of (1.13)-(1.15). Then

$$\sup_{\Omega\times[0,\infty)} |u^\varepsilon(x,t)| \leq M_0, \tag{3.1}$$

$$\sup_{\Omega\times[0,\infty)} |u_t^\varepsilon(x,t)|\ (x,t) \leq M, \tag{3.2}$$

where M is a constant depending on the C^2-norms of u_0 and ϕ, and

$$\int_\Omega (|Du^\varepsilon| + \varepsilon|Du^\varepsilon|^2)dx \leq C_3 \text{ for all } t \geq 0. \tag{3.3}$$

Proof. The inequality (3.1) is a consequence of the maximum principle applied to (1.13) - (1.15). The inequality (3.2) also follows from the maximum principle applied to the differentiated in t equation (1.13); one needs to observe here that $u_t^\varepsilon = 0$ on $\partial\Omega \times [0,\infty)$ and

$$u_t^\varepsilon = \sqrt{1+|D(\phi+u_0)|^2}[H(\phi+u_0)+\varepsilon\Delta(u_0+\phi)] \text{ in } \Omega\times\{0\}.$$

In order to verify (3.3), note first that by (1.13)

$$\int_\Omega \left(\frac{u_t^\varepsilon \eta + u_i^\varepsilon \eta_i}{\sqrt{1+|Du^\varepsilon|^2}} + \varepsilon u_i^\varepsilon \eta_i \right) dx = 0 \tag{3.4}$$

for any $\eta \in H_0^1(\Omega)$ and $t \geq 0$, where $\eta_i = \partial\eta/\partial x_i$. Take $\eta = u^\varepsilon - \phi$ and substitute in (3.4). Then, taking into account (3.1) and (3.2), and after some straight forward manipulations, we obtain (3.3). The proposition is proved.

Once the inequalities (3.1)-(3.3) are established we can apply Theorem C in [OU] and conclude that for any subdomain $\Omega' \subset\subset \Omega$ and any $T > 0$ there exists a constant $\overline{C} = \overline{C}(dist(\Omega', \partial\Omega), T)$ such that

$$\sup_{\Omega'}|Du^{\varepsilon}(x,t)| \leq C_4 \text{ for } t \in [0,T]. \tag{3.5}$$

This estimate, (3.2) and standard results on uniformly parabolic equations imply that there is a subsequence of $\{u^{\varepsilon}\}$ converging in $C^k(\overline{\Omega}' \times [0,T])$, for any $k \geq 0$, to some function $u \in C^{\infty}(\Omega \times [0,\infty))$. The function u satisfies (1.5), (1.7), and it follows from (3.1) - (3.3) that $u_t \in L_{\infty}(\Omega \times [0,\infty))$ and $u \in W^{1,1}(\Omega)$ for all $t \geq 0$. It is shown, as in [OU], section 4.13 (cf. [LT], section 3.2), that u satisfies (1.11).

Proof of Theorem 2. It follows from (1.14) and (2.11) by standard arguments that

$$|D(u^{\varepsilon}(x,t) - \phi(x))| \leq C_5 e^{-\mu t} \text{ in } \partial\Omega \times [T,\infty).$$

By Theorem C′ in [OU], this inequality and (3.1)-(3.3) imply that there exists some $\bar{t} \geq T$ such that

$$|Du^{\varepsilon}(x,t)| \leq C_6 \text{ on } \overline{\Omega} \times [\bar{t},\infty), \tag{3.6}$$

where $C_6 = C_6(\bar{t}, C_5)$. Then the solutions $\{u^{\varepsilon}\}$ and all their derivatives admit uniform bounds in $\overline{\Omega} \times [\bar{t},T']$ independent on ε for any $T' > \bar{t}$ (see [LSU], ch. IV). Therefore, the generalized solution $u(x,t)$ of (1.5)-(1.7) is in $C^1(\overline{\Omega} \times [\bar{t},T'])$ and by general results on uniformly parabolic PDE's, it is in $C^{\infty}(\overline{\Omega} \times [\bar{t},\infty))$. Consequently, (1.16) and (1.17) are satisfied. The inequality (1.18) follows now from (2.1) after passing to the limit in ε for $t > \bar{t}$ and, if necessary, replacing C by a larger constant. Theorem 2 is proved.

4. Proof of Theorem 3

The proof of this theorem follows the same steps as the proofs of Theorems 1 and 2.

For each $\varepsilon > 0$ the equation (1.30) is uniformly parabolic and, therefore, the problem (1.30)-(1.32) admits a solution u^{ε} of class $C^{\infty}(\overline{\Omega} \times [0,\infty))$. It follows from the maximum principle that u^{ε} is a unique solution.

We begin by establishing C^0-estimates for all time.

(a) We may assume that the origin of the coordinate system is inside Ω. Let B_R be a ball of radius R centered at O and containing Ω strictly inside.

Let

$$v(x,t) = m\left(2R^2 - \frac{1}{2}x^2\right)e^{-\mu t},$$

where $m = \sup_{\Omega}|u_0(x)|R^{-2}$, and μ a positive constant to be determined. Using (1.21), we obtain

$$\begin{aligned} L^{\varepsilon}v &= -\mu v + \sqrt{1+m^2|x|^2e^{-2\mu t}}\left(\sum_{i=1}^{n} F_{ii}\right)me^{-\mu t} + \varepsilon n\sqrt{1+m^2|x|^2e^{-2\mu t}}\, me^{-\mu t} \\ &\geq \left[-2\mu R^2 + \alpha_3(1+m^2|x|^2)^{-3/2}\right]me^{-\mu t} \\ &\geq \left[-2\mu R^2 + \alpha_3(1+m^2R^2)^{-3/2}\right]me^{-\mu t}. \end{aligned}$$

Let $\mu = 3^{-1}R^{-2}\alpha_3(1+m^2R^2)^{-3/2}$. Then $L^{\varepsilon}v > 0$ in $\overline{\Omega}\times[0,\infty)$ and, clearly, $\pm L^{\varepsilon}u^{\varepsilon} = 0 < L^{\varepsilon}v$ in $\overline{\Omega}\times[0,\infty)$. Obviously, $|u_0(x)| \leq v(x,0)$ in Ω and $\pm u^{\varepsilon} = 0 < v$ on $\partial\Omega\times[0,\infty)$. By the maximum principle

$$|u^{\varepsilon}(x,t)| \leq v(x,t) \text{ in } \overline{\Omega}\times[0,\infty). \tag{4.1}$$

(b) Next we improve the estimate (4.1) for large t.

Let $\Omega_{\delta} = \{x\in\Omega \mid d(x) < \delta\}$ where $d(x) \equiv dist(x,\partial\Omega)$ and $\delta > 0$ is such that $d \in C^{\infty}(\overline{\Omega}_{\delta})$. Put $f(d(x)) = (1/k)(1-e^{-kd(x)})$, $x\in\overline{\Omega}_{\delta}$, where $k = const > 0$ is to be chosen later.

Consider the function

$$w(x,t) = c[f(d(x)) + g(t)]e^{-\mu t}$$

where $g(t) = [(T-t)_+]^2$, μ as in (4.1), and $c > 0$ a constant to be chosen. Put also $d_i = \partial d/\partial x_i$ and $d_{ij} = \partial^2/\partial x_i\partial x_j$ and note that $|Dd| = 1$.

We have

$$\begin{aligned} L^{\varepsilon}w &= -\mu w + cg_te^{-\mu t} - \sqrt{1+|Dw|^2} \\ &\quad \times [F_{ij}(f''d_id_j + f'd_{ij}) + \varepsilon(f''+f'\Delta d)]ce^{-\mu t} \quad (g_t := \partial g/\partial t). \end{aligned}$$

Since $f'' < 0$, we have, using (1.21),

$$
\begin{aligned}
-f''F_{ij}(Dw)d_i d_j &= -\ f''e^{2\mu t}\frac{F_{ij}(Dw)\,(\partial w/\partial x_i)\,(\partial w/\partial x_j)}{c^2 f'^2} \\
&\geq -\ \frac{f''\alpha_3}{(1+|Dw|^2)^{3/2}}.
\end{aligned}
$$

Using the inequality on the right hand side of (1.21) we get

$$
|F_{ij}(Dw)d_{ij}| \leq \frac{C_1\alpha_4}{\sqrt{1+|Dw|^2}},
$$

where C_1 is a constant depending on dimension n and C^2-norm of $d(x)$ in $\overline{\Omega}_\delta$. Recall that $d(x) \in C^2(\overline{\Omega}_\delta)$ for any $\delta < \delta_0$ where δ_0 is the same as at the end of the proof of Lemma 2.2.

Noting that $g_t' \geq -2$ and $f' > 0$, we obtain

$$
\begin{aligned}
L^\varepsilon w \ &\geq [-\mu(f+g) - 2 - \frac{f''\alpha_3}{1+c^2 f'^2 e^{-2\mu t}} - f'C_1\alpha_4 - \\
&-\varepsilon\sqrt{1+c^2 e^{-2\mu t} f'^2}(f'' + f'\Delta d)]ce^{-\mu t}.
\end{aligned} \tag{4.2}
$$

We now want to show that one can choose $T \geq 1$, c, k, δ, and ε_0 so that

$$
L^\varepsilon w \ \geq\ 0 \ \text{ for all } \ \varepsilon \leq \varepsilon_0 \ \text{ in } \ \Omega_\delta \times (T-1, \infty), \tag{4.3}
$$

$$
w(x, T-1) \ \geq\ 2me^{-\mu(T-1)}, \tag{4.4}
$$

$$
w(x,t) \ \geq\ 2me^{-\mu t} \ \text{ in } \ \{x \in \Omega \mid d(x) = \delta\} \times [T-1, \infty). \tag{4.5}
$$

In order to establish (4.3)-(4.5), we proceed as follows. First, restrict c and $T \geq 1$ by the requirement

$$
ce^{-\mu(T-1)} \leq 1. \tag{4.6}
$$

Obviously, $ce^{-\mu t} \leq 1$ for all $t \geq T-1$. Also, $g(t) \leq 1$ for all $t \geq T-1$. Next, we note that $f(d(x)) \leq \delta_0$ when $x \in \overline{\Omega}_\delta$, $\delta < \delta_0$. Now we obtain from (4.2)

$$
\begin{aligned}
L^\varepsilon w \ &\geq [(2^{-1}k\alpha_3 - C_1\alpha_4)e^{-kd} - 2 - \mu(1+\delta_0) - \\
&- \varepsilon\sqrt{2}\left(k + \sup_{\overline{\Omega}_\delta}|\Delta d|\right)]ce^{-\mu t} \ \text{ in } \ \Omega_\delta \times [T-1, \infty).
\end{aligned}
$$

In this inequality the constant C_1 depends on the C^2-norm of w in $\overline{\Omega}_\delta$. Assuming that $\delta \le \delta_0/2$, we may suppose that C_1 is determined by the bound on the C^2-norm of w in $\overline{\Omega}_{\delta_0/2}$.

Choose k and δ so that

$$(2^{-1}k\alpha_3 - C_1\alpha_4)e^{-1} - 2 - \mu(1+\delta_0) > 0 \text{ and } \delta = \min\{1/k, \delta_0/2\}.$$

Next let

$$c = \max\{2m,\ 2me\delta^{-1}\},$$

and define T as the smallest $T \ge 1$ satisfying (4.6). Then

$$w(x,T-1) = c[f+g]e^{-\mu(T-1)} \ge ce^{-\mu(T-1)} \ge 2me^{-\mu(T-1)},$$

and (4.4) is satisfied. Further,

$$w(x,t)\,|_{d(x)=\delta} = c(f+g)e^{-\mu t}\,|_{d(x)=\delta} \ge c\delta e^{-k\delta}e^{-\mu t}$$

$$\ge 2me^{-\mu t} \text{ for } t \ge T-1,$$

and (4.5) is also satisfied. On the other hand, $w(x,t)\,|_{\partial\Omega} \ge 0 = u^\varepsilon(x,t)$. This, (4.3)-(4.5), and the maximum principle imply that for $\varepsilon \le \varepsilon_0$

$$|u^\varepsilon(x,t)| \le w(x,t) \text{ in } \overline{\Omega}_\delta \times [T-1,\infty).$$

For the rest of this section it is assumed without further reminding that $\varepsilon \le \varepsilon_0$.

Since $g(t) = 0$ for $t \ge T$ we conclude that

$$|u^\varepsilon(x,t)| \le C_2 d(x)e^{-\mu t} \text{ in } \overline{\Omega}_\delta \times [T,\infty),$$

where $C_2 = ce^{-1}$. Combining the last inequality with (4.1) and adjusting the constant C_2 if necessary we conclude that

$$|u^\varepsilon(x,t)| \le C_2 d(x)e^{-\mu t} \text{ in } \overline{\Omega} \times [T,\infty). \tag{4.7}$$

(c) A standard argument shows now that

$$\left|\frac{\partial u^\varepsilon}{\partial \nu}\right| \le C_2 e^{-\mu t} \text{ in } \partial\Omega \times [T,\infty),$$

where ν is the exterior unit normal vector field on $\partial\Omega$. Since $u^\varepsilon(x,t) =$ on $\partial\Omega$, we conclude that

$$|Du^\varepsilon(x,t)| \le C_2 e^{-\mu t} \text{ in } \partial\Omega \times [T,\infty). \tag{4.8}$$

(d) We have the following analogue of Proposition 3.1. *Let* $M_0 = \sup_\Omega |u_0|$. *Then*

$$\sup_{\Omega\times[0,\infty)} |u^\varepsilon(x,t)| \le M_0, \tag{4.9}$$

$$\sup_{\Omega\times[0,\infty)} |u_t^\varepsilon(x,t)| \le M, \tag{4.10}$$

where M is a constant depending on the C^2*-norm of* u_0. *Also,*

$$\int_\Omega (|Du^\varepsilon| + \varepsilon|Du^\varepsilon|^2)dx \le c_0 \quad \textit{for all} \quad t \ge 0, \tag{4.11}$$

where $c_0 = c_0(M_0, M, \alpha_0)$.

The inequalities (4.9)-(4.11) are established similarly to (3.1)-(3.3) except that when proving (4.11) one needs to take into account (1.19).

As in [LU] and [OU], section 4.4, we will use the tangential operator δ defined on $C^1(\Omega)$ as

$$\delta g = \nabla g - \langle \nabla g, N\rangle N,$$

where $g \in C^1(\Omega)$, $\nabla g = (g_1, g_2, \dots, g_n, 0)$, $g_i = \partial g/\partial x_i$, $N = (-Du^\varepsilon, 1)/\nu$, and $\nu = \sqrt{1+|Du^\varepsilon|^2}$. Evidently,

$$|\delta g|^2 \ge |Dg|^2(1-|Du^\varepsilon|^2/\nu^2) = |Dg|^2/\nu^2, \tag{4.12}$$

and using (1.21) we get

$$\alpha_3 \nu^{-1}|\delta g|^2 \le F_{ij} g_i g_j \le \alpha_4 \nu^{-1}|Dg|^2. \tag{4.13}$$

Rewrite the equation (1.30) as

$$u_t^\varepsilon - \nu \frac{dF_i^\varepsilon}{dx_i} = 0 \text{ in } \Omega \times [0,\infty), \tag{4.14}$$

where $F_i^\varepsilon = F_i + \varepsilon p_i$ and $p \equiv Du^\varepsilon$. Here and below in this section we omit the argument Du^ε at F and its derivatives.

Applying the operator $(p_k/\mathrm{v})d/dx_k$ to (4.14) we obtain

$$\mathrm{v}_t - \mathrm{v}\frac{d}{d}x_i\,(F^{\varepsilon}_{ij}\mathrm{v}_j) + \mathrm{v}\Lambda = \left(\frac{p_k \mathrm{v}_k}{\mathrm{v}}\right)\frac{dF^{\varepsilon}_i}{dx_i}, \tag{4.15}$$

where $\mathrm{v}_t = \partial \mathrm{v}/\partial t$, $\mathrm{v}_i = \partial \mathrm{v}/\partial x_i$ and

$$\Lambda \equiv \left[\frac{d}{dx_j}\left(\frac{p_k}{\mathrm{v}}\right)\right]\frac{dF^{\varepsilon}_i}{dx_k} = \frac{\partial}{\partial p_s}\left(\frac{p_k}{\mathrm{v}}\right)F^{\varepsilon}_{ij}u^{\varepsilon}_{si}u^{\varepsilon}_{jk}.$$

Since

$$\frac{\partial}{\partial p_s}\left(\frac{p_k}{\mathrm{v}}\right)\xi_s\xi_k \geq \mathrm{v}^{-3}|\xi|^2 \quad \text{for all} \quad \xi \in \mathbf{R}^n,$$

and, because of (1.21), $\Lambda \geq 0$.

Multiply (4.15) by $\eta \mathrm{v}^{-1}$, $\eta \in H^1_0(\Omega)$, and integrate over Ω. After integration by parts on the left, and estimating the right hand side, using (4.14), (4.12), and the inequality $p^2/\mathrm{v}^2 < 1$, we obtain

$$\int_\Omega \left(\frac{\mathrm{v}_t}{\mathrm{v}}\eta + F^{\varepsilon}_{ij}\mathrm{v}_i\eta_j + \Lambda\eta\right)dx \leq M\int_\Omega \frac{|\delta \mathrm{v}|}{\mathrm{v}}|\eta|\,dx \quad \text{for all } t \geq 0, \tag{4.16}$$

where M is the constant from the inequality (4.10).

This inequality replaces the inequality (4.22) in [OU]. Formally both look the same, but the function F here replaces the function $\sqrt{1+p^2}$ in [OU] (also denoted by F in [OU]). Using (4.16) and (3.1)-(3.3) one derives exact analogues of Theorems C and C′ in [OU] for the function F satisfying conditions (1.19)-(1.21). The arguments used in [OU] to prove Theorems C and C′ carry over to our case with only minor modifications; one only needs to use the structure conditions (1.19)-(1.21), inequalities (4.9)-(4.11), and (4.13) instead of the corresponding properties of the function $\sqrt{1+p^2}$ and its derivatives. The details are lengthy but straight forward and we will not repeat them here.

Once analogues of Theorems C and C′ in [OU] are established we have an estimate of $|Du^{\varepsilon}(x,t)|$ in $\Omega' \times [0,T]$ for any $\Omega' \subset\subset \Omega$ and $T > 0$, and, on the account of (4.8), we have an estimate of $|Du^{\varepsilon}(x,t)|$ in $\overline{\Omega} \times [\bar{t},\infty)$ for some $\bar{t} \geq 0$. These estimates are similar to (3.5) and (3.6) in section 3 and the proof of Theorem 3 is completed in the same way as the proofs of Theorems 2 and 3.

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Department of Mathematics
and Computer Science
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Department of Mathematics
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Leningrad, USSR

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