YOUNG MEASURES IN A NONLOCAL PHASE TRANSITION PROBLEM

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IMA Preprint Series # 1365

November 1995

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Young Measures in a Nonlocal Phase Transition Problem

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October 12, 1995

Abstract

A nonlocal variational problem modeling phase transitions is studied under the framework of Young measures. The existence of global minima with desired internal layers on an infinite tube is proved by combining a weak convergence result of Young measures and the principle of concentration-compactness. The regularity of such global minima is discussed, and the nonlocal variational problem is also considered on asymptotic tubes.

1 Introduction

We are concerned with a variational problem modeling phase transitions where a nonlocal term is involved. We assume that u is a phase field parameter defined in an unbounded tube Ω in R^d . $u = -1$ and $u = 1$ represent two configurations of a perfect crystal and $u \in (-1, 1)$ represents a mixture. We associate to u its free energy

$$E(u) = \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x-y)(u(x) - u(y))^2 dx dy + \int_{\Omega} W(u) dx. \quad (1.1)$$

The function W in the second part of (1.1) is a balanced double well potential (see Figure 1) describing the bulk energy density. The double integral in (1.1) characterizes u 's self interaction energy. Here J is a nonnegative function with $J(-\xi) = J(\xi)$ and is normalized so that $\int_{R^d} J(\xi) d\xi = 1$. $J(x-y)$ measures

Figure 1: The graph of W .

the intensity of interaction between sites x and y . It takes both short and long range interaction into account, and therefore the double integral differs from the traditional way of describing the interaction energy by $\int_{\Omega} |\nabla u|^2 dx$ that ignores interaction at distance. The dependence of J on x and y through $x - y$ suggests that the material is homogeneous. Since we do not ask $J(\xi)$ to depend only on $|\xi|$, the material is allowed to be anisotropic.

We refer to Bates, Fife, Ren and Wang [3] for more about this model. The Euler-Lagrange equation of (1.1) is

$$\int_{\Omega} J(x - y)u(y)dy - K(x)u(x) - f(u(x)) = 0 \text{ in } \Omega \quad (1.2)$$

calculated in Fife [12] where $K(x) = \int_{\Omega} J(x - y)dy$ and $f(u) = W'(u)$ (see Figure 2). We ask u to be close to -1 at one end of the tube Ω and close to 1 at the other end so that a transition of phases occurs somewhere in Ω . The precise meaning of this boundary condition at infinity is given in (2.10). (1.2) is a nonlinear integral equation. It differs from the standard Allen-Cahn equation

$$\begin{cases} \Delta u - f(u) = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

which is often used to model phase transitions in that the first term of (1.2) is nonlocal. This nonlocal term together with $-K(x)u(x)$ comes from the double integral term in (1.1).

In [3] traveling waves in R^1 of the gradient flow of (1.1) are studied for unnecessarily balanced W , meaning that the two wells of W may have different depths, via a homotopy argument. One result there says that in the case when W is balanced (1.2) has a stable solution in R^1 connecting -1 to 1 . The argument

Figure 2: The graph of f .

relies on the monotonicity of the solutions and can not be generalized to treat less regular domains in R^d .

Bates *et al* approached (1.2) by truncation and formal approximation by higher order differential, hence local, equations. Though the higher order equations can be studied via geometric singular perturbation method ([1, 2, 13]), concentration-compactness method ([4, 5]) and deformation method ([16]), the convergence to the nonlocal equation remains open.

In this paper we study (1.2) by directly minimizing the energy functional (1.1). Being a minimization problem on an unbounded domain, our work involves the concentration-compactness principle of P.L. Lions [14].

The main difficulty of our approach to the existence question arises from the lack of pointwise convergence of minimizing sequences. It is easy to show that a sequence $\{u_n\}$ of bounded functions with more and more rapid oscillations can have uniformly bounded energy. Therefore a minimizing sequence does not immediately converge pointwise. The weak- $*$ limit in $L^\infty(\Omega)$ is not clearly an energy minimum. To fix this problem we study (1.1) in a much wider class of functions. We allow measure-valued functions to enter the admissible set. Such generalized functions are known as families of Young measures. In this new class the energy attains its minimum at a measure-valued function. A key fact proved in Section 2 is that a sequence of measure-valued functions converges locally in some weak sense to a measure-valued function whose energy is not greater than those of the functions in the sequence. Then in Section 3 we show that this fact together with the concentration-compactness principle gives a global minimum in “thin” periodic tubes.

In Section 4 we discuss the regularity of the measure-valued solutions under various conditions on J and W . Since our solutions are measure-valued func-

tions. The first question to ask is whether they are usual functions. If they are, are they continuous? We show that sometimes the global minima are usual functions, and sometimes the global minima can not be continuous, should they be usual functions. Unfortunately most questions concerning the regularity of global minima are open.

Finally in Section 5 we indicate how to prove the same result as that in Section 3 for “fat” tubes. We also discuss tubes that are only asymptotically periodic, for which a sufficient condition for the existence is given.

Other nonlocal problems in the literature include [6, 16] of Brandon *et al* who studied ferromagnetic materials with the help of measure-valued functions, and [7, 8, 9, 10, 15] of De Masi *et al* who studied a phase transition problem derived from statistical mechanics, for which stable traveling waves are found by perturbation method.

We use C to denote a generic constant that may change from line to line. The domain Ω is always assumed smooth. We often fail to mention passing to subsequences when handling sequences.

2 Preliminaries

We assume throughout this paper that J and W satisfy

H-1 J is bounded, Borel measurable and nonnegative on $\mathbb{R}^d$ with $J(-\xi) = J(\xi)$, $\int_{\mathbb{R}^d} J(\xi) d\xi = 1$ and $J(\xi) > c > 0$ when $|\xi| < \eta$ for some $\eta > 0$.

H-2 W is a C^2 function that has exactly two global minima at -1 and 1 where $W(-1) = W(1) = 0$, and $W''(-1) > 0$, $W''(1) > 0$.

Definition 2.1 A measure-valued function ν with range $[-1, 1]$ is a family of probability measures $\{\nu_x\}$ on $[-1, 1]$ parametrized by $x \in \bar{\Omega}$ such that for each Borel set $E \subset [-1, 1]$ $\nu_x(E)$ is Borel measurable with respect to $x \in \bar{\Omega}$.

General references for measure-valued functions include [11, 18, 19]. A measure-valued function is a usual function if $\nu_x = \delta(u(x))$ for some Borel measurable function u . Here $\delta(u)$ denotes the Dirac delta function with its support at point u . Denote the set of all the measure-valued functions with range $[-1, 1]$ by $\mathcal{M}(\bar{\Omega})$.

As a remark about the range $[-1, 1]$ one could start with measure-valued functions with a larger range, say $[-2, 2]$. For such a measure-valued function ν' define a new measure-valued function ν with range $[-1, 1]$ by

$$\nu_x = \nu'_x|_{[-1,1]} + \nu'_x([-2, -1])\delta(-1) + \nu'_x((1, 2])\delta(1)$$

where $\nu'_x|_{[-1,1]}$ denotes the restriction measure of ν'_x on $[-1, 1]$, i.e., for each Borel set $A \subset [-2, 2]$

$$\nu'_x|_{[-1,1]}(A) = \nu'_x(A \cap [-1, 1]).$$

It easy to verify $E(\nu) \leq E(\nu')$, so we only consider measure-valued functions with range $[-1, 1]$.

Lemma 2.2 *On every open set $\Omega \subset \mathbb{R}^d$ if $\{\nu_n\} \subset \mathcal{M}(\overline{\Omega})$, there exists $\nu \in \mathcal{M}(\overline{\Omega})$ such that along a subsequence of $\{\nu_n\}$ again labeled by n for each continuous F defined on $\overline{\Omega}^2 \times [-1, 1]^2$ and continuous G defined on $\overline{\Omega} \times [-1, 1]$ both with compact supports*

$$\begin{aligned} \lim_{n \rightarrow \infty} & \left[\int_{\overline{\Omega}^2} \int_{[-1, 1]^2} F(x, y, a, b) d\nu_{n,x} d\nu_{n,y} dx dy + \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_{n,x} dx \right] \\ &= \int_{\overline{\Omega}^2} \int_{[-1, 1]^2} F(x, y, a, b) d\nu_x d\nu_y dx dy + \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_x dx. \end{aligned}$$

Note we write $d\nu_{n,x}$ for $d\nu_{n,x}(a)$ and $d\nu_{n,y}$ for $d\nu_{n,y}(b)$ suppressing a and b .

Proof. Define measures σ_n on $\overline{\Omega} \times [-1, 1]$ by

$$\sigma_n(E) = \int_{\overline{\Omega}} \nu_{n,x}(\{a : (x, a) \in E\}) dx$$

for each Borel set $E \subset \overline{\Omega} \times [-1, 1]$. Note $\{a : (x, a) \in E\}$ is the x -slice of E . Clearly $\{\sigma_n\}$ is a sequence of radon measures uniformly bounded on each compact subset of $\overline{\Omega} \times [-1, 1]$. Then there exists a Radon measure σ on $\overline{\Omega} \times [-1, 1]$ such that for each continuous function G on $\overline{\Omega} \times [-1, 1]$ with compact support

$$\lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_{n,x} dx = \lim_{n \rightarrow \infty} \int_{\overline{\Omega} \times [-1, 1]} G d\sigma_n = \int_{\overline{\Omega} \times [-1, 1]} G d\sigma.$$

Applying Theorem 10 and the second step of the proof of Theorem 11 in [11], Chapter 1, we find a measure-valued function ν such that

$$\int_{\overline{\Omega} \times [-1, 1]} G d\sigma = \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_x dx.$$

Therefore

$$\lim_{n \rightarrow \infty} \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_{n,x} dx = \int_{\overline{\Omega}} \int_{[-1, 1]} G(x, a) d\nu_x dx. \quad (2.1)$$

Now take Σ_n to be the product measure $\sigma_n \times \sigma_n$. Identify $\overline{\Omega}^2 \times [-1, 1]^2$ with $(\overline{\Omega} \times [-1, 1])^2$. Then $\Sigma_n \rightarrow \Sigma = \sigma \times \sigma$ in the sense that for every continuous F on $\overline{\Omega}^2 \times [-1, 1]^2$ with compact support

$$\lim_{n \rightarrow \infty} \int_{\overline{\Omega}^2 \times [-1, 1]^2} F d\Sigma_n = \int_{\overline{\Omega}^2 \times [-1, 1]^2} F d\Sigma.$$

Figure 3: An example of tubes.

Then by Fubini's theorem

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\bar{\Omega}^2} \int_{[-1,1]^2} F(x, y, a, b) d\nu_{n,x} d\nu_{n,y} dx dy &= \lim_{n \rightarrow \infty} \int_{\bar{\Omega}^2 \times [-1,1]^2} F d\Sigma_n \\ &= \int_{\bar{\Omega}^2 \times [-1,1]^2} F d\Sigma = \int_{\bar{\Omega}^2} \int_{[-1,1]^2} F(x, y, a, b) d\nu_x d\nu_y dx dy \end{aligned}$$

which together with (2.1) proves the lemma. $\square$

It is an easy exercise to show

Corollary 2.3 *In Lemma 2.2 F and G need only to be bounded and Borel measurable with compact support.*

For each $\nu \in \mathcal{M}(\bar{\Omega})$ set

$$\langle \nu_x \rangle := \int_{[-1,1]} a d\nu_x \quad (2.2)$$

which stands for the mean of ν_x . Define an energy functional

$$E(\nu) = \frac{1}{4} \int_{\bar{\Omega}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy + \int_{\bar{\Omega}} \int_{[-1,1]} W(a) d\nu_x dx \quad (2.3)$$

for each $\nu \in \mathcal{M}(\bar{\Omega})$, which generalizes (1.1). Note the energy of ν is not necessarily finite. We set

$$\mathcal{A} = \{\nu : E(\nu) < \infty\} \quad (2.4)$$

so E is a functional from $\mathcal{A}$ to the set of nonnegative real numbers.

Definition 2.4 Ω is a tube if there exists $K > 0$ so that $\Omega \subset \mathbb{R}^1 \times [-K, K]^{d-1} \subset \mathbb{R}^d$ and there exists $T \in \mathbb{R}^d$ so that $x + kT \in \Omega$ if and only if $x \in \Omega$ for all integer k (see Figure 3 where the two ends are denoted by e_1 and e_2).

We single out a simple case when the tube is “thin” compared to J .

Definition 2.5 Ω is a thin tube if K in Definition 2.4 can be chosen so that $J(\xi) > c > 0$ for all $\xi \in [-K, K]$.

A segment $S(z^1, r)$ of Ω is defined by

$$S(z^1, r) = \{x \in \bar{\Omega} : |x^1 - z^1| < r\} \quad (2.5)$$

where $x = (x^1, x')$.

We now take Ω to be a thin tube and set

$$\hat{\nu}(x^1) = \frac{1}{|S(x^1, \eta)|} \int_{z \in S(x^1, \eta)} \langle \nu_z \rangle dz \quad (2.6)$$

for $x^1 \in R^1$. Here $\eta > 0$ is chosen that $J(\xi) > c(\eta) > 0$ for all $|\xi| < 2\eta$, and $|S(x^1, \eta)|$ is the Lebesgue measure of $S(x^1, \eta)$. Clearly $\hat{\nu}$ is continuous.

Lemma 2.6 Suppose Ω is a thin tube. For every $\epsilon_1 > 0$ there is $\theta > 0$ such that for every $\nu \in \mathcal{A}$ $z^1 \in R^1$

$$\frac{1}{4} \int_{S^2(z^1, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy + \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx < \theta$$

implies that either $|\hat{\nu}(z^1) + 1| < \epsilon_1$ or $|\hat{\nu}(z^1) - 1| < \epsilon_1$.

Proof. For each $\nu \in \mathcal{A}$ $z^1 \in R^1$ we estimate $W(\hat{\nu}(z^1))$ as follows.

$$\begin{aligned} W(\hat{\nu}(z^1)) &= \frac{1}{|S(z^1, \eta)|} \int_{S(z^1, \eta)} W(\hat{\nu}(z^1)) dx \\ &= \frac{1}{|S(z^1, \eta)|} \int_{S(z^1, \eta)} \int_{[-1,1]} W(a + \hat{\nu}(z^1) - a) d\nu_x dx \\ &= \frac{1}{|S|} \int_{S(z^1, \eta)} \int_{[-1,1]} [W(a) + f(a)(\hat{\nu}(z^1) - a) + \frac{f'(\cdot)}{2}(\hat{\nu}(z^1) - a)^2] d\nu_x dx \\ &\leq \frac{1}{|S|} \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx + \frac{1}{|S|} \left[\int_{S(z^1, \eta)} \int_{[-1,1]} f^2(a) d\nu_x dx \right]^{1/2} \\ &\quad \left[\int_{S(z^1, \eta)} \int_{[-1,1]} (a - \hat{\nu}(z^1))^2 d\nu_x dx \right]^{1/2} + C \int_{S(z^1, \eta)} \int_{[-1,1]} (a - \hat{\nu}(z^1))^2 d\nu_x dx \\ &\leq \frac{1}{|S|} \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx + \frac{1}{|S|} \left[\int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx \right]^{1/2} \\ &\quad \left[\int_{S(z^1, \eta)} \int_{[-1,1]} (a - \hat{\nu}(z^1))^2 d\nu_x dx \right]^{1/2} + C \int_{S(z^1, \eta)} \int_{[-1,1]} (a - \hat{\nu}(z^1))^2 d\nu_x dx \\ &\leq C \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx + C \int_{S(z^1, \eta)} \int_{[-1,1]} (a - \hat{\nu}(z^1))^2 d\nu_x dx \quad (2.7) \end{aligned}$$

by Hölder's inequality and H-2. On the other hand by the definition of η

$$\int_{S^2(z^1, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy \geq C \int_{S^2} \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy$$

(Here we used the condition that Ω is thin)

$$\begin{aligned} &= C \int_{S^2} \int_{[-1,1]^2} [(a - \widehat{\nu}(z^1))^2 + (b - \widehat{\nu}(z^1))^2 - 2(a - \widehat{\nu}(z^1))(b - \widehat{\nu}(z^1))] d\nu_x d\nu_y dx dy \\ &= C \int_{S^2(z^1, \eta)} \int_{[-1,1]^2} [(a - \widehat{\nu}(z^1))^2 + (b - \widehat{\nu}(z^1))^2] d\nu_x d\nu_y dx dy \\ &\geq C \int_{S(z^1, \eta)} \int_{[-1,1]} (a - \widehat{\nu}(z^1))^2 d\nu_x dx. \end{aligned}$$

In short we have

$$\begin{aligned} &\int_{S(z^1, \eta)} \int_{[-1,1]} (a - \widehat{\nu}(z^1))^2 d\nu_x dx \\ &\leq C \int_{S^2(z^1, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy. \end{aligned} \quad (2.8)$$

Combing (2.8) with (2.7) we find

$$\begin{aligned} W(\widehat{\nu}(z^1)) &\leq C \left[\frac{1}{4} \int_{S^2(z^1, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy \right. \\ &\quad \left. + \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx \right], \end{aligned} \quad (2.9)$$

which immediately implies the lemma. $\square$

Corollary 2.7 *For each $\nu \in \mathcal{A}$ on a thin tube $\widehat{\nu}$ has limits -1 or 1 at $-\infty$ or ∞ .*

We then set

$$\mathcal{A}_{-1}^1 = \{\nu \in \mathcal{A} : \lim_{x^1 \rightarrow -\infty} \widehat{\nu}(x^1) = -1 \text{ and } \lim_{x^1 \rightarrow \infty} \widehat{\nu}(x^1) = 1\}. \quad (2.10)$$

We define $\mathcal{A}_{-1}^{-1}$, $\mathcal{A}_1^{-1}$ and $\mathcal{A}_1^1$ in similar ways, and clearly these subclasses are mutually disjoint and

$$\mathcal{A} = \mathcal{A}_{-1}^{-1} \cup \mathcal{A}_{-1}^1 \cup \mathcal{A}_1^{-1} \cup \mathcal{A}_1^1.$$

We look for global minima of E in each of the four subclasses. Clearly $\nu_x = \delta(-1)$ is the only global minimum in $\mathcal{A}_{-1}^{-1}$ and $\nu_x = \delta(1)$ is the only global minimum in $\mathcal{A}_1^1$, so we only consider $\mathcal{A}_{-1}^1$ and the subclass $\mathcal{A}_1^{-1}$ can be treated in the same way.

3 Existence of Measure-valued Solutions

We now state our main theorem about the existence. We consider thin tubes (see Definition 2.5) in this section. General tubes are treated in Section 5.

Theorem 3.1 *On each thin tube Ω there exists $\mu \in \mathcal{A}_{-1}^1$ such that $E(\mu) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$.*

Let $\{\nu_n\} \subset \mathcal{A}_{-1}^1$ be a minimizing sequence, i.e.,

$$\lim_{n \rightarrow \infty} E(\nu_n) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu). \quad (3.1)$$

Remark 3.2 *One can assume that the ν_n 's are usual functions (see Theorem 1, [19]).*

We will construct a minimum from $\{\nu_n\}$. To this end we prove a few lemmas.

Lemma 3.3 *There exists $c > 0$ such that $E(\nu) > c$ for all $\nu \in \mathcal{A}_{-1}^1$.*

Proof. Suppose there exist $\nu_n \in \mathcal{A}_{-1}^1$ such that $\lim_{n \rightarrow \infty} E(\nu_n) = 0$. Then Lemma 2.6 implies that for every small $\epsilon > 0$ $|\hat{\nu}_n(x^1) + 1| < \epsilon$ or $|\hat{\nu}_n(x^1) - 1| < \epsilon$ for all $x^1 \in \mathbb{R}^1$ and large n . But $\hat{\nu}_n(x_n^1) = 0$ for some x_n^1 since

$$\lim_{x^1 \rightarrow -\infty} \hat{\nu}_n(x^1) = -1 \text{ and } \lim_{x^1 \rightarrow \infty} \hat{\nu}_n(x^1) = 1.$$

This contradiction proves the lemma. $\square$

To handle the possible loss of compactness due to the unboundedness of Ω we proceed as in [4, 5]. Define Lévy concentration functions Q_n associated with ν_n by

$$Q_n(t) = \sup_{z^1 \in \mathbb{R}^1} \left[\frac{1}{4} \int_{S^2(z^1, t)} J(x - y) \int_{[-1, 1]^2} (a - b)^2 d\nu_{n,x} d\nu_{n,y} dx dy + \int_{S(z^1, t)} \int_{[-1, 1]} W(a) d\nu_{n,x} dx \right] \quad (3.2)$$

for $t \geq 0$. Clearly $\{Q_n\}$ is a sequence of nonnegative nondecreasing functions such that

$$\lim_{n \rightarrow \infty} \lim_{t \rightarrow \infty} Q_n(t) = \lim_{n \rightarrow \infty} E(\nu_n) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) > 0. \quad (3.3)$$

Note the last inequality follows from Lemma 3.3. By a classical lemma there exists a nonnegative nondecreasing function $Q(t)$ such that along a suitable subsequence of $\{\nu_n\}$, again denoted by $\{\nu_n\}$,

$$\lim_{n \rightarrow \infty} Q_n(t) = Q(t) \quad (3.4)$$

for all $t \geq 0$. Set $\lambda = \lim_{t \rightarrow \infty} Q(t)$. Clearly $0 \leq \lambda \leq \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$. By (3.2), (3.3) and (3.4) we make the following three statements (see [14] for details).

I If $\lambda = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$, there exist a subsequence of $\{\nu_n\}$ labeled by the same index and $z_n^1 \in R^1$ so that for every $\epsilon > 0$ there exists $r > 0$ satisfying

$$\begin{aligned} & \frac{1}{4} \int_{S^2(z_n^1, r)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} \nu_{n,y} dx dy \\ & + \int_{S(z_n^1, r)} \int_{[-1,1]} W(a) d\nu_{n,x} dx > \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \epsilon. \end{aligned}$$

II If $\lambda = 0$, then for all $r > 0$

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sup_{z^1 \in R^1} \left[\frac{1}{4} \int_{S^2(z^1, r)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \right. \\ & \left. + \int_{S(z^1, r)} \int_{[-1,1]} W(a) d\nu_{n,x} dx \right] = 0. \end{aligned}$$

III If $0 < \lambda < \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$, there exist a subsequence of $\{\nu_n\}$ labeled by the same index such that for every $\epsilon > 0$ there exist $r > 0$, $r_n \rightarrow \infty$, and a positive integer n_1 such that for all $n > n_1$

$$\lambda - \epsilon < Q_n(r), \quad Q_n(r_n) < \lambda + \epsilon.$$

Lemma 3.4 *The case $\lambda = 0$ can not occur.*

Proof. The proof is the same as that of Lemma 3.3. $\square$

Lemma 3.5 *The case $0 < \lambda < \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$ can not occur.*

We postpone the proof of this lemma to the end of this section.

We now deduce from Statement I and Lemma 2.6 that there exist integers k_n such that for every $\epsilon_1 > 0$ there exists $r > 0$ so that for all n

$$|\widehat{\nu}_n(x^1 - k_n T^1) + 1| < \epsilon_1 \text{ for } x^1 < -r, \quad |\widehat{\nu}_n(x^1 - k_n T^1) - 1| < \epsilon_1 \text{ for } x^1 > r. \quad (3.5)$$

We introduce a shift operator τ so that

$$(\tau_k \nu)_x = \nu_{x-kT} \quad (3.6)$$

where k is an integer. Consider the shifted sequence $\{\tau_{k_n} \nu_n\}$. (3.5) becomes

$$|\widehat{\tau_{k_n} \nu_n}(x^1) + 1| < \epsilon_1 \text{ for } x^1 < -r, \quad |\widehat{\tau_{k_n} \nu_n}(x^1) - 1| < \epsilon_1 \text{ for } x^1 > r. \quad (3.7)$$

Set $\mu \in \mathcal{M}(\overline{\Omega})$ to be the limit of $\{\tau_{k_n} \nu_n\}$ in the sense of Lemma 2.2. Corollary 2.3 and (3.7) imply for $x^1 < -r$

$$\begin{aligned}\widehat{\mu}(x^1) &= \frac{1}{|S(x^1, \eta)|} \int_{S(x^1, \eta)} \int_{[-1, 1]} a \, d\mu_y dy \\ &= \lim_{n \rightarrow \infty} \frac{1}{|S(x^1, \eta)|} \int_{S(x^1, \eta)} \int_{[-1, 1]} a \, d(\tau_{k_n} \nu_n)_y dy \in (-1 - \epsilon_1, -1 + \epsilon_1).\end{aligned}$$

The same argument shows that for $x^1 > r$ $\widehat{\mu}(x^1) \in (1 - \epsilon_1, 1 + \epsilon_1)$. Therefore

$$\lim_{x^1 \rightarrow -\infty} \widehat{\mu}(x^1) = -1 \text{ and } \lim_{x^1 \rightarrow \infty} \widehat{\mu}(x^1) = 1. \quad (3.8)$$

Corollary 2.3 implies

$$\begin{aligned}& \frac{1}{4} \int_{S^2(0, r)} J(x - y) \int_{[-1, 1]^2} (a - b)^2 d\mu_x d\mu_y dx dy + \int_{S(0, r)} \int_{[-1, 1]} W(a) d\mu_x dx \\ &= \lim_{n \rightarrow \infty} \left[\frac{1}{4} \int_{S^2(0, r)} J(x - y) \int_{[-1, 1]^2} (a - b)^2 d(\tau_{k_n} \nu_n)_x d(\tau_{k_n} \nu_n)_y dx dy \right. \\ & \quad \left. + \int_{S(0, r)} \int_{[-1, 1]} W(a) d(\tau_{k_n} \nu_n)_x dx \right] \leq \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu).\end{aligned}$$

By sending $r \rightarrow \infty$ we find $E(\mu) \leq \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$. This and (3.8) imply $\mu \in \mathcal{A}_{-1}^1$ and $E(\mu) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$. This proves Theorem 3.1.

We close this section by proving Lemma 3.5.

Proof of Lemma 3.5. It follows from Statement III that for every $\epsilon > 0$ we have $r > 0$ and a sequence $r_n \rightarrow \infty$, such that $Q_n(r)$, $Q_n(r_n) \in (\lambda - \epsilon, \lambda + \epsilon)$ for large n . Therefore we can find z_n^1 so that

$$\begin{aligned}& \frac{1}{4} \int_{S^2(z_n^1, r)} J(x - y) \int_{[-1, 1]^2} (a - b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \\ & + \int_{S(z_n^1, r)} \int_{[-1, 1]} W(a) d\nu_{n,x} dx \in (\lambda - \epsilon, \lambda + \epsilon),\end{aligned} \quad (3.9)$$

$$\begin{aligned}& \frac{1}{4} \int_{(\overline{\Omega} \times \overline{\Omega}) \setminus (S(z_n^1, r_n) \times S(z_n^1, r_n))} J(x - y) \int_{[-1, 1]^2} (a - b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \\ & + \int_{\overline{\Omega} \setminus S(z_n^1, r_n)} W(a) d\nu_{n,x} dx \in \left(\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \lambda - \epsilon, \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \lambda + \epsilon \right)\end{aligned} \quad (3.10)$$

for large n .

Let us denote the two segments of $S(z_n^1, r_n) \setminus S(z_n^1, r)$ by D_{1n} and D_{2n} with the first one lying to the left of the second one. Note $D_{1n} = S(z_n^1 - (r_n + r)/2, (r_n - r)/2)$ and $D_{2n} = S(z_n^1 + (r_n + r)/2, (r_n - r)/2)$. Clearly $|D_{1n}|, |D_{2n}| \rightarrow \infty$ as $n \rightarrow \infty$. Also denote $\bar{\Omega} \setminus S(z_n^1, r)$ by M_{1n} and M_{2n} with $D_{1n} \subset M_{1n}$ and $D_{2n} \subset M_{2n}$. For large n

$$\frac{1}{4} \int_{D_{1n}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy + \int_{D_{1n}} \int_{[-1,1]} W d\nu_{n,x} dx < 2\epsilon$$

and a similar inequality holds for D_{2n} . Then Lemma 2.6 implies that the $\hat{\nu}_n(x^1)$'s stay close to -1 or 1 on D_{1n} and D_{2n} . We consider three cases

Case 1: Suppose, choosing a subsequence if necessary, the $\hat{\nu}_n(x^1)$'s stay close to -1 on D_{2n} in the sense of Lemma 2.6.

We will show that in this case the ν_n 's already have energies at least not much less than $\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$ in M_{2n} , so the total energy is by (3.9) is at least not much less than $\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) + \lambda$. This shows that ν_n can not be a minimizing sequence.

To this end we truncate ν_n at $z_n^1 + r$. Set

$$\nu'_{n,x} = \begin{cases} \nu_{n,x} & \text{if } x^1 \geq z_n^1 + r \\ \delta(-1) & \text{if } x^1 < z_n^1 + r \end{cases} \quad (3.11)$$

where $x = (x^1, x')$. Clearly $\nu'_n \in \mathcal{A}_{-1}^1$, so

$$E(\nu'_n) \geq \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu). \quad (3.12)$$

We proceed to show that $E(\nu'_n)$ and

$$\frac{1}{4} \int_{M_{2n}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy + \int_{M_{2n}} \int_{[-1,1]} W(a) d\nu_{n,x} dx$$

are comparable. First we note by (3.11)

$$\int_{\bar{\Omega}} \int_{[-1,1]} W(a) d\nu'_{n,x} dx = \int_{M_{2n}} \int_{[-1,1]} W(a) d\nu_{n,x} dx. \quad (3.13)$$

The other part of $E(\nu'_n)$ is estimated as follows. Write, denoting $\{x \in \bar{\Omega} : x^1 < z_n^1 + r\}$ by $\bar{\Omega}^1$ where $x = (x^1, x')$,

$$\int_{\bar{\Omega}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu'_{n,x} d\nu'_{n,y} dx dy$$

$$\begin{aligned}
&= \int_{M_{2n}^2} \dots + \int_{M_{2n}} \int_{y^1 < z_n^1 + r} \dots + \int_{x^1 < z_n^1 + r} \int_{M_{2n}} \dots + \int_{x^1 < z_n^1 + r} \int_{y^1 < z_n^1 + r} \dots \\
&= \int_{M_{2n}^2} \dots + \int_{M_{2n}} \int_{y^1 < z_n^1 + r} \dots + \int_{x^1 < z_n^1 + r} \int_{M_{2n}} \dots \quad (3.14)
\end{aligned}$$

since

$$\int_{x^1 < z_n^1 + r} \int_{y^1 < z_n^1 + r} \int_{[-1,1]^2} (a-b)^2 d\nu'_{n,x} d\nu'_{n,y} = 0$$

by (3.11). We need to show that the last two terms are small (of order ϵ) as $n \rightarrow \infty$. Since they are symmetric, we only consider

$$\begin{aligned}
&\int_{M_{2n}} \int_{y^1 < z_n^1 + r} \dots = \int_{x^1 \geq z_n^1 + r_n} \int_{y^1 < z_n^1 + r} \dots + \int_{D_{2n}} \int_{y^1 < z_n^1 + r} \dots \\
&\leq C \int_{x^1 \geq z_n^1} \int_{y^1 < z_n^1 + r} J(x-y) dx dy + \int_{D_{2n}} \int_{y^1 < z_n^1 + r} \dots
\end{aligned}$$

The first term approaches 0 as $n \rightarrow \infty$ since $r_n \rightarrow \infty$. The last line is therefore bounded by

$$\begin{aligned}
&\int_{D_{2n}} \int_{y^1 < z_n^1 + r} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\delta(-1) dx dy + o(1) \\
&\leq C \int_{D_{2n}} \int_{[-1,1]} (a+1)^2 d\nu_{n,x} dx + o(1)
\end{aligned}$$

since function $x \rightarrow \int_{y^1 < z_n^1 + r} J(x-y) dy \leq 1$ by H-2 is uniformly bounded.

Cover D_{2n} with m many mutually disjoint $S(t_{n,1}, \eta), S(t_{n,2}, \eta), \dots, S(t_{n,m}, \eta)$ with $D_{2n} = \cup_{i=1}^m S(t_{n,i}, \eta)$. This can be done if we choose r_n appropriately in the first place. Then

$$\begin{aligned}
&C \int_{D_{2n}} \int_{[-1,1]} (a+1)^2 d\nu_{n,x} dx + o(1) \\
&= C \sum_{i=1}^m \int_{S(t_{n,i}, \eta)} \int_{[-1,1]} (a+1)^2 d\nu_{n,x} dx + o(1) \\
&= C \sum_{i=1}^m \int_{S(t_{n,i}, \eta)} \int_{[-1,1]} (a - \hat{\nu}(t_{n,i}) + \hat{\nu}(t_{n,i}) + 1)^2 d\nu_{n,x} dx + o(1) \\
&= C \sum_{i=1}^m \left[\int_{S(t_{n,i}, \eta)} \int_{[-1,1]} (a - \hat{\nu}(t_{n,i}))^2 d\nu_{n,x} dx + |S(t_{n,i}, \eta)| (\hat{\nu}(t_{n,i}) + 1)^2 \right] + o(1) \\
&\leq C \sum_{i=1}^m \left[\int_{S(t_{n,i}, \eta)} \int_{[-1,1]} (a - \hat{\nu}(t_{n,i}))^2 d\nu_{n,x} dx + W(\hat{\nu}(t_{n,i})) \right] + o(1)
\end{aligned}$$

by H-1 since $\widehat{\nu}(t_{n,i})$ is close to -1 on D_{2n} . Applying (2.8) and (2.9) we find that the last line is bounded by

$$C \sum_{i=1}^m \left[\frac{1}{4} \int_{S^2(t_{n,i}, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \right. \\ \left. + \int_{S(t_{n,i}, \eta)} \int_{[-1,1]} W(a) d\nu_{n,x} dx \right] + o(1) \leq C \left[\frac{1}{4} \int_{D_{2n}^2} \dots + \int_{D_{2n}} \dots \right] + o(1).$$

Applying (3.9) and (3.10) we find

$$\int_{M_{2n}} \int_{y^1 < z_n^1 + r} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \leq 3\epsilon \quad (3.15)$$

for large n . We deduce from (3.13), (3.14) and (3.15) that

$$E(\nu'_n) \leq \frac{1}{4} \int_{M_{2n}^2} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \\ + \int_{M_{2n}} \int_{[-1,1]} W(a) d\nu_{n,x} dx + 6\epsilon$$

for large n . This, (3.9) and (3.12) imply

$$E(\nu_n) \geq \lambda - \epsilon + \left[\frac{1}{4} \int_{\overline{\Omega}^2 \setminus S^2(z_n^1, r)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy \right. \\ \left. + \int_{\overline{\Omega} \setminus S(z_n^1, r)} \int_{[-1,1]} W(a) d\nu_{n,x} dx \right] \geq \lambda - \epsilon + \left[\frac{1}{4} \int_{M_{2n}^2} \dots + \int_{M_{2n}} \dots \right] \\ \geq \lambda - \epsilon + E(\nu'_n) - 6\epsilon \geq \lambda - 7\epsilon + \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu).$$

By choosing $7\epsilon < \lambda/2$ we find that $\{\nu_n\}$ can not be a minimizing sequence in $\mathcal{A}_{-1}^1$. This rules out case 1.

Case 2: Suppose, choosing a subsequence if necessary, the $\widehat{\nu}_n(x^1)$'s stay close to 1 on D_{1n} in the sense of Lemma 2.6.

This case is handled along the same line. One shows that ν_n already has large energy on M_{1n} .

Case 3: Suppose, choosing a subsequence if necessary, the $\widehat{\nu}_n(x^1)$'s stay close to -1 on D_{1n} and close to 1 on D_{2n} in the sense of Lemma 2.6.

We will show that ν_n has large energy on $S(z_n^1, r_n)$. To this end we truncate ν_n at $z_n^1 - r_n$ and $z_n^1 + r_n$. Set

$$\nu'_{n,x} = \begin{cases} \delta(-1) & \text{if } x^1 < z_n^1 - r_n \\ \nu_{n,x} & \text{if } z_n^1 - r_n \leq x^1 \leq z_n^1 + r_n \\ \delta(1) & \text{if } x^1 > z_n^1 + r_n \end{cases} \quad (3.16)$$

where $x = (x^1, x')$. Clearly $\nu'_n \in \mathcal{A}_{-1}^1$, so

$$E(\nu'_n) \geq \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu). \quad (3.17)$$

We compare $E(\nu'_n)$ with

$$\frac{1}{4} \int_{S^2(z_n^1, r_n)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy + \int_S \int_{[-1,1]} W(a) d\nu_{n,x} dx.$$

Clearly by (3.16)

$$\int_{\bar{\Omega}} \int_{[-1,1]} W(a) d\nu'_{n,x} dx = \int_{S(z_n^1, r_n)} \int_{[-1,1]} W(a) d\nu_{n,x} dx \quad (3.18)$$

The other part of $E(\nu'_n)$ is estimated as follows.

$$\begin{aligned} \int_{\bar{\Omega}^2} \dots &= \int_{S^2(z_n^1, r_n)} \dots + \int_{x^1 < z_n^1 - r_n} \int_{y^1 < z_n^1 - r_n} \dots \\ &+ \int_{x^1 > z_n^1 + r_n} \int_{y^1 > z_n^1 + r_n} \dots + \int_{x^1 < z_n^1 - r_n} \int_{y^1 > z_n^1 + r_n} \dots + \int_{x^1 > z_n^1 + r_n} \int_{y^1 < z_n^1 - r_n} \dots \\ &+ \int_S \int_{y^1 < z_n^1 - r_n} \dots + \int_S \int_{y^1 > z_n^1 + r_n} \dots + \int_{x^1 < z_n^1 - r_n} \int_S \dots + \int_{x^1 > z_n^1 + r_n} \int_S \dots \\ &= \int_{S^2} \dots + \int_{x^1 < z_n^1 - r_n} \int_{y^1 > z_n^1 + r_n} \dots + \int_{x^1 > z_n^1 + r_n} \int_{y^1 < z_n^1 - r_n} \dots \\ &+ \int_S \int_{y^1 < z_n^1 - r_n} \dots + \int_S \int_{y^1 > z_n^1 + r_n} \dots + \int_{x^1 < z_n^1 - r_n} \int_S \dots + \int_{x^1 > z_n^1 + r_n} \int_S \dots \end{aligned}$$

by (3.16). The second and third terms are again symmetric, so we only consider

$$\int_{x^1 < z_n^1 - r_n} \int_{y^1 > z_n^1 + r_n} \dots \leq C \int_{x^1 < z_n^1 - r_n} \int_{y^1 > z_n^1 + r_n} J(x-y) dx dy \rightarrow 0 \quad (3.19)$$

as $n \rightarrow \infty$ since $r_n \rightarrow \infty$. The last four terms are estimated in a way similar to that leading to (3.15) and each of them is bounded by, say, 3ϵ . Together with (3.18) and (3.19) we find

$$E(\nu'_n) \leq \frac{1}{4} \int_{S^2(z_n^1, r_n)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_{n,x} d\nu_{n,y} dx dy$$

$$+ \int_{S(z_n^1, r_n)} \int_{[-1,1]} W(a) d\nu_{n,x} dx + \epsilon + 12\epsilon$$

for large n . This, (3.10) and (3.17) imply

$$E(\nu_n) \geq (\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \lambda) - \epsilon + E(\nu_n') - 13\epsilon \geq (\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \lambda) + \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - 14\epsilon.$$

By choosing $14\epsilon < (\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) - \lambda)/2$ we deduce that $\{\nu_n\}$ can not be a minimizing sequence in $\mathcal{A}_{-1}^1$. This rules out case 3 and the proof of Lemma 3.5 is complete. $\square$

4 Regularity of Global Minima

In this section we assume Ω is a thin tube and μ is a global minimum of E in $\mathcal{A}_{-1}^1$. The “thinness” condition on Ω will be removed in the next section. We first derive an alternative formula of $E(\nu)$ for each $\nu \in \mathcal{A}$. Note for each $x \in \bar{\Omega}$

$$\begin{aligned} & \int_{[-1,1]^2} (a - b)^2 d\nu_x d\nu_y \\ &= \int_{[-1,1]^2} [(a - \langle \nu_x \rangle) - (b - \langle \nu_y \rangle) + (\langle \nu_x \rangle - \langle \nu_y \rangle)]^2 d\nu_x d\nu_y \\ &= (\langle \nu_x \rangle - \langle \nu_y \rangle)^2 + 2 \int_{[-1,1]} (a - \langle \nu_x \rangle)^2 d\nu_x \\ &= (\langle \nu_x \rangle - \langle \nu_y \rangle)^2 + 2 \int_{[-1,1]} a^2 d\nu_x - 2 \langle \nu_x \rangle^2. \end{aligned}$$

We deduce

$$\begin{aligned} E(\nu) &= \frac{1}{4} \int_{\bar{\Omega}^2} J(x - y) (\langle \nu_x \rangle - \langle \nu_y \rangle)^2 dx dy \\ &+ \int_{\bar{\Omega}} \left[\int_{[-1,1]} (W(a) + \frac{a^2}{2}) d\nu_x - \frac{\langle \nu_x \rangle^2}{2} \right] dx. \end{aligned} \quad (4.1)$$

An immediate consequence of (4.1) is

Proposition 4.1 *If $W(a) + a^2/2$ is convex in $a \in [-1, 1]$, there is a usual function as global minimum. If $W(a) + a^2/2$ is strictly convex, every global minimum is a usual function.*

$W(a) + a^2/2$ is strictly convex if $W((a+b)/2) + ((a+b)/2)^2/2 < [W(a) + a^2/2 + W(b) + b^2/2]/2$ for $-1 \leq a < b \leq 1$,

Figure 4: The graph of $W(a) + a^2/2$.

Proof. Suppose μ is a global minimum constructed in Section 3. Set $\mu' = \delta(<\mu_x>)$. Clearly $\mu' \in \mathcal{A}_{-1}^1$ and by (4.1)

$$E(\mu') - E(\mu) = \int_{\Omega} [W(<\mu_x>) + \frac{<\mu_x>^2}{2} - \int_{[-1,1]} (W(a) + \frac{a^2}{2}) d\mu_x] dx \leq 0 \quad (4.2)$$

since by the convexity

$$W(<\mu_x>) + \frac{<\mu_x>^2}{2} \leq \int_{[-1,1]} (W(a) + \frac{a^2}{2}) d\mu_x.$$

Therefore the usual function μ' is a global minimum.

If $W(a) + a^2/2$ is strictly convex and μ is a global minimum, then the equality in (4.2) holds if and only if $\mu_x = \delta(<\mu_x>)$ for almost every $x \in \bar{\Omega}$, which implies that μ is a usual function. $\square$

Next we consider a case where $W(a) + a^2/2$ is not convex in a .

H-3 Assume $W(a) + a^2/2$ achieves its minimal value exactly at p_c and q_c with $-1 < p_c < q_c < 1$ (see Figure 4) and $W(a) + a^2/2$ is strictly convex on $[-1, p_c]$ and $[q_c, 1]$.

Let μ be a global minimum from $\mathcal{A}_{-1}^1$. Define a new measure-valued function μ' by

$$\begin{aligned} \mu'_x &= \mu_x|_{[-1, p_c]} + \mu_x|_{[q_c, 1]} \\ &+ \frac{\mu((p_c, q_c))q_c - \int_{(p_c, q_c)} a d\mu_x}{q_c - p_c} \delta(p_c) + \frac{\int_{(p_c, q_c)} a d\mu_x - \mu((p_c, q_c))p_c}{q_c - p_c} \delta(q_c) \end{aligned} \quad (4.3)$$

for each $x \in \overline{\Omega}$. Clearly $\langle \mu'_x \rangle = \langle \mu_x \rangle$, $E(\mu') \leq E(\mu)$ by (4.1) and the equality holds if and only if $\mu_x = \mu'_x$ for almost every $x \in \Omega$. We have proved

Proposition 4.2 *Under H-3 for each global minimum $\mu \in \mathcal{A}_{-1}^1$*

$$\text{supp } \mu_x \cap (p_c, q_c) = \emptyset$$

for almost every $x \in \overline{\Omega}$ where $\text{supp } \mu_x$ denotes the support of measure μ_x .

A consequence is the following irregularity result that has already appeared in a simpler case in [3]. Here we interpret it from an energy viewpoint.

Corollary 4.3 *Under H-3 if $u(x)$ is a usual function minimizing E in $\mathcal{A}_{-1}^1$, u has at least a discontinuous point jumping from p_c to q_c .*

Since for a.e. x $\text{supp } \mu_x \subset [-1, p_c] \cup [q_c, 1]$, we further simplify μ with the help of the convexity of $W(a) + a^2/2$ on $[-1, p_c]$ and $[q_c, 1]$. Set μ' so that

$$\mu'_x = \mu_x([-1, p_c])\delta(p(x)) + \mu_x([q_c, 1])\delta(q(x))$$

where

$$p(x) = \begin{cases} \int_{[-1, p_c]} ad\mu_x / \mu([-1, p_c]) & \text{if } \mu([-1, p_c]) \neq 0 \\ p_c & \text{if } \mu([-1, p_c]) = 0 \end{cases}$$

and

$$q(x) = \begin{cases} \int_{[q_c, 1]} ad\mu_x / \mu([q_c, 1]) & \text{if } \mu([q_c, 1]) \neq 0 \\ q_c & \text{if } \mu([q_c, 1]) = 0. \end{cases}$$

Again by (4.1) $E(\mu') < E(\mu)$ unless $\mu'_x = \mu_x$ for a.e. x . Therefore we may write

$$\mu_x = (1 - \theta(x))\delta(p(x)) + \theta(x)\delta(q(x)) \quad (4.4)$$

at a.e. x for some $\theta(x) \in [0, 1]$, $p(x) \in [-1, p_c]$ and $q(x) \in [q_c, 1]$.

Finally we consider the one dimensional case.

H-4 In addition to H-3 assume $\Omega = R^1$.

Let μ written in the form (4.4) be a global minimum in $\mathcal{A}_{-1}^1$. We derive the Euler-Lagrange equation it satisfies. Fix two smooth functions ϕ and ψ on R^1 with compact supports. Set a deformation μ_t of μ by

$$\mu_{t,x} = (1 - \theta(x))\delta(p(x) + t\phi(x)) + \theta(x)\delta(q(x) + t\psi(x)) \quad (4.5)$$

for t near 0. Clearly $\mu_0 = \mu$. Since μ_t is a measure-valued function with range, say $[-2, 2]$ (note the remark before Lemma 2.2), we have $E(\mu_0) \leq E(\mu_t)$. Therefore after some calculation

$$0 = \frac{dE(\mu_t)}{dt} \Big|_{t=0} = \int_{R^1} \{ (1 - \theta(x)) [p(x) - J * ((1 - \theta)p + \theta q)(x) + f(p(x))] \phi(x) \\ + \theta(x) [q(x) - J * ((1 - \theta)p + \theta q)(x) + f(q(x))] \psi(x) \} dx$$

where $J * ((1 - \theta)p + \theta q)$ is the convolution of J and $(1 - \theta)p + \theta q$. Since this is true for all ϕ and ψ , we conclude

$$\begin{cases} (1 - \theta(x)) [J * ((1 - \theta)p + \theta q)(x) - p(x) - f(p(x))] = 0 \\ \theta(x) [J * ((1 - \theta)p + \theta q)(x) - q(x) - f(q(x))] = 0 \end{cases} \quad (4.6)$$

for a.e. $x \in \overline{\Omega}$. If $\theta(x) \in (0, 1)$ then

$$\begin{cases} J * ((1 - \theta)p + \theta q)(x) - p(x) - f(p(x)) = 0 \\ J * ((1 - \theta)p + \theta q)(x) - q(x) - f(q(x)) = 0 \end{cases}$$

which implies

$$p(x) + f(p(x)) = q(x) + f(q(x)).$$

According to H-3 and the fact $p(x) \in [-1, p_c]$ and $q(x) \in [q_c, 1]$, we have

$$p(x) + f(p(x)) \leq 0 \text{ and } q(x) + f(q(x)) \geq 0.$$

Therefore $p(x) + f(p(x)) = q(x) + f(q(x)) = 0$ that implies

Proposition 4.4 *Under H-4 the representation (4.4) of a global minimum μ satisfies $p(x) = p_c$, $q(x) = q_c$ when $\theta(x) \in (0, 1)$ for a.e. $x \in \overline{\Omega}$.*

We note $((1 - \theta)p + \theta q)(x) = \mu_x$ and

Claim $J * ((1 - \theta)p + \theta q)(x) \rightarrow -1$ as $x \rightarrow -\infty$.

To see this we first approximate J in L^1 norm by step functions. Then the claim holds if it holds when J is replaced by step functions. A step function can be a linear combination of characteristic functions of small intervals. For each characteristic function χ_I of interval I Corollary 2.7 implies

$$\lim_{x \rightarrow -\infty} \chi_I * ((1 - \theta)p + \theta q)(x) = -|I|.$$

The claim is then proved by the usual approximation argument.

The claim and the second equation of (4.6) imply that $\theta(x) = 0$ for x sufficiently close to $-\infty$, i.e., $\mu_x = \delta(p(x))$ there. Now note for such x

$$J * ((1 - \theta)p + \theta q)(x) - p(x) - f(p(x)) = 0.$$

Since $p(x) \in [-1, p_c]$ and $a \rightarrow a + f(a)$ is continuously invertible by H-3, we find that $p(x)$ is continuous with $p(x) \rightarrow -1$ as $x \rightarrow -\infty$. Applying the same argument to $x \rightarrow \infty$ we deduce the following about the continuity of μ near infinity.

Proposition 4.5 *Under H-4 for x sufficiently close to $-\infty$ (∞ , respectively) $\mu_x = \delta(p(x))$ ($\mu_x = \delta(q(x))$, respectively) where $p(x)$ ($q(x)$, respectively) is continuous with $p(x) \rightarrow -1$ ($q(x) \rightarrow 1$, respectively) as $x \rightarrow -\infty$ (∞ , respectively).*

5 Further Results

We first extend Theorem 3.1 to general tubes. Given a tube Ω let us divide it into the union of thin tubes $\Omega_1, \Omega_2, \dots, \Omega_l$ so that $\bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \cup \dots \cup \bar{\Omega}_l$ and $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. On each Ω_i a segment $S_i(z^1, r)$ is defined to be $\{x \in \Omega_i : |x^1 - z^1| < r\}$. Define for each $\nu \in \mathcal{A}$

$$\hat{\nu}_i(x^1) = \frac{1}{|S_i(x^1, \eta)|} \int_{S_i(x^1, \eta)} \langle \nu_x \rangle dz.$$

Analogous to Lemma 2.6 we have

Lemma 5.1 *In a tube Ω for every given ϵ_1 there is $\theta > 0$ such that for every $\nu \in \mathcal{A}$ $z^1 \in R^1$*

$$\frac{1}{4} \int_{S^2(z^1, \eta)} J(x-y) \int_{[-1,1]^2} (a-b)^2 d\nu_x d\nu_y dx dy + \int_{S(z^1, \eta)} \int_{[-1,1]} W(a) d\nu_x dx < \theta$$

implies either $|\hat{\nu}_i(z^1) + 1| < \epsilon_1$ for all $i = 1, 2, \dots, l$, or $|\hat{\nu}_i(z^1) - 1| < \epsilon_1$ for all $i = 1, 2, \dots, l$.

Proof. Applying Lemma 2.6 on each Ω_i we find that given ϵ_1

$$|\hat{\nu}_i(z^1) + 1| < \epsilon_1 \text{ or } |\hat{\nu}_i(z^1) - 1| < \epsilon_1$$

if θ is chosen small enough. Suppose without the loss of generality that $|\hat{\nu}_1(z^1) + 1| < \epsilon_1$. We need to show that $|\hat{\nu}_i(z^1) + 1| < \epsilon_1$ for $i = 2, 3, \dots, l$. To this end choose a continuous deformation from Ω_1 to Ω_2 , i.e., for each $t \in [1, 2]$ find a thin tube $\Omega_{(t)}$ with $\Omega_{(1)} = \Omega_1$ and $\Omega_{(2)} = \Omega_2$. On each $\Omega_{(t)}$ set

$$S_{(t)}(z^1, \eta) = \{x \in \bar{\Omega}_{(t)} : |x^1 - z^1| < \eta\} \text{ and}$$

$$\hat{\nu}_{(t)}(z^1) = \frac{1}{|S_{(t)}(z^1, \eta)|} \int_{S_{(t)}(z^1, \eta)} \langle \nu_x \rangle dx.$$

By Lemma 2.6 $|\hat{\nu}_{(t)}(z^1) + 1| < \epsilon_1$ or $|\hat{\nu}_{(t)}(z^1) - 1| < \epsilon_1$. But since $\hat{\nu}_{(t)}(z^1)$ is continuous in t , we find $|\hat{\nu}_2(z^1) + 1| = |\hat{\nu}_{(2)}(z^1) + 1| < \epsilon_1$. Repeating this argument we find $|\hat{\nu}_i(z^1) + 1| < \epsilon_1$ for $i = 2, 3, \dots, l$. This proves Lemma 5.1. $\square$

Corollary 5.2 *For each $\nu \in \mathcal{A}$ either $\lim_{z^1 \rightarrow -\infty} \hat{\nu}_i(z^1) = -1$ for all $i = 1, 2, \dots, l$ or $\lim_{z^1 \rightarrow -\infty} \hat{\nu}_i(z^1) = 1$ for all $i = 1, 2, \dots, l$. A similar statement holds for $z^1 \rightarrow \infty$.*

We can now define unambiguously $\mathcal{A}_{-1}^1$ to be

$$\{\nu \in \mathcal{A} : \lim_{z^1 \rightarrow -\infty} \widehat{\nu}_i(z^1) = -1, i = 1, \dots, l \text{ and } \lim_{z^1 \rightarrow \infty} \widehat{\nu}_i(z^1) = 1, i = 1, \dots, l\}. \quad (5.1)$$

Theorem 5.3 *On each tube Ω there is $\mu \in \mathcal{A}_{-1}^1$ such that $E(\mu) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$.*

To prove this theorem one follows the argument in Section 3 using Lemma 5.1 instead of Lemma 2.6. One need to modify the argument leading to (3.15) by modifying (2.8) and (2.9) accordingly. We leave the detail to the reader.

Next we relax the periodicity condition on Ω a little to consider asymptotically periodic Ω .

Definition 5.4 *Ω is an asymptotic tube if it can be divided into three disjoint open domains, Ω_1 , Ω_2 and Ω_0 , such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2 \cup \overline{\Omega}_0$, Ω_0 is bounded and Ω_1 and Ω_2 are “half periodic”.*

That Ω_m , $m = 1, 2$, is half periodic means there exists $T_m \in R^d$, $m = 1, 2$, such that for every $x_m \in \Omega_m$, $x_m + T_m \in \Omega_m$, and for every $x_m \in \Omega_m$ there exist $y_m \in \Omega_0$ and a non-negative integer n_m satisfying $x_m = y_m + n_m T_m$. Ω_1 and Ω_2 are indeed the two half periodic tails of Ω . I associate the end e_1 to Ω_1 and the other end e_2 to Ω_2 (see Figure 5).

Define $\mathcal{A}$ by (2.4) and $\mathcal{A}_{-1}^1$ by a formula similar to (5.1). Set G_m , $m = 1, 2$, to be the periodic extension of Ω_m , i.e.,

$$G_m = \{x \in R^d : x + n_m T_m \in \Omega_m \text{ for some integer } n_m\}.$$

Denote the other end of G_m by e'_m (see Figure 5). Define two auxiliary functionals E_m by (2.3) on G_m . Take $\mathcal{A}_{-1}^1(m)$ to be

$$\{\nu \in \mathcal{M}(\overline{G_m}) : E_m(\nu) < \infty, \lim_{x \rightarrow e_m} \nu(x) = (-1)^m, \lim_{x \rightarrow e'_m} \nu(x) = (-1)^{m+1}\}$$

where the limits are interpreted in a way similar to (5.1). The following theorem gives a sufficient condition for the existence of a global minimum in $\mathcal{A}_{-1}^1$.

Theorem 5.5 *If Ω is an asymptotic tube, then*

$$\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) \leq \min\left\{\inf_{\nu \in \mathcal{A}_{-1}^1(1)} E_1(\nu), \inf_{\nu \in \mathcal{A}_{-1}^1(2)} E_2(\nu)\right\}.$$

If the strict inequality

$$\inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu) < \min\left\{\inf_{\nu \in \mathcal{A}_{-1}^1(1)} E_1(\nu), \inf_{\nu \in \mathcal{A}_{-1}^1(2)} E_2(\nu)\right\}$$

holds, then there is $\mu \in \mathcal{A}_{-1}^1$ such that $E(\mu) = \inf_{\nu \in \mathcal{A}_{-1}^1} E(\nu)$.

We omit the proof of this theorem, which is a combination of the techniques in this paper and the ones in [5]. Applications of this theorem can also be found there.

Figure 5: An example of Ω , G_1 and G_2 .

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