**RELATIVISTIC THEORY OF SUPERPOTENTIALS
FOR A NONHOMOGENEOUS, SPATIALLY
ISOTROPIC MEDIUM**

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RELATIVISTIC THEORY OF SUPERPOTENTIALS FOR A NONHOMOGENEOUS, SPATIALLY ISOTROPIC MEDIUM

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Introduction. In optics and electrodynamics, situations involving materials with nonhomogeneous electromagnetic material properties often arise. In those situations where relativistic effects are negligible, problem solutions are generally obtained via a theoretical framework that disregards the fact that the structure and properties of the electromagnetic field, constitutive laws, and field equations must be invariant with respect to the choice of coordinates in space-time. In this work, we initiate the idea that new insight can be gained through the development of a covariant theory of electromagnetic wave propagation in a nonhomogeneous medium. It is our hope that utilization of the power of relativistic continuum theories of materials will bring greater understanding of the optics of nonhomogeneous media.

In this paper, we establish the existence of a new superpotential for the electromagnetic field and derive the governing field equation for the superpotential in covariant form. Several significant results are found along the path towards this goal, some of which are simply alternate but heretofore unstated ways of understanding the relativistic theory, and some of which are brand new.

We adopt the axiomatic approach of postulating the general covariant formulation of the Maxwell equations with the appropriate and tensorially consistent constitutive laws for linear media. Natural decomposition of these constitutive laws and the field tensors with respect to the Fermi frames immediately leads to the standard constitutive relations for the proper fields, including the optical rotation tensor. The notion that existence and consistence of the optical rotation tensor is a direct result of natural decomposition of the correct constitutive laws for the field tensors appears to be new.

For a spatially isotropic medium (with respect to the generalized 4-velocity), the constitutive laws can be written in terms of three scalars, the spatial projector, the generalized 4-velocity, and one of the spatial Levi-Civita tensor Δ -densities (see §1). For optically inactive materials, one of the scalars can be dropped. This results in a simple relation, adopted sequentially from §2.

The theory is then developed in terms of the electromagnetic potential and a new gauge condition which is a generalization of the Lorentz gauge condition commonly used

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in relativistic electrodynamics in vacuum. In the limiting case of vacuum, the new gauge condition takes the exact form of the classical Lorentz condition. We put the governing field equation for the electromagnetic potential in a form that reveals the nature and contributonal origins of the various terms. If, for example, we consider some, any, or all of the specific cases that the motion of the material is rigid, rotation-free, or acceleration-free, that the medium is completely isotropic, or that the medium is homogeneous, certain terms in the special form of the governing field equation are readily seen to vanish.

The governing field equation for the electromagnetic potential forms a consistent set of coupled partial differential equations whose solutions exist only if the new gauge condition is satisfied simultaneously. To alleviate the problem of admissibility, we introduce superpotentials which satisfy the new gauge condition identically. With the help of two new tensor identities involving the curvature tensor, we derive the governing field equation for the superpotential in covariant form. The nature of this (hyperbolic) system of partial differential equations is then seen to be quite simple and beautiful.

§1. Preliminaries. We carry out our analysis within the framework of the (general) relativistic Eulerian theory of continuous media. In that respect we adhere primarily to the work of Synge [1] and Bressan [2] while for geometric principles, language, and notation we follow the work of Schouten [3] and Flanders [4]. We begin by recalling several definitions, conventions, and formulae important to the ensuing development.

After Schouten [3], a V_n is defined to be an n -dimensional geometric manifold endowed with a symmetric linear connection $\Gamma_{\alpha\beta}^\mu = 2!\Gamma_{(\alpha\beta)}^\mu$ and a symmetric covariant constant¹ real fundamental tensor $g_{\alpha\beta}$. An R_n is a V_n in which the curvature tensor vanishes. We consider space-time to be a V_4 with an indefinite fundamental tensor, and the associated fundamental quadratic form, or *space-time metric*

$$(1.1) \quad ds^2 = -g_{\alpha\beta} dx^\alpha dx^\beta$$

has signature -2 . The space-time V_4 is therefore not ordinary.

Henceforth we follow the standard convention that Greek indices run from 1 to 4 and Latin indices run from 1 to 3. We remark that this convention does not apply to fixed labels on indices.

After Synge [1] and Bressan [2], we consider a smoothly moving 3-dimensional continuous body Ω and exclude the possibility of irregular motions such as sliding, fracture, or collision with another body. In that case we can think of Ω as a set of material points $\Omega = \cup_p^\star$ and the union $W_\Omega = \cup W_p^\star$ of the world lines $W_p^\star$ of the material points is called the *world tube* occupied by Ω . W_Ω represents a (4-dimensional) domain in the space-time V_4 .

¹A tensor is a *covariant constant* if its covariant derivative is equal to zero.

The *generalized 4-velocity* u^α at the event x^α is defined to be the tangent to the world line W_p parameterized by the arc-length s defined by (1.1). In that case

$$(1.2) \quad u^\alpha = \frac{dx^\alpha}{ds}.$$

The absolute derivative of a differentiable tensor $T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$ is given by²

$$(1.3) \quad \frac{\delta}{\delta s} T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = u^\gamma \nabla_\gamma T^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q},$$

where the symbol ∇_γ denotes the covariant derivative. The *intrinsic acceleration* a^α is defined as

$$(1.4) \quad a^\alpha \stackrel{\text{def}}{=} \frac{\delta u^\alpha}{\delta s}.$$

The following trivial formulae arise in the analysis

$$(1.5) \quad u^\alpha u_\alpha = -1,$$

$$(1.6) \quad u^\alpha \nabla_\beta u_\alpha = 0,$$

$$(1.7) \quad u^\alpha a_\alpha = 0.$$

We introduce the usual *spatial projector*

$$(1.8) \quad \overset{\perp}{g}_{\alpha\beta} \stackrel{\text{def}}{=} g_{\alpha\beta} + u_\alpha u_\beta,$$

and write³

$$(1.9a) \quad \overset{\perp}{T}^{\alpha_1 \alpha_2 \dots \alpha_p}_{\beta_1 \dots \beta_q} = \overset{\perp}{g}^{\alpha_1 \cdot \gamma} T^{\gamma \alpha_2 \dots \alpha_p}_{\beta_1 \dots \beta_q},$$

$$(1.9b) \quad T^{\alpha_1 \dots \alpha_p}_{\overset{\perp}{\beta}_1 \beta_2 \dots \beta_q} = \overset{\perp}{g}_{\beta_1 \cdot \gamma} T^{\alpha_1 \dots \alpha_p}_{\gamma \beta_2 \dots \beta_q},$$

$$(1.9c) \quad \overset{\perp}{T}^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} = T^{\overset{\perp}{\alpha}_1 \dots \overset{\perp}{\alpha}_p}_{\overset{\perp}{\beta}_1 \dots \overset{\perp}{\beta}_q}.$$

The index α of the tensor $T^{\alpha \alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$ is called *spatial* if

$$(1.10) \quad u_\alpha T^{\alpha \alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q} \equiv 0,$$

²Synge [1].

³We remark that both the metric tensor $g_{\alpha\beta}$ and the spatial projector $\overset{\perp}{g}_{\alpha\beta}$ are symmetric tensors and the use of dots when raising or lowering indices is therefore optional. For example $\overset{\perp}{g}^\beta_\alpha = \overset{\perp}{g}^\beta \cdot_\alpha$ and one can write $\overset{\perp}{g}^\beta_\alpha$ without causing any confusion. We further note that $g^\beta_\alpha = \delta^\beta_\alpha$, where δ^β_α is the Kronecker delta.

and any geometric quantity is called *spatial* whenever all of its indices are spatial. Note that $\overset{\perp}{g}_{\alpha\beta}$ is a spatial tensor and relation (1.10) is equivalent to the statement $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = T_{\beta_1 \dots \beta_q}^{\perp \alpha_1 \alpha_2 \dots \alpha_p}$.

The *natural decomposition* of a tensor $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ with respect to the index α_1 is defined by

$$(1.11) \quad T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = T_{\beta_1 \dots \beta_q}^{\perp \alpha_1 \dots \alpha_p} - u^{\alpha_1} u_\gamma T_{\beta_1 \dots \beta_q}^{\gamma \alpha_2 \dots \alpha_p}.$$

A similar procedure can be applied to repeatedly decompose the tensors on the right hand side of (1.11) until all indices of $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ are exhausted. The representation of $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$ as the sum of the 2^{p+q} individual terms thus obtained comprises the *natural decomposition of the tensor* $T_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p}$.

Finally, two important geometric quantities, the *spatial alternating tensor Δ -densities* or *spatial Levi-Civita tensor Δ -densities*⁴ of weight +1 and -1, respectively, are defined by

$$(1.12a) \quad \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} u_\rho \tilde{\varepsilon}^{\rho\alpha\beta\gamma},$$

$$(1.12b) \quad \underset{\sim}{\tilde{\varepsilon}}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} u^\rho \underset{\sim}{\varepsilon}_{\rho\alpha\beta\gamma},$$

where the *Levi-Civita tensor Δ -densities* $\tilde{\varepsilon}^{\alpha\beta\gamma\mu}$ and $\underset{\sim}{\varepsilon}_{\alpha\beta\gamma\mu}$ are alternating tensor Δ -densities of weight +1 and -1, respectively, with $\tilde{\varepsilon}^{1234} = 1 = \underset{\sim}{\varepsilon}_{1234}$. We adopt the convention that a tilde ($\sim$) directly over a kernel denotes Δ -density of weight +1 and a tilde directly under a kernel denotes Δ -density of weight -1.⁵ The following important properties⁶ arise in the ensuing analysis

$$(1.13) \quad \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma\mu} \equiv 0, \quad \overset{\perp}{\underset{\sim}{\varepsilon}}_{\alpha\beta\gamma\mu} \equiv 0,$$

$$(1.14a) \quad \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} \overset{\perp}{\underset{\sim}{\varepsilon}}_{\sigma\mu\gamma} = \overset{\perp}{g}_\sigma \cdot \overset{\perp}{\alpha} \overset{\perp}{g}_\mu \cdot \overset{\perp}{\beta} - \overset{\perp}{g}_\sigma \cdot \overset{\perp}{\beta} \overset{\perp}{g}_\mu \cdot \overset{\perp}{\alpha},$$

$$(1.14b) \quad \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} \overset{\perp}{\underset{\sim}{\varepsilon}}_{\alpha\beta\mu} = 2! \overset{\perp}{g}_\mu \cdot \overset{\perp}{\gamma},$$

$$(1.14c) \quad \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} \overset{\perp}{\underset{\sim}{\varepsilon}}_{\alpha\beta\gamma} = 3!.$$

⁴Bressan calls these quantities the spatial Ricci tensor Δ -densities. We refrain from this appellation in order to avoid confusion with the standard nomenclature designating the Ricci tensor as $R_{\lambda\mu} \stackrel{\text{def}}{=} R_{\beta\lambda\mu}^{\cdot\cdot\cdot\beta}$, where $R_{\alpha\beta\gamma}^{\cdot\cdot\cdot\mu}$ is the curvature tensor of the V_4 .

⁵Under a coordinate transformation $x^{\kappa'} = x^{\kappa'}(x^\kappa)$, a Δ -density of weight w transforms with a weight factor Δ^{-w} , where Δ is the Jacobian, $\Delta \stackrel{\text{def}}{=} \det [\partial x^{\kappa'} / \partial x^\kappa]$, and the sign of the determinant is retained. For example $\tilde{\varepsilon}^{\alpha'\beta'\gamma'\mu'} = \Delta^{-1} \tilde{\varepsilon}^{\alpha\beta\gamma\mu}$. See Schouten [3], p. 12.

⁶Bressan [2], p. 51.

§2. Maxwell Field Equations and Polarization Tensors. In terms of the electromagnetic field bivectors $F_{\alpha\beta}$ and $G_{\alpha\beta}$ and the generalized current Δ -density $\tilde{j}^\alpha$, the Maxwell equations in the space-time V_4 take the form⁷

$$(2.1) \quad \partial_{[\alpha} F_{\beta\gamma]} = 0,$$

$$(2.2) \quad \partial_\beta \tilde{G}^{\alpha\beta} = \tilde{j}^\alpha,$$

where the process of *alternation* over the indices is indicated by a pair of square brackets and where $\tilde{G}^{\alpha\beta}$ denotes the *dual representation* of $G_{\alpha\beta}$ in terms of a contravariant bivector Δ -density of weight +1. The tensorial homeomorphism between the two sets of quantities $G_{\alpha\beta}$ and $\tilde{G}^{\alpha\beta}$ is given by⁸

$$(2.3) \quad \tilde{G}^{\alpha\beta} = \frac{1}{2!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} G_{\gamma\mu}.$$

The *standard dual representation* $*G_{\alpha\beta}$, of $G_{\alpha\beta}$ is defined by

$$(2.4) \quad *G_{\alpha\beta} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}} \frac{1}{2!} \tilde{\varepsilon}_{\alpha\beta}{}^{\kappa\sigma} G_{\kappa\sigma},$$

where g denotes the determinant of the metric tensor $g_{\alpha\beta}$ and is a scalar Δ -density, and can thus be written

$$(2.5) \quad *G_{\alpha\beta} = \frac{1}{\sqrt{g}} g_{\alpha\gamma} g_{\beta\mu} \tilde{G}^{\gamma\mu}.$$

Consider the natural decomposition of the field tensor $F_{\alpha\beta}$

$$(2.6) \quad F_{\alpha\beta} = \overset{\perp}{F}_{\alpha\beta} + 2! u_{[\alpha} F_{\beta]},$$

where

$$(2.7a) \quad \overset{\perp}{F}_{\alpha\beta} = \overset{\perp}{g}_{\alpha}{}^{\gamma} \overset{\perp}{g}_{\beta}{}^{\mu} F_{\gamma\mu},$$

$$(2.7b) \quad F_{\beta} = \overset{\perp}{g}_{\beta}{}^{\gamma} F_{\gamma\mu} u^{\mu}.$$

Similarly

$$(2.8) \quad G_{\alpha\beta} = \overset{\perp}{G}_{\alpha\beta} + 2! u_{[\alpha} G_{\beta]},$$

⁷Barut [5], p. 93 ff.

⁸Schouten [3], p. 26.

and

$$(2.9) \quad \tilde{j}^\alpha = \tilde{j}^\alpha - u^\alpha u_\gamma \tilde{j}^\gamma.$$

Clearly $\overset{\perp}{F}_{\alpha\beta}$ and $\overset{\perp}{G}_{\alpha\beta}$ are spatial bivectors, $F_{(1)\beta}$ and $G_{(1)\beta}$ are spatial vectors, $\tilde{j}^\alpha$ is a spatial vector Δ -density of weight +1 and $u_\gamma \tilde{j}^\gamma$ is a scalar Δ -density of weight +1. We associate with these the geometric quantities

$$(2.10) \quad \begin{aligned} cB_{\alpha\beta} &\stackrel{\text{def}}{=} \overset{\perp}{F}_{\alpha\beta}, & E_\beta &\stackrel{\text{def}}{=} F_{(1)\beta}, \\ cD_{\alpha\beta} &\stackrel{\text{def}}{=} \overset{\perp}{G}_{\alpha\beta}, & H_\beta &\stackrel{\text{def}}{=} -G_{(1)\beta}, \\ \tilde{J}^\alpha &\stackrel{\text{def}}{=} \tilde{j}^\alpha, & c\tilde{\rho} &\stackrel{\text{def}}{=} -u_\gamma \tilde{j}^\gamma, \end{aligned}$$

where c is the speed of light in vacuum. The kernels B, E, D, H, J and ρ denote in MKS units the proper (i.e. measured with respect to an observer moving with the material point P) magnetic induction, electric field, electric displacement, magnetic field, true electric current, and true electric charge, respectively.

We remark that there exists a tensorial homeomorphism between *spatially dual* sets of spatial quantities, given by

$$(2.11) \quad \begin{aligned} E_\alpha &= \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{E}^{\beta\gamma}, & \tilde{E}^{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} E_\gamma, \\ B_{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{B}^\gamma, & \tilde{B}^\alpha &= \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} B_{\beta\gamma}, \\ D_{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{D}^\gamma, & \tilde{D}^\alpha &= \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} D_{\beta\gamma}, \\ H_\alpha &= \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{H}^{\beta\gamma}, & \tilde{H}^{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} H_\gamma, \\ J_{\alpha\beta} &= \overset{\perp}{\tilde{\varepsilon}}_{\alpha\beta\gamma} \tilde{J}^\gamma, & \tilde{J}^\alpha &= \frac{1}{2!} \overset{\perp}{\tilde{\varepsilon}}^{\alpha\beta\gamma} J_{\beta\gamma}. \end{aligned}$$

We may further note that *integrability* of the Maxwell equations requires the condition

$$(2.12) \quad \partial_\alpha \tilde{j}^\alpha = 0,$$

which in terms of the proper true current and charge densities comprises a statement of the punctual law of conservation of electric charge.

We introduce the *polarization tensor* $P_{\alpha\beta}$ via the relation

$$(2.13) \quad z_0 G_{\alpha\beta} = *(F_{\alpha\beta} + P_{\alpha\beta}),$$

where z_0 is the *intrinsic impedance of the vacuum*. In the absence of matter it follows that

$$(2.14) \quad z_0 G_{\alpha\beta} = {}^*F_{\alpha\beta} = \frac{1}{2!} \frac{1}{\sqrt{g}} g_{\alpha\gamma} g_{\beta\mu} \tilde{\varepsilon}^{\gamma\mu\kappa\sigma} F_{\kappa\sigma}.$$

Taking the natural decompositions of $\tilde{\varepsilon}^{\gamma\mu\kappa\sigma}$ and $F_{\kappa\sigma}$ and using (2.10) and (2.11) we find

$$(2.15) \quad \begin{aligned} \frac{1}{2!} \tilde{\varepsilon}^{\gamma\mu\kappa\sigma} F_{\kappa\sigma} &= - \left(u^{[\gamma} \tilde{\varepsilon}^{\perp] \mu] \kappa \sigma} - u^{[\kappa} \tilde{\varepsilon}^{\perp \sigma] \gamma \mu} \right) (c B_{\kappa\sigma} + 2! u_{[\kappa} E_{\sigma]}) \\ &= \left(\tilde{E}^{\gamma\mu} - 2! c u^{[\gamma} \tilde{B}^{\mu]} \right). \end{aligned}$$

Substituting this in (2.14) and using (2.8) and (2.9), we obtain the free-space relations

$$(2.16) \quad z_0 c D_{\alpha\beta} = E_{\alpha\beta},$$

$$(2.17) \quad z_0 H_\alpha = c B_\alpha.$$

We define the *permittivity* ε_0 and *permeability* μ_0 of the vacuum by

$$(2.18) \quad \varepsilon_0 \stackrel{\text{def}}{=} \frac{1}{z_0 c}, \quad \mu_0 \stackrel{\text{def}}{=} \frac{z_0}{c},$$

and thus obtain the familiar relations

$$(2.19) \quad \mu_0 \varepsilon_0 = \frac{1}{c^2}, \quad z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}.$$

In the presence of (polarizable) matter, we take the natural decomposition of the polarization bivector $P_{\alpha\beta}$ (compare with 2.6, 2.7a,b)

$$(2.20) \quad P_{\alpha\beta} = \tilde{P}_{\alpha\beta} + 2! u_{[\alpha} P_{\beta]}.$$

The tensor $\tilde{P}_{\alpha\beta}$ is a spatial bivector and P_{β} is a spatial vector, and we associate these with the geometric quantities

$$(2.21a) \quad M_{\alpha\beta} \stackrel{\text{def}}{=} -\frac{1}{c} P_{\alpha\beta},$$

$$(2.21b) \quad P_{\beta} \stackrel{\text{def}}{=} \varepsilon_0 P_{\beta}.$$

The kernel M denotes the proper magnetic polarization, or *magnetization* and the kernel P denotes the proper electric polarization.

§3. Constitutive Relations for a Linear, Memoryless Material. Even with the generalized velocity u^α prescribed, eqns. (2.1), (2.2) and (2.13) for the field and polarization tensors remain underdetermined. We thus consider the simplest practical example and complete the system of equations with the following linear constitutive relation for a memoryless material with no permanent polarization

$$(3.1) \quad P_{\alpha\beta} = \chi_{\alpha\beta}^{\cdot\cdot\gamma\mu} F_{\gamma\mu}.$$

Equation (2.13) then takes the form

$$(3.2) \quad z_0 G_{\alpha\beta} = * [Q_{\alpha\beta}^{\cdot\cdot\gamma\mu} F_{\gamma\mu}],$$

where

$$(3.3) \quad Q_{\alpha\beta}^{\cdot\cdot\gamma\mu} \stackrel{\text{def}}{=} g_{\alpha}^{\cdot\cdot[\gamma} g_{\beta}^{\cdot\cdot]\mu} + \chi_{\alpha\beta}^{\cdot\cdot[\gamma\mu]}.$$

Using (2.4) and the fact that $**T_{\alpha\beta} = -T_{\alpha\beta}$ in a V_4 , eqn. (3.2) can be written as

$$(3.4) \quad z_0 \tilde{G}^{\alpha\beta} = -\tilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu},$$

where

$$(3.5) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \sqrt{g} g^{\alpha\kappa} g^{\beta\sigma} Q_{\kappa\sigma}^{\cdot\cdot\gamma\mu}.$$

Due to the antisymmetry property of the bivectors $\tilde{G}^{\alpha\beta}$ and $F_{\alpha\beta}$ it is obvious that the tensor Δ -density $\tilde{Q}^{\alpha\beta\gamma\mu}$ is antisymmetric with respect to the pairs of indices $\alpha\beta$ and $\gamma\mu$. Furthermore $z_0 \tilde{G}^{\alpha\beta} F_{\alpha\beta} = -\tilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu} F_{\alpha\beta} = \tilde{Q}^{\gamma\mu\alpha\beta} F_{\alpha\beta} F_{\gamma\mu}$ and upon factoring $F_{\alpha\beta}$ we find that (3.4) can be written in the form

$$z_0 \tilde{G}^{\alpha\beta} = -\tilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu} = \tilde{Q}^{\gamma\mu\alpha\beta} F_{\gamma\mu}.$$

From these considerations it follows that the tensor Δ -density $\tilde{Q}^{\alpha\beta\gamma\mu}$ must have the symmetry properties

$$(3.6) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \tilde{Q}^{\gamma\mu\alpha\beta} = -\tilde{Q}^{\mu\gamma\alpha\beta}.$$

These symmetry properties can also be derived from Green's thermodynamic principle.

Taking the natural decomposition and using (3.6) the quantity $\tilde{Q}^{\alpha\beta\gamma\mu}$ can be expressed in the form

$$(3.7) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \tilde{Q}^{\perp\alpha\beta\gamma\mu} + 2! \left(u_{(1)}^{[\alpha} \tilde{Q}^{\beta]\gamma\mu} + u_{(1)}^{[\gamma} \tilde{Q}^{\mu]\alpha\beta} \right) + 4u_{(2)}^{[\alpha} \tilde{Q}^{\beta][\gamma} u^{\mu]},$$

where

$$(3.8) \quad \underset{(1)}{\tilde{Q}}^{\alpha\beta\gamma} \stackrel{\text{def}}{=} -u_\sigma \underset{(1)}{\tilde{Q}}^{\sigma\frac{1}{2}\alpha\frac{1}{2}\beta\frac{1}{2}\gamma},$$

$$(3.9) \quad \underset{(2)}{\tilde{Q}}^{\alpha\beta} \stackrel{\text{def}}{=} -u_\kappa u_\sigma \underset{(2)}{\tilde{Q}}^{\kappa\frac{1}{2}\alpha\frac{1}{2}\sigma\frac{1}{2}\beta}.$$

Using the decomposition (3.7) and the fact that

$$(3.10) \quad \tilde{G}^{\alpha\beta} = \frac{1}{2!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} G_{\gamma\mu} = -(\tilde{H}^{\alpha\beta} + 2! c u^{[\alpha} \tilde{G}^{\beta]}),$$

(cf. 2.10) eqn. (3.4) can be written in terms of the proper fields as

$$(3.11a) \quad z_0 \tilde{H}^{\alpha\beta} = \underset{(1)}{\tilde{Q}}^{\alpha\beta\gamma\mu} (c B_{\gamma\mu}) + 2! u^{[\gamma} \underset{(1)}{\tilde{Q}}^{\mu]\alpha\beta} (2! u_{[\gamma} E_{\mu]}),$$

$$(3.11b) \quad c z_0 \tilde{D}^\beta = 2 \underset{(2)}{\tilde{Q}}^{\beta[\gamma} u^{\mu]} (2! u_{[\gamma} E_{\mu]}) + \underset{(1)}{\tilde{Q}}^{\beta\gamma\mu} (c B_{\gamma\mu}).$$

A simple form of a complete and consistent relation can be obtained by dropping the cross-coupling terms, i.e. by setting the *optical rotation tensor* Δ -density

$$(3.12) \quad \underset{(1)}{\tilde{Q}}^{\alpha\beta\gamma} \equiv 0.$$

In that case, using (2.18), the relations (3.11a,b) take the form

$$(3.13a) \quad \mu_0 \tilde{H}^{\alpha\beta} = \underset{(1)}{\tilde{Q}}^{\alpha\beta\gamma\mu} B_{\gamma\mu},$$

$$(3.13b) \quad \tilde{D}^\beta = \varepsilon_0 (2 \underset{(2)}{\tilde{Q}}^{\beta\gamma} E_\gamma).$$

We take the spatial dual of the relation (3.13a) and use (2.10) to get

$$(3.14) \quad \mu_0 H_\alpha = \frac{1}{2!} \underset{\sim}{\varepsilon}^{\frac{1}{2}\alpha\beta\gamma\frac{1}{2}} \underset{\sim}{\tilde{Q}}^{\beta\gamma\kappa\sigma} \underset{\sim}{\varepsilon}^{\frac{1}{2}\kappa\sigma\mu\frac{1}{2}} \tilde{B}^\mu = \underset{(r)}{\nu}_{\alpha\mu} \tilde{B}^\mu,$$

where

$$(3.15) \quad \underset{(r)}{\nu}_{\alpha\mu} \stackrel{\text{def}}{=} \frac{1}{2!} \underset{\sim}{\varepsilon}^{\frac{1}{2}\alpha\beta\gamma\frac{1}{2}} \underset{\sim}{\tilde{Q}}^{\beta\gamma\kappa\sigma} \underset{\sim}{\varepsilon}^{\frac{1}{2}\kappa\sigma\mu\frac{1}{2}}.$$

The relation (3.13a) and its spatial dual (3.14) are required to be quasi-invertible in the sense that there exists a symmetric, spatial tensor Δ -density $\tilde{\mu}^{\alpha\beta}_{(r)}$ of weight +1 such that

$$(3.16) \quad \nu_{\alpha\beta} \tilde{\mu}^{\beta\gamma}_{(r)} = \frac{1}{g} g_{\alpha}^{\gamma}.$$

A second symmetric, spatial tensor Δ -density of weight +1 is defined by

$$(3.17) \quad \tilde{\varepsilon}^{\alpha\beta}_{(r)} \stackrel{\text{def}}{=} 2\tilde{Q}^{\alpha\beta}_{(2)}.$$

Relations (3.13a,b) can then be written in the form

$$(3.18a) \quad \tilde{B}^{\alpha} = \tilde{\mu}^{\alpha\beta}_{(r)} \mu_0 H_{\beta},$$

$$(3.18b) \quad \tilde{D}^{\alpha} = \tilde{\varepsilon}^{\alpha\beta}_{(r)} \varepsilon_0 E_{\beta}.$$

The quantity $\tilde{\mu}^{\alpha\beta}_{(r)}$ is called the *relative permeability tensor Δ -density* and the quantity $\tilde{\varepsilon}^{\alpha\beta}_{(r)}$ is called the *relative permittivity tensor Δ -density*.

We remark that we could have postulated (3.18a,b) as the constitutive relations for the proper fields at the outset and worked backwards to (3.1)–(3.4). However the relations (3.1)–(3.4) better facilitate the succeeding analysis and they immediately reveal the possibility of materials exhibiting an optical rotation tensor⁹ and the consistency of such a model. In addition, macroscopic theories of more complicated effects and materials almost always require general constitutive relations in terms of the polarization that cannot be reduced to simple linear relations between the proper fields.¹⁰

A medium characterized by the constitutive law (3.4) is called (*electromagnetically*) *spatially isotropic* if $\tilde{Q}^{\alpha\beta\gamma\mu}$ remains invariant for spatial rotations relative to the generalized 4-velocity u^{α} . In that case, by considering the natural decomposition (3.7), one can readily show that

$$(3.19a) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \tilde{q}_{(0)} \left(\frac{1}{g} g^{\alpha\gamma} \frac{1}{g} g^{\beta\mu} - \frac{1}{g} g^{\alpha\mu} \frac{1}{g} g^{\beta\gamma} \right),$$

$$(3.19b) \quad \tilde{Q}^{\alpha\beta\gamma}_{(1)} = q_{(1)} \tilde{\varepsilon}^{\alpha\beta\gamma}_{(1)},$$

$$(3.19c) \quad \tilde{Q}^{\alpha\beta}_{(2)} = \tilde{q}_{(2)} \frac{1}{g} g^{\alpha\beta},$$

⁹Optical rotation tensors are discussed in Azzam and Bashara [6].

¹⁰Many examples occur in the theory of nonlinear optics and include second-harmonic generation, parametric oscillation and frequency tuning, the electro-optic effect, and the Faraday and Kerr effects (see for example Bloembergen [7], Yariv [8], and Toupin [9]).

where

$$(3.20a) \quad \tilde{q}_{(0)} = \frac{1}{3!} \tilde{Q}^{\alpha\beta}{}_{\alpha\beta},$$

$$(3.20b) \quad q_{(1)} = \frac{1}{3!} \tilde{\varepsilon}_{\alpha\beta\gamma} \tilde{Q}^{\alpha\beta\gamma}{}_{(1)},$$

$$(3.20c) \quad \tilde{q}_{(2)} = \frac{1}{3} \tilde{Q}^{\alpha}{}_{(2)\alpha},$$

and by convention (3.12) we take $q_{(1)} \equiv 0$. We remark that the factors of 3 rather than factors of 4 arise in (3.20a,b,c) due the spatial nature of $\tilde{Q}^{\alpha\beta\gamma\mu}{}_{(1)}$, $\tilde{Q}^{\alpha\beta\gamma}{}_{(1)}$, and $\tilde{Q}^{\alpha\beta}{}_{(2)}$. In addition $\tilde{q}_{(0)}$ and $\tilde{q}_{(2)}$ are required to be positive.

We henceforth consider only spatially isotropic media. In that case $\tilde{Q}^{\alpha\beta\gamma\mu}$ takes the form

$$(3.21) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \tilde{q}_{(0)} \left(\frac{1}{g} \alpha^\gamma \frac{1}{g} \beta^\mu - \frac{1}{g} \alpha^\mu \frac{1}{g} \beta^\gamma \right) - \tilde{q}_{(2)} 4u^{[\alpha} \frac{1}{g} \beta^{\gamma]} [\gamma u^{\mu]}.$$

From (3.3) it follows that in vacuum

$$(3.22) \quad \tilde{q}_{(0)} = \tilde{q}_{(2)} = \frac{1}{2} \sqrt{g}.$$

We have an additional relation between the proper current density and proper electric field

$$(3.23) \quad \tilde{J}^\alpha = \sigma_{(0)(r)} \tilde{\sigma}^{\alpha\beta} E_\beta,$$

where the base *conductivity* $\sigma_{(0)}$ and the *relative conductivity tensor* Δ -density $\tilde{\sigma}^{\alpha\beta}{}_{(r)}$ are related to the electronic charge and to the density, effective mass, and mean free time between collisions of the charge carriers. This linear relation, called *Ohm's Law*, describes the effect of drift of charge carriers under the proper electric field¹¹ and is derived by considering the generalized Lorentz force on the charge carriers when they are slowly moving and $B_{\alpha\beta} \approx 0$. More general relations arise through consideration of the contribution of the magnetic induction and charge carrier velocity (not necessarily u_α) to the Lorentz force.

For perfect dielectrics $\sigma_{(0)} \equiv 0$. In that case, using the fact that $\tilde{j}^\alpha$ is a contravariant vector Δ -density of weight +1, substituting (3.23) in eqn. (2.12), and using (2.10) we get

$$(3.24) \quad \nabla_\alpha (u^\alpha \tilde{\rho}) = 0.$$

¹¹More appropriately, it describes the displacement of the Fermi sphere in momentum space. See Kittel [10], p. 140ff.

If the motion of matter is rigid,¹² expansion of (3.24) and application of (1.6) and (1.3) yields

$$(3.25) \quad \frac{\delta \tilde{\rho}}{\delta s} = 0,$$

everywhere in W_Ω . Equation (3.15) is just a statement of the fact that the electric charge remains constant along the world lines.

The Maxwell equations for a linear, memoryless, spatially isotropic medium thus take the form

$$(3.26) \quad \partial_{[\alpha} F_{\beta\gamma]} = 0,$$

$$(3.27) \quad \partial_\beta (\tilde{Q}^{\alpha\beta\gamma\mu} F_{\gamma\mu}) = -z_0 \tilde{j}^\alpha,$$

where $\tilde{Q}^{\alpha\beta\gamma\mu}$ is given by (3.21) and (3.20 a,c).

§4. Electromagnetic Potentials. Equation (3.26) and the converse of Poincaré's Lemma for a 2-form imply the existence of a C^1 vector A_β such that

$$(4.1) \quad F_{\alpha\beta} = 2! \partial_{[\alpha} A_{\beta]}.$$

The vector A_β is not unique since $F_{\alpha\beta}$ remains invariant under the gauge transformation

$$(4.2) \quad A_\beta \mapsto A_\beta + \partial_\beta \theta,$$

where θ is an arbitrary C^2 absolute scalar. The vector A_β is called the *electromagnetic potential* associated with the electromagnetic field tensor $F_{\alpha\beta}$.

Substituting (4.1) in (3.27b) we arrive at the governing field equation for the electromagnetic potential

$$(4.3) \quad 2! \partial_\beta \{ \tilde{Q}^{\alpha\beta\gamma\mu} \partial_{[\gamma} A_{\mu]} \} = -z_0 \tilde{j}^\alpha.$$

To take advantage of the gauge freedom, we expand eqn. (4.3) and seek a coordinate invariant condition that leaves the field equation in a tractable and enlightening form.

Using the facts that the bracketed expression in eqn. (4.3) represents a bivector Δ -density of weight +1 and that the space-time manifold is a V_4 , the equation can be written

$$(4.4) \quad 2! \nabla_\beta \{ \tilde{Q}^{\alpha\beta\gamma\mu} \nabla_{[\gamma} A_{\mu]} \} = -z_0 \tilde{j}^\alpha.$$

¹²The motion of matter is called *rigid* if the strain rate tensor $e_{\alpha\beta} \stackrel{\text{def}}{=} \nabla_{(\alpha} u_{\beta)} \equiv 0$ everywhere in W_Ω . The process of *symmetrization* over the indices is indicated by a pair of parentheses. See Bressan [2], pp. 142 ff.

The following definitions facilitate the analysis

$$(4.5a) \quad \tilde{T}_{(1)}^{\alpha\beta} \stackrel{\text{def}}{=} \tilde{q}_{(0)} (\frac{1}{g}{}^{\alpha\gamma} \frac{1}{g}{}^{\beta\mu} - \frac{1}{g}{}^{\alpha\mu} \frac{1}{g}{}^{\beta\gamma}) \nabla_{[\gamma} A_{\mu]} = 2! \tilde{q}_{(0)} \frac{1}{g}{}^{\mu[\beta} \nabla^{\frac{1}{\alpha}}] A_{\mu},$$

$$(4.5b) \quad \tilde{T}_{(2)}^{\alpha\beta} \stackrel{\text{def}}{=} 2! \tilde{q}_{(2)} u^{[\alpha} \frac{1}{g}{}^{\beta]\gamma} u^{\mu} \nabla_{\gamma} A_{\mu},$$

$$(4.5c) \quad \tilde{T}_{(3)}^{\alpha\beta} \stackrel{\text{def}}{=} -2! \tilde{q}_{(2)} u^{[\alpha} \frac{1}{g}{}^{\beta]\gamma} u^{\mu} \nabla_{\mu} A_{\gamma}.$$

Equation (4.4) can evidently be written as

$$(4.6) \quad \nabla_{\beta} (\tilde{T}_{(1)}^{\alpha\beta} + \tilde{T}_{(2)}^{\alpha\beta} + \tilde{T}_{(3)}^{\alpha\beta}) = -\frac{z_0}{2!} \tilde{j}^{\alpha}.$$

We now examine each term in eqn. (4.6) separately.

The first term in eqn. (4.6) can be expanded as

$$(4.7) \quad \nabla_{\beta} \tilde{T}_{(1)}^{\alpha\beta} = 2! \nabla_{\beta} \{ \frac{1}{g}{}^{\mu[\beta} \nabla^{\frac{1}{\alpha}}] (\tilde{q}_{(0)} A_{\mu}) - q_{(0)}^{\frac{1}{\alpha}} \tilde{A}_{(1)}^{\beta]} \},$$

where

$$(4.8) \quad q_{(0)}^{\alpha} \stackrel{\text{def}}{=} \frac{1}{\tilde{q}_{(0)}} \nabla^{\alpha} \tilde{q}_{(0)},$$

and

$$(4.9) \quad \tilde{A}_{(1)}^{\beta} \stackrel{\text{def}}{=} \tilde{q}_{(0)} \frac{1}{g}{}^{\beta\mu} A_{\mu}.$$

For any covariant vector Δ -density $\tilde{V}_{\gamma}$, the identity

$$(4.10) \quad \frac{1}{g}{}^{\beta\gamma} \nabla^{\frac{1}{\alpha}} \tilde{V}_{\gamma} \equiv \nabla^{\frac{1}{\alpha}} \tilde{V}^{\frac{1}{\beta}} - \tilde{V}_{\gamma} \nabla^{\frac{1}{\alpha}} (u^{\beta} u^{\gamma})$$

arises trivially from the definition of the spatial projector. Using this identity and definition (4.9), eqn. (4.7) can be written as

$$(4.11) \quad \nabla_{\beta} \tilde{T}_{(1)}^{\alpha\beta} = \nabla_{\beta} \{ \nabla^{\frac{1}{\alpha}} \tilde{A}_{(1)}^{\beta} - \nabla^{\frac{1}{\beta}} \tilde{A}_{(1)}^{\alpha} - 2! (q_{(0)} A_{\gamma} \nabla^{\frac{1}{\alpha}} (u^{\beta]} u^{\gamma}) + q_{(0)}^{\frac{1}{\alpha}} \tilde{A}_{(1)}^{\beta]} \}.$$

Upon adding and subtracting $\nabla^{\alpha} \nabla_{\beta} \tilde{A}_{(1)}^{\beta}$, (4.11) takes the form

$$(4.12) \quad \begin{aligned} \nabla_{\beta} \tilde{T}_{(1)}^{\alpha\beta} = & \{ (\nabla_{\beta} \nabla^{\frac{1}{\alpha}} - \nabla^{\alpha} \nabla_{\beta}) \tilde{A}_{(1)}^{\beta} + \nabla^{\alpha} \nabla_{\beta} \tilde{A}_{(1)}^{\beta} - \nabla_{\beta} \nabla^{\frac{1}{\beta}} \tilde{A}_{(1)}^{\alpha} \\ & - 2! \nabla_{\beta} \{ q_{(0)} A_{\gamma} \nabla^{\frac{1}{\alpha}} (u^{\beta]} u^{\gamma}) + q_{(0)}^{\frac{1}{\alpha}} \tilde{A}_{(1)}^{\beta]} \}. \end{aligned}$$

Again using the definition of the spatial projector, we write

$$\nabla_\beta \nabla_{(1)}^\perp \tilde{A}^\beta = \nabla_\beta \nabla_{(1)}^\alpha \tilde{A}^\beta + \nabla_\beta (u^\alpha u^\gamma \nabla_{(1)} \tilde{A}^\beta),$$

and

$$\nabla_\beta \nabla_{(1)}^\perp \tilde{A}^\alpha = \nabla_\beta \nabla_{(1)}^\beta \tilde{A}^\alpha + \nabla_\beta (u^\beta u^\gamma \nabla_{(1)} \tilde{A}^\alpha).$$

Substituting these in (4.12) we get

$$\begin{aligned} \nabla_\beta \tilde{T}^{\alpha\beta} &= R_{\beta(1)}^\alpha \tilde{A}^\beta + \nabla^\alpha \nabla_{\beta(1)} \tilde{A}^\beta - \nabla_\beta \nabla_{(1)}^\beta \tilde{A}^\alpha \\ (4.13) \quad &+ 2! \nabla_\beta \{ u^\gamma u^{[\alpha} \nabla_{(1)} \tilde{A}^{\beta]} - \tilde{q}_{(0)} A_\gamma \nabla^{[\alpha} (u^{\beta]} u^\gamma) - q_{(0)}^{[\alpha} \tilde{A}^{\beta]} \}, \end{aligned}$$

where $R_{\alpha\beta} \stackrel{\text{def}}{=} R_{\gamma\alpha\beta}^\gamma$ is the Ricci tensor of the space-time V_4 .

Expansion of the second term in (4.6) yields

$$\begin{aligned} \nabla_\beta \tilde{T}^{\alpha\beta} &= \nabla_\beta \{ \tilde{q}_{(2)} (u^\alpha \tilde{g}^{\perp\beta\gamma} u^\mu - u^\beta \tilde{g}^{\perp\alpha\gamma} u^\mu) \nabla_\gamma A_\mu \}, \\ (4.14) \quad &= \nabla_\beta \{ \tilde{q}_{(2)} (u^\alpha u^\mu \nabla_{(2)}^\perp A_\mu - u^\beta u^\mu \nabla_{(2)}^\alpha A_\mu) \}, \\ &= \nabla_\beta \{ \nabla_{(2)}^\perp \tilde{A}^\alpha - \nabla_{(2)}^\alpha \tilde{A}^\beta + 2! q_{(2)}^{[\alpha} \tilde{A}^{\beta]} + 2! \tilde{q}_{(2)} A_\mu \nabla^{[\alpha} (u^{\beta]} u^\mu) \}, \end{aligned}$$

where

$$(4.15) \quad \tilde{A}^{\alpha(2)} \stackrel{\text{def}}{=} \tilde{q}_{(2)} u^\alpha u^\mu A_\mu,$$

and

$$(4.16) \quad q_{(2)}^\alpha \stackrel{\text{def}}{=} \frac{1}{\tilde{q}_{(2)}} \nabla^\alpha \tilde{q}_{(2)}.$$

Adding and subtracting $\nabla^\alpha \nabla_{\beta(2)} \tilde{A}^\beta$, (4.14) takes the form

$$\begin{aligned} \nabla_\beta \tilde{T}^{\alpha\beta} &= -(\nabla_\beta \nabla_{(2)}^\perp - \nabla^\alpha \nabla_{\beta(2)}) \tilde{A}^\beta + \nabla_\beta \nabla_{(2)}^\perp \tilde{A}^\alpha - \nabla^\alpha \nabla_{\beta(2)} \tilde{A}^\beta \\ (4.17) \quad &+ 2! \nabla_\beta \{ q_{(2)}^{[\alpha} \tilde{A}^{\beta]} + \tilde{q}_{(2)} A_\mu \nabla^{[\alpha} (u^{\beta]} u^\mu) \}. \end{aligned}$$

In a manner similar to that used in (4.12), eqn. (4.17) can be rewritten as

$$(4.18) \quad \begin{aligned} \nabla_\beta \tilde{T}_{(2)}^{\alpha\beta} &= -R_{\dot{\beta}}^\alpha \tilde{A}_{(2)}^\beta - \nabla^\alpha \nabla_\beta \tilde{A}_{(2)}^\beta + \nabla_\beta \nabla^\beta \tilde{A}_{(2)}^\alpha \\ &\quad - 2! \nabla_\beta \{ u^\gamma u^{[\alpha} \nabla_\gamma \tilde{A}_{(2)}^{\beta]} - \tilde{q}_{(2)} A_\gamma \nabla^{[\alpha} u^{\beta]} u^\gamma - q_{(2)}^{[\alpha} \tilde{A}_{(2)}^{\beta]} \}. \end{aligned}$$

Finally, expanding the last term in eqn. (4.6) yields the result

$$(4.19) \quad \begin{aligned} \nabla_\beta \tilde{T}_{(3)}^{\alpha\beta} &= \nabla_\beta \{ \tilde{q}_{(2)} (u^\beta \tilde{g}^{\perp\alpha\gamma} u^\mu - u^\alpha \tilde{g}^{\perp\beta\gamma} u^\mu) \nabla_\mu A_\gamma \}, \\ &= \nabla_\beta \{ \tilde{q}_{(2)} (u^\beta u^\mu g^{\alpha\gamma} - u^\alpha u^\mu g^{\beta\gamma}) \nabla_\mu A_\gamma \}, \\ &= 2! \nabla_\beta \{ u^\mu q_{(2)\mu} u^{[\alpha} \tilde{q}_{(2)} A^{\beta]} - u^\gamma u^{[\alpha} \nabla_\gamma (\tilde{q}_{(2)} A^{\beta]} \} \}. \end{aligned}$$

We define a new contravariant vector Δ -density of weight +1

$$(4.20) \quad \tilde{A}_{(3)}^\beta \stackrel{\text{def}}{=} \tilde{A}_{(1)}^\beta - \tilde{A}_{(2)}^\beta = \tilde{q}_{(0)} A^\beta - \tilde{q}_{(2)} u^\beta u_\gamma A^\gamma.$$

Using this definition and (4.6), (4.13), (4.18) and (4.19), the field equation (4.6) can be written

$$(4.21) \quad \begin{aligned} &\nabla_\beta \nabla^\beta \tilde{A}_{(3)}^\alpha - R_{\dot{\beta}}^\alpha \tilde{A}_{(3)}^\beta - \nabla^\alpha \nabla_\beta \tilde{A}_{(3)}^\beta + 2! \nabla_\beta \{ (\tilde{q}_{(0)} - \tilde{q}_{(2)}) A_\gamma \nabla^{[\alpha} (u^{\beta]} u^\gamma) \} \\ &\quad - 2! \nabla_\beta \{ u^\gamma u^{[\alpha} \nabla_\gamma \tilde{A}_{(3)}^{\beta]} - u^\gamma u^{[\alpha} \nabla_\gamma (\tilde{q}_{(2)} A^{\beta]} \} \\ &\quad - 2! \nabla_\beta \{ q_{(2)}^{[\alpha} \tilde{A}_{(2)}^{\beta]} - q_{(0)}^{[\alpha} \tilde{A}_{(1)}^{\beta]} + u_\mu q_{(2)}^\mu u^{[\alpha} \tilde{q}_{(2)} A^{\beta]} \} = \frac{z_0}{2!} j^\alpha. \end{aligned}$$

After some lengthy algebraic details and division by $\sqrt{g}$, eqn. (4.21) can be shown to have the coordinate-invariant form

$$(4.22) \quad \begin{aligned} &\nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\dot{\beta}}^\alpha A_{(3)}^\beta - \nabla^\alpha \nabla_\beta A_{(3)}^\beta + \nabla_\beta \{ 2! q_{(2)}^{[\alpha} A_{(3)}^{\beta]} \} + 2! \nabla_\beta \left\{ \left(q_{(0)}^{[\alpha} - q_{(2)}^{[\alpha} \right) A_{(1)}^{\beta]} \right\} \\ &\quad + 2! \nabla_\beta \left\{ \left(q_{(0)} - q_{(2)} \right) \left(A_\gamma \nabla^{[\alpha} (u^{\beta]} u^\gamma) - u^{[\alpha} \frac{\delta}{\delta s} A_{(1)}^{\beta]} \right) \right\} = \frac{z_0}{2!} j^\alpha, \end{aligned}$$

where $A_{(3)}^\beta = g^{-1/2} \tilde{A}_{(3)}^\beta$, $A_{(1)}^\beta = g^{-1/2} \tilde{A}_{(1)}^\beta$, etc.

At first glance eqn. (4.22) appears involved, but further analysis reveals its simplicity. In the case when $\tilde{Q}^{\alpha\beta\gamma\mu}$ is an isotropic tensor Δ -density of weight +1 rather than a spatially isotropic tensor Δ -density of weight +1 we have

$$(4.23) \quad q_{(0)} \equiv q_{(2)} \stackrel{\text{def}}{=} q(x^\alpha),$$

$$(4.24) \quad q_{(0)}^\alpha \equiv q_{(0)}^\alpha \stackrel{\text{def}}{=} q^\alpha,$$

$$(4.25) \quad \tilde{Q}^{\alpha\beta\gamma\mu} = \sqrt{g} q (g^{\alpha\gamma} g^{\beta\mu} - g^{\alpha\mu} g^{\beta\gamma}),$$

and

$$(4.26) \quad A_{(3)}^\beta = q(\frac{1}{g}{}^\beta{}_\gamma A^\gamma - u^\beta u^\gamma A_\gamma) = qA^\beta.$$

Equation (4.22) therefore immediately reduces to

$$(4.27) \quad \nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\beta(3)}^\alpha A_{(3)}^\beta - \nabla^\alpha \nabla_\beta A_{(3)}^\beta + 2! \nabla_\beta \{q^{[\alpha} A_{(3)}^{\beta]} \} = \frac{z_0}{2!} j^\alpha.$$

This can be verified by substituting (4.25) in (4.4) to get (after dividing by $\sqrt{g}$)

$$2! \nabla_\beta \{q \nabla^{[\alpha} A^{\beta]} \} = -\frac{z_0}{2!} j^\alpha,$$

or

$$(4.28) \quad -2! \nabla_\beta \{ \nabla^{[\alpha} (q A^{\beta]}) - q^{[\alpha} A^{\beta]} \} = \frac{z_0}{2!} j^\alpha.$$

Adding and subtracting $\nabla^\alpha \nabla_\beta (q A^\beta)$ and using (4.26), (4.28) can be written as

$$\nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\beta(3)}^\alpha A_{(3)}^\beta - \nabla^\alpha \nabla_\beta A_{(3)}^\beta + 2! \nabla_\beta \{q^{[\alpha} A_{(3)}^{\beta]} \} = \frac{z_0}{2!} j^\alpha,$$

which is in accord with eqn. (4.27). If the medium is homogeneous as well as isotropic, q^α vanishes, q is a covariant constant, and eqn. (4.22) simplifies to

$$(4.29) \quad \nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\beta(3)}^\alpha A_{(3)}^\beta - \nabla^\alpha \nabla_\beta A_{(3)}^\beta = \frac{z_0}{2!} j^\alpha.$$

In vacuum, $q = 1/2$ and thus eqn. (4.29) takes the familiar form

$$(4.30) \quad \nabla_\beta \nabla^\beta A^\alpha - R_{\beta}^\alpha A^\beta - \nabla^\alpha \nabla_\beta A^\beta = z_0 j^\alpha.$$

Note that in most normalized systems of units (not MKS) $z_0 \equiv 1$.

We remark that the analysis of this section is independent of any assumptions concerning rigidity, angular velocity and/or intrinsic acceleration. Consider the sixth term on the left hand side of eqn. (4.22)

$$(4.31) \quad 2! \nabla_\beta \{ (\frac{q}{(0)} - \frac{q}{(2)}) A_\gamma \nabla^{[\alpha} (u^{\beta]} u^\gamma) \} = \nabla_\beta \{ (\frac{q}{(0)} - \frac{q}{(2)}) A_\gamma (u^\gamma \omega^{\alpha\beta} - 2! u^{[\alpha} e^{\beta]\gamma} - 2! u^{[\alpha} \omega^{\beta]\gamma}) \},$$

where $\omega_{\alpha\beta}$ is the *local angular velocity (or average rate of rotation)*,

$$(4.32) \quad \omega_{\alpha\beta} \stackrel{\text{def}}{=} \nabla_{[\alpha} \frac{1}{\alpha} u_{\beta]},$$

and $e_{\alpha\beta}$ is the *strain rate tensor*

$$(4.33) \quad e_{\alpha\beta} \stackrel{\text{def}}{=} \nabla_{(\alpha} \frac{1}{\alpha} u_{\beta)}.$$

Thus if the motion of matter is both rigid ($e_{\alpha\beta} \equiv 0$) and rotation free¹³ ($\omega_{\alpha\beta} \equiv 0$), then the term (4.31) in eqn. (4.22) vanishes.

¹³The physical interpretation of the statement $\omega_{\alpha\beta} \equiv 0$ is that the motion of matter is such that in a neighborhood of a world line of a material point the medium does not rotate with respect to an orthonormal tetrad carried along the world line by Fermi-Walker transport. See Synge [1], p. 173.

§5. Generalized Lorentz Gauge. Equation (4.22) can be simplified by fixing the *generalized Lorentz Gauge*

$$(5.1) \quad \nabla_{\beta} A_{(3)}^{\beta} = 0.$$

This coordinate invariant condition immediately reduces to the classical Lorentz condition in the case of isotropic homogeneous media. In terms of tensor Δ -densities, the generalized Lorentz Gauge can be written

$$(5.2) \quad \frac{1}{\sqrt{g}} \partial_{\beta} \tilde{A}_{(3)}^{\beta} = 0,$$

and therefore in terms of the dual representation

$$(5.3) \quad A_{(3)\alpha\beta\gamma} \stackrel{\text{def}}{=} \varepsilon_{\alpha\beta\gamma\mu} \tilde{A}_{(3)}^{\mu},$$

it has the alternate coordinate invariant form

$$(5.4) \quad \partial_{[\alpha} A_{\beta\gamma\mu]} = 0.$$

We remark that fixing the generalized Lorentz gauge does not uniquely specify the vector A_{β} . In fact, form invariance of (5.1) requires that in the gauge transformation (4.2) the additive scalar θ may be any solution of the equation

$$(5.5) \quad \nabla_{\beta} \left(q_{(0)} \nabla^{\beta} \theta \right) - \nabla_{\beta} \left(q_{(2)} u^{\beta} \frac{\delta \theta}{\delta s} \right) = 0.$$

By considering eqn. (5.5) in a locally natural and proper frame and using the fact that $q_{(0)}$ and $q_{(2)}$ are positive, one can easily see that the equation is hyperbolic. This means that the solution depends only on the initial conditions for θ , which are completely arbitrary.

Substituting (5.1) in eqn. (4.22), the governing field equation for the electromagnetic potential takes the form

$$(5.6) \quad \begin{aligned} \nabla_{\beta} \nabla^{\beta} \tilde{A}_{(3)}^{\alpha} - R_{\beta}^{\alpha} \tilde{A}_{(3)}^{\beta} + 2! \nabla_{\beta} \left\{ q_{(0)}^{[\alpha} A_{(1)}^{\beta]} - q_{(2)}^{[\alpha} A_{(2)}^{\beta]} \right\} \\ + 2! \nabla_{\beta} \left\{ \left(q_{(0)} - q_{(2)} \right) \left(A_{\gamma} \nabla^{[\alpha} (u^{\beta]} u^{\gamma}) - u^{[\alpha} \frac{\delta}{\delta s} A_{(2)}^{\beta]} \right) \right\} = \frac{z_0}{2!} j^{\alpha}, \end{aligned}$$

subject to the condition (5.1), and where we have used the fact that

$$(5.7) \quad q_{(2)}^{[\alpha} A_{(3)}^{\beta]} + \left(q_{(0)}^{[\alpha} - q_{(2)}^{[\alpha} \right) A_{(1)}^{\beta]} = q_{(0)}^{[\alpha} A_{(1)}^{\beta]} - q_{(2)}^{[\alpha} A_{(2)}^{\beta]}.$$

For rigid motion that is locally rotation-free, eqn. (5.6) simplifies to

$$(5.8) \quad \nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\beta(3)}^\alpha A_{(3)}^\beta + 2! \nabla_\beta \left\{ q_{(0)}^{[\alpha} A_{(1)}^{\beta]} - q_{(2)}^{[\alpha} A_{(2)}^{\beta]} - (q_{(0)} - q_{(2)}) u^{[\alpha} \frac{\delta}{\delta s} A_{(2)}^{\beta]} \right\} = \frac{z_0}{2!} j^\alpha.$$

For homogeneous isotropic media, equation (5.1) immediately reduces to the classical Lorentz gauge condition and equation (5.6) becomes

$$(5.9) \quad \nabla_\beta \nabla^\beta A_{(3)}^\alpha - R_{\beta(3)}^\alpha A_{(3)}^\beta = \frac{z_0}{2!} j^\alpha,$$

which in vacuum takes the standard form

$$(5.10) \quad \nabla_\beta \nabla^\beta A^\alpha - R_{\beta}^\alpha A^\beta = z_0 j^\alpha.$$

§6. Superpotentials. The governing field equation (5.6) for the electromagnetic potention A_γ represents a coupled system of partial differential equations which have to be solved in conjunction with the generalized Lorentz condition (5.1). Equation (5.6) forms a consistent set of partial differential equations and existence of a solution of this system is guaranteed only if (5.1) holds simultaneously. Finding a solution is therefore difficult because only those solutions of eqn. (5.6) which simultaneously satisfy condition (5.1) are admissible. To alleviate this problem, we now introduce superpotentials which satisfy the generalized Lorentz condition identically.

Equation (5.4) and the converse of Poincaré's Lemma for a 3-form in V_4 imply that locally there exists a C^2 covariant bivector $\Pi_{\alpha\beta}$ such that

$$(6.1) \quad A_{(3)\alpha\beta\gamma} = 3! \partial_{[\alpha} \Pi_{\beta\gamma]}.$$

The bivector $\Pi_{\alpha\beta}$ is not unique since the representation (6.1) admits the gauge transformation

$$(6.2) \quad \Pi_{\alpha\beta} \mapsto \Pi_{\alpha\beta} + 2! \partial_{[\alpha} \psi_{\beta]},$$

where ψ_β is an arbitrary C^3 covariant vector. In terms of the dual representations

$$(6.3) \quad \tilde{A}_{(3)}^\alpha = \frac{1}{3!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} A_{\beta\gamma\mu},$$

$$(6.4) \quad \tilde{\Pi}^{\alpha\beta} = \frac{1}{2!} \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \Pi_{\gamma\mu},$$

eqn. (6.1) can be written as

$$(6.5) \quad \tilde{A}_{(3)}^\alpha = 2! \nabla_\beta \tilde{\Pi}^{\alpha\beta}.$$

The bivector $\Pi_{\alpha\beta}$ is called the (*electromagnetic*) *superpotential associated with the electromagnetic potential* A_γ .

For the sake of simplicity in deriving the governing field equation for the superpotential $\Pi_{\alpha\beta}$, we assume that the motion of the material is locally rotation-free ($\omega_{\alpha\beta} = \nabla_{[\alpha} u_{\beta]} \equiv 0$) and that the medium is a *perfect (spatially isotropic) dielectric*, that is

$$(6.6) \quad \underset{(0)}{\sigma} \equiv 0, \quad \underset{(0)}{q}^\alpha \equiv 0, \quad e_{\alpha\beta} = \nabla_{(\alpha} u_{\beta)} \equiv 0.$$

In the event that $\omega_{\alpha\beta} \neq 0$ and/or any part of (6.6) does not hold, derivation of the governing field equation for $\Pi_{\alpha\beta}$ follows from our analysis in an obvious and straightforward manner.

Since $\omega_{\alpha\beta} \equiv 0$, it follows that

$$(6.7) \quad u_{[\alpha} \nabla_\beta u_{\gamma]} = u_{[\alpha} u_\beta a_{\gamma]} = 0,$$

and therefore due to the time-like nature of the world lines (meaning the true square $-u^\alpha u_\alpha > 0$), the vector field u_α is *V₃-forming*,¹⁴ in other words, there exists a family of V_3 's to which the world lines are orthogonal. Thus the stream-lines form a *normal congruence*.¹⁵ The family of V_3 's represent cross-sections of the world tube W_Ω and can be parameterized by s . We assume that there is a cross-section indicated by $s = s_0$ for which $\tilde{\rho}(x^\alpha(s_0)) \equiv 0$. Then (6.6) and eqn. (3.25) imply that $\tilde{\rho} \equiv 0$ everywhere in W_Ω , and hence $\tilde{j}^\alpha \equiv 0$ everywhere in W_Ω .

Substituting the representation (6.5) in eqn. (5.6) and using the above simplifying assumptions and identity (A1.1) from the Appendix we readily obtain

$$(6.8) \quad \frac{1}{\sqrt{g}} \nabla_\beta \left\{ \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h) u^{[\alpha} \frac{\delta}{\delta s} (\frac{1}{g})^{\beta]}_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} \right. \\ \left. - 2! \underset{(2)}{q}^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} - 3R_{\gamma\sigma}^{\cdot\cdot[\alpha\beta} \tilde{\Pi}^{\gamma]\sigma} \right\} = 0,$$

where

$$(6.9) \quad h \stackrel{\text{def}}{=} \underset{(2)}{q} / \underset{(0)}{q}.$$

Evidently, the quantity within the braces is a contravariant bivector Δ -density of weight +1, and the covariant derivative can thus be replaced by a partial derivative. It immediately follows from the converse of Poincare's Lemma for a 2-form in V_4 that there exists a C^1 covariant vector ϕ_β such that

$$(6.10) \quad \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h) u^{[\alpha} \frac{\delta}{\delta s} (\frac{1}{g})^{\beta]}_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} - 2! \underset{(2)}{q}^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} \\ - 3R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta} \tilde{\Pi}^{\gamma]\sigma} = \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \partial_{[\gamma} \phi_{\mu]}.$$

¹⁴Schouten [3], p. 244 ff.

¹⁵Synge [1], p. 173.

The dual representation of the gauge transformation (6.2) can be written as

$$(6.11) \quad \tilde{\Pi}^{\alpha\beta} \mapsto \tilde{\Pi}^{\alpha\beta} + \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \partial_{[\gamma} \psi_{\mu]} = \tilde{\Pi}^{\alpha\beta} + 2! \partial_\gamma \tilde{\psi}^{\alpha\beta\gamma},$$

where

$$(6.12) \quad \tilde{\psi}^{\alpha\beta\gamma} = \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \psi_\mu.$$

Using this in (6.10) and applying identity (A1.6) from the Appendix we get

$$(6.13) \quad \begin{aligned} \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h)u^{[\alpha} \frac{\delta}{\delta s} (\frac{1}{g}{}^\beta{}_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma}) - 2! q^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} \\ - 3R_{\kappa\sigma}{}^{\cdot\cdot[\alpha\beta} \tilde{\Pi}^{\kappa]\sigma} = -2! \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\sigma}{}^{\cdot\cdot[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} \right. \\ \left. - 3R_{\kappa\sigma}{}^{\cdot\cdot[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} - \tilde{\phi}^{\alpha\beta\gamma} \right\}, \end{aligned}$$

where

$$(6.14) \quad \tilde{\phi}^{\alpha\beta\gamma} = \tilde{\varepsilon}^{\alpha\beta\gamma\mu} \phi_\mu.$$

The freedom to choose ψ_β (or $\tilde{\psi}^{\alpha\beta\gamma}$) and any attached initial conditions then allows us to suppress the unknown ϕ_β (or $\tilde{\phi}^{\alpha\beta\gamma}$) from the preceding analysis simply by noting that a solution of the equation

$$(6.15) \quad \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\sigma}{}^{\cdot\cdot[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} - 3R_{\kappa\sigma}{}^{\cdot\cdot[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} = \tilde{\phi}^{\alpha\beta\gamma}$$

exists. The ensuing field equation for the superpotential $\tilde{\Pi}^{\alpha\beta}$ thus takes the simple form

$$(6.16) \quad \begin{aligned} \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 2!(1-h)u^{[\alpha} \frac{\delta}{\delta s} (\frac{1}{g}{}^\beta{}_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma}) \\ - 2! q^{[\alpha} u^{\beta]} u_\gamma \nabla_\sigma \tilde{\Pi}^{\gamma\sigma} - 3R_{\kappa\sigma}{}^{\cdot\cdot[\alpha\beta} \tilde{\Pi}^{\kappa]\sigma} = 0. \end{aligned}$$

In the case of homogeneous, isotropic media, eqn. (6.16) reduces to

$$(6.17) \quad \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - 3R_{\kappa\sigma}{}^{\cdot\cdot[\alpha\beta} \tilde{\Pi}^{\kappa]\sigma} = 0.$$

We remark that in classical, non-relativistic theory, the curvature tensor vanishes and $\nabla_\alpha u_\beta \equiv 0$, and eqn. (6.17) thus takes the limiting form

$$(6.18) \quad \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} = 0.$$

By taking the natural decomposition of $\tilde{\Pi}^{\alpha\beta}$

$$(6.19) \quad \tilde{\Pi}^{\alpha\beta} = \tilde{\Pi}_{(m)}^{\alpha\beta} + 2! \sqrt{g} u^{[\alpha} \Pi_{(e)}^{\beta]},$$

where

$$(6.20a) \quad \tilde{\Pi}_{(m)}^{\alpha\beta} \stackrel{\text{def}}{=} \tilde{\Pi}^{\perp\alpha\beta} = \frac{1}{\varepsilon} \alpha^{\beta\gamma} \Pi_{(m)\gamma},$$

$$(6.20b) \quad \Pi_{(e)}^{\alpha} \stackrel{\text{def}}{=} \frac{1}{\sqrt{g}} \frac{1}{\beta} \tilde{\Pi}^{\beta\gamma} u_{\gamma},$$

eqn. (6.18) yields one equation for the spatial covariant vector $\Pi_{(m)\gamma}$ and one for the spatial contravariant vector $\Pi_{(e)}^{\gamma}$, namely

$$(6.21) \quad \nabla_{\sigma} \nabla^{\sigma} \Pi_{(m)\gamma} = 0,$$

$$(6.22) \quad \nabla_{\sigma} \nabla^{\sigma} \Pi_{(e)}^{\gamma} = 0.$$

Evidently, these two spatial vectors are related to the Hertz vectors of classical electrodynamics¹⁶, and the superpotential $\tilde{\Pi}^{\alpha\beta}$ is therefore of the Hertzian type. It is interesting to note that natural decomposition of the gauge transformation (6.2) gives rise to a form which is at variance with that of the standard gauge transformations for the classical Hertz vectors.¹⁷ This strongly leads us to the conclusion that the standard gauge transformations used in classical electrodynamics are geometrically and relativistically inconsistent, and will be the topic of a forthcoming paper.

Appendix 1. In this appendix, two tensor identities invoked in §6 are proved. Derivation of each identity is motivated by the desire to write a certain geometric quantity as the divergence of another quantity that contains only the Laplace-Beltrami operator¹⁸ acting on a contravariant p -vector Δ -density of weight $+1$ plus transvections of the curvature tensor with the same p -vector Δ -density.

IDENTITY 1. Let $\tilde{\Pi}^{\alpha\beta}$ be a contravariant bivector Δ -density of weight $+1$ in a V_n . Then

$$(A1.1) \quad \nabla_{\beta} \nabla^{\beta} \nabla_{\sigma} \tilde{\Pi}^{\alpha\sigma} - R_{\beta}^{\alpha} \nabla_{\sigma} \tilde{\Pi}^{\beta\sigma} \equiv \nabla_{\beta} \left\{ \nabla_{\sigma} \nabla^{\sigma} \tilde{\Pi}^{\alpha\beta} - 3 R_{\kappa\sigma}^{\cdot\cdot} [\alpha\beta \tilde{\Pi}^{\kappa}\sigma] \right\},$$

where $R_{\alpha\beta\gamma}^{\cdot\cdot\mu}$ is the curvature tensor of the V_n and R_{β}^{α} is the Ricci tensor

$$(A1.2) \quad R_{\beta}^{\alpha} \stackrel{\text{def}}{=} R_{\gamma\beta}^{\cdot\cdot\alpha\gamma}.$$

¹⁶Stratton [11], p. 28 ff.

¹⁷Born and Wolf [12], p. 81.

¹⁸Note that the operator is actually of the D'Alembertian type due to the indefinite nature of the metric tensor.

Proof. Expanding the left-hand side of (A1.1) and using the fact that $\tilde{\Pi}^{\alpha\beta}$ is a contravariant bivector Δ -density of weight +1 in a V_n , we get

$$\begin{aligned}
\nabla_\beta \nabla^\beta \nabla_\sigma \tilde{\Pi}^{\alpha\sigma} - R_{\dot{\beta}}^{\dot{\alpha}} \nabla_\sigma \tilde{\Pi}^{\beta\sigma} &= \nabla_\beta (\nabla^\beta \nabla_\sigma \tilde{\Pi}^{\alpha\sigma} - \nabla^\alpha \nabla_\sigma \tilde{\Pi}^{\beta\sigma}) \\
&= \nabla_\beta \left\{ (\nabla^\beta \nabla_\sigma - \nabla_\sigma \nabla^\beta) \tilde{\Pi}^{\alpha\sigma} - (\nabla^\alpha \nabla_\sigma - \nabla_\sigma \nabla^\alpha) \tilde{\Pi}^{\beta\sigma} \right. \\
&\quad \left. + \nabla_\sigma \nabla^\beta \tilde{\Pi}^{\alpha\sigma} - \nabla_\sigma \nabla^\alpha \tilde{\Pi}^{\beta\sigma} \right\}, \\
(A1.3) \quad &= \nabla_\beta \left\{ g^{\beta\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\beta\sigma} \right. \\
&\quad \left. + \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} - \nabla_\sigma (\nabla^\alpha \tilde{\Pi}^{\beta\sigma} + \nabla^\beta \tilde{\Pi}^{\sigma\alpha} - \nabla^\sigma \tilde{\Pi}^{\beta\alpha}) \right\}, \\
&= \nabla_\beta \left\{ g^{\beta\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\beta\sigma} \right. \\
&\quad \left. + \nabla_\sigma \nabla^\sigma \tilde{\Pi}^{\alpha\beta} \right\} - 3 \nabla_\beta \nabla_\sigma (\nabla^{[\alpha} \tilde{\Pi}^{\beta\sigma]}).
\end{aligned}$$

Evidently, since $\nabla^{[\alpha} \tilde{\Pi}^{\beta\sigma]}$ is a contravariant trivector Δ -density of weight +1, we have

$$(A1.4) \quad \nabla_\beta \nabla_\sigma (\nabla^{[\alpha} \tilde{\Pi}^{\beta\sigma]}) = \nabla_\beta \partial_\sigma (\nabla^{[\alpha} \tilde{\Pi}^{\beta\sigma]}) = \partial_\beta \partial_\sigma (\nabla^{[\alpha} \tilde{\Pi}^{\beta\sigma]}) \equiv 0.$$

Also

$$\begin{aligned}
&g^{\beta\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\alpha\sigma} - g^{\alpha\gamma} 2! \nabla_{[\gamma} \nabla_{\sigma]} \tilde{\Pi}^{\beta\sigma} \\
&= R_{\dot{\sigma}\dot{\kappa}}^{\dot{\beta}\dot{\alpha}} \tilde{\Pi}^{\kappa\sigma} + R_{\dot{\sigma}\dot{\kappa}}^{\dot{\beta}\dot{\sigma}} \tilde{\Pi}^{\alpha\kappa} - R_{\dot{\sigma}\dot{\kappa}}^{\dot{\alpha}\dot{\beta}} \tilde{\Pi}^{\kappa\sigma} - R_{\dot{\sigma}\dot{\kappa}}^{\dot{\alpha}\dot{\sigma}} \tilde{\Pi}^{\beta\kappa}, \\
(A1.5) \quad &= (R_{\dot{\sigma}\dot{\kappa}}^{\dot{\beta}\dot{\alpha}} - R_{\dot{\sigma}\dot{\kappa}}^{\dot{\alpha}\dot{\beta}}) \tilde{\Pi}^{\kappa\sigma} + R_{\dot{\sigma}\dot{\kappa}}^{\dot{\beta}\dot{\sigma}} \tilde{\Pi}^{\alpha\kappa} - R_{\dot{\sigma}\dot{\kappa}}^{\dot{\alpha}\dot{\sigma}} \tilde{\Pi}^{\beta\kappa}, \\
&= -R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\beta}} \tilde{\Pi}^{\kappa\sigma} + R_{\dot{\kappa}\dot{\sigma}}^{\dot{\sigma}\dot{\beta}} \tilde{\Pi}^{\alpha\kappa} - R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\sigma}} \tilde{\Pi}^{\beta\kappa}, \\
&= -R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\beta}} \tilde{\Pi}^{\kappa\sigma} - R_{\dot{\kappa}\dot{\sigma}}^{\dot{\sigma}\dot{\beta}} \tilde{\Pi}^{\alpha\sigma} + R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\kappa}} \tilde{\Pi}^{\beta\sigma}, \\
&= -3 R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\beta}} [\tilde{\Pi}^{\kappa\sigma}]^{\dot{\alpha}\dot{\beta}},
\end{aligned}$$

where we have used the symmetry properties of the curvature tensor¹⁹ and relabelled various dummy indices where required.

Substituting (A1.4) and (A1.5) in (A1.3), identity (A1.1) results immediately. $\blacksquare$

IDENTITY 2. Let $\tilde{\psi}^{\alpha\beta\gamma}$ be a contravariant trivector Δ -density of weight +1 in a V_n . Then

$$\begin{aligned}
(A1.6) \quad &\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\beta}} [\nabla_\gamma \tilde{\psi}^{\kappa\sigma}]^{\dot{\alpha}\dot{\beta}} \\
&\equiv \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\dot{\sigma}}^{[\alpha} \tilde{\psi}^{\beta\gamma]\dot{\sigma}} - 3 R_{\dot{\kappa}\dot{\sigma}}^{\dot{\alpha}\dot{\beta}} [\tilde{\psi}^{\gamma\kappa}]^{\dot{\alpha}\dot{\beta}} \right\}.
\end{aligned}$$

¹⁹Schouten [3], pp. 144 ff.

Proof. Expansion of the left-hand side of (A1.6) yields

$$\begin{aligned}
& \nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\cdot\cdot[\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} \\
&= \nabla_\gamma \{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} \} + \nabla^\sigma \{ 2! \nabla_{[\sigma} \nabla_{\gamma]} \tilde{\psi}^{\alpha\beta\gamma} \} \\
&\quad + 2! g^{\sigma\mu} \nabla_{[\mu} \nabla_{\gamma]} \nabla_\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\cdot\cdot[\alpha\beta} \nabla_{|\gamma|} \tilde{\psi}^{\kappa]\sigma\gamma}, \\
&= \nabla_\gamma \{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + g^{\gamma\mu} 2! \nabla_{[\mu} \nabla_{\sigma]} \tilde{\psi}^{\alpha\beta\sigma} \} \\
&\quad + 2! g^{\sigma\mu} \nabla_{[\mu} \nabla_{\gamma]} \nabla_\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\cdot\cdot[\alpha\beta} \nabla_{|\gamma|} \tilde{\psi}^{\kappa]\sigma\gamma},
\end{aligned}
\tag{A1.7}$$

where a pair of bars around an index indicate that the index is to be excluded from the process of alternation or symmetrization. For the sake of economy, in order to obviate extensive algebra, we define

$$\tilde{\Psi}^{\alpha\beta} \stackrel{\text{def}}{=} 2! g^{\sigma\mu} \nabla_{[\mu} \nabla_{\gamma]} \nabla_\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3R_{\kappa\sigma}^{\cdot\cdot\cdot[\alpha\beta} \nabla_{|\gamma|} \tilde{\psi}^{\kappa]\sigma\gamma}.
\tag{A1.8}$$

Expanding (A1.8), we get

$$\begin{aligned}
\tilde{\Psi}^{\alpha\beta} &= -R_{\gamma\sigma}^{\sigma\cdot\cdot\cdot\kappa} \nabla_\kappa \tilde{\psi}^{\alpha\beta\gamma} + R_{\gamma\kappa}^{\sigma\cdot\cdot\cdot\alpha} \nabla_\sigma \tilde{\psi}^{\kappa\beta\gamma} + R_{\gamma\kappa}^{\sigma\cdot\cdot\cdot\beta} \nabla_\sigma \tilde{\psi}^{\alpha\kappa\gamma} \\
&\quad + R_{\gamma\kappa}^{\sigma\cdot\cdot\cdot\gamma} \nabla_\sigma \tilde{\psi}^{\alpha\beta\kappa} - R_{\kappa\sigma}^{\cdot\cdot\cdot\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa\sigma\gamma} - R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\alpha} \nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\beta} \nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned}$$

Upon relabelling dummy indices and collecting terms, this becomes

$$\begin{aligned}
\tilde{\Psi}^{\alpha\beta} &= R_{\sigma\kappa}^{\gamma\cdot\cdot\cdot\alpha} \nabla_\gamma \tilde{\psi}^{\kappa\beta\sigma} + R_{\sigma\kappa}^{\gamma\cdot\cdot\cdot\beta} \nabla_\gamma \tilde{\psi}^{\alpha\kappa\sigma} \\
&\quad - R_{\kappa\sigma}^{\cdot\cdot\cdot\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa\sigma\gamma} - R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\alpha} \nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\beta} \nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned}
\tag{A1.9}$$

Using the cyclic identity for the curvature tensor and the fact that $\tilde{\psi}^{\alpha\beta\gamma}$ is a completely antisymmetric quantity, we can write

$$R_{\sigma\kappa}^{\mu\cdot\cdot\cdot\nu} \nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = (R_{\sigma\kappa}^{\nu\cdot\cdot\cdot\mu} + R_{\kappa\sigma}^{\cdot\cdot\cdot\mu\nu}) \nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = (-R_{\sigma\kappa}^{\mu\cdot\cdot\cdot\nu} + R_{\kappa\sigma}^{\cdot\cdot\cdot\mu\nu}) \nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma},$$

and therefore

$$R_{\sigma\kappa}^{\mu\cdot\cdot\cdot\nu} \nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma} = \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\cdot\mu\nu} \nabla_\gamma \tilde{\psi}^{\lambda\kappa\sigma}.
\tag{A1.10}$$

This can be used to rewrite (A1.9) as

$$\begin{aligned}
\tilde{\Psi}^{\alpha\beta} &= \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\cdot\gamma\alpha} \nabla_\gamma \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\cdot\gamma\beta} \nabla_\gamma \tilde{\psi}^{\alpha\kappa\sigma} \\
&\quad - R_{\kappa\sigma}^{\cdot\cdot\cdot\alpha\beta} \nabla_\gamma \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\alpha} \nabla_\gamma \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\cdot\kappa\beta} \nabla_\gamma \tilde{\psi}^{\alpha\sigma\gamma}.
\end{aligned}
\tag{A1.11}$$

Factoring out the covariant derivative in (A1.11) yields

$$\begin{aligned}
\tilde{\Psi}^{\alpha\beta} = \nabla_\gamma \left\{ \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} \right. \\
\left. + R_{\kappa\sigma}^{\cdot\cdot\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \right\} - \left\{ \frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha}) \tilde{\psi}^{\kappa\beta\sigma} + \frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\gamma\beta}) \tilde{\psi}^{\alpha\kappa\sigma} \right. \\
\left. - (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} - (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha}) \tilde{\psi}^{\beta\sigma\gamma} + (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\kappa\beta}) \tilde{\psi}^{\alpha\sigma\gamma} \right\}
\end{aligned}
\tag{A1.12}$$

Application of the Bianchi identity and the complete antisymmetry of $\tilde{\psi}^{\kappa\beta\sigma}$ gives (after relabelling dummy indices)

$$\begin{aligned}
\frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha}) \tilde{\psi}^{\kappa\beta\sigma} &= (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha}) \tilde{\psi}^{\beta\sigma\gamma}, \\
\frac{1}{2} (\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\gamma\beta}) \tilde{\psi}^{\alpha\kappa\sigma} &= -(\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\kappa\beta}) \tilde{\psi}^{\alpha\sigma\gamma}, \\
(\nabla_\gamma R_{\kappa\sigma}^{\cdot\cdot\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} &= (\nabla_{[\gamma} R_{\kappa\sigma]}^{\cdot\cdot\alpha\beta}) \tilde{\psi}^{\gamma\kappa\sigma} \equiv 0.
\end{aligned}
\tag{A1.13}$$

The second bracketed term in (A1.12) therefore vanishes and

$$\begin{aligned}
\tilde{\Psi}^{\alpha\beta} = \nabla_\gamma \left\{ \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} \right. \\
\left. - R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \right\}.
\end{aligned}
\tag{A1.14}$$

Equation (A1.7) thus takes the form

$$\begin{aligned}
\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} &= \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + R_{\sigma\kappa}^{\gamma\cdot\cdot\alpha} \tilde{\psi}^{\kappa\beta\sigma} \right. \\
&+ R_{\sigma\kappa}^{\gamma\cdot\cdot\beta} \tilde{\psi}^{\alpha\kappa\sigma} + R_{\sigma\kappa}^{\gamma\cdot\cdot\sigma} \tilde{\psi}^{\alpha\beta\kappa} + \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} - \frac{1}{2} R_{\kappa\sigma}^{\cdot\cdot\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} \\
&\left. - R_{\kappa\sigma}^{\cdot\cdot\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} \right\}.
\end{aligned}$$

Using (A1.10) this can be written as

$$\begin{aligned}
\nabla_\sigma \nabla^\sigma \nabla_\gamma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta} \nabla_\gamma \tilde{\psi}^{\kappa]\sigma\gamma} &= \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + R_{\kappa\sigma}^{\cdot\cdot\gamma\alpha} \tilde{\psi}^{\kappa\beta\sigma} + \right. \\
&+ R_{\kappa\sigma}^{\cdot\cdot\gamma\beta} \tilde{\psi}^{\alpha\kappa\sigma} - R_{\kappa\sigma}^{\cdot\cdot\alpha\beta} \tilde{\psi}^{\gamma\kappa\sigma} + R_{\kappa\sigma}^{\cdot\cdot\kappa\alpha} \tilde{\psi}^{\beta\sigma\gamma} - R_{\kappa\sigma}^{\cdot\cdot\kappa\beta} \tilde{\psi}^{\alpha\sigma\gamma} + R_{\kappa\sigma}^{\cdot\cdot\kappa\gamma} \tilde{\psi}^{\alpha\beta\sigma} \left. \right\}, \\
&= \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} + 3 R_{\kappa\sigma}^{\cdot\cdot\kappa[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} - 3 R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} \right\}, \\
&= \nabla_\gamma \left\{ \nabla_\sigma \nabla^\sigma \tilde{\psi}^{\alpha\beta\gamma} - 3 R_{\sigma}^{[\alpha} \tilde{\psi}^{\beta\gamma]\sigma} - 3 R_{\kappa\sigma}^{\cdot\cdot[\alpha\beta} \tilde{\psi}^{\gamma]\kappa\sigma} \right\},
\end{aligned}$$

which is the required result. $\blacksquare$

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