IMPLEMENTING A GENERIC SMOOTH FUNCTION

BY

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IMPLEMENTING A GENERIC SMOOTH FUNCTION

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Abstract

We define a realization of a smooth, real-valued function in a manner which is based upon Hurwicz's model of privacy-preserving economic mechanisms. Realizations are distinguished by "type" — a measure of the kind of information that the realization requires from each agent in order to operate. Necessary conditions for the existence of a realization of a particular type are given, and the types of realizations which exist for a generic smooth function are classified.

Most of the mathematical techniques in this paper are commonly used in differential geometry and in the study of smooth mappings. Some results from the theory of algebraic equations are also employed.

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1. Introduction

In recent years, economists have begun a formal study of decentralization in economic mechanisms. This paper will follow the particular line of research that was initiated by Hurwicz (1960).

The most notable achievement of this theory thus far is the proof that the price mechanism is "informationally efficient" among all decentralized mechanisms which assign Pareto-optimal allocations to a sufficiently large class of environments (see Hurwicz (1975), Osana (1978), and Mount and Reiter (1974)). In the price mechanism, the vast amount of information which is needed to completely determine any agent's characteristics is encoded in a relatively small number of signals — the agent's excess demand function. Despite the size of the space of environments, a finite-dimensional message space is all that is necessary in order to "realize"¹ (or compute in a decentralized manner) the Walrasian allocations.

This leads to the following question: what properties enable the mapping which assigns the Walrasian allocations to environments to be realized in such an efficient manner? A more abstract version of this question is given by the following problem. For $1 \leq i \leq N$, let E_i denote the space whose elements determine the i^{th} agent's characteristics. In what ways can an arbitrary real-valued function F on $E_1 \times E_2 \times \dots \times E_N$ be decentralized? This is relevant to the first problem, since in order to realize any given mapping which assigns allocations to economic environments, it is necessary to decentralize several real-valued functions. It is important in its own right to the theory of economic decentralization, since this theory encompasses many other economic problems besides the simple trading model.

In this paper, we will study this second problem in the case where $E_1, E_2, \dots, E_N$ and the message space M are Euclidean, F is C^{∞} , and the communication process is assumed to have certain regularity properties. We will use the dimension of M as our primary measure of informational efficiency.

¹ This term (and its definition) is due to Mount and Reiter (1974).

The mechanism design problem under this particular set of hypotheses has been extensively studied by Hurwicz et al. (1978). Though these hypotheses may seem overly restrictive, our results are still meaningful, since many proofs in the theory of decentralization involve a restriction to a subclass of environments which is Euclidean (i.e., the Cobb-Douglas utility functions). A thorough analysis of this case should prove to be a valuable tool. The primary reason for working in this special setting, however, is to facilitate our analysis. Hopefully, we will gain insight into more meaningful economic models by working first in this idealized setting.

We will now discuss the problem in greater detail. One way to realize F requires each agent to communicate a complete description of his characteristics to the message space. This is the direct revelation mechanism. It requires the use of a message space of dimension $\sum_{i=1}^N \dim E_i$. This is a very inefficient way to realize F , since there is no encoding of information about the environment with this mechanism. It does provide us, however, with an a priori upper bound on the size of the message space which is needed to realize F .

Hurwicz has shown that for each $1 \leq j \leq N$, it is possible to realize F with a mechanism which uses a message space of dimension $1 + \sum_{i \neq j} \dim E_i$. These are the parameter transfer mechanisms, which will be illustrated in Section 2. These mechanisms require all but one of the agents to provide a complete description of his characteristics to the message space. Using this information, the exceptional agent is then able to compute F .

Our primary objective in this paper is to determine when there exist alternative mechanisms to direct revelation and parameter transfer which realize a given function F . The dimension of the message space alone will not adequately distinguish between mechanisms for this purpose. We will therefore distinguish mechanisms by "type," which measures not only the total amount of information that is used by the mechanism, but also the amount which is acquired from each individual agent. Our results are mostly negative: for a generic F , the types

of mechanisms which realize F are essentially given by direct revelation and by certain generalizations of parameter transfer (which will be discussed in Section 5).

Before our results can be described in greater detail, we must first formally present our model. This will be the content of the next section. Section 3 contains an economic interpretation of this model. We will then summarize the results of this paper in Section 4.

2. The Model

Mathematically, this paper studies a particular kind of local factorization of a smooth (C^∞), real-valued function that is defined on an open subset of a Euclidean space. The main goal of this paper is to classify the "types" of these factorizations which exist for a generic smooth function.

The problem can be represented by the Mount-Reiter diagram (Figure 1). We will begin to explain this factorization at the bottom of the diagram. For each i between 1 and N , E_i is an open subset of $\mathbb{R}^{k_i}$. We let θ_i denote an arbitrary element of $\mathbb{R}^{k_i}$, and $\theta = (\theta_1, \dots, \theta_N)$ represents an arbitrary element of $\mathbb{R}^K$ (where $K = \sum_{i=1}^N k_i$). Later, we will need to write $\theta_i = (\theta_i^t)$, where $\theta_i^t \in \mathbb{R}$ and $1 \leq t \leq k_i$.

The point $p = (p_1, \dots, p_N)$ is a given, fixed element of $\prod_{i=1}^N E_i$. F is a smooth, real-valued function whose domain is an open subset of $\mathbb{R}^K$ that contains the product $\prod_{i=1}^N E_i$. For some number $C \leq K$, M is an open subset of $\mathbb{R}^C$. Each μ_i is a smooth mapping which sends $E_i \times M$ into $\mathbb{R}^{c_i}$, where $1 \leq c_i \leq k_i$ and $\sum_{i=1}^N c_i = C$. Finally, define the mapping μ from $\prod_{i=1}^N E_i \times M$ into $\mathbb{R}^C$ by $\mu = (\mu_1, \dots, \mu_N)$.

We insist that these mappings satisfy the following conditions:

for each $\theta \in \prod_{i=1}^N E_i$, there exists a unique $m \in M$ such that $\mu(\theta, m) = 0$; (1)

$\text{rank } D_{\theta_i} \mu_i(p_i, m) = c_i$ at any m that satisfies $\mu_i(p_i, m) = 0$. (2)

Additionally, the smooth function h must be chosen so that the diagram has the following commutativity property:

$$h(m) = F(\theta) \text{ if } \theta \text{ and } m \text{ satisfy } \mu(\theta, m) = 0. \quad (3)$$

When a smooth function F has a factorization of this form on some product open set $\prod_{j=1}^N E_i$ that contains the point p , we will say that F has a (privacy-pre-serving) realization of type $(c_1, \dots, c_N)$ at p .

Since all of our arguments will be local in nature, the regularity conditions (1) and (2) are not overly restrictive. They are imposed mostly to facilitate our discussion.

An example will help to clarify the mathematics of this factorization.

Example. Let $E_1 = E_2 = \mathbb{R}^{2+}$, the positive orthant of $\mathbb{R}^2$. For $i = 1, 2$, represent $\theta_i \in E_i$ as $\theta_i = (\theta_i^1, \theta_i^2)$. Since M is an open subset of $\mathbb{R}^C$, we will let $(m_1, \dots, m_C)$ denote an element of M .

In this example, we will discuss several methods of realizing $P(\theta_1, \theta_2) = (\theta_2^2 - \theta_1^2) / (\theta_1^1 + \theta_2^1)$ at any point in $E_1 \times E_2$. Normally, however, one can only construct a realization of a smooth function in a neighborhood of a given point.

Any smooth function F whose domain is $E_1 \times E_2$ has a trivial realization of type $(2, 2)$ at each point in $E_1 \times E_2$ which is given below:

$$\mu_1(\theta_1, m) = (m_1 - \theta_1^1, m_2 - \theta_1^2) = 0$$

$$\mu_2(\theta_2, m) = (m_3 - \theta_2^1, m_4 - \theta_2^2) = 0$$

$$h(m) = F(m_1, m_2, m_3, m_4) \quad .$$

Since P satisfies $(\partial P / \partial \theta_1^2) \neq 0$ on $E_1 \times E_2$, the following equations determine a realization of type $(1, 2)$ of P at any point in $E_1 \times E_2$:

$$\mu_1(\theta_1, m) = P(\theta_1^1, \theta_1^2, m_2, m_3) - m_1 = 0$$

$$\mu_2(\theta_2, m) = (m_2 - \theta_2^1, m_3 - \theta_2^2) = 0$$

$$h(m) = m_1 \quad .$$

The condition $(\partial P / \partial \theta_1^2) \neq 0$ insures that property (2) of a realization is satisfied. Since $(\partial P / \partial \theta_2^2) \neq 0$ on $E_1 \times E_2$, we could also construct in a similar manner a realization of type (2, 1) of P at any point in $E_1 \times E_2$. These are two versions of parameter transfer, which was mentioned in Section 1.

In this paper, we will show that a generic smooth function on $E_1 \times E_2$ ($= \mathbb{R}^{2+} \times \mathbb{R}^{2+}$) only has realizations at a given point of these three types. This particular function P , however, also has the following realization of type (1, 1) at each point of $E_1 \times E_2$:

$$\mu_1(\theta_1, m) = \theta_1^2 + \theta_1^1 m_1 - m_2 = 0$$

$$\mu_2(\theta_2, m) = \theta_2^2 - \theta_2^1 m_1 - m_2 = 0$$

$$h(m) = m_1 \quad .$$

3. Economic Interpretation

We will now briefly discuss an economic interpretation of this model. Each element of the set E_i determines a profile of characteristics for the i^{th} agent. For instance, an element of E_i might contain the exponents of the i^{th} agent's Cobb-Douglas utility function, and it might also contain a vector which determines the i^{th} agent's initial endowment. The function F (the performance standard) represents some observed (or desired) outcome for the economy. What is observed or desired will typically depend upon the environment, whose range of values is given by $\prod_{i=1}^n E_i$.

These ideas are illustrated by our example in the last section, which was developed by Hurwicz (1980). This example depicts a two-agent, two-good economy. Let x and z represent amounts of the two goods. We assume that agent i 's initial endowment of good x is fixed at $\bar{x}_i$. The i^{th} agent's utility function is given by the following formula:

$$u_i(x, z) = (\theta_i^2 - \theta_i^1 \bar{x}_i)x + (\theta_i^2/2)x^2 + z .$$

A precise value of the i^{th} agent's utility function is determined by choosing any element (θ_i^1, θ_i^2) of E_i . The performance standard P in this example computes the amount of good x which must be traded from the second agent to the first in order to achieve a Walrasian allocation for the economy.

The Mount-Reiter diagram depicts the equilibrium state of a communication process. This can be shown in the following manner. Let $M = \prod_{i=1}^N M_i$, where each M_i is an open subset of $\mathbb{R}^{c_i}$, and represent an element m of M as $m = (m_1, m_2, \dots, m_N)$. The elements of M_i represent the possible messages of the i^{th} agent. We imagine that the i^{th} agent receives a message $m^t = (m_1^t, m_2^t, \dots, m_N^t)$ from all of the agents at time t , and then issues a new message m_i^{t+1} based upon m^t and his characteristics θ_i , according to some rule g_i :

$$m_i^{t+1} = g_i(m^t, \theta_i) .$$

The communication process is in equilibrium at m if the following equation is satisfied for each value of i :

$$g_i(m, \theta_i) = m_i .$$

Letting $\mu_i(m, \theta_i) \equiv g_i(m, \theta_i) - m_i$, we obtain the equations that are given in the Mount-Reiter diagram. The function h (the outcome function) must satisfy (3), else the mechanism would not realize the function F when the communication process is in equilibrium.

This paper is focussed solely on the equilibrium state of the communication process. Note, however, that the issue of incentives can be incorporated into this

model by making assumptions about the adjustment rules g_i . This is part of Hurwicz's original formulation of the model, and it has been extensively studied elsewhere. Also, the dynamic stability of communication processes of this form has recently been studied by Jordan (1982), Mount and Reiter (1983), and Williams (1983).

In a realization of F of type $(c_1, \dots, c_N)$, the number c_i reflects the amount of information that is taken from the i^{th} agent when the system is in equilibrium, while $C = \sum_{i=1}^N c_i$ measures the total amount of information that is used to compute F . These numbers can be used to compare the efficiency and equity of different methods of processing a given performance standard F . They can also be used to evaluate alternative performance standards, depending upon the types of realizations which exist for each of the alternatives.

The model that we are using originated in Hurwicz (1960), and it was further developed in Mount and Reiter (1974). This paper is especially indebted to the work of Hurwicz et al. (1978). Each of these papers discusses the economic motivation for this model in further detail. A more general introduction to the aims of this theory can be found in Reiter (1977).

4. Summary of Results

In Section 5, we will construct a set of realizations at p for any smooth function F which satisfies $\text{rank } D_{\theta_i} F(p) = 1$ for each value of i . The types of realizations which we will construct in this section will be very limited: each will satisfy $c_i = k_i$ for all but at most one value of the index i . These will essentially be versions of parameter transfer.

We will then begin our main task — to characterize the conditions that a smooth function F must satisfy if it has a realization at p of type $(c_1, \dots, c_N)$. For all natural numbers k and for each different type of realization, we will prove the existence of a set of systems of polynomial equations and strict polynomial inequalities on the k -jet space of smooth functions which has the following property: the k -jet of F must be a solution to at least one of the systems of this set if F has

a realization at p of this particular type. When $c_i \neq k_i$ for at least two distinct values of the index i , we will show that for k sufficiently large each of the systems on the k -jet space must contain at least one polynomial equation.

In the main result of this paper (Theorem 4), we will combine a set of polynomial equations on the k -jet spaces that are obtained by this procedure with the regularity conditions from Section 5 to determine an open, dense subset Ω of the space of smooth functions at p . It will be shown that an element of Ω has a realization of type $(c_1, \dots, c_N)$ at p if and only if $c_i = k_i$ for all but at most one value of the index i . A generic function F therefore only has realizations of the types that will be described in Section 5.

We use the theory of algebraic equations in this paper in order to make the statement of Theorem 4 as strong as possible. If we restricted ourselves to the techniques of differential topology, we could only conclude that the k -jets of the set of functions which have realizations of types other than those that are described in Section 5 must be of measure zero in the k -jet space of functions at p . This result follows from the Morse-Sard Theorem (Hirsch, 1975, p. 69), and it plays a key role in the proof of our Theorem 3. Our use of algebra allows us to go beyond this to achieve a much more precise description of the set of functions at p which have nontrivial types of realizations.

There has also been some interest in determining necessary conditions on F for the existence of a realization of type $(c_1, \dots, c_N)$ at p . This topic will be discussed in Section 10.

5. The Types of Realizations which Exist for a Generic F

We will now show how to construct a realization of type $(k_1, \dots, c_j, \dots, k_N)$ of F at p for any value of c_j between 1 and k_j , provided that F satisfies $\text{rank } D_{\theta_j} F(p) = 1$. First, choose $r \equiv c_j - 1$ elements from the set $\{d\theta_j^t \mid 1 \leq t \leq k_j\}$ whose span at p does not contain $\sum_{t=1}^{k_j} \partial P / \partial \theta_j^t d\theta_j^t$; for convenience, we assume $\{d\theta_j^1, \dots, d\theta_j^r\}$ has this property. If we write an element m of M as

$m = (m_1, \dots, m_N)$, where $m_i \in \mathbb{R}^{k_i}$ for all $i \neq j$ and $m_j \in \mathbb{R}^{c_j}$, then a realization of F is determined in a neighborhood of p by the following equations:

$$\mu_i(\theta_i, m) = m_i - \theta_i = 0, \quad \text{for all } i \neq j$$

$$\mu_j(\theta_j, m) = (\theta_j^1, \dots, \theta_j^r, F(m_1, \dots, \theta_j, \dots, m_N)) - m_j = 0$$

$$h(m) = m_j^{r+1}, \quad \text{where } m_j = (m_j^1, \dots, m_j^{r+1}).$$

As in the preceding example, the condition $\text{rank } D_{\theta_j} F(p) = 1$ insures that μ_j satisfies property (2) of a realization.

6. Necessary and Sufficient Conditions for the Existence of a Realization

Since the results of this paper are local in nature, they will be stated in terms of germs of smooth mappings. The germ of a smooth function f at p is the equivalence class of f under the following equivalence relation on the set of all smooth functions which are defined near p : $f \sim g$ if and only if $f = g$ on some (arbitrarily small) open set containing p . A similar equivalence relation can be defined on the set of all diffeomorphisms which are defined near p . See Golubitsky and Guillemin (1973) for further information.

Let $C_p^\infty(\mathbb{R}^K)$ denote the space of germs of smooth functions at p , and let G_p represent the space of germs of diffeomorphisms which send p to p ; in each of these cases, we are using the Whitney C^∞ topology. J_p^k and G_p^k are the respective k -jet spaces, and j^k and j_G^k are the corresponding k -jet mappings. For $F \in C_p^\infty(\mathbb{R}^K)$ and $x \in G_p$, we let $F^k \equiv j^k(F)$ and $x^k \equiv j_G^k(x)$.

Our first theorem equates the existence of a realization of type $(c_1, \dots, c_N)$ of F at p with the existence of an element of G_p that has a special form and that has a certain relationship with F . The results of this paper are obtained by studying this type of coordinate system and this relationship between G_p and $C_p^\infty(\mathbb{R}^K)$, rather than by working directly with the Mount-Reiter diagram. The proof of this theorem uses several techniques which were first developed in Hurwicz et al. (1978).

In the statement of this theorem, we will use $((m_1, n_1), \dots, (m_N, n_N))$ to represent an arbitrary element in the range of any x in G_p , where $m_i \in \mathbb{R}^{c_i}$ and $n_i \in \mathbb{R}^{k_i - c_i}$.

THEOREM 1. $F \in C_p^\infty(\mathbb{R}^K)$ has a realization of type $(c_1, \dots, c_N)$ at p if and only if there exists an element x of G_p which has the following properties:

$$i) \quad F \circ x^{-1} \text{ depends only upon the variables } m_1, \dots, m_N; \quad (4)$$

$$ii) \quad \text{if we write } x^{-1} \equiv u \equiv (u_1, \dots, u_N), \text{ where the range of } u_j \text{ is a subset of } \mathbb{R}^{k_j}, \text{ then each mapping } u_j \text{ does not depend upon the variable } n_i \text{ for all } i \neq j. \quad (5)$$

The proof of this theorem is lengthy, and it does not provide any insight that would be useful in the remainder of this paper; for these reasons, it will be deferred to the very last section (Section 11).

7. $U_p^k(c_1, \dots, c_N)$

Let $U_p(c_1, \dots, c_N)$ denote the subset of $G_p \times C_p^\infty(\mathbb{R}^K)$ consisting of all pairs (x, F) which together satisfy properties (4) and (5). If we let pr denote the standard projection mapping from $G_p \times C_p^\infty(\mathbb{R}^K)$ onto $C_p^\infty(\mathbb{R}^K)$, then by Theorem 1, the set of all smooth functions which have realizations of type $(c_1, \dots, c_N)$ at p is equal to $\text{pr}(U_p(c_1, \dots, c_N))$.

Let $U_p^k(c_1, \dots, c_N) \equiv (j_G^k \times j_p^k)(U_p(c_1, \dots, c_N))$, and pr_k denote the standard projection mapping from $G_p^k \times J_p^k$ onto J_p^k . Since $\text{pr}_k \circ (j_G^k \times j_p^k) = j^k \circ \text{pr}$, a smooth function can have a realization of type $(c_1, \dots, c_N)$ at p only if its k -jet lies in $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ for all values of k .

The main result of this paper will be obtained by studying the structure of $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ as a subset of J_p^k . To do this, we must first determine the structure of $U_p^k(c_1, \dots, c_N)$ inside $G_p^k \times J_p^k$. This will be the content of the next theorem.

For any natural numbers n and k , $d(n,k)$ will denote the dimension over $\mathbf{R}$ of the vector space of polynomials in n variables which have degree less than or equal to k . Since all of the results in this paper are stated with respect to the standard coordinate system in $\mathbf{R}^K$, we can identify J_p^k with a Euclidean space of dimension $d(K,k)$, and G_p^k with an open subset of a Euclidean space of dimension $K(d(K,k) - 1)$.

THEOREM 2. $U_p^k(c_1, \dots, c_N)$ is a smooth submanifold of $G_p^k \times J_p^k$ with

$$\dim U_p^k(c_1, \dots, c_N) = \sum_{i=1}^N [k_i(d(C+k_i - c_i, k) - 1)] + d(C, k).$$

It is determined as a subset of $G_p^k \times J_p^k$ by a system of algebraic equations.

PROOF. Let $V_p(c_1, \dots, c_N)$ denote the subset of G_p that consists of all diffeomorphisms u which have the form described in (5) of Theorem 1: if we write $u = (u_1, \dots, u_N)$, then each u_j does not depend upon the variable n_i for all $i \neq j$. Also, define B_p as the set of all elements F of $C_p^\infty(\mathbf{R}^K)$ which depend only upon the variables $m_1, \dots, m_N$.

By the definition of $U_p(c_1, \dots, c_N)$, an element (x, F) of $G_p \times C_p^\infty(\mathbf{R}^K)$ is in $U_p(c_1, \dots, c_N)$ if and only if $(x^{-1}, F \circ x^{-1})$ is an element of $V_p(c_1, \dots, c_N) \times B_p$. If we define $V_p^k(c_1, \dots, c_N)$ to equal $j_G^k(V_p(c_1, \dots, c_N))$ and let B_p^k equal $j^k(B_p)$, then the following statement must be true: (x^k, F^k) is an element of $U_p^k(c_1, \dots, c_N)$ if and only if $((x^{-1})^k, (F \circ x^{-1})^k)$ is in $V_p^k(c_1, \dots, c_N) \times B_p^k$.

The results of this theorem will now be obtained by studying the mapping T^k from $G_p^k \times J_p^k$ into itself that is defined by $T^k(x^k, F^k) = ((x^{-1})^k, (F \circ x^{-1})^k)$. T^k is an invertible mapping which is given by $\dim(G_p^k \times J_p^k)$ rational functions in the coordinates of $G_p^k \times J_p^k$.

It is easy to see that $V_p^k(c_1, \dots, c_N) \times B_p^k$ is a submanifold of $G_p^k \times J_p^k$ which is determined by a set of algebraic equations. $U_p^k(c_1, \dots, c_N)$ must have this form as a subset of $G_p^k \times J_p^k$, since

$$U_p^k(c_1, \dots, c_N) = (T^k)^{-1} (V_p^k(c_1, \dots, c_N) \times B_p^k) .$$

The dimensions of these two submanifolds must also be equal. We can easily determine that $\dim(V_p^k(c_1, \dots, c_N))$ is equal to $\sum_{i=1}^N k_i (d(C+k_i - c_i, k) - 1)$, and $\dim B_p^k$ equals $d(C, k)$, from which the formula for $\dim U_p^k(c_1, \dots, c_N)$ in the statement of the theorem follows immediately.

Q. E. D.

8. The Structure of $\text{pr}_k(U_p^k(c_1, \dots, c_N))$

By using the information that we have just obtained about $U_p^k(c_1, \dots, c_N)$, we can now characterize the structure of $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ inside J_p^k . This characterization provides a set of necessary conditions for the existence of a realization of type $(c_1, \dots, c_N)$ at p .

THEOREM 3. For each natural number k , there exists a finite set $\{\psi_t^k\}$ of finite systems of polynomial equations and strict polynomial inequalities in the variables of J_p^k which has the following property:

an element of J_p^k lies in $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ if and only if that element is a solution to at least one of the systems ψ_s^k of the set $\{\psi_t^k\}$.

If $c_i \neq k_i$ for at least two distinct values of the index i , then for k sufficiently large each of the systems of the set $\{\psi_t^k\}$ will contain at least one polynomial equation.

PROOF. The existence of a set of finite systems of polynomial equations and inequalities which determine $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ is guaranteed by Tarski's generalization of Sturm's Theorem (see Jacobson (1975, Theorem 16, p. 312)). Each of these systems can be broken down into a finite set of finite systems of polynomial equations and strict polynomial inequalities, from which we obtain the set of systems $\{\psi_t^k\}$ described in the statement of the theorem. We can assume that any system which does not have a solution has been removed from $\{\psi_t^k\}$.

The solution set of a system ψ_s^k is an open subset of J_p^k if and only if ψ_s^k consists solely of strict inequalities. Since $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ is the union of the solution sets of the systems in $\{\psi_t^k\}$, each of these systems must contain at least one polynomial equation whenever $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ does not contain any open subsets of J_p^k . This will be the case if the dimension of $U_p^k(c_1, \dots, c_N)$ is less than the dimension of J_p^k : when this inequality is true, by the Morse-Sard Theorem (Hirsch, 1975, p. 69) $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ must have measure zero as a subset of J_p^k , and hence it could not contain any open subsets of J_p^k .

It follows that we can complete the proof of the theorem by showing that if $c_i \neq k_i$ for at least two distinct values of the index i , then the inequality given below must be true for values of k that are sufficiently large:

$$\begin{aligned} \dim J_p^k &= d(K, k) > \sum_{i=1}^N [k_i (d(C + k_i - c_i, k) - 1)] + d(C, k) \\ &= \dim U_p^k(c_1, \dots, c_N) . \end{aligned} \quad (6)$$

This will be proven by using the following property of $d(n, k)$: if a and b are natural numbers with $a > b$, then

$$\lim_{k \rightarrow \infty} d(a, k) / d(b, k) = \infty .$$

We can apply this limit to show that the inequality (6) is true for k sufficiently large provided we first show that under the hypothesis, we have $K > C$ and $K > C + k_i - c_i$ for each i . This is easily verified, for when $k_i \neq c_i$ for at least two values of i , the following inequalities are true for each j :

$$K - C \geq K - (C + k_i - c_j) = \sum_{i \neq j} k_i - c_i > 0 .$$

Q. E. D.

9. Proof of Main Result

The proof of our main result is a simple consequence of the preceding theorem together with the results of Section 5.

THEOREM 4. There exists an open, dense subset Ω of $C_p^\infty(\mathbb{R}^K)$ which has the following property: an element F of Ω has a realization of type $(c_1, \dots, c_N)$ at p if and only if $c_i \neq k_i$ for at most one value of the index i .

PROOF. For each of the types $(c_1, \dots, c_N)$ of realization which satisfy $c_i \neq k_i$ for at least two distinct values of i , choose a finite set of polynomial equations in the variables of J_p^k for some value of k so that this set has the following property: F^k must satisfy at least one of the equations in this set if F has a realization at p of that particular type. Form the subset Ω^* of $C_p^\infty(\mathbb{R}^K)$ which consists of all germs at p whose jets do not satisfy any of the polynomial equations that are obtained in this manner: an element of Ω^* does not have a realization of any type $(c_1, \dots, c_N)$ in which $c_i \neq k_i$ for more than one value of i .

We impose the following additional conditions on the 1-jet of an element F of Ω^* : $\text{rank } D_{\theta_i} F(p) = 1$ for all $1 \leq i \leq N$. As discussed in Section 5, this guarantees that F has a realization at p of each type $(c_1, \dots, c_N)$ in which $c_i \neq k_i$ for at most one value of the index i .

The subset Ω of Ω^* which consists of those germs whose 1-jets satisfy all N of these additional regularity conditions is an open, dense subset of $C_p^\infty(\mathbb{R}^K)$ which has the property given in the statement of the theorem.

Q. E. D.

10. The Computation of Necessary Conditions

In the proof of Theorem 3, Tarski's generalization of Sturm's Theorem is applied to prove the existence of a finite set of finite systems of polynomial equations and inequalities which determine $\text{pr}_k(U_p^k(c_1, \dots, c_N))$ as a subset of J_p^k . There exists an algorithm (due to Tarski and to Seidenberg) which can be used to derive this set of systems from the equations which determine $U_p^k(c_1, \dots, c_N)$ as

a subset of $G_p^k \times J_p^k$ (Jacobson, 1975, pp. 295-316). A simpler approach is to notice that the equations which determine $U_p^k(c_1, \dots, c_N)$ are homogeneous both in the variables of G_p^k and in the variables of J_p^k (Williams, 1982, Theorem 7, p. 30). There is an algorithm which uses this property of $U_p^k(c_1, \dots, c_N)$ to determine a set of homogeneous polynomial equations on J_p^k whose solution set contains $\text{pr}_k(U_p^k(c_1, \dots, c_N))$; a simplified version is described in vanderWaerden (1950, pp. 6-9), and in most classical books on systems of algebraic equations. It should be noted, however, that for most realization problems, neither of these algorithms is very practical for computation by hand.

Some necessary conditions have been derived in the two-agent case by other methods. Let $D_{1,2} F$ denote a $k_1 \times k_2$ matrix whose (t,s) entry is $\partial^2 F / \partial \theta_1^t \partial \theta_2^s$. F can have a realization of type (c_1, c_2) at p with $c_1 + c_2 \leq \min(k_1, k_2)$ only if the following rank condition is satisfied in a neighborhood of p :

$$\text{rank} \begin{vmatrix} 0 & D_{\theta_2} F \\ (D_{\theta_1} F)^T & D_{1,2} F \end{vmatrix} \leq \min(k_1, k_2) .$$

This condition was first established by Hurwicz (1979) in the case of $k_1 = k_2 = 3$. It was later generalized to arbitrary values of k_1 and k_2 by Williams (1982, pp. 40-48). Hurwicz has shown by an example that this condition is not sufficient to guarantee that F has a realization with the form described above.

11. Proof of Theorem 1

Sufficiency. The mapping x can be represented as $x \equiv (x_i^t)_{1 \leq i \leq N, 1 \leq t \leq k_i}$, where $x_i^t \in C_p^\infty(\mathbb{R}^K)$. We will construct a realization of F from the functions in the set $\{x_i^t \mid 1 \leq i \leq N, 1 \leq t \leq c_i\}$. For each $1 \leq i \leq N$, define $x_i : \mathbb{R}^K \rightarrow \mathbb{R}^{c_i}$ by $x_i \equiv (x_i^t)_{1 \leq t \leq c_i}$.

Recall that a variable in the domain of u is represented as

$$((m_1, n_1), \dots, (m_1, n_1), \dots, (m_N, n_N)) ,$$

where $m_i \in \mathbb{R}^{c_i}$ and $n_i \in \mathbb{R}^{k_i - c_i}$. Also, we will write $m_i \equiv (m_i^t)_{1 \leq t \leq c_i}$, and $n_i \equiv (n_i^t)_{c_i < t \leq k_i}$.

By the following argument, we will be able to assume that in some neighborhood of p ,

$$\text{rank } D_{\theta_j}(x_j) = c_j \quad (7)$$

for each $1 \leq j \leq N$. Property (5) in the statement of the theorem implies that the vector field $\partial/\partial x_j^s = u_*(\partial/\partial n_j^s)$ lies in the span of $\{\partial/\partial \theta_j^t \mid 1 \leq t \leq k_j\}$ for each $c_j < s \leq k_j$. Since $\{\partial/\partial x_i^t \mid 1 \leq i \leq N, c_i < t \leq k_i\}$ spans the kernel of $D_{\theta}((x_i)_{1 \leq i \leq N})$, we must have $\text{rank } D_{\theta_j}((x_i)_{1 \leq i \leq N}) = c_j$ for each j . If for some j we have $\text{rank } D_{\theta_j}(x_j) < c_j$, then by applying a sequence of linear operations in the functions in $\{x_i^t \mid 1 \leq i \leq N, 1 \leq t \leq c_j\}$ we can construct an element of G_p from x that has the desired regularity property for each j , and that still satisfied properties (4) and (5) of the theorem.

We are now ready to define the realization of F . The space M will be some (sufficiently small) open subset of $\prod_{i=1}^N \mathbb{R}^{c_i}$ that contains the point

$(p_i^t)_{1 \leq i \leq N, 1 \leq t \leq c_i}$; let $m \equiv (m_1, \dots, m_N)$ denote an element of M . With this

notation, each u_j is a function of m and n_j , so we will write $u_j(m, n_j)$. For convenience, write $p_i^* \equiv (p_i^t)_{c_i < t \leq k_i}$ for each i . The realization is given by the following equations:

for $1 \leq i \leq N$,

$$\mu_i(\theta_i, m) = x_i(u_1(m, p_1^*), \dots, \theta_i, \dots, u_N(m, p_N^*)) - m_i = 0,$$

and

$$h(m) = F \circ u((m_1, p_1^*), \dots, (m_N, p_N^*)).$$

Each mapping μ_i depends only upon the variables θ_i and m , since $p_1^*, p_2^*, \dots, p_N^*$ are fixed, constant vectors in this proof. Similarly, h depends only upon the variable m .

We must verify that these mappings satisfy properties (1) - (3) of a realization. The regularity condition in property (2) — rank $D_{\theta} \mu_i(p_i, m) = c_i$ at any m that satisfies $\mu_i(p_i, m) = 0$ — follows from our earlier work which allowed us to assume that rank $D_{\theta_j} x_j = c_j$ for each j .

We now direct our attention to property (1). Let $q = (q_1, \dots, q_N)$ be a point near p . We will first show that there exists an m which satisfies $\mu_i(q_i, m) = 0$ for each $1 \leq i \leq N$. Let $x(q) \equiv ((\tilde{m}_1, \tilde{n}_1), \dots, (\tilde{m}_N, \tilde{n}_N))$, and let $\tilde{m} \equiv (\tilde{m}_1, \dots, \tilde{m}_N)$. Since $x = u^{-1}$, the following equations hold for each value of i :

$$\begin{aligned} \tilde{m}_i &= x_i \circ u((\tilde{m}_1, p_1^*), \dots, (\tilde{m}_i, \tilde{n}_i), \dots, (\tilde{m}_N, p_N^*)) \\ &= x_i(u_1(\tilde{m}, p_1^*), \dots, u_i(\tilde{m}, \tilde{n}_i), \dots, u_N(\tilde{m}, p_N^*)) \\ &= x_i(u_1(\tilde{m}, p_1^*), \dots, q_i, \dots, u_N(\tilde{m}, p_N^*)) \quad . \end{aligned}$$

Therefore, $\tilde{m} \equiv (x_1(q), \dots, x_N(q))$ satisfies $\mu_i(q_i, m) = 0$ for each i .

To complete our verification of (1), we need to show that m is the unique value of m that satisfies $\mu_i(q_i, m) = 0$ for each i . This will be accomplished by proving that $D_m \mu(\theta, m)$ has (maximal) rank C when $\theta = p$ and $m = (x_1, \dots, x_N)(p)$. Let A denote the $C \times K$ matrix which for each i has $D_{\theta_i} x_i$ in the $c_i \times k_i$ block that corresponds to the variables m_i and θ_i , and which is zero everywhere else. Letting id denote the $C \times C$ identity matrix, we can compute $D_m \mu$ as follows:

$$D_m \mu = (D_{\theta}(x_1, \dots, x_N) - A) \cdot D_m u - \text{id} \quad .$$

Since $x = u^{-1}$, we have $D_{\theta}(x_1, \dots, x_N) \cdot D_m u = \text{id}$ when $\theta = p$ and $m = (x_1, \dots, x_N)(p)$; therefore, $D_m \mu = -A \cdot D_m u$ for these values of m and θ . The matrix A has rank C (by (7)), and $D_m u$ is nonsingular. This implies that the rank of $D_m \mu(p, (x_1, \dots, x_N)(p))$ must be equal to C .

We will continue to write $x(q) = ((\tilde{m}_1, \tilde{n}_1), \dots, (\tilde{m}_N, \tilde{n}_N))$ as we verify that (3) holds. This property follows immediately from (5):

$$\begin{aligned}
h(\tilde{m}) &= F \circ u((\tilde{m}_1, p_1^*), \dots, (\tilde{m}_N, p_N^*)) \\
&= F \circ u((\tilde{m}_1, \tilde{n}_1), \dots, (\tilde{m}_N, \tilde{n}_N)) \\
&= F(q) .
\end{aligned}$$

Necessity. We assume that F has a realization of type $(c_1, \dots, c_N)$ at p , and we will construct an element x in G_p that satisfies (4) and (5). Our proof of this section of the theorem is modelled on Spivak's proof of the Frobenius Integrability Theorem (Spivak, 1979, Theorem 5, p. 262).

By property (1) of a realization, we know that for each θ near p there exists a unique m in M that satisfies $\mu(\theta, m) = 0$. Let $H(\theta)$ denote the value of m that has this property.

We will first define a set of integrable distributions near p that will be used to construct the diffeomorphism x . For any point $q = (q_1, \dots, q_N)$ near p and for each value of i , property (2) guarantees that the set

$$\{(q_1, \dots, \theta_i, \dots, q_N) \mid \mu_i(\theta_i, H(q)) = 0\}$$

is a smooth manifold in some neighborhood of p . As q varies, this defines a foliation of $\mathbf{R}^K$ near p . Let $\mathcal{Q}_i$ denote the integrable distribution which determines this foliation. Note that $\dim \mathcal{Q}_i(\theta) = k_i - c_i$ at each point θ near p , and $\mathcal{Q}_i(\theta)$ is a subspace of the span of $\{\partial/\partial\theta_i^t \mid 1 \leq t \leq k_i\}$ at θ .

By a similar argument, we can show that the set $\{\theta \mid \mu(\theta, H(q)) = 0\}$ is a smooth manifold in some neighborhood of p . As q varies, this also defines a foliation of $\mathbf{R}^K$ near p . We will let $\mathcal{Q}$ denote the distribution that is associated with the foliation. Note that

$$\mathcal{Q} = \mathcal{Q}_1 + \mathcal{Q}_2 + \dots + \mathcal{Q}_N ,$$

and that the dimension of $\mathcal{Q}$ at each point is $\sum_{i=1}^N (k_i - c_i)$.

By the following argument, it suffices that we construct a diffeomorphism $x = (x_i^t)$ which satisfies $\partial/\partial x_i^t \in \mathcal{Q}_i$ for all $1 \leq i \leq N$ and $c_i < t \leq k_i$. If we can find

an x which has this property, then the equations $x_i^t = m_i^t$ for $1 \leq i \leq N$ and $1 \leq t \leq c_i$ determine the integral manifolds of $\mathcal{Q}$ as $(m_1, \dots, m_N)$ varies. By property (3) of a realization, F must be constant on each of these manifolds; property (4) in the theorem follows immediately. Property (5) follows by noting that

$$u_*(\partial/\partial n_i^t) = \partial/\partial x_i^t \in \mathcal{Q}_i$$

for $c_i < t \leq k_i$: since $\mathcal{Q}_i$ is a subspace of the span of $\{\partial/\partial \theta_i^t \mid 1 \leq t \leq k_i\}$, this implies that variations in n_i^t for $c_i < t \leq k_i$ can only affect the value of u_i .

To complete the proof of this part of the theorem, we will now construct a diffeomorphism x at p that has the property described above. For convenience, we can assume that $\mathcal{Q}_i(p)$ is spanned by $\{\partial/\partial \theta_i^t \mid c_i < t \leq k_i\}$. For each $1 \leq j \leq N$, let

$$\Pi_j: \prod_{i=1}^N \mathbb{R}^{k_i} \rightarrow \mathbb{R}^{k_j - c_j} \subset \mathbb{R}^{k_j}$$

denote the standard projection onto the last $k_j - c_j$ coordinates of $\mathbb{R}^{k_j}$, and define the mapping Π by

$$\Pi \equiv (\Pi_1, \dots, \Pi_N): \prod_{i=1}^N \mathbb{R}^{k_i} \rightarrow \prod_{i=1}^N \mathbb{R}^{k_i - c_i}.$$

For $1 \leq i \leq N$, Π_{i*} at p defines an isomorphism from $\mathcal{Q}_i(p)$ to the tangent space to $\mathbb{R}^{k_i - c_i}$ at $\Pi_i(p)$; also, the restriction of Π_* at p to $\mathcal{Q}(p)$ is an isomorphism onto $T_{\Pi(p)} \left(\prod_{i=1}^N \mathbb{R}^{k_i - c_i} \right)$. By continuity, each of the Π_{i*} and also Π_* are isomorphisms on $\mathcal{Q}_i(\theta)$ and $\mathcal{Q}(\theta)$ (respectively) for all θ in some neighborhood of p . Therefore, near p there exist $\sum_{i=1}^N (k_i - c_i)$ smooth vector fields X_i^t (where $1 \leq i \leq N$ and $c_i < t \leq k_i$) which have the following properties:

- i) $X_i^t \in \mathcal{Q}_i$;
- ii) $\Pi_*(X_i^t) = \partial/\partial \theta_i^t$.

We will have a diffeomorphism of the desired form if we can find an $x = (x_i^t)$ in G_p which satisfies $\partial/\partial x_i^t = X_i^t$ for all $1 \leq i \leq N$ and $c_i < t \leq k_i$. To prove that there exists an x with this property, we only need to show that $[X_u^s, X_v^t] = 0$ whenever $u \neq v$ or $s \neq t$. The second property listed above for the vector fields X_i^t implies the following equation:

$$\Pi_*[X_u^s, X_v^t] = [\partial/\partial \theta_u^s, \partial/\partial \theta_v^t] = 0 .$$

Since $\mathcal{Q}$ is integrable, $[X_u^s, X_v^t] \in \mathcal{Q}$; we must then have $[X_u^s, X_v^t] = 0$, since Π_* is an isomorphism on $\mathcal{Q}(T)$ for θ near p .

Q. E. D.

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Mount-Reiter Diagram.

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70	S. Williams,	Sufficient Conditions for Nash Implementation			
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