**LINEAR OPERATORS THAT STRONGLY PRESERVE
THE INDEX OF IMPRIMITIVITY**

By

LeRoy B. Beasley

and

Norman J. Pullman

IMA Preprint Series # 889

October 1991

LINEAR OPERATORS THAT STRONGLY PRESERVE
THE INDEX OF IMPRIMITIVITY

by

LeRoy B. Beasley
Utah State University
Logan, Utah
USA 84322-3900

and

Norman J. Pullman
Queen's University
Kingston Ontario
CANADA K7L 3N6

ABSTRACT Let $\mathcal{M}$ denote the square nonnegative matrices and $d(X)$ the index of imprimitivity of an irreducible matrix. For each $2 \leq k \leq n$, we characterize those linear operators, T , on $\mathcal{M}$ for which $d(T(X)) = k$ if and only if $d(X) = k$ for all irreducible X in $\mathcal{M}$. Similar results are given for binary Boolean and fuzzy matrices, and extended to matrices over integral domains, fields and division rings.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant OGP0004041.

§1. INTRODUCTION AND SUMMARY.

Suppose $\mathbb{K}$ is a field and $\mathcal{M} = \mathcal{M}_{m,n}(\mathbb{K})$ is the set of all $m \times n$ matrices over $\mathbb{K}$. If T is a linear operator on $\mathcal{M}$ and f is a function defined on $\mathcal{M}$, then T preserves f if $f(T(A)) = f(A)$ for all A in $\mathcal{M}$. Further, if $\mathcal{L}$ is a subset of $\mathcal{M}$ and $T(X) \in \mathcal{L}$ for each X in $\mathcal{L}$ we say that T preserves $\mathcal{L}$.

Frobenius (1897,[8]), Marcus and Moyls (1959, [14] and [15]), Marcus and May (1962, [13]), Marcus and Purves (1959, [16]), characterized those linear operators on $\mathcal{M}$ that preserve: determinant and characteristic polynomial, rank, permanent, and the r^{th} elementary symmetric function ($r \geq 4$), respectively. In 1983, McDonald [12] found that the characterizations of the first three functions were valid over more general rings. Typically, the first operations that come to mind for preserving f turn out to be the only ones. For example, T preserves the characteristic polynomial if and only if T is a similarity transformation, transposition or composition of such operations ([8]). In 1987, Hershkowitz [10] characterized certain operators on $\mathcal{M}_{n,n}(\mathbb{K})$ where $\mathbb{K}$ is a field of characteristic p ($p = 0$ or $p > n^2 - n$), that preserve those matrices whose graphs (resp. digraphs) contain no cycle of length $\ell \geq k$. Operators preserving clique covering numbers of graphs were described in 1985 in [17].

In 1984 and 1985 analogues of Marcus and Moyls' work on rank were obtained by Beasley, Gregory and Pullman ([1], [2], [3]) for certain types of semirings. In 1987, Beasley and Pullman ([4]) characterized operators that preserve the term rank, those that preserve the permanent and those that preserve the rook polynomial for such semirings. These semirings include the nonnegative reals and such combinatorially significant systems as the nonnegative integers and the Boolean algebra of two elements. Their results also apply to fuzzy matrices. Recently they characterized those linear

operators T on $\mathcal{M}$, the set of $n \times n$ matrices over a semiring, that *strongly* preserve various sets of matrices in $\mathcal{M}$, that is, those T such that $T(X)$ is in $\mathcal{F}$ if and only if X is in $\mathcal{F}$. For example, in [5] and [6], they characterized those operators on the square nonnegative matrices that strongly preserve the irreducible, and those that strongly preserve the primitive, matrices.

Irreducible matrices play a central rôle in the theory of nonnegative matrices. They are (entrywise) nonnegative $n \times n$ matrices A for which $n = 1$ or the sum of the first n powers of A has no zero entries. If A is irreducible, then there is a unique positive integer $d = d(A)$ such that for some permutation matrix P ,

$$P^{-1}AP = \begin{bmatrix} 0 & A_1 & 0 & & 0 \\ 0 & 0 & A_2 & & 0 \\ & & & \ddots & \\ 0 & 0 & & & A_{d-1} \\ A_d & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the i -th zero block on the diagonal is $n_i \times n_i$, and for some integer q , $P^{-1}A^{dq}P$ is a direct sum of d (entrywise) positive $n_i \times n_i$ matrices. The integer $d(A)$ is called the *index of imprimitivity* of A ; if $d(A) = 1$, then some power of A is positive, i.e. A is *primitive*.

The index of imprimitivity has several interpretations. For instance the spectral radius of A (the maximum of the moduli of the eigenvalues of A) is an eigenvalue of A of algebraic multiplicity $d(A)$. The directed graph of A , $\mathcal{D} = \mathcal{D}(A)$, is strongly connected if and only if A is irreducible, in which case $d(A)$ is the greatest common divisor of the cycle lengths of $\mathcal{D}(A)$. References, further details about this index and other preliminaries are given in Section 2.

We extend the results of [6] on primitivity ($d(A) = 1$) and refine the results of [5] on irreducibility in this paper. For example, for each $2 \leq k$

$\leq n$, we characterize those linear operators on the square nonnegative matrices that strongly preserve the set $\mathcal{F}_k$ of nonnegative matrices with index of imprimitivity k . The following three operations strongly preserve $\mathcal{F}_k$:

- (a) transposition ($X \longrightarrow X^t$);
- (b) permutation similarity ($X \longrightarrow PXP^t$ for some fixed permutation matrix P); and
- (c) positive entrywise scaling ($X \longrightarrow [x_{ij} a_{ij}]$ for some fixed matrix A with all $a_{ij} > 0$).

It is an immediate consequence of Theorem 5, in Section 4, that these operations (and their compositions) are the only ones on the $n \times n$ nonnegative matrices with zero diagonal that strongly preserve $\mathcal{F}_k$ unless $n = 3$ and $k = 2$. Then another operation must be added to the list to generate the required group of operators;

$$(d) \quad \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & c & b \\ a & 0 & d \\ e & f & 0 \end{bmatrix}.$$

In Section 2 we list most of the definitions and preliminary results.

In Section 3, Theorem 4, we characterize the linear operators on the square matrices (even those with non-zero diagonal) over the 2-element Boolean algebra that strongly preserve index of imprimitivity k .

In Section 4, Theorem 5, we extend that characterization to all antinegative zero-divisor-free semirings (such as the nonnegative reals, the nonnegative integers and the fuzzy matrices).

In Section 5, Theorem 6, by a method similar to Hershkowitz [10], we extend the results of Section 3 to cover operators on $n \times n$ matrices over cancellation semirings (such as integral domains, division rings, etc.). In these we find the same sort of results as quoted above, except we require the semiring to have at least n^2+1 elements.

§2. PRELIMINARIES

2.1 Semirings.

A *semiring* (see e.g., Gregory and Pullman [9] or Kim [11]) is a binary system $(S, +, \times)$ such that $(S, +)$ is an abelian monoid (identity 0), $(S, \times)$ is a monoid (identity 1), $\times$ distributes over $+$, $0 \times s = s \times 0 = 0$ for all s in S , and $1 \neq 0$. Usually S denotes the system and $\times$ is denoted by juxtaposition. If $(S, \times)$ is abelian then S is *commutative*. If 0 is the only element to have an additive inverse then S is *antinegative*. All rings with unity are semirings but no such ring is antinegative.

Algebraic terms such as *unit* and *zero divisor* are defined for semirings as for rings. Algebraic operations on matrices over a semiring and such notions as *linearity* and *invertibility* are also defined as if the underlying scalars were in a field.

If S is not commutative, unless otherwise noted, we will take the operation of multiplication by scalars to be left multiplication: $(s, A) \rightarrow sA$.

Here are some examples of semirings which occur in combinatorics. They are all commutative and antinegative. Let H be any Boolean algebra, then $(H, \cup, \cap)$ is a semiring. Let C be any chain with lower bound 0 and upper bound 1, then $(C, \max, \min)$ is a semiring (a *chain semiring*). In particular, if F is the real interval $[0,1]$, then $(F, \max, \min)$ is a semiring, the *fuzzy semiring*. The nonnegative members of any subring of the reals, (under real addition and multiplication), is a semiring. In particular, the nonnegative reals and the nonnegative integers are semirings.

Let $M_n(S)$ denote the $n \times n$ matrices over S . The $n \times n$ identity matrix, I_n , and the $m \times n$ zero matrix, $O_{m,n}$, are defined as if S were a field. The $n \times n$ matrix all of whose entries are zero except its $(i,j)^{th}$ which is 1, is

denoted E_{ij} . We call E_{ij} a *cell*; it is a *link* if $i \neq j$ and a *loop* if $i = j$.

We denote the $m \times n$ matrix all of whose entries are 1 by $J_{m,n}$.

Linearity of transformations is defined as for vector spaces over fields. A linear transformation on M is completely determined by its behavior on the set of cells in M .

2.2 The directed graph of a matrix.

Suppose $A \in M$. The *directed graph*, $\mathfrak{D}(A)$, associated with A has n vertices $v_1, v_2, \dots, v_n$, with $v_i v_j$ being an arc if and only if $a_{ij} \neq 0$.

A family $\mathcal{F}$ of matrices in M is *graphical* if for all $A \in \mathcal{F}$, all the matrices whose directed graphs are isomorphic to $\mathfrak{D}(A)$ are also in $\mathcal{F}$. Thus, membership in $\mathcal{F}$ depends upon the properties of the di(irected)graph of a matrix. For example, an $n \times n$ nonnegative matrix is irreducible if and only if its digraph is strongly connected. So the irreducible $n \times n$ nonnegative matrices are a graphical family. The index of imprimitivity of such a matrix is the greatest common divisor of the cycle lengths of its digraph (see Section 2.4), an isomorphism-invariant. Therefore the family of all irreducible $n \times n$ nonnegative matrices with index of imprimitivity k , is graphical for each k .

In describing the digraph of a matrix A , we will often disregard its isolated vertices and limit our description to the digraph induced by the arcs of $\mathfrak{D}(A)$. For example, if E is a link, then $\mathfrak{D}(E + E^t)$ really consists of a *digon* (i.e. an oriented 2-cycle) and $n-2$ isolated vertices, but we will refer to it as a "digon". We also refer to $E + E^t$ as a *digon-matrix*. An *s-star matrix* is the sum of s links all in the same line (row or column). For example, $\sum_{j=2}^4 E_{1j}$ is a 3-star matrix. Its digraph is (except for isolated vertices) an orientation of the complete bipartite graph $K_{1,3}$ with v_1 as a

source and v_2, v_3 , and v_4 as sinks. As another example, we refer to $\mathfrak{D}(E_{12} + E_{32})$ and $\mathfrak{D}(E_{12} + E_{23})$ as *orientations of a 2-path*, ignoring the $n-3$ isolated vertices. The 2-path in question is the (undirected) graph with vertex-set $\{v_1, v_2, v_3\}$ and two edges $v_1 v_2, v_2 v_3$; a replica of $K_{1,2}$.

2.3 The pattern homomorphism; irreducibility and primitivity.

The *pattern*, $\bar{A}$, of a matrix A in $M_n(\mathbb{S})$ is the $(0,1)$ -matrix whose $(i,j)^{\text{th}}$ entry is 0 if and only if $a_{ij} = 0$. We will also assume that $\bar{A}$ is in $M_n(\mathbb{B})$ where $\mathbb{B}$ is the Boolean algebra of two elements 0 and 1. Here, the rules of arithmetic are the same as if $\mathbb{B}$ were a ring except that $1 + 1 = 1$.

The mapping accomplished by associating each matrix, A , in $M(\mathbb{S})$ with its pattern, $\bar{A}$, in $M(\mathbb{B})$ is a semiring homomorphism when $\mathbb{S}$ is antinegative and zero-divisor-free.

If T is a linear operator on $M(\mathbb{S})$, let $\bar{T}$ its *pattern*, be the operator on $M(\mathbb{B})$ defined by $\bar{T}(\bar{E}_{ij}) = \overline{T(E_{ij})}$ for all (i,j) . Then

$$\overline{\bar{T}(A)} \leq \bar{T}(\bar{A}) \text{ for all } A \text{ in } M(\mathbb{S}). \quad (2.1)$$

Equality holds in (2.1) if $\mathbb{S}$ is an antinegative semiring having no zero divisors. Therefore we have the following lemma.

LEMMA 1. The mapping $T \longrightarrow \bar{T}$ is a homomorphism of the semiring of linear operators on $M(\mathbb{S})$ onto the semiring of linear operators on $M(\mathbb{B})$ if $\mathbb{S}$ is antinegative and free of zero-divisors.

A matrix M in $M_n(\mathbb{S})$ is *reducible* if $n > 1$ and for some permutation matrix P , $P^{-1}MP = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ where X is $m \times m$ and $1 \leq m < n$. Otherwise M is *irreducible*.

It can be shown (see e.g.[11, Theorem 5.2.9]) that if M is in $\mathcal{M}_n(\mathbb{B})$ and $n > 1$, then M is irreducible if and only if there exist a permutation matrix P and a positive integer $d = d(A)$ such that

$$P^{-1}AP = \begin{bmatrix} 0 & A_1 & 0 & & 0 \\ 0 & 0 & A_2 & & 0 \\ & & & \ddots & \\ 0 & 0 & & & A_{d-1} \\ A_d & 0 & 0 & 0 & 0 \end{bmatrix} \quad (2.2)$$

where for $1 \leq i \leq d$, the i -th zero block on the diagonal is $n_i \times n_i$, and for some integer q , $P^{-1}A^{dq}P$ is the direct sum $\bigoplus_{i=1}^d J_{n_i n_i}$. The integer $d(A)$ is uniquely determined by A , it is called the *index of imprimitivity* of A ; if $d(A) = 1$, then some power $A^m = J_{n,n}$, i.e. A is *primitive*. We denote the family of all irreducible matrices in $\mathcal{M}_n(\mathbb{B})$ of index of imprimitivity k by $\mathcal{F}_k(\mathbb{B})$. If $k > 1$, then no member of $\mathcal{F}_k(\mathbb{B})$ has a diagonal entry equal to 1, by (2.2). Therefore,

$$\mathcal{F}_k(\mathbb{B}) \subseteq \mathcal{K} \quad \text{when} \quad k > 1. \quad (2.3)$$

2.4 Graphical interpretation of irreducibility.

We refer the reader to Bruualdi and Ryser [7], pp. 68-88 for proofs of the assertions we make in this section.

Suppose $\mathcal{D} = \mathcal{D}(A)$ is the directed graph of the (0,1)-matrix A . Then $\mathcal{D}$ is strongly connected if and only if A is irreducible. Now suppose A is irreducible and $d = d(A)$, the index of imprimitivity of A . Then d is the greatest common divisor of the lengths of the (directed) cycles in $\mathcal{D}$. Moreover the vertex-set of $\mathcal{D}$ is the union of d mutually disjoint nonempty sets $V_1, V_2, \dots, V_d$ with the following properties. If uv is an arbitrary arc of $\mathcal{D}$ from vertex u to vertex v , then for some index i with $1 \leq i \leq d$, u is a vertex of V_i and v is a vertex of V_{i+1} (here, $V_{d+1} = V_1$). The sets V_i are

called *cyclically transferring classes* by some authors; Brualdi and Ryser [7] call them *sets of imprimitivity*. Some examples of strongly connected digraphs are displayed in Figures 1, 2a and 2b of Section 3, below; their cyclically transferring classes are described in the proof of Lemma 2.

2.5 Other terminology and notation.

The number of nonzero entries in a matrix is denoted $|A|$. The number elements in a set is denoted $|\mathcal{P}|$ too. We trust this will cause no confusion.

We denote the *Schur product* of A and B in $\mathcal{M}$ by $A \circ B$. That is, $C = A \circ B$ if and only if $c_{ij} = a_{ij} b_{ij}$ for all i and j . Some authors call this the *Hadamard product*.

When X and Y are in $\mathcal{M}_{n,n}(\mathbb{B})$ we define $X \setminus Y$ to be the matrix Z such that $z_{ij} = 1$ if and only if $x_{ij} = 1$ and $y_{ij} = 0$. For example, the matrix in $\mathcal{M}_{n,n}(\mathbb{S})$ having all off diagonal entries 1 and all diagonal entries 0 is denoted K_n , or just K when n is understood. When $\mathbb{S} = \mathbb{B}$, $K = J \setminus I$. We also denote $\{X \in \mathcal{M}_{n,n}(\mathbb{S}) : X \leq K\}$ by $\mathcal{K}$. Thus $\mathcal{K}$ is the set of all matrices in $\mathcal{M}$ with all diagonal entries equal to zero.

If A and B are in $\mathcal{M}$, we say B *covers* A (written $B \geq A$ or $A \leq B$) if $b_{ij} = 0$ implies $a_{ij} = 0$ for all i, j . This provides a reflexive, transitive relation on $\mathcal{M}$.

Suppose $\mathcal{F}$ is a family of matrices in $\mathcal{M}$. We say $\mathcal{F}$ *covers* a matrix A in $\mathcal{M}$ if A is covered by some member of $\mathcal{F}$. A member X of $\mathcal{F}$ is *critical* if X covers no other member of $\mathcal{F}$. We say that $\mathcal{F}$ covers a family of matrices $\mathcal{G}$ *selectively* if whenever a member of $\mathcal{G}$ is covered by $\mathcal{F}$, it is covered by a critical member of $\mathcal{F}$.

§ 3. MATRICES OVER THE 2-ELEMENT BOOLEAN ALGEBRA.

3.1 Operators on $\mathcal{K}$.

We are going to use two general theorems. The first is an immediate consequence of Lemma 2.3 of [5].

THEOREM 1. Any linear operator on $\mathcal{K}$ that is bijective on the links and preserves 2-star matrices is in the group of operators on $\mathcal{K}$ generated by transposition and the permutation similarity operators.

THEOREM 2. Suppose $\mathcal{F}$ is a family of matrices in $\mathcal{K}$ that covers the set of links and selectively covers the set of matrices having at most 2 nonzero entries, then any linear operator on $\mathcal{K}$ that strongly preserves $\mathcal{F}$ is bijective on the links.

After proving Theorem 2, we will show that $\mathcal{F}_k$ satisfies the hypothesis of Theorem 2 when $2 \leq k \leq n$, and that unless $k = 2$ and $n = 3$, any linear operator strongly preserving $\mathcal{F}_k$ preserves 2-star matrices. Our main result on operators on $\mathcal{K}$, Theorem 3, then follows from Theorems 1 and 2.

Proof of Theorem 2. Let L be a linear operator on $\mathcal{K}$ that strongly preserves $\mathcal{F}$. There is a positive integer p for which L^p is idempotent since $\mathcal{K}$ is a finite set; let $Q = L^p$. Then Q , as a power of L , also strongly preserves $\mathcal{F}$.

First, we will show that Q is *nonsingular*, i.e. that $Q(X) = 0$ only if $X = 0$. Suppose $Q(X) = 0$ and there is a link $E \leq X$. Since $\mathcal{F}$ covers the links, selectivity implies that some critical M in $\mathcal{F}$ covers E . Then $Q(M) = Q(M \setminus E)$ because Q is linear and $Q(E) = 0$. But that contradicts the

fact that Q strongly preserves $\mathcal{F}$, because $M \setminus E \notin \mathcal{F}$ since M is a critical member of $\mathcal{F}$.

Next we will show that Q fixes each link. Let E be any link, then for some link F , $Q(E) \geq F$ because Q is nonsingular. Suppose $F \neq E$ and let $H = E + F$. We have $E \leq M$ for some member and hence some critical member M of $\mathcal{F}$. Then $Q(M) = Q(M + E) = Q(M) + Q^2(E) \geq Q(M + F) \geq Q(M)$ because Q is idempotent and $Q(E) \geq F$. But then $Q(M) = Q(M + F)$, so $M + F \in \mathcal{F}$ because Q preserves $\mathcal{F}$ strongly. Therefore H is covered by $\mathcal{F}$. By selectivity, $H \leq C$ for some critical member $C \in \mathcal{F}$. Let $N = C \setminus F$, then $N \notin \mathcal{F}$ and $Q(C) \geq Q(N) = Q(N + E) = Q(N) + Q^2(E) \geq Q(N + F) = Q(C)$. Therefore $Q(N) = Q(C)$, contradicting the fact that Q preserves $\mathcal{F}$ strongly. Therefore $F = E$ and hence $Q(E) = E$.

Finally we will show that L is bijective on the links. Let E be any link, then $L(E) \neq 0$ because Q is linear and nonsingular. Therefore $L(E) \geq F$ for some link F and hence $E = L^p(E) \geq L^{p-1}(F)$. But $L^{p-1}(F) \neq 0$ because Q is nonsingular. Consequently $E = L^{p-1}(F)$ and hence $L(E) = F$, a link. If $L(G) = L(E)$ for some link G , then $Q(G) = Q(E)$ and hence $G = E$. ■

Now we will establish that the $\mathcal{F}_k$ satisfy the hypotheses of Theorem 2 when $k \geq 2$.

LEMMA 2. For all $2 \leq k \leq n$, $\mathcal{F}_k$ covers the links and selectively covers $\mathcal{K}_2$, the set of matrices in $\mathcal{K}$ having at most two nonzero entries. Specifically, $\mathcal{F}_2$ covers each A in $\mathcal{K}_2$ and, when $k > 2$, $\mathcal{F}_k$ covers each A in $\mathcal{K}_2$ except for the digon-matrices.

Proof. We will define a family $\mathcal{G}(n, k)$ of strong digraphs on n vertices with k cyclically transferring classes; each digraph has the property that the

removal of any of its arcs produces a digraph that is not strong. Thus their matrices will be critical members of $\mathcal{F}_k$.

Case 1: $n \geq k > 2$. The diagram in Figure 1 below describes a specific member $\mathcal{D}_1(n,k)$ of $\mathcal{G}(n,k)$; the parameter $p = n - k + 1$. The k cyclically transferring classes for the digraph $\mathcal{D}_1(n,k)$ are: $V_1 = \{1,2,\dots,p\}$, $V_2 = \{p+1\}, \dots, V_{k-1} = \{p + i - 1\}, \dots, V_{k-1} = \{n - 1\}$, $V_k = \{n\}$. The rest of the members of $\mathcal{G}(n,k)$ with $k > 2$ are obtained by relabelling the vertices of $\mathcal{D}_1(n,k)$. Every arc, every orientation of a 2-path (i.e. an undirected path with two edges), and every independent pair of arcs can be embedded in a member of $\mathcal{G}(n,k)$ by appropriately relabelling the vertices of the digraph of Figure 1. Since $\mathcal{F}_k$ covers no digon-matrix when $k > 2$, and the matrix of each member of $\mathcal{G}(n,k)$ is a critical member of $\mathcal{F}_k$, the lemma follows.

Notice that when $k = 2$, the digraph $\mathcal{D}_1(n,2)$ of Figure 1 is a union of $n-1$ digons with a common vertex $\{n\}$. If $n \geq 4$, $\mathcal{F}_2$ covers matrices whose digraphs contain independent pairs of arcs. For those n , $\mathcal{D}_1(n,2)$ and hence each of its vertex-relabellings, contains no independent pair of arcs and so they are unsuitable for use in proving this lemma. For that reason we use a different class of digraphs to define $\mathcal{G}(n,k)$ when $k=2$.

FIGURE 1

Case 2: $k = 2$. For odd $n \geq 3$, the diagram in Figure 2a below describes a specific member $\mathcal{D}_a(n,2)$ of $\mathcal{G}(n,2)$. The two cyclically transferring classes for the digraph $\mathcal{D}_a(n,2)$ are: $W_1 = \{2,4,6,8,\dots,n-3,n-1\}$ and $W_2 = \{1,3,5,7,\dots,n-2,n\}$. For even $n \geq 4$, the diagram in Figure 2b below describes a member $\mathcal{D}_b(n,2)$ of $\mathcal{G}(n,2)$. The two cyclically transferring classes for the digraph $\mathcal{D}_b(n,2)$ are: $U_1 = \{2,4,6,8,\dots,n-2,n\}$ and $U_2 = \{1,3,5,\dots,n-3,n-1\}$. When $n > 2$, the rest of the members of $\mathcal{G}(n,2)$ are obtained by relabelling the vertices of $\mathcal{D}_a(n,2)$ if n is odd and $\mathcal{D}_b(n,2)$ if n is even. Let $\mathcal{G}(2,2)$ consist of a single digon. Every arc, and the digraph of any matrix in $\mathcal{K}_2$ can be embedded in some member of $\mathcal{G}(n,2)$. Since the matrices of all the members of $\mathcal{G}(n,2)$ are critical members of $\mathcal{F}_2$, we have completed the proof of Lemma 2. ■

FIGURE 2a

FIGURE 2b

3.2 Operators on $\mathcal{K}$ strongly preserving $\mathcal{F}_k(\mathbb{B})$ for $k \geq 2$.

We noted in (2.3) that $\mathcal{F}_k(\mathbb{B}) \leq \mathcal{K}$. Let T denote an arbitrary linear operator on $\mathcal{K}$ strongly preserving $\mathcal{F}_k = \mathcal{F}_k(\mathbb{B})$; we will assume $2 \leq k \leq n$.

It follows from Lemma 2 and Theorem 2 that T is bijective on the links;

so to satisfy the hypotheses of Theorem 1, we must show that T preserves 2-star matrices. We will do that in the next sequence of lemmas.

Note first that T^{-1} strongly preserves $\mathcal{F}_k$ and that for all X , X is covered by $\mathcal{F}_k$ if and only if $T(X)$ is covered by $\mathcal{F}_k$.

LEMMA 3. Suppose $S \in \mathcal{K}$.

- a) If $\mathcal{D}(S)$ is an orientation of a 2-path, then so is $\mathcal{D}(T(S))$.
- b) If $\mathcal{D}(S)$ is a digon, so is $\mathcal{D}(T(S))$.
- c) If $\mathcal{D}(S)$ is an independent pair of arcs, so is $\mathcal{D}(T(S))$.

Proof. a) Suppose $\mathcal{D}(S)$ is an orientation of a 2-path. Then there is a link G such that $\mathcal{D}(G + S)$ is an acyclic orientation of a triangle. It follows that $G + S$ is not covered by $\mathcal{F}_k$. But S is covered by $\mathcal{F}_k$ (by Lemma 2), so $T(S)$ is covered by $\mathcal{F}_k$. Therefore, when $k > 2$, $T(S)$ cannot be a digon-matrix.

Suppose $k = 2$. If $T(S)$ were a digon-matrix, then $\mathcal{D}(T(G + S))$ would be the union of a digon and an arc, and hence, embeddable in a member of $\mathcal{G}(n, 2)$. But then, $T(G + S)$ and hence $G + S$ would be covered by $\mathcal{F}_2$, a contradiction. Therefore, for all $2 \leq k \leq n$, $T(S)$ is not a digon-matrix.

By applying the argument of the previous paragraph to the other two orientations of 2-paths contained in $\mathcal{D}(G + S)$, it can be shown that $T(G + S)$ covers no digon-matrices.

So $\mathcal{D}(T(G + S))$ has no digon-matrices and it has three arcs. It is either an orientation of a triangle or an orientation of an acyclic undirected graph with three edges. The latter type of digraphs can be embedded in some member of $\mathcal{G}(n, k)$, so their matrices are covered by $\mathcal{F}_k$. Since $T(G + S)$ is not covered by $\mathcal{F}_k$, its digraph must be an orientation of a triangle. Consequently, $\mathcal{D}(T(S))$ is an orientation of a 2-path.

- b) Suppose $\mathcal{D}(S)$ is a digon. Since a) applies to T^{-1} , as well as to T ,

$\mathfrak{D}(T(S))$ is either a digon or an independent pair of arcs. If the latter, then $n > 3$ and $\mathfrak{D}(T(S))$ can be embedded in one of the digraphs in $\mathcal{G}(n,k)$. Therefore $T(S)$, and hence S is covered by $\mathcal{F}_k$.

If $k > 2$, then that is impossible, so we can assume that $k = 2$. The orientations of triangles are the only digraphs with three arcs that are subgraphs of no strong digraph of index two. That follows from the fact (see Section 2.4) that the index of imprimitivity is the greatest common divisor of the directed cycle lengths. Let E be a link covered by S and choose links F and G so that the digraph of $X = E + F + G$ is an orientation of a triangle. So X is not covered by $\mathcal{F}_2$. Then $\mathfrak{D}(T(X))$ must be an orientation of a triangle. Otherwise it would be a subgraph of some strong digraph of index 2 and so $T(X)$, and hence X , would be covered by $\mathcal{F}_2$.

Similarly, $Y = E^t + F + G$ and hence $\mathfrak{D}(T(Y))$ is an orientation of a triangle. Thus the oriented triangles $\mathfrak{D}(T(X))$ and $\mathfrak{D}(T(Y))$ both contain the oriented 2-path, $T(F + G)$. Therefore, $\mathfrak{D}(T(E))$ and $\mathfrak{D}(T(E^t))$ are orientations of the same edge, so $\mathfrak{D}(T(S))$ is a digon.

Since digons, orientations of 2-paths and independent pairs of arcs are the only types of digraphs that $\mathfrak{D}(T(S))$ can be, a) and b) imply c). ■

LEMMA 4. If $3 \leq k \leq n$ and S is a 2-star matrix, then so is $T(S)$.

Proof. Suppose $S \leq Y$ and Y is a p -star matrix for $2 \leq p \leq n-1$. First we will show that $\mathfrak{D}(T(Y))$ is an orientation of a star with p edges. Lemma 3a) implies that fact directly if $p = 2$.

If $p \geq 3$, the same lemma implies that the arc $\mathfrak{D}(T(E))$ shares exactly one vertex with each of the two arcs of the directed 2-path $\mathfrak{D}(T(S))$, for each link E of $Y \setminus S$.

If $p \geq 4$, $\mathfrak{D}(T(E + S))$ cannot be an orientation of a triangle since $\mathfrak{D}(E +$

S) is a subgraph of some member of $\mathcal{G}(n,k)$ and no matrix of an oriented triangle is covered by $\mathcal{F}_k$ when $k \neq 3$.

It follows that for all $E \leq Y$, $\mathcal{D}(T(E))$ shares the common vertex of the arcs of $\mathcal{D}(T(S))$; i.e. $\mathcal{D}(T(Y))$ is an oriented star with p edges. If $p = 3$, then $Y = S + E$ and $\mathcal{D}(Y)$ is an orientation of a star with three edges. Then Y is covered by some member of $\mathcal{F}_3$. If $\mathcal{D}(T(Y))$ is an oriented triangle, then it is a 3-cycle because the acyclically oriented triangle is not a subgraph of any strong digraph of index three. There is a link F such that $\mathcal{D}(S + F)$ is an acyclic orientation of a triangle, so $S + F$ is not covered by $\mathcal{F}_3$. The arc $\mathcal{D}(T(F))$ has exactly one vertex in common with $\mathcal{D}(T(Y))$, by Lemma 3. It follows that $\mathcal{D}(T(Y + F))$ is a subgraph of a graph in $\mathcal{G}(n,3)$. Thus, $T(Y + F) \leq M$ for some $M \in \mathcal{F}_3$, and hence $S + F \leq T^{-1}(M) \in \mathcal{F}_3$, a contradiction.

We have shown that $\mathcal{D}(T(Y))$ is an orientation of a star with p edges for all $2 \leq p \leq n-1$; let c be its center. Suppose $T(S)$ is not a 2-star matrix. Then the in- and out-degrees of c are each positive. Choose $p = n-k+2$. Then Y is not covered by $\mathcal{F}_k$. For if $Y \leq M \in \mathcal{F}_k$, then the cyclic classes in $\mathcal{D}(M)$, other than the one containing c and the two classes containing the in-set and out-set of c , each contain at least one vertex for a total of at least $(k+2) + 1 + p > n$ vertices, which is impossible. On the other hand, since the in- and out-degrees of c are positive there is a member $A \in \mathcal{F}_k$ covering $T(Y)$. Its digraph is illustrated in Figure 3 if $k = 3$, and in Figure 4 if $k > 3$. Therefore, $Y \leq T^{-1}(A) \in \mathcal{F}_k$, a contradiction. Consequently, $T(S)$ is a 2-star matrix. ■

FIGURE 3

FIGURE 4

LEMMA 5. If S is a 2-star matrix, $n \geq 4$ and $k = 2$, then $T(S)$ is also a 2-star matrix.

Proof. We may assume, without loss of generality, that $S = E_{12} + E_{13}$. Let $X = E_{24} + E_{42} + \sum_{j=4}^n (E_{3j} + E_{j3}) + S$. Suppose $T(S)$ is not a 2-star matrix.

Then, by Lemma 3, $\mathfrak{D}(T(S))$ is a directed path of length 2. Lemma 3 implies that $\mathfrak{D}(T(E_{24} + E_{42}))$ is a digon incident with $\mathfrak{D}(T(E_{12}))$, but not with $\mathfrak{D}(T(E_{13}))$ and $\mathfrak{D}(T(E_{3j} + E_{j3}))$ is a digon incident with $\mathfrak{D}(T(E_{13}))$ but not with $\mathfrak{D}(T(E_{12}))$ for all $4 \leq j \leq n$. The digraphs $\mathfrak{D}(X)$ and $\mathfrak{D}(Y)$, where Y or its transpose is $T(X)$, are illustrated in Figures 5 and 6, below. Clearly, $\mathfrak{D}(X)$ is not strongly connected, but $\mathfrak{D}(T(X))$ is and it has 2 cyclically transferring classes because the greatest common divisor of its cycle lengths is 2. This contradiction completes the proof of the lemma. ■

FIGURE 5

FIGURE 6

THEOREM 3. The group $\Sigma(n,k)$ of linear operators on $\mathcal{K}$ strongly preserving the irreducible matrices of index of imprimitivity k is the group generated by transposition and the permutation similarity operators if

i) $3 \leq k \leq n$,

or ii) $k = 2$ and $n \neq 3$.

The group $\Sigma(3,2)$ is generated by transposition, the similarity operators and

the operator L :
$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & c & b \\ a & 0 & d \\ e & f & 0 \end{bmatrix}.$$

Proof. If i) or ii) hold, then the theorem follows from Theorem 2 and the lemmas. So we now assume that $k = 2$ and $n = 3$.

According to Lemma 2, Theorem 2, and Lemma 3(b), each linear operator on $\mathcal{M} = \mathcal{M}_{3,3}(\mathbb{B})$ strongly preserving $\mathcal{F}_2$, is invertible on $\mathcal{K}$ and permutes the digon-matrices. Now $\mathcal{F}_2$ consists of three matrices: $A = \begin{bmatrix} 011 \\ 100 \\ 100 \end{bmatrix}$, $B = \begin{bmatrix} 010 \\ 101 \\ 010 \end{bmatrix}$ and $C = \begin{bmatrix} 001 \\ 001 \\ 110 \end{bmatrix}$. Each of them is a sum of two of the three digon-matrices: $D_1 = \begin{bmatrix} 010 \\ 100 \\ 000 \end{bmatrix}$, $D_2 = \begin{bmatrix} 001 \\ 000 \\ 100 \end{bmatrix}$ and $D_3 = \begin{bmatrix} 000 \\ 001 \\ 010 \end{bmatrix}$. Therefore each invertible operator on $\mathcal{K}$ that

permutes the digon-matrices also permutes $\mathcal{F}_2$. Consequently $\Sigma = \Sigma(3,2)$ is the group of invertible operators on $\mathcal{K}$ that permute the digon-matrices. Let $\mathcal{P}$ denote the group of linear operators on $\mathcal{K}$ generated by the permutation similarities, transposition and the special operator L . Evidently $\mathcal{P} \subseteq \Sigma$, since the members of $\mathcal{P}$ are invertible and permute the digon-matrices.

Now to complete the proof, we'll show that $\mathcal{P}$ contains a set of operators generating Σ . For each permutation matrix P , let T_P denote the operator $X \longrightarrow PXP^T$. Put $L_1 = L$, $L_2 = T_Q L T_Q$ and $L_3: X \longrightarrow L_1 L_2 (X^T)$, where $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Then the operator L_1 interchanges E_{12} with E_{21} and fixes each other cell of $\mathcal{K}$; L_2 interchanges E_{13} with E_{31} and fixes the other cells of $\mathcal{K}$; and L_3 interchanges E_{23} with E_{32} and fixes the other cells of $\mathcal{K}$.

Recall that Σ consists of the invertible operators that permute the digon-matrices. Let $\mathcal{N}$ denote the subgroup of linear operators $N \in \Sigma$ such that $N(D_i) = D_i$ for $i = 1, 2, 3$. If $N \in \mathcal{N}$, then for each cell $E \in \mathcal{K}$, either

$$(a) \quad N(E) = E \text{ and } N(E^T) = E^T$$

$$\text{or } (b) \quad N(E) = E^T \text{ and } N(E^T) = E.$$

It follows from the previous paragraph that N is a product of some of the L_i and hence the operators L_i generate $\mathcal{N}$.

Now suppose $T \in \Sigma$, the strong preservers of $\mathcal{F}_2 = \{A, B, C\}$. Then $T(A) = A, B$ or C . Let $R = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then, by direct calculation, $T_R(A) = B$ and $T_R^{-1}(A) = C$. Therefore, for some permutation similarity operator T_P , $T_P T(A) = A$. Let $S = T_P T$. Then $S \in \Sigma$ and $S(A) = A$. Now $A = D_1 + D_2$, and, as noted in the first paragraph, the members of Σ permute the digon-matrices. Therefore either (a) $S(D_1) = D_1$ and $S(D_2) = D_2$, or (b) $S(D_1) = D_2$ and $S(D_2) = D_1$. In either case, $S(D_3) = D_3$. If (a) holds, then $S \in \mathcal{N}$. If (b) holds, then $T_Q S \in \mathcal{N}$ because T_Q interchanges D_1 with D_2 and fixes D_3 ; so $S = T_Q^{-1} N$ for some $N \in \mathcal{N}$. Consequently, $T = T_P^{-1} N$ or $T_P^{-1} T_Q^{-1} N$ for some $N \in \mathcal{N}$. Therefore, by the

previous paragraph, T is a product of permutation similarities and at most three of the L_i . But the latter are products of L , some permutation similarities and transposition, and so we have shown that $T \in S$, completing the proof. ■

3.3 Operators on $M_n(\mathbb{B})$ strongly preserving $\mathcal{F}_k(\mathbb{B})$, $k \geq 2$.

Let $M = M_n(\mathbb{B})$ and $2 \leq k \leq n$. If s is a linear transformation mapping the diagonal members $\mathcal{D}$ of M into M , let $L_s(X) = X \circ K + s(X \circ I)$, for all X in M . If L is any linear operator on M , let $L^*(X) = L(X \circ K) + X \circ I$, for all X in M .

LEMMA 6. Suppose L strongly preserves $\mathcal{F}_k$. Then

(a) $L(K) \subseteq K$

and (b) for all nonzero A in $\mathcal{D}$, $L(A)$ is not covered by $\mathcal{F}_k$.

Proof. Part (a). Suppose $L(E) \geq F$ for some link E and some cell F . By Lemma 2, the link E , and hence $L(E)$, is covered by $\mathcal{F}_k$. Therefore F is covered by $\mathcal{F}_k$. It follows from (2.3) that F is a link since $k \geq 2$.

Part(b). Suppose $L(A) \leq M \in \mathcal{F}_k$. Then $L(A) + X = M$, for some $X \leq K$. By Theorem 2, Lemma 2 and part (a) of this lemma, $X = L(Y)$ for some $Y \leq K$. Therefore $L(A + Y)$, $A + Y$, and hence A is covered by $\mathcal{F}_k$. Therefore $A \in K$ by (2.3). ■

We say that L_s is *compatible* (with $\mathcal{F}_k$) if $s(A)$ is not covered by $\mathcal{F}_k$ for all diagonal nonzero A .

LEMMA 7. L_s strongly preserves $\mathcal{F}_k$ if and only if L_s is compatible.

Proof. If D is diagonal, then $L_s(D) = s(D)$. Suppose L_s strongly preserves $\mathcal{F}_k$. Then Lemma 6(b) implies that $D = 0$ if $s(D)$ is covered by $\mathcal{F}_k$.

Conversely, suppose L_s is compatible. If $X \in \mathcal{F}_k$, then $X \in \mathcal{K}$ by (2.3), so $L_s(X) = X \in \mathcal{F}_k$. If $L_s(X) \in \mathcal{F}_k$, then $I \circ X = 0$ by compatibility, because $s(I \circ X) \leq L_s(X)$. Thus $X = X \circ K = L_s(X) \in \mathcal{F}_k$. ■

Similar straight-forward arguments establish the next lemma.

LEMMA 8. If L strongly preserves $\mathcal{F}_k$, define a linear operator s mapping $\mathcal{D}$ into $\mathcal{M}$ by $s(D) = L(D)$, all D in $\mathcal{D}$. Then L_s and L^* strongly preserve $\mathcal{F}_k$ and $L = L_s L^*$.

If a linear operator L on $\mathcal{M}$ strongly preserves $\mathcal{F}_k$, then so does L^* and L^* maps $\mathcal{K}$ into $\mathcal{K}$. So Theorem 3 implies that, unless $k = 2$ and $n = 3$, $L^*(X \circ K) = P^{-1}(\tau(X \circ K))P$ for some permutation P ; τ is either the identity map or transposition. But similarity and transposition are linear and fix the diagonal of each matrix, so $L^*(X) = P^{-1}(\tau(X))P$. Therefore L^* is in the group of operators on $\mathcal{M}$ generated by transposition and similarity. Then Theorem 3 and the Lemmas of this section imply the following theorem.

THEOREM 4. If $2 \leq k \leq n$, then the semigroup of linear operators on $\mathcal{M}_n(\mathbb{B})$ strongly preserving $\mathcal{F}_k(\mathbb{B})$ is generated by transposition, the permutation similarity operators and the compatible operators, unless $k = 2$ and $n = 3$. In that case it is generated by those operators and the special operator defined by

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & c & b \\ a & 0 & d \\ e & f & 0 \end{bmatrix}.$$

§4. MATRICES OVER ANTINEGATIVE, ZERO-DIVISOR-FREE SEMIRINGS.

As we noted in §2, the nonnegative reals, the nonnegative integers and the fuzzy scalars are all examples of antinegative semirings with no zero divisors.

When $\mathcal{S}$ is an antinegative semiring without zero-divisors, then the pattern homomorphism ensures that A in $\mathcal{M}(\mathcal{S})$ is irreducible if and only if its pattern $\bar{A}$ in $\mathcal{M}(\mathbb{B})$ is irreducible, that (2.2) holds for A and that $d(A) = d(\bar{A})$ for all irreducible A in $\mathcal{M}(\mathcal{S})$. Moreover, Lemma 1 implies that P is invertible in $\mathcal{M}(\mathcal{S})$ if and only if $\bar{P}$ is invertible in $\mathcal{M}(\mathbb{B})$ (i.e. $\bar{P}$ is a permutation matrix).

THEOREM 5. Suppose $\mathcal{S}$ is an antinegative semiring with no zero divisors. If $2 \leq k \leq n$, unless $k = 2$ and $n = 3$, the semigroup of linear operators on $\mathcal{M} = \mathcal{M}_n(\mathcal{S})$ strongly preserving $\mathcal{F}_k = \mathcal{F}_k(\mathcal{S})$ is generated by: transposition, the similarity operators ($X \longrightarrow PXP^t$, for fixed invertible P), the compatible diagonal replacement operators ($X \longrightarrow X \circ K + s(X \circ I)$ for some fixed linear mapping s of the diagonal matrices of $\mathcal{M}$ into $\mathcal{M}$ such that $s(A)$ is not covered by $\mathcal{F}_k$ for any diagonal $A \neq 0$), and the nonzero scaling operators, $X \longrightarrow A \circ X \circ B$, for some A, B in $\mathcal{M}$ with no zero entries (we can take $X \longrightarrow A \circ X$ when $\mathcal{S}$ is commutative). If $n = 3$ and $k = 2$, it is generated by the above operators and the special operator defined by

$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & c & b \\ a & 0 & d \\ e & f & 0 \end{bmatrix}.$$

Note that "similarity operator" in the statement of Theorem 5 could be replaced by "permutation similarity operator" because each invertible matrix P in $\mathcal{M}(\mathcal{S})$ is a Schur product $Q \circ (JP)$ where Q is the permutation matrix in $\mathcal{M}(\mathcal{S})$

whose pattern in $\mathcal{M}(\mathbb{B})$ is $\bar{P}$ and J is the $n \times n$ matrix of ones. And so each similarity operator is a composition of a permutation similarity with a nonzero scaling operator.

§5. MATRICES OVER CANCELLATION SEMIRINGS.

In this section, $\mathbb{S}$ is a *cancellation semiring*, that is, for all a, b in $\mathbb{S}$ if $x \neq 0$ and either $ax = bx$ or $xa = xb$ then $a = b$. For example, $\mathbb{S}$ could be the integers, any field or any division ring. Note that cancellation semirings are zero-divisor-free.

The following lemma is proven in [6, p.212].

LEMMA 9. If $\mathbb{S}$ is infinite or has $s > m$ elements and A is an $m \times r$ matrix over $\mathbb{S}$ with no zero rows, then for some $x \in \mathbb{S}^r$, Ax has no zero entries.

The proof of the next lemma closely parallels that of Lemma 5.2 in [6].

LEMMA 10. The linear operator T on $\mathcal{M} = \mathcal{M}_n(\mathbb{S})$ strongly preserves the $\mathcal{F}_k(\mathbb{S})$ only if its pattern, $\bar{T}$, strongly preserves $\mathcal{F}_k(\mathbb{B})$, whenever the cancellation semiring $\mathbb{S}$ has at least $n^2 + 1$ elements.

We now have the following theorem from Theorem 3 and the lemmas of this section. The definition of scaling is as in Theorem 4 to allow for the possibility of $\mathbb{S}$ being noncommutative.

THEOREM 6. Suppose $\mathbb{S}$ is a cancellation semiring with at least n^2+1 elements. Unless $n = 3$ and $k = 2$, the semigroup of linear operators on $M_n(\mathbb{S})$ strongly preserving $\mathcal{F}_k(\mathbb{S})$ is generated by the operators of the following types:

- (a) transposition,
- (b) similarity,
- (c) nonzero entrywise scaling,
- (d) compatible diagonal replacement.

If $n = 3$ and $k = 2$, then the semigroup is generated by the above types of

operators and the operator defined by
$$\begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & c & b \\ a & 0 & d \\ e & f & 0 \end{bmatrix}.$$

We note, as in Theorem 5, that "similarity" can be replaced by "permutation similarity" in the statement of Theorem 6 because each similarity operator is a composition of a permutation similarity and a nonzero entrywise scaling operator.

ACKNOWLEDGEMENT.

This article was completed while the first author was visiting the Institute for Mathematics and its Applications at the University of Minnesota. The authors wish to thank this Institute for the support recieved on this project.

REFERENCES

1. L. B. Beasley and N. J. Pullman, Boolean rank preserving operators and Boolean rank-1 spaces, *Linear Algebra Appl.*, 65(1984), 55-77.
2. L. B. Beasley and N. J. Pullman, Fuzzy rank-preserving operators, *Linear Algebra Appl.*, 73(1986), 197-211.

3. L. B. Beasley, D. A. Gregory, and N. J. Pullman, Nonnegative rank preserving operators, *Linear Algebra Appl.*, 65(1985), 207-223
4. L. B. Beasley and N. J. Pullman, Term rank, permanent and rook polynomial preservers, *Linear Algebra Appl.* 90(1987) 33-46.
5. L. B. Beasley and N. J. Pullman, Linear operators that strongly preserve graphical properties of matrices, *Discrete Math.* (to appear).
6. L. B. Beasley and N. J. Pullman, Linear operators that strongly preserve primitivity, *Linear and Multilinear Algebra*, 25(1989), 205-213
7. R. A. Brualdi and H. J. Ryser, "Combinatorial Matrix Theory", Vol.39, Encyclopedia of Mathematics and its Applications, Cambridge U. Press, Cambridge, 1991.
8. G. Frobenius, Über die Darstellung der endlicher Gruppen durch linear Substitutionen: 1, *Sitzungsbericht der Preuss. Acad. Wiss. zu Berlin*, (1897), 994-1015.
9. D. A. Gregory and N. J. Pullman, Semiring rank: Boolean rank and nonnegative factorizations, *J. Comb. Info. & Syst. Sci.*, 8(1983), 223-233.
10. D. Hershkowitz, Linear mappings which preserve acyclicity properties of graphs and digraphs and applications to matrices, *Discrete Math.* 64(1987), 157-190.
11. K. H. Kim, *Boolean Matrix Theory and Applications*, Pure and Applied Mathematics, Vol. 70, Marcel Dekker, New York, 1982.
12. B. R. McDonald, R -linear endomorphisms of $(R)^n$ preserving invariants, *Mem. Amer. Math. Soc.*, 46, No. 287 (first of two numbers) (1983).
13. M. Marcus and F. May, The permanent function, *Canad. J. Math.*, 14(1962), 177-189.
14. M. Marcus and B. N. Moyls, Linear transformations on algebras of matrices, *Canad. J. Math.*, 11(1959), 61-66.
15. M. Marcus and B. N. Moyls, Transformations on tensor spaces, *Pacific J. Math.*, 9(1959), 1215-1221.
16. M. Marcus and R. Purves, Linear transformations on algebras of matrices, II: the invariance of the elementary symmetric functions, *Canad. J. Math.*, 11(1959), 383-396.
17. N. J. Pullman, Linear operators that strongly preserve clique covering numbers, *Ars Combinatoria*, 19A(1985) 27-36.

#	Author/s	Title
801	Hi Jun Choe,	Regularity for solutions of nonlinear variational inequalities with gradient constraints
802	Peter Shi and Yongzhi Xu,	Quasistatic linear thermoelasticity on the unit disk
803	Satyanad Kichenassamy and Peter J. Olver,	Existence and non-existence of solitary wave solutions to higher order model evolution equations
804	Dening Li,	Regularity of solutions for a two-phase degenerate Stefan Problem
805	Marek Fila, Bernhard Kawohl and Howard A. Levine,	Quenching for quasilinear equations
806	Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii,	Global existence of weak solutions for interface equations coupled with diffusion equations
807	Mark J. Friedman and Eusebius J. Doedel,	Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study
808	Mark J. Friedman,	Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
809	Peter W. Bates and Songmu Zheng,	Inertial manifolds and inertial sets for the phase-field equations
810	J. López Gómez, V. Márquez and N. Wolanski,	Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
811	Xinfu Chen and Fahuai Yi,	Regularity of the free boundary of a continuous casting problem
812	Eden, A., Foias, C., Nicolaenko, B. and Temam, R.,	Inertial sets for dissipative evolution equations Part I: Construction and applications
813	Jose-Francisco Rodrigues and Boris Zaltzman,	On classical solutions of the two-phase steady-state Stefan problem in strips
814	Viorel Barbu and Srdjan Stojanovic,	Controlling the free boundary of elliptic variational inequalities on a variable domain
815	Viorel Barbu and Srdjan Stojanovic,	A variational approach to a free boundary problem arising in electrophotography
816	B.H. Gilding and R. Kersner,	Diffusion-convection-reaction, free boundaries, and an integral equation
817	Shoshana Kamin, Lambertus A. Peletier and Juan Luis Vazquez,	On the Barenblatt equation of elastoplastic filtration
818	Avner Friedman and Bei Hu,	The Stefan problem with kinetic condition at the free boundary
819	M.A. Grinfeld,	The stress driven instabilities in crystals: mathematical models and physical manifestations
820	Bei Hu and Lihe Wang,	A free boundary problem arising in electrophotography: solutions with connected toner region
821	Yongzhi Xu, T. Craig Poling, and Trent Brundage,	Direct and inverse scattering of time harmonic acoustic waves in an inhomogeneous shallow ocean
822	Steven J. Altschuler,	Singularities of the curve shrinking flow for space curves
823	Steven J. Altschuler and Matthew A. Grayson,	Shortening space curves and flow through singularities
824	Tong Li,	On the Riemann problem of a combustion model
825	L.A. Peletier & W.C. Troy,	Self-similar solutions for diffusion in semiconductors
826	C.J. van Duijn, L.A. Peletier & R.J. Schotting,	On the analysis of brine transport in porous media
827	Minkyu Kwak,	Finite dimensional description of convective reaction-diffusion equations
828	Minkyu Kwak,	Finite dimensional inertial forms for the 2D Navier-Stokes equations
829	Victor A. Galaktionov and Sergey A. Posashkov,	On some monotonicity in time properties for a quasilinear parabolic equation with source
830	Victor A. Galaktionov,	Remark on the fast diffusion equation in a ball
831	Hi Jun Choe and Lihe Wang,	A regularity theory for degenerate vector valued variational inequalities
832	Vladimir I. Oliker and Nina N. Uraltseva,	Evolution of nonparametric surfaces with speed depending on curvature, II. The mean curvature case.
833	S. Kamin and W. Liu,	Large time behavior of a nonlinear diffusion equation with a source
834	Shoshana Kamin and Juan Luis Vazquez,	Singular solutions of some nonlinear parabolic equations
835	Bernhard Kawohl and Robert Kersner,	On degenerate diffusion with very strong absorption
836	Avner Friedman and Fernando Reitich,	Parameter identification in reaction-diffusion models
837	E.G. Kalnins, H.L. Manocha and Willard Miller, Jr.,	Models of q -algebra representations I. Tensor products of special unitary and oscillator algebras
838	Robert J. Sacker and George R. Sell,	Dichotomies for linear evolutionary equations in Banach spaces
839	Oscar P. Bruno and Fernando Reitich,	Numerical solution of diffraction problems: a method of variation of boundaries
840	Oscar P. Bruno and Fernando Reitich,	Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain
841	Victor A. Galaktionov and Juan L. Vazquez,	Asymptotic behaviour for an equation of superslow diffusion. The Cauchy problem
842	Josephus Hulshof and Juan Luis Vazquez,	The Dipole solution for the porous medium equation in several

- 843 **Shoshana Kamin and Juan Luis Vazquez**, The propagation of turbulent bursts
- 844 **Miguel Escobedo, Juan Luis Vazquez and Enrike Zuazua**, Source-type solutions and asymptotic behaviour for a diffusion-convection equation
- 845 **Marco Biroli and Umberto Mosco**, Discontinuous media and Dirichlet forms of diffusion type
- 846 **Stathis Filippas and Jong-Shenq Guo**, Quenching profiles for one-dimensional semilinear heat equations
- 847 **H. Scott Dumas**, A Nekhoroshev-like theory of classical particle channeling in perfect crystals
- 848 **R. Natalini and A. Tesei**, On a class of perturbed conservation laws
- 849 **Paul K. Newton and Shinya Watanabe**, The geometry of nonlinear Schrödinger standing waves
- 850 **S.S. Sritharan**, On the nonsmooth verification technique for the dynamic programming of viscous flow
- 851 **Mario Taboada and Yuncheng You**, Global attractor, inertial manifolds and stabilization of nonlinear damped beam equations
- 852 **Shigeru Sakaguchi**, Critical points of solutions to the obstacle problem in the plane
- 853 **F. Abergel, D. Hilhorst and F. Issard-Roch**, On a dissolution-growth problem with surface tension in the neighborhood of a stationary solution
- 854 **Erasmus Langer**, Numerical simulation of MOS transistors
- 855 **Haim Brezis and Shoshana Kamin**, Sublinear elliptic equations in $\mathbb{R}^n$
- 856 **Johannes C.C. Nitsche**, Boundary value problems for variational integrals involving surface curvatures
- 857 **Chao-Nien Chen**, Multiple solutions for a semilinear elliptic equation on $\mathbb{R}^N$ with nonlinear dependence on the gradient
- 858 **D. Brochet, X. Chen and D. Hilhorst**, Finite dimensional exponential attractor for the phase field model
- 859 **Joseph D. Fehribach**, Mullins-Sekerka stability analysis for melting-freezing waves in helium-4
- 860 **Walter Schempp**, Quantum holography and neurocomputer architectures
- 861 **D.V. Anosov**, An introduction to Hilbert's 21st problem
- 862 **Herbert E Huppert and M Grae Worster**, Vigorous motions in magma chambers and lava lakes
- 863 **Robert L. Pego and Michael I. Weinstein**, A class of eigenvalue problems, with applications to instability of solitary waves
- 864 **Mahmoud Affouf**, Numerical study of a singular system of conservation laws arising in enhanced oil reservoirs
- 865 **Darin Beigie, Anthony Leonard and Stephen Wiggins**, The dynamics associated with the chaotic of tangles two dimensional quasiperiodic vector fields: theory and applications
- 866 **Gui-Qiang Chen and Tai-Ping Liu**, Zero relaxation and dissipation limits for hyperbolic conservation laws
- 867 **Gui-Qiang Chen and Jian-Guo Liu**, Convergence of second-order schemes for isentropic gas dynamics
- 868 **Aleksander M. Simon and Zbigniew J. Grzywna**, On the Larché-Cahn theory for stress-induced diffusion
- 869 **Jerzy Luczka, Adam Gadomski and Zbigniew J. Grzywna**, Growth driven by diffusion
- 870 **Mitchell Luskin and Tsorng-Whay Pan**, Nonplanar shear flows for nonaligning nematic liquid crystals
- 871 **Mahmoud Affouf**, Unique global solutions of initial-boundary value problems for thermodynamic phase transitions
- 872 **Richard A. Brualdi, Keith L. Chavey and Bryan L. Shader**, Rectangular L -matrices
- 873 **Xinfu Chen, Avner Friedman and Bei Hu**, The thermistor problem with zero-one conductivity II
- 874 **Raoul LePage**, Controlling a diffusion toward a large goal and the Kelly principle
- 875 **Raoul LePage**, Controlling for optimum growth with time dependent returns
- 876 **Marc Hallin and Madan L. Puri**, Rank tests for time series analysis a survey
- 877 **V.A. Solonnikov**, Solvability of an evolution problem of thermocapillary convection in an infinite time interval
- 878 **Horia I. Ene and Bogdan Vernescu**, Viscosity dependent behaviour of viscoelastic porous media
- 879 **Kaushik Bhattacharya**, Self-accommodation in martensite
- 880 **D. Lewis, T. Ratiu, J.C. Simo and J.E. Marsden**, The heavy top: a geometric treatment
- 881 **Leonid V. Kalachev**, Some applications of asymptotic methods in semiconductor device modeling
- 882 **David C. Dobson**, Phase reconstruction via nonlinear least-squares
- 883 **Patricio Aviles and Yoshikazu Giga**, Minimal currents, geodesics and relaxation of variational integrals on mappings of bounded variation
- 884 **Patricio Aviles and Yoshikazu Giga**, Partial regularity of least gradient mappings
- 885 **Charles R. Johnson and Michael Lundquist**, Operator matrices with chordal inverse patterns
- 886 **B.J. Bayly**, Infinitely conducting dynamos and other horrible eigenproblems
- 887 **Charles M. Elliott and Stefan Luckhaus**, 'A generalised diffusion equation for phase separation of a multi-component mixture with interfacial free energy'
- 888 **Christian Schmeiser and Andreas Unterreiter**, The derivation of analytic device models by asymptotic methods
- 889 **LeRoy B. Beasley and Norman J. Pullman**, Linear operators that strongly preserve the index of imprimitivity
- 890 **Jerry Donato**, The Boltzmann equation with lie and cartan
- 891 **Thomas R. Hoffend Jr., Peter Smereka and Roger J. Anderson**, Method for resolving the laser induced local heating of moving magneto-optical recording media