NECESSARY AND SUFFICIENT CONDITIONS FOR THE
EXISTENCE OF A LOCALLY STABLE MESSAGE PROCESS*⁺

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Local Stability of Message Processes

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1. INTRODUCTION

In 1960, Hurwicz [5] introduced an abstract model of the collection and processing of information in an economy. This model initiated the development of a theoretical framework for the evaluation and comparison of alternative economic mechanisms. Besides being used in the economic literature, the model (in its more abstract forms) has itself been the subject of intensive study.

Hurwicz originally modelled an informationally decentralized dynamic process. Most of the research on this model, however, has focused not on the dynamic process, but on the nature of the equilibrium state of the model. This is true partly because there is much to be learned by studying the equilibrium state, and partly because this "simpler" static version of the model has itself posed some difficult mathematical problems, many of which are still poorly understood.

Much of the research on the static version of the model has centered on the problem of measuring the amount and nature of the information which must be collected from the agents in the economy in order to compute some desired objective function. In particular, the minimum amount of information which is needed to (statically) implement the objective has been extensively studied. In the last few years, these issues have been reexamined in the context of Hurwicz's original dynamic model: the scope of the questions has been broadened to encompass the dynamic process itself, and the amount and nature of the information which is needed to insure that the dynamic process actually achieves the objective over time. Reiter [8] first raised these issues with an example of an objective function which assigns a desired equilibrium value to each possible environment in an economy. In this example, he shows that the total amount of information which is collected by a given static process when it computes the equilibrium

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When an equilibrium message correspondence is specified for each agent in an economy, there is an infinite number of sets of equations which can be used to represent these correspondences, each of which defines a different dynamic message process for the economy. By working only with C^1 message processes, this paper first explains the relationship between all of the message processes which determine the same equilibrium message correspondences for the agents. We then turn to the problem of local stability: our main goal is to describe when a set of equilibrium message correspondences can be achieved over time by a locally stable message process. By analyzing derivatives, conditions are found under which this both can and cannot be done. A procedure is outlined which can be used to construct a locally stable message process from the equations which define the given equilibrium correspondences when certain regularity conditions are satisfied, and several examples are analyzed.

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value is strictly less than the amount which would be needed by any dynamic process to actually find the equilibrium value over time.

By pointing out that a discrepancy can exist between the minimum amount of information needed for a dynamic implementation, as opposed to the minimum needed for a static implementation of a given objective function, Reiter's example has renewed interest in the dynamic nature of the original model. Jordan's paper [6] represents the first attempt to study in an abstract manner the informational requirements of dynamic processes which converge to implement a given objective, and Mount and Reiter [7] have also written on this subject. The emphasis in both of these papers is on the differences between the informational needs of the static model, and of the convergent, dynamic model.

Each of these papers describes the background of this problem in detail, so it will not be discussed further here. We will now then turn to the aims of this paper.

Much of the research on the static version of the Hurwicz model has focused on the set of states of the system at which a given agent (with a fixed profile of characteristics) would not instigate any change in the system. This defines a correspondence between elements of the set of all possible characteristics for this agent, and subsets of the set of states of the system--the agent's equilibrium message correspondence. In a sense, the value of this correspondence at a particular profile of characteristics consists of all states which the agent would then find acceptable. Given an environment for the economy (i.e., a profile of characteristics for each agent), the equilibrium states of the entire system are obtained by forming the intersection of the sets which are determined separately by the individual agents. Most of the work on the problem of existence of mechanisms which statically implement a given objective and on the problem of determining the minimum amount of information that is needed to

implement the objective has had these sets and the way they vary under changes in the environment as its primary concern.

We will build on this approach in this paper. Each agent's equilibrium message correspondence is presumed to be given (perhaps observed), invariant aspect of the problem. Our goal is to study those decentralized dynamic processes for which the sets of equilibria that are determined separately by each agent, the rule by which he can influence the state of the system, and a choice of an environment exactly matches the sets which are determined by the given correspondences. In particular, we want to learn when the sets which are determined by the given correspondences can be achieved over time by a dynamic process, and alternatively, when there is something inherently unstable about the given equilibrium message correspondences. Reiter's example shows that each of these cases can happen.

Since most of our knowledge of the static model is based upon the agent's equilibrium correspondences, this is the most direct approach to understanding the difference between the information needed by a static implementation of an objective function, and the informational requirements of a convergent, dynamic implementation. It is also a meaningful approach from the economic viewpoint, because in most uses of the static model the agents' equilibrium message correspondences have a concrete economic interpretation. It is worthwhile to try to build on these correspondences to construct a convergent, dynamic process which implements some objective, because in this way the economic interpretation of the static implementation will not be lost.

Before we summarize the results of this paper, we must first precisely define our terminology and the problem. This will be done in the next section.

2. DEFINITIONS AND NOTATION

We will assume that there are n agents in the economy. For each $1 \leq i \leq n$, the i^{th} agent's characteristics is determined by a point in a C^1 manifold E^i , and the space of environments for the economy is $\prod_{i=1}^n E^i$. We will often use $e = (e_i)$ to represent an environment for the economy. Let the dimension of E^i equal k_i , where $k_i \geq 1$.

We begin our analysis with a given set of correspondences $\mu_1, \mu_2, \dots, \mu_n$, where μ_i assigns subsets of M to the elements of E^i . The correspondence μ_i is the i^{th} agent's equilibrium message correspondence. We restrict ourselves to correspondences $\mu_1, \mu_2, \dots, \mu_n$ which have the following special property: for each i , there exists a C^1 mapping g_i sending $E^i \times M$ into some Euclidean space $\mathbb{R}^{c_i}$ such that the value of μ_i at any element e_i in E^i is equal to the solution set in M of the equation $g_i(e_i, \cdot) = 0$ (here, "0" denotes the zero vector of the Euclidean space). We also insist that the mappings $g_1, g_2, \dots, g_n$ jointly define a " C^1 message process," which we now define.

DEFINITION. Let $c_1, c_2, \dots, c_n$ be any positive numbers which satisfy $1 \leq c_i \leq k_i$ for each value of i , and let M be an open subset of $\mathbb{R}^c$, where $c = \sum_{i=1}^n c_i$. Represent an element m of M as $m = (m_1, \dots, m_n)$, where m_i is an element of $\mathbb{R}^{c_i}$. A C^1 message process of type $(c_1, \dots, c_n)$ on $\prod_{i=1}^n E^i$ is a set of differential equations $\dot{m} = G(e, m)$ in the variables of M which are parametrized by the elements of $\prod_{i=1}^n E^i$, and which have the following properties:

- i) for each $1 \leq i \leq n$, the adjustment rule for the variable m_i has the form $\dot{m}_i = g_i(e_i, m)$, where g_i is a C^1 mapping on $E^i \times M$ with values in $\mathbb{R}^{c_i}$;
- ii) for every e in some open subset of $\prod_{i=1}^n E^i$, there exists an m in M such that $G(e, m) = 0$.

The space M is the message space for the message process. If each g_i also satisfies $\text{rank } Dg_i = c_i$ at every point in $(g_i)^{-1}(0)$, then $\dot{m} = G$ is a regular C^1 message process.

When the value of μ_i at any e_i in E^i is given by the solution set of the equation $g_i(e_i, \cdot) = 0$, we will say that the equation $g_i = 0$ represents the equilibrium message correspondence μ_i . Similarly, if this holds for each $1 \leq i \leq n$, then the message process $\dot{m} = G = (g_i)$ represents (in equilibrium) the correspondences $\mu_1, \mu_2, \dots, \mu_n$.

To be more accurate, a C^1 message process should be called an informationally decentralized (or "privacy-preserving") message process: any changes in the state of the system that are instigated by the i^{th} agent by means of the rule g_i are based solely upon the value e_i of his own characteristics and the current state of the system, and not upon the value of any other agent's characteristics. We will sometimes write $(\dot{m}_i) = G(e, m) = (g_i(e_i, m))$ to emphasize that the process is informationally decentralized.

Note also that each agent has exclusive control over the adjustment of several of the message variables. This matches a standard hypothesis in elementary game theory, wherein each player is presumed to have complete freedom to choose his own strategy, according to his own preferences and information. It forces us to use a message space M which is a Cartesian product of n C^1 manifolds; for simplicity's sake, we assume that M is an open subset of $\mathbb{R}^C$. Unlike several papers on the static version of the Hurwicz model, we cannot let M be an arbitrary manifold.

The agent's equilibrium message correspondences are the invariant in our approach, and we will study the properties of the different message processes which can be used to represent these correspondences. When the i^{th} agent's

equilibrium message correspondence is represented by $g_i(e_i, \cdot) = 0$, where g_i is a C^1 mapping, there exists an infinite number of C^1 mappings f_i on $E^i \times M$ with the property that the equation $f_i(e_i, \cdot) = 0$ represents exactly the same correspondence between elements of E^i and subsets of M as $g_i(e_i, \cdot) = 0$. Our goal is to learn when we can choose from among all of these alternatives a set of C^1 mappings $\{f_1, f_2, \dots, f_n\}$ such that the message process that is given by the n equations $\dot{m}_i = f_i(e_i, m)$ is locally stable, in a sense which we shall define shortly. Note that if the equilibrium message correspondences were originally constructed to statically implement a given objective function, then any of the message processes that we will be considering will also statically implement the objective, since neither the message space nor the equilibrium message correspondences will be varied. It is the adjustment processes alone which we will manipulate, with the goal of constructing a locally stable message process.

Though the equilibrium message correspondences are the invariant in our approach, our notation and most of our analysis will be in terms of the message processes which represent these correspondences. We choose to analyze the problem in this manner, because it is simpler to work with the message processes than with the intrinsic geometry of the equilibrium message correspondences, both notationally and in terms of the analysis. To avoid confusion about the nature of our results, we will therefore explain how our definitions and results reflect the properties of the equilibrium message correspondences, rather than the message process which has been chosen to represent them.

We will now define what we mean by "local stability" of message processes. Given a C^1 message process $\dot{m} = G$, for any elements $e' = (e'_i)$ of $\prod_{i=1}^n E^i$

and m' of M there exists a unique C^1 mapping $m^*(e', m', \cdot): \mathbb{R} \rightarrow M$ which satisfies $m^*(e', m', 0) = m'$, and $d/dt m^*(e', m', t') = G(e', m^*(e', m', t'))$ at each $t' \in \mathbb{R}$ [1, p.50-55]. This is the solution to the differential equation $\dot{m} = G$ with initial conditions (e', m') . For any $m' \in M$ and for each value of i , we will let $E_m^i(g_i)$ denote the set of all elements e_i of E^i which satisfy the equation $g_i(e_i, m') = 0$, while $E_m(G)$ will denote the set of all elements of $\prod_{i=1}^n E^i$ which satisfy $G(\cdot, m') = 0$. Note that $E_m(G)$ is equal to the Cartesian product $\prod_{i=1}^n E_m^i(g_i)$.

DEFINITION. Let $(\dot{m}_i) = G = (g_i)$ be a C^1 message process on $\prod_{i=1}^n E^i \times M$. If (e', m') is an element of $\prod_{i=1}^n E^i \times M$ which satisfies $G(e', m') = 0$, then $\dot{m} = G$ is locally stable at (e', m') if and only if:

i) for each neighborhood V of m' , there exists a neighborhood U of m' such that $m^*(e', m, t) \in V$ for any $m \in U$ and all $t \in [0, \infty)$.
(Lyapunov stability).

ii) $\lim_{t \rightarrow \infty} m^*(e', m, t)$ exists for all $m \in U$, and it satisfies the equation (1)
 $G(e', \cdot) = 0$. (asymptotic stability).

The message process $\dot{m} = G$ is locally stable at m' if it is locally stable at every element (e, m') of $E_m(G) \times \{m'\}$, and $\dot{m} = G$ locally stable on M if it is locally stable at every $m' \in M$.

REMARK. When we begin with equilibrium message correspondences $\mu_1, \mu_2, \dots, \mu_n$ and choose a C^1 message process $\dot{m} = G = (g_i)$ to represent them, the sets $E_m^i(g_i)$ are truly independent of the message process $\dot{m} = G$ which has been chosen, for they are uniquely defined by the correspondences themselves. The sets $E_m(G)$ have the same property. The meaning of "local stability at m' " is therefore determined by the equilibrium message correspondences themselves, inde-

pendently of their representation, and one checks a message process $\dot{m} = G$ which represents these correspondences to see whether or not it has this property. Obviously, similar remarks can also be made about "local stability at (e', m') ," and "local stability on M ."

Note that in (1) we do not insist that $\lim_{t \rightarrow \infty} m^*(e', m, t)$ equal m' , even when m is chosen arbitrarily close to m' : our definition only requires that some equilibrium be achieved over time. For some message processes $\dot{m} = G$, there can exist solutions to the equation $\dot{m} = G(e', \cdot)$ in every deleted neighborhood of m' , so the solution curves of the differential equation $\dot{m} = G(e', \cdot)$ might approach some other equilibrium message value for e' . This kind of "degeneracy" in the solution set of $G(e', \cdot) = 0$ is in fact the source of most examples of equilibrium message correspondences which cannot be achieved by message processes which are stable at m' , including Reiter's original example. We will discuss this topic further in Section 6.

The difference between our definition of local stability and Jordan's is that we do not impose any condition on the behavior of $m^*(e, m, t)$ for e near e' , while Jordan insists that $m^*(e, m, t)$ be an element of V for all $m \in U$, $t \in [0, \infty)$, and e near e' , and that $\lim_{t \rightarrow \infty} m^*(e, m, t)$ exists and satisfies $G(e, \cdot) = 0$. While this additional requirement for stability is attractive from the viewpoint of economic theory, it does impose a limitation on the kind of message process that can be studied: for any e', m' which satisfy $G(e', m') = 0$, Jordan's definition implicitly presupposes that there are neighborhoods U_1 of e' and U_2 of m' such that $G(e, \cdot) = 0$ has a solution in U_2 for any e in U_1 . We will not insist that G have this property.

Most of our analysis will concern local stability at elements m' of M . In this regard, it will resemble the approach taken by Mount and Reiter [7]. Mount and Reiter, however, studied the Lyapunov stability of message processes, and they did not insist that the solution curves of $\dot{m} = G(e', m)$ be asymptotically stable.

The following classical result from the theory of differential equations provides us with a computational method for evaluating the stability of message processes.

THEOREM 1. A C^1 message process $\dot{m} = G$ is locally stable at a point (e', m') in $G^{-1}(0)$ if each eigenvalue of $D_m G(e', m')$ has negative real part. If at least one eigenvalue of $D_m G(e', m')$ has negative real part then G is unstable at (e', m') . If each eigenvalue of $D_m G(e', m')$ has nonpositive real part and at least one eigenvalue has real part equal to zero, then the local stability or instability of G at (e', m') cannot be determined from the value of $D_m G(e', m')$ alone.

A proof and further discussion of this result can be found both in Hirsch and Smale [4, Ch. 9, Sec. 2] and in Arnold [1, Ch.3, Sec. 23].

3. SUMMARY OF RESULTS

In our next section, we will characterize the relationship that exists between the mappings f_i and g_i if the equations $f_i(e_i, \cdot) = 0$ and $g_i(e_i, \cdot) = 0$ represent exactly the same equilibrium message correspondence. At this point, we will need to assume that the given equilibrium message correspondence can be represented by a mapping g_i which satisfied a certain regularity condition. This result will define all of the mathematical manipulations that can be performed on the mapping g_i without changing the i^{th} agent's message correspondence.

We will then review several classical algebraic results concerning the eigenvalues of real matrices: we will describe conditions on a matrix which insure that all of its eigenvalues have negative real part, and we will also describe conditions under which the matrix must have at least one eigenvalue with positive real part. In light of Theorem 1, our general procedure in this paper is now obvious: starting with regular a C^1 message process $\dot{m} = G$ which represents a given set of equilibrium message correspondences and any (e', m') in $G^{-1}(0)$, we will either try to manipulate G to a form where each eigenvalue of $D_m G(e', m')$ has negative real part, or we will try to show that however G is manipulated, the derivative of the resulting message process with respect to m will have at least one eigenvalue with positive real part.

After we review the algebraic results that will be needed in this paper, we will reanalyze Reiter's example of a pair of equilibrium message correspondences which cannot be represented by a locally stable message process. As mentioned earlier, Reiter's original example depends upon the existence of a "degeneracy" in the equilibrium message correspondences: for some environment (e_1^1, e_2^1) the value of the first agent's equilibrium message correspondence at e_1^1 exactly equals the value of the second agent's equilibrium message correspondence at e_2^1 . We will interpret this degeneracy from an analytic viewpoint -- as a singularity of $D_m G$, where $\dot{m} = G$ is any C^1 message process which represent the given equilibrium message correspondences. We will then give an example of a pair of correspondences which do not have this degeneracy at any environment, and yet which also cannot be represented by a locally stable message process.

In Section 8, we will abstract from these two examples a set of conditions on the principal minors of $D_m G$ which guarantees that the correspondences that are given by $g_1(e, \cdot) = 0, g_2(e_2, \cdot) = 0, \dots, g_n(e_n, \cdot) = 0$ cannot be represented

by a C^1 message process which is locally stable on M . The conditions are of the following nature: when the set of points in $G^{-1}(0)$ at which the determinant of $D_m G$ changes sign fails to have a certain special shape (Theorem 3), or when certain principal minors of $D_m G$ change sign over $G^{-1}(0)$ in a particular fashion (Theorem 5), then every message process which represents the same equilibrium message correspondences as $\dot{m} = G$ must be unstable at some equilibrium pair (e', m') .

We will turn this problem around in Theorem 6 of Section 9. Here, we will impose regularity conditions on the principal minors of the matrix $D_m G$ which enable us to construct a locally stable message process from the mappings $g_1, g_2, \dots, g_n$, without disturbing the equilibrium message correspondences which they represent. A procedure will be outlined for actually carrying out this construction on a C^1 message process.

The conditions on the principal minors of $D_m G$ in Theorem 6 are sufficient, but not necessary, for the existence of a locally stable C^1 message process which represents the same equilibrium message correspondences as $\dot{m} = G$. An example will be presented at the end of this paper, however, which illustrates the difficulties that are involved in weakening the hypotheses of this theorem. This example is of interest by itself: in all other known examples of this kind, the obstacle to local stability on M can be characterized by the peculiar behavior of the equilibrium message correspondences at some point in $\prod_{i=1}^n E^i$, while this example depends upon the behavior of these correspondences over the entire space of environments.

The major procedural difference between this paper and the papers by Reiter [8], Mount and Reiter [7], and Jordan [6] is that we assume that the message process $\dot{m} = G$ which represents the equilibrium message correspondences is C^1 ,

while these earlier papers only assume that it is continuous (with appropriate Lipschitz conditions to insure the existence of solutions). Our analysis is focused almost entirely on the matrix $D_m G$. We thereby lose some of the generality of this earlier work, but we are also able to make a more extensive analysis of the problem of local stability for message processes.

Much of the methodology of this paper was inspired by Saari and Simon's "Effective Price Mechanisms" [9]. This is to be expected, since the problems that we will be studying here are basically a generalization of the problems that were solved by Saari and Simon: they studied the informational requirements of convergent price mechanisms, while we are studying the information requirements of stability for a broader, more abstract class of mechanisms. The mathematical differences between the two papers arise from our insistence that each agent has complete control over the adjustment of several of the message variables, which jointly characterize the state of the system. In Saari and Simon's paper, the state of the system is given by a set of prices, and prices are adjusted according to the values of the aggregate excess demand function and its derivatives. This additional requirement is the source of most of our difficulties in constructing locally stable message processes. For instance, it prevents us from using Newton's Method as a general procedure for finding the equilibrium states, as Saari and Simon were able to do. The reader should keep in mind that the problems we will discuss are trivial in the absence of the constraint of decentralization.

4. "ALLOWABLE" MANIPULATIONS OF MESSAGE PROCESSES

Let f_i and g_i be C^1 mappings from $E^i \times M$ into $\mathbb{R}^{C_i}$ such that the equations $f_i = 0$ and $g_i = 0$ represent the equilibrium message correspondence for the i^{th} agent. Our first theorem will show that if g_i satisfies a regularity condition, then f_i can be expressed in terms of g_i on some open subset of $E^i \times M$ which contains $g_i^{-1}(0)$. This result will form the basis for most of the analysis in this paper.

THEOREM 2. Let g_i be a C^1 mapping from $E^i \times M$ into $\mathbb{R}^{c_i}$ which satisfies $\text{rank } Dg_i = c_i$ at each point in $g_i^{-1}(0)$. Given any C^1 mapping f_i from $E^i \times M$ into $\mathbb{R}^{c_i}$ for which $f_i^{-1}(0) = g_i^{-1}(0)$ as subsets of $E^i \times M$, there exists a C^1 mapping A_i from an open subset of $E^i \times M$ which contains $g_i^{-1}(0)$ into $L(\mathbb{R}^{c_i}, \mathbb{R}^{c_i})$ such that $f_i = A_i \cdot g_i$ at all points in the domain of A_i .

PROOF. Since $\text{rank } Dg_i = c_i$ on $g_i^{-1}(0)$, at any point in $g_i^{-1}(0)$ we can supplement the c_i real-valued C^1 functions given by g_i with $\dim(E^i \times M) - c_i$ additional C^1 functions to form a coordinate system for $E^i \times M$ in a neighborhood of the point. In each of these neighborhoods, Hadamard's Lemma (see below) can be applied to construct a $c_i \times c_i$ matrix A_i' of C^1 functions such that $f_i = A_i' \cdot g_i$ is true at each point in the neighborhood. A partition of unity [3, p.43] can then be used to construct A_i from these local solutions. Q.E.D.

A proof of Hadamard's Lemma can be found in Saari and Simon [9, Lemma 1, p. 1104]. We are using this lemma exactly as they used it--to reduce the study of a large class of mechanisms to a matrix multiplication problem. Because this result is so critical to our analysis, we must henceforth restrict our attention to those equilibrium message correspondences which can be given by an equation $g_i(e_i, \cdot) = 0$ that satisfies $\text{rank } Dg_i = c_i$ at each solution (e_i, m) . This explains our interest in regular message processes.

Note that the value of g_i cannot lie in $\ker A_i$ (the kernel of A_i) if g_i is nonzero, else $g_i^{-1}(0)$ would be a proper subset of $f_i^{-1}(0)$. A_i need not be nonsingular at elements of $g_i^{-1}(0)$, but this can create problems which will be discussed shortly. It is easy to verify that given any C^1 mappings

$A_1, A_2, \dots, A_n$ where A_i sends $E_i \times M$ into $L(\mathbb{R}^{c_i}, \mathbb{R}^{c_i})$ and $\ker A_i$ does not contain g_i at any point where g_i is nonzero, the message process $(\dot{m}_i) = (A_i \cdot g_i)$ will represent the same equilibrium message correspondences for the agents as $(\dot{m}_i) = g_i$. Theorem 2 states that every message process which represents these correspondences must have this form near $G^{-1}(0)$, so we have completely characterized the ways that G can be manipulated in order to achieve local stability.

If we let $A \equiv (A_i)$ denote a $c \times c$ block-diagonal matrix which has matrices $A_1, A_2, \dots, A_n$ down its main diagonal, where each A_i is a $c_i \times c_i$ matrix of functions on $E^i \times M$ such that g_i is not in $\ker A_i$ when g_i is nonzero, then we are interested in all message processes of the form $\dot{m} = A \cdot G$. By Theorem 1, we will want to study the eigenvalues of $D_m(A \cdot G)$ at elements of $G^{-1}(0)$. The following formula can be computed by the product rule:

$$D_m(A \cdot G) = D_m A \cdot G + A \cdot D_m G.$$

At elements of $G^{-1}(0)$, this formula simplifies to the following form:

$$D_m(A \cdot G) = A \cdot D_m G. \quad (2)$$

The problem of local stability for the equilibrium message correspondences that are given by a regular C^1 message process $\dot{m} = G$ has been reduced to the problem of finding a particular kind of $c \times c$ matrix A such that each eigenvalue of $A \cdot D_m G$ at a point in $G^{-1}(0)$ has nonpositive real part. Ideally, we would want the real part of each eigenvalue of $A \cdot D_m G$ at any point in $G^{-1}(0)$ to be negative: the local stability of $\dot{m} = A \cdot G$ at a point in $G^{-1}(0)$ where some of the eigenvalues of $A \cdot D_m G$ are purely imaginary or even zero can be difficult to determine, especially if higher order derivatives of $A \cdot G$ do not

exist. Also, even if $\dot{m} = A \cdot G$ is locally stable at such a point, the convergence rate of the message process to an equilibrium for some nearby choices of the initial conditions would be undesirably slow. This problem cannot be avoided if there are singular points of $D_m G$ in $G^{-1}(0)$: as (2) shows, the presence of these points is an intrinsic property of the equilibrium message correspondences that are determined by G , and they cannot be eliminated by changing the equations which determine these correspondences. We can, however, insist that the matrix A should not introduce "many" singularities into the message process, in a sense which we shall specify in the following definition. Note that A can be singular; this is the problem with singularities of A_i which was alluded to earlier.

This definition will focus on local stability at a value m' of M , since most of the analysis in this paper concerns this kind of stability. For a message process $\dot{m} = F$, $S_{m'}(F)$ will denote the set of all points e in $E_{m'}(F)$ at which $D_m F(\cdot, m')$ is singular. In this definition, it should be presumed that each set $E_{m'}(F)$ is given the induced topology from $\prod_{i=1}^n E^i$.

DEFINITION. Let $\dot{m} = G = (g_i)$ be a regular C^1 message process of type $(c_1, \dots, c_n)$, and let $\dot{m} = F = (f_i)$ be a C^1 message process of this same type, where $g_i^{-1}(0)$ for each value of i . If $S_{m'}(F) \setminus S_{m'}(G)$ is nowhere dense in $E_{m'}(G)$ for all $m' \in M$, then F is an effective message process. $EMP(G)$ will denote the set of all effective message processes which determine the same equilibrium message correspondences as $\dot{m} = G$.

Note that the criterion for deciding if F is effective depends upon the equilibrium message correspondences that are determined by both F and G , but not upon the message process G itself: as mentioned before, the singular

points of $D_m G$ in $G^{-1}(0)$ are an intrinsic characteristic of these equilibrium message correspondences, since $\dot{m} = G$ is a regular message process. This definition could be given in terms of the equilibrium message correspondences themselves, but it is simpler to use G .

Note also that if $\dot{m} = G$ and $\dot{m} = G^*$ are regular message processes which represent the same equilibrium message correspondences $\mu_1, \mu_2, \dots, \mu_n$, then $S_{m'}(G) = S_{m'}(G^*)$ at every m' in M . As a consequence, $EMP(G)$ must equal $EMP(G^*)$. The set $EMP(G)$ is therefore defined by the correspondences $\mu_1, \mu_2, \dots, \mu_n$ themselves: it is independent of the choice of the representation G . Perhaps we should write " $EMP(\mu_1, \mu_2, \dots, \mu_n)$ " instead of " $EMP(G)$," but we will use the later form because it is simpler, and also to emphasize that we are assuming $\mu_1, \mu_2, \dots, \mu_n$ can be represented by a regular message process. In sum, each set $EMP(G)$ uniquely defines a set of equilibrium message correspondences by means of the equations $g_i = 0$, and each set of equilibrium message correspondences which can be represented with a regular message process uniquely defines a set $EMP(G)$. All of our results will be stated in terms of $EMP(G)$, and with this one-to-one relationship we are thereby proving theorems about the equilibrium message correspondences themselves.

5. REVIEW OF ALGEBRAIC RESULTS

We will now review several classical results which concern the signs of the real parts of the eigenvalues of a given (real) matrix. We want to be able to determine by looking at the matrix if each of its eigenvalues has negative real part, or if at least one must have positive real part. The more general problem of determining the signs of the real parts of the roots of an arbitrary real

polynomial is solved by the Routh-Hurwitz Criterion [2, p.194].

ROUTH-HURWITZ CRITERION. Given a real polynomial $f(z) = b_0 z^p + b_1 z^{p-1} + \dots + b_p$, for $j > p$ let $b_j \equiv 0$, and for $1 \leq i \leq p$ define Δ_i to equal the determinant of the $i \times i$ matrix that is given in Figure 1. Each root of f has negative real part if and only if $(b_0)^i \Delta_i > 0$ for all values of i , and if $(b_0)^i \Delta_i < 0$ for at least one value of i , then f will have at least one root with positive real part.

The determinant Δ_k is called the k^{th} Hurwitz determinant. A comprehensive treatment of this subject can be found in Chapter 15 of Gantmacher [2].

We will now develop the notation that we will need for applying the Routh-Hurwitz Criterion to the characteristic polynomials of matrices. Define the mapping ψ_p from $L(\mathbb{R}^p, \mathbb{R}^p)$ into $\mathbb{R}^{p+1}$ by the following rule: for $D \in L(\mathbb{R}^p, \mathbb{R}^p)$, $\psi_p(D) = ((-1)^p, b_1, b_2, \dots, b_p)$ if and only if $(-1)^p \lambda^p + b_1 \lambda^{p-1} + \dots + b_p$ is the characteristic polynomial of D . If we regard each of the Hurwitz determinants as a function whose domain is $\mathbb{R}^{p+1}$, then the Routh-Hurwitz Criterion implies the following statement: each eigenvalue of a $p \times p$ matrix D has negative real part if and only if $(-1)^{pi} (\Delta_i \circ \psi_p(D)) > 0$ for all $1 \leq i \leq p$, and at least one eigenvalue of D will have positive real part if $(-1)^{pi} (\Delta_i \circ \psi_p(D)) < 0$ for some value of i .

We will also need the following formula-- $\Delta_p = b_p \Delta_{p-1}$. This formula is an immediate consequence of the form of the matrix which is used to compute Δ_p : the p^{th} column is identically zero except for the (p,p) entry, which is b_p , and the determinant of the principal minor formed by the first $p-1$ rows and columns is Δ_{p-1} . Since the constant term of the characteristic polynomial of D equals the determinant of D , this formula has the following form when it is applied to matrices:

$$\Delta_p \circ \psi_p(D) = \det D(\Delta_{p-1} \circ \psi_p(D)). \quad (3)$$

Most of the examples in this paper are of equilibrium message correspondences which cannot be determined by a message process which is locally stable on M . Our explanation of these examples will not require the use of the Routh-Hurwitz Criterion, for in these cases, a simpler test can be applied: given a $p \times p$ matrix D with $\psi_p(D) = ((-1)^p, b_1, b_2, \dots, b_p)$, the real part of each eigenvalue of D can be nonpositive only if $(-1)^p b_i \geq 0$ is true for each value of i . This test can be proven by studying the factorization of the characteristic polynomial of D into irreducible linear and quadratic terms.

6. REITER'S EXAMPLE

Much of Mount and Reiter [7] is devoted to an analysis of the two agent problem, with a two dimensional message space; for $i = 1, 2$, the i^{th} agent adjusts the i^{th} message variable according to the rule $\dot{m}_i = g_i(e_i, m)$, where g_i is a continuous function. As before, let $\dot{m} = G = (g_1, g_2)$. Given a point m' in M , the main result of their paper describes a relationship between the solution sets of $g_1(e_1', \cdot) = 0$ and $g_2(e_2', \cdot) = 0$ inside M such that, when this relationship is satisfied at some (e_1', e_2') in $E_{m'}(G)$, then no C^0 message process which determines the same equilibrium message correspondences as $\dot{m} = G$ can be locally stable at m' . The first requirement is that the solution sets of $g_1(e_1', \cdot) = 0$ and $g_2(e_2', \cdot) = 0$ are identically equal. Mount and Reiter also assume that the solution sets of $g_1(e_1, \cdot) = 0$ are both C^0 curves through m' for all (e_1, e_2) in some neighborhood of (e_1', e_2') inside $E_{m'}(G)$, and that these curves can be made to "cross over" each other by varying (e_1, e_2) in a neighborhood of (e_1', e_2') (see Figure 2).

In sum, Mount and Reiter have characterized a kind of degeneracy of the equilibrium message correspondences at a point in $E_{m'}(G)$ which prevents any

message process which determines these correspondences from being locally stable at m' . If this result is interpreted with the added assumption that the message process $\dot{m} = G$ is C^1 , then the degeneracy which Mount and Reiter have isolated is a type of singularity of G : (e_1^1, e_2^1) is a point at which the restriction of $\det D_m G(\cdot, m')$ to $E_m(G)$ changes sign. This interpretation is briefly discussed in the very last section of their paper.

We will now use this approach to analyze Reiter's original example. For $i = 1, 2$, the i^{th} agent's space of characteristics E^i is $\mathbb{R}^2$, and an element e_i of E^i will be denoted by $e_i = (x_i, z_i)$. Our message space M is $\mathbb{R}^2$. The C^1 message process $\dot{m} = G$ that we will be studying is given by the following formulas:

$$\dot{m}_1 = g_1((x_1, z_1), m) = z_1 - m_1 - x_1 m_2,$$

$$\dot{m}_2 = g_2((x_2, z_2), m) = z_2 - x_2 m_1 - m_2.$$

Note that $\dot{m} = G$ is regular.

We will now show that any effective C^1 message process which determines the same equilibrium message correspondences as $\dot{m} = G$ cannot be locally stable at any m' in M . Given a value m' in M and any x_1, x_2 in $\mathbb{R}$, the value of z_i which satisfies $g_i((x_i, z_i), m') = 0$ is uniquely determined as a function of x_i and m' . We can therefore identify $E_m(G)$ with the x_1, x_2 plane. Most of our analysis in this example will concern the variation of $\det D_m G$ on $E_m(G)$. Since $\det D_m G = 1 - x_1 x_2$ does not depend upon z_1 or z_2 , we do not have to concern ourselves with the variations in z_1 and z_2 over $E_m(G)$.

Theorem 2 states that any effective C^1 message process $\dot{m} = F$ which determines the same equilibrium message correspondences as $\dot{m} = G$ must have

the form $\dot{m}_i = a_i(e_i, m)g_i(e_i, m)$ for $i = 1, 2$, where each a_i is a C^1 function which can equal zero only when g_i is zero. On $E_{m'}(G)$, a_i can be thought of as a function of x_i alone, so we will write $a_i(x_i)$ when we are discussing the restriction of a_i to $E_{m'}(G)$.

As before, let A denote the 2×2 diagonal matrix whose (i, i) diagonal entry is a_i ; on $E_{m'}(G)$, we have $D_m F = A \cdot D_m G$. From Section 5, we know that $D_m F$ has at least one eigenvalue with positive real part whenever $\det D_m F$ is negative, so we must have $\det D_m F = a_1(x_1) a_2(x_2)(1 - x_1 x_2) \geq 0$ at each point in $E_{m'}(G)$ if F is stable at m' . Also, note that since F is an effective message process, the zero set of each $a_i(x_i)$ must consist only of isolated points on the x_i axis.

If $a_1(x_1) a_2(x_2)(1 - x_1 x_2) \geq 0$ on $E_{m'}(G)$, then $a_1(x_1) a_2(x_2)$ must change sign across the curve $x_1 x_2 = 1$. Figure 3 indicates why this is impossible: the set $x_1 x_2 = 1$ is a hyperbola, while the solution set of $a_1(x_1) a_2(x_2) = 0$ (illustrated with dotted lines) is rectilinear. Because the zeros of each $a_i(x_i)$ are isolated, there must exist an open region in $E_{m'}(G)$ on which $\det D_m F$ is negative. Therefore, F cannot be locally stable at m' .

The point of this example is that no element of $\text{EMP}(G)$ can be locally stable at m' because of the shape of the singular set of $D_m G$ in $E_{m'}(G)$. For the moment, we will not go beyond this graphical argument. In Section 8, we will return to this issue in our proof of Theorem 3.

7. AN EXTENSION OF REITER'S EXAMPLE

We will now modify the preceding example to show that even if $\det D_m G$ is nonzero on $E_{m'}(G)$, there still might not exist an element of $\text{EMP}(G)$ which is locally stable at m' . Here, we will study the coefficient of the characteristic polynomial of $D_m F$ which was neglected in the last example-- $\text{tr } D_m F$,

the trace of $D_m F$. The reader should note that in this example, the equilibrium message correspondences will not degenerate in any obvious fashion, at any value of the environment. For any environment, there will in fact be a unique message value which lies in the value of each agent's equilibrium message correspondence at that environment.

We must first change the message process $\dot{m} = G$ and the space of environments. For $i = 1, 2$, the i^{th} agent's characteristics is given by the real numbers y_i and z_i , where $-1 < y_i < 1$ and z_i is unbounded. $\mathbb{R}^2$ will continue to serve as the message space. The message process $\dot{m} = G$ is given by the following equations:

$$\dot{m}_1 = z_1 - y_1 m_1 - m_2, \quad (6)$$

$$\dot{m}_2 = z_2 - m_1 - y_2 m_2. \quad (7)$$

Note that $\det D_m G = y_1 y_2 - 1$ is negative on $(E^1 \times E^2) \times M$, so $\dot{m} = G$ is regular.

We will now show that each element F of $\text{EMP}(G)$ must be unstable at every element m' of M . As in the preceding example, we can identify $E_{m'}(G)$ with a subset of a plane--here, a square (without boundary) in the y_1, y_2 plane. We again represent F as $F = A \cdot G$, where A is a 2×2 diagonal matrix whose (i, i) entry is the C^1 function $a_i((y_i, z_i), m)$, and on $E_{m'}(G)$ we will use $a_i(y_i)$ to denote this function.

If F is locally stable at m' , the following inequalities must be true at each point in $E_{m'}(G)$:

$$-\text{tr } D_m F = y_1 a_1(y_1) + y_2 a_2(y_2) \geq 0, \quad (4)$$

$$\det D_m F = a_1(y_1) a_2(y_2) \det D_m G \geq 0. \quad (5)$$

Since $\det D_m G < 0$ on $E_{m'}(G)$, (5) simplifies to $a_1(y_1) a_2(y_2) \leq 0$.

If we regard $(a_1(y_1), a_2(y_2))$ as a vector at the origin in the y_1, y_2 plane, then (4) implies that the radian measure of the angle between (y_1, y_2) and $(a_1(y_1), a_2(y_2))$ must be less than or equal to $\pi/2$. Since F is effective, we can choose a point (y_1', y_2') in $E_{m'}(G) \cap \mathbb{R}^{2+}$ at which $a_1(y_1')$ and $a_2(y_2')$ are both nonzero. The remainder of our argument will be illustrated by the graphs in Figure 4. Since $a_1(y_1) a_2(y_2) \leq 0$ in $E_{m'}(G)$ if F is locally stable at m' , the vector $(a_1(y_1), a_2(y_2))$ must stay within the shaded areas on the graphs. We first assume that $a_1(y_1') < 0$ and $a_2(y_2') > 0$ (see (i) of Figure 4). Hold y_1 equal to y_1' , and move y_2 from y_2' to $y_2'' < 0$; as this is done, $a_1(y)$ does not change, so $a_2(y_2)$ must be nonnegative, else $(a_1(y_1'), a_2(y_2''))$ would not lie in the shaded region. Therefore, the angle between (y_1', y_2'') and $(a_1(y_1'), a_2(y_2''))$ is obtuse, so (4) is violated at (y_1', y_2'') .

A similar argument can be made if $a_1(y_1') > 0$ and $a_2(y_2') < 0$ (see (ii) of Figure 4). In this case, y_2 is set equal to y_2' , and y_1 is varied from y_1' to $y_1'' < 0$. The inequality (4) must be false at (y_1'', y_2') .

In this example, it is the existence of a point in $E_{m'}(G)$ where each of the diagonal entries of $D_m G$ changes sign which forces every element of $\text{EMP}(G)$ to be unstable at m' . In our next section, we will abstract several general results on this topic from the preceding examples.

8. OBSTACLES TO STABILITY WHICH ARE REVEALED BY $D_m G$

Our next theorem concerns the property of $D_m G$ on $E_{m'}(G)$ which in our

first example prevented any element of $\text{EMP}(G)$ from being locally stable at m' --the shape of $S_m(G)$, its singularity set in $E_m(G)$. To be more precise, it is not $S_m(G)$ which is at issue here: it is the shape of the set of points in $E_m(G)$ at which the restriction of $\det D_m G$ to $E_m(G)$ actually changes sign which is important. We will call this set $C_m(G)$, and its elements can be specified in the following manner:

$$C_m(G) = \text{Cl} \{e \in E_m(G) \mid \det D_m G(e, m') > 0\} \cap \quad (6)$$

$$\text{Cl} \{e \in E_m(G) \mid \det D_m G(e, m') < 0\} .$$

Let $S(G)$ denote the set of points in G^{-1} at which $D_m G$ is singular. To obtain a global result, we will need to use the set of points in $G^{-1}(0)$ at which the restriction of $\det D_m G$ to $G^{-1}(0)$ changes sign. This set will be called $C(G)$, and it is defined below:

$$C(G) = \text{Cl} \{(e, m) \in G^{-1}(0) \mid \det D_m G(e, m) > 0\} \cap$$

$$\text{Cl} \{(e, m) \in G^{-1}(0) \mid \det D_m G(e, m) < 0\} .$$

Clearly, $C_m(G) \subset S_m(G)$, and $C(G) \subset S(G)$.

Theorem 3 characterizes the structure of $C(G)$ under the assumption that some element of $\text{EMP}(G)$ is locally stable on M , and it also characterizes the structure of $C_m(G)$ when at least one element of $\text{EMP}(G)$ is locally stable at m' . Though this is the simplest way to state and prove this theorem, it is the contrapositives of these results which should be most useful in practice. This is illustrated by our analysis of Reiter's example: when $C_m(G)$ did not have

the required form, we were able to conclude that no element of $\text{EMP}(G)$ could be locally stable at m' . It will be possible to apply our characterization of $C(G)$ in a similar manner.

THEOREM 3. Let $\dot{m} = G$ be a regular C^1 message process of type $(c_1, \dots, c_n)$. If some element of $\text{EMP}(G)$ is locally stable on M , then there exists C^1 functions $\phi_1, \phi_2, \dots, \phi_n$ defined on $\prod_{i=1}^n E^i \times M$ with the following properties:

$$\text{i) each } \phi_i \text{ depends only upon the variables } e_i \in E^i \text{ and } m \in M; \quad (7)$$

$$\text{ii) } C(G) \subset \bigcup_{i=1}^n \phi_i^{-1}(0), \text{ and } \bigcup_{i=1}^n \phi_i^{-1}(0) \setminus S(G) \text{ is nowhere dense in } G^{-1}(0). \quad (8)$$

Similarly, if some element of $\text{EMP}(G)$ is locally stable at m' , then there exists C^1 functions $\gamma_1, \gamma_2, \dots, \gamma_n$ defined on $\prod_{i=1}^n E^i$ such that

$$\text{i) each } \gamma_i \text{ depends only upon the variable } e_i \in E^i;$$

$$\text{ii) } C_{m'}(G) \subset \bigcup_{i=1}^n \gamma_i^{-1}(0), \text{ and } \bigcup_{i=1}^n \gamma_i^{-1}(0) \setminus S_{m'}(G) \text{ is nowhere dense in } E_{m'}(G).$$

PROOF. As before, let $c = \dim M$. Assume that $A \cdot G$ is locally stable on M , where $A = (A_i)$. For $1 \leq i \leq n$, define $\phi_i(e_i, m)$ by the equation $\phi_i(e_i, m) \equiv \det A_i$. The set $\bigcup_{i=1}^n \phi_i^{-1}(0) \setminus S(G)$ must be nowhere dense in $G^{-1}(0)$, because $A \cdot G$ is an effective message process.

Since $A \cdot G$ is locally stable on M , the following inequality must be true at each point in $G^{-1}(0)$:

$$(-1)^c (\det A) \det D_m G = (-1)^c \phi_1 \phi_2 \dots \phi_n \cdot \det D_m G \geq 0. \quad (9)$$

This inequality can be used to show that $C(G)$ is a subset of $\bigcup_{i=1}^n \phi_i^{-1}(0)$. If

each ϕ_i were nonzero at some element (e', m') of $C(G)$, then each would be nonzero in some neighborhood of (e', m') in $\prod_{i=1}^n E^i \times M$. By the definition of $C(G)$, there would have to exist some element of $G^{-1}(0)$ near (e', m') at which (6) does not hold.

The second part of the theorem can be proven by a similar argument. An element $A \cdot G$ of $EMP(G)$ is chosen which is locally stable at m' , where A is again equal to (A_i) . For $1 \leq i \leq n$, define $\gamma_i(e_i)$ by the formula $\gamma_i(e_i) = \det A_i(e_i, m')$. As in the proof of the first part of the theorem, the inequality $(-1)^C(\det A) \det D_m G \geq 0$ and the assumption that $A \cdot G$ is effective can then be applied to achieve the desired results. Q.E.D.

Recall that $E_{m'}(G)$ can be written as a Cartesian product, $E_{m'}(G) = \prod_{i=1}^n E_{m'}^i(g_i)$. Theorem 3 states that if some element of $EMP(G)$ is locally stable at m' , then $C_{m'}(G)$ is a subset of a finite union of sets which have the following form:

$$E_{m'}^1(g_1) \times E_{m'}^2(g_2) \times \dots \times (\gamma_i^{-1}(0) \cap E_{m'}^i(g_i)) \times \dots \times E_{m'}^n(g_n).$$

Moreover, assume that there is a point e' in $C_{m'}(G)$ which is not in the closure of $S_{m'}(G) \cap C_{m'}(G)$. In some neighborhood of e' in $E_{m'}(G)$, the difference between this finite union and $C_{m'}(G)$ is nowhere dense. In this sense, Theorem 3 characterizes the local structure of $C_{m'}(G)$ when some element of $EMP(G)$ is locally stable at m' . As illustrated by our example, it imposes a very strong condition on $C_{m'}(G)$, so long as $C_{m'}(G)$ is nonempty and $c_i \neq k_i$ for at least two values of the index i .

We can be more precise on this matter. Let $\zeta(c_1, \dots, c_n)$ denote the set of regular C^1 message processes of type $(c_1, \dots, c_n)$ on $\prod_{i=1}^n E^i \times M$ which is defined by the following conditions:

i) $c_i \neq k_i$ for at least two values of the index i ; (10)

ii) a regular C^1 message process $\dot{m} = F$ of type $(c_1, \dots, c_n)$ is in $\zeta(c_1, \dots, c_n)$ if and only if $S(F)$ is nonempty. (11)

The set $\zeta(c_1, \dots, c_n)$ can be equipped with the Whitney C^1 topology. For all elements $\dot{m} = G$ of an open, dense subset of $\zeta(c_1, \dots, c_n)$, the set $C(G)$ will also be nonempty, and it will not have the structure that is described in (7) - (8). This shows that for the types of message processes that are characterized in (10) - (11), the presence of singularities of $D_m G$ in $G^{-1}(0)$ is usually sufficient to insure that no element of $EMP(G)$ can be locally stable on M . Finally, we again emphasize that for a regular message process $\dot{m} = G$, the singularities of $D_m G$ in $G^{-1}(0)$ are intrinsic geometric property of the equilibrium message correspondences which $\dot{m} = G$ represents, and they will appear in any C^1 message process which is chosen to represent these correspondences.

Before we prove a result (Theorem 5) which is based upon the example in Section 7, we must first prove the following theorem. This theorem will be needed for our proof of Theorem 5, but it should also be valuable for other purposes.

THEOREM 4. Let $A \cdot G \in EMP(G)$, where $A = (A_i)$ and $G = (g_i)$, and let m' be an element of M at which $A \cdot G$ is locally stable. If $\det D_m G$ is nonzero on some topological component of $E_m(G)$, then for each value of i the function $\det A_i$ cannot be both positive and negative on that component.

PROOF. As before, we can write $E_m(G) = \prod_{i=1}^n E_m^i(g_i)$. It is sufficient that we prove the theorem when $E_m(G)$ itself is connected.

Let $\pi(\det A_i)$ denote the product of the n functions $\det A_1, \det A_2, \dots, \det A_n$.

Since $A \cdot G$ is locally stable at m' , the following inequality will be true at each point in $E_{m'}(G)$:

$$(-1)^c(\det A)(\det D_m G) = (-1)^c(\pi(\det A_i))(\det D_m G) \geq 0. \quad (12)$$

We will now show that if $\det A_j$ were both positive and negative on $E_{m'}(G)$, then (12) could not be true at all points in $E_{m'}(G)$ at which each of the functions $\det A_1, \det A_2, \dots, \det A_n$ is nonzero. Recall that each function $\det A_i$ depends only upon the value of the variable $e_i \in E_{m'}^i(g_i)$. By varying e_j from e_j' in $E_{m'}^j(g_j)$ while holding $e_i = e_i'$ for all $i \neq j$, the value of $\det A_j$ can be made to change sign, while the value of $\det A_i$ for $i \neq j$ would remain constant. The sign of $\det D_m G$ would also remain the same as e_j is varied, since $\det D_m G$ is by hypothesis nonzero on $E_{m'}(G)$, and $E_{m'}(G)$ is connected. Therefore, (12) would be violated somewhere on $E_{m'}(G)$. The conclusion follows immediately. Q.E.D.

We are now able to prove a theorem which concerns the obstacles to stability that was illustrated in Section 7. The determinant of $A \cdot D_m G$ is just one of the coefficients of the characteristic polynomial of $A \cdot D_m G$. Our next result will show that the behavior of the other coefficients on $E_{m'}(G)$ can also prevent $A \cdot G$ from being locally stable at a point m' in M .

This theorem only concerns C^1 message processes of type $(c_1, \dots, c_n)$, where each $c_i = 1$: each agent uses a single C^1 function to adjust only one of the variables of M , and $c = \dim M = n$. In this case, each element of F of $\text{EMP}(G)$ can be represented as $F = A \cdot G$, where A is a $c \times c$ diagonal matrix whose (i,i) entry is a function on $E^i \times M$. Here, we will write $A = (a_i)$, where $a_i(e_i, m)$ is the (i,i) entry of A . We restrict ourselves to

this special case so that the coefficients of the characteristic polynomial of $A \cdot D_m G$ will have a special form: for each $1 \leq p \leq c$, the coefficient of λ^{c-p} in the characteristic polynomial of $A \cdot D_m G$ will itself be a sum of monomials of degree p in the variables $a_1, a_2, \dots, a_c$. For instance, the coefficient of λ^0 is $\det D_m G \cdot a_1 a_2 \dots a_c$, and the coefficient of λ^{c-1} is $(-1)^{c-1} \sum_{i=1}^c (\partial g_i / \partial m_i) a_i$.

We will study the coefficient of λ^{c-p} in the characteristic polynomial of $A \cdot D_m G$ as a homogeneous polynomial of degree p in $a_1, a_2, \dots, a_c$. Let $b_p(a_1, \dots, a_c)$ denote this polynomial. In order to discuss the coefficients of $b_p(a_1, \dots, a_c)$, we need to introduce some new notation: given any $k \times k$ matrix H and a subset $\{\pi(1), \pi(2), \dots, \pi(p)\}$ of $\{1, 2, \dots, k\}$, $H(\pi(1), \dots, \pi(p))$ will denote the principal minor of H of order p whose entries lie in the rows (and columns) that are indexed by the numbers $\pi(1), \pi(2), \dots, \pi(p)$. In this new notation, $(-1)^{c-p} \det D_m G(\pi(1), \dots, \pi(p))$ is the coefficient of the monomial $a_{\pi(1)} \dots a_{\pi(p)}$ in $b_p(a_1, \dots, a_c)$.

If $\det D_m G$ is nonzero some component of $E_m(G)$, then it is sometimes possible to show that no element of $EMP(G)$ is locally stable at m' simply by observing the changes in sign of the determinants of the principal minors of $D_m G$ over the component in question. By Theorem 4, each a_i could not change sign on the component, so it might be impossible to choose functions $a_1, a_2, \dots, a_c$ that will make $(-1)^{c-p} b_p$ nonnegative on $E_m(G)$. This was the case in the example of Section 7: because we could choose points in $E_m(G)$ to make $\det D_m G(1)$ and $\det D_m G(2)$ (i.e., the diagonal entries of $D_m G$) have whatever signs we wanted, there could not exist functions $a_1(e_1)$ and $a_2(e_2)$ which made $b_1(a_1, a_2)$ nonnegative at all points in $E_m(G)$.

Theorem 5 will clarify the preceding argument. For each $1 \leq p \leq c-1$, there are $k(p) = \binom{c}{p}$ principal minors of order p of $D_m G$. An assignment of a sign $(+,-)$ to each of the determinants of these minors will be referred to as a sign pattern for the principal minors of $D_m G$ of order p . Clearly, there are $2^{k(p)}$ possible sign patterns for the principal minors of order p . We will say that $D_m G$ achieves a particular sign pattern of order p on a given subset of $E_m(G)$ if at some point in that set the sign of each determinant of a principal minor of order p of $D_m G$ matches the sign given to it by the sign pattern.

THEOREM 5. Let $\dot{m} = G$ be a regular C^1 message process of type $(1, 1, \dots, 1)$, and let m' be an element of M such that $\det D_m G$ is nonzero on some component of $E_m(G)$. If for some $1 \leq p \leq c-1$ the matrix $D_m G$ achieves each of the possible sign patterns of order p on that component of $E_m(G)$, then no element of $EMP(G)$ can be locally stable at m' .

PROOF. It is sufficient that we prove this theorem in the case where $E_m(G)$ itself is connected. We assume that there is an element $A \cdot G$ of $EMP(G)$ which is stable at m' , and we will achieve a contradiction. As in the discussion which precedes this theorem, we let $A = (a_i)$. By Theorem 4, each diagonal entry a_i of A cannot be both positive and negative on $E_m(G)$. Clearly, any monomial in $a_1, a_2, \dots, a_c$ will also have this property.

Since $A \cdot G$ is locally stable at m' , the coefficient $b_p(a_1, \dots, a_c)$ of λ^{c-p} in the characteristic polynomial of $A \cdot D_m G$ must satisfy $(-1)^c b_p \geq 0$ at all points in $E_m(G)$. We can now complete the proof of the theorem by showing that there must exist a point in $E_m(G)$ at which this inequality is false. The matrix $D_m G$ achieves every sign pattern of order p on $E_m(G)$, so we can move to a point in $E_m(G)$ where the following inequality is true for each subset $\{\pi(1), \pi(2), \dots, \pi(p)\}$ of $\{1, 2, \dots, c\}$:

$$(-1)^{2c-p} \det D_m G(\pi(1), \dots, \pi(p)) \cdot a_{\pi(1)} a_{\pi(2)} \dots a_{\pi(p)} \leq 0. \quad (13)$$

Moreover, since $A \cdot G \in \text{EMP}(G)$ and $\det D_m G \neq 0$ on $E_m(G)$, we can make (13) a strict inequality for each subset $\{\pi(1), \pi(2), \dots, \pi(p)\}$ of $\{1, 2, \dots, c\}$ by moving to a nearby point in $E_m(G)$. Since $(-1)^c b_p(a_1, \dots, a_c)$ is determined by summing the values which appear on the left-hand side of (13) as $\{\pi(1), \dots, \pi(p)\}$ varies over all subsets of $\{1, 2, \dots, c\}$ of order p , we have found a point at which $(-1)^c b_p < 0$. Q.E.D.

REMARK. We can define the term "sign pattern" for the set of monomials of order p in $a_1, a_2, \dots, a_c$ in an obvious way, and we say that this set of monomials achieves a given sign pattern on $E_m(G)$ if there is some point on $E_m(G)$ at which the sign of each monomial matches the sign that is given to it by the sign pattern. Note that for $1 < p < c$, certain sign patterns for the set of monomials of order p can never be achieved, no matter what signs are assumed by $a_1, a_2, \dots, a_c$: for instance, when $c = 3$ and $p = 2$, it is impossible to find real values of a_1, a_2 , and a_3 so that the product of any two of these numbers is negative. This means that by observing the sign patterns which the set of monomials of order p can actually achieve on $E_m(G)$, it is sometimes possible to show that no element of $\text{EMP}(G)$ can be locally stable at m' even if $D_m G$ fails to achieve every sign pattern of order p on a component of $E_m(G)$.

9. A PROCEDURE FOR CONSTRUCTING A LOCALLY STABLE C^1 MESSAGE PROCESS

We now turn to the problem of determining when some element of $\text{EMP}(G)$ is locally stable on M . Our results in Section 8 indicate that this can occur

only if $D_m G$ and its principal minors satisfy some sort of regularity conditions on $G^{-1}(0)$. Theorem 6 will give such a set of conditions on $D_m G$. Before we state this theorem, we will first work an example which both motivates its proof and also illustrates a procedure for actually constructing an element of $EMP(G)$ which is locally stable on M from G .

EXAMPLE. Let D be a 3×3 matrix with constant entries. In this example, we will show how to find a 3×3 diagonal matrix A such that each eigenvalue of $A \cdot D$ has negative real part. A set of conditions on D which enables us to find a matrix A with this property will be developed as we proceed with the example.

As before, we let a_i denote the (i,i) entry of A , and $b_i(a_1, a_2, a_3)$ will denote the coefficient of λ^{3-i} in the characteristic polynomial of $A \cdot D$:

$$b_1 = a_1 \det D(1) + a_2 \det D(2) + a_3 \det D(3)$$

$$b_2 = -(a_1 a_2 \det D(1,2) + a_1 a_3 \det D(1,3) + a_2 a_3 \det D(2,3))$$

$$b_3 = a_1 a_2 a_3 \det D.$$

The Routh-Hurwitz Criterion states that each eigenvalue of $A \cdot D$ will have negative real part if and only if $-b_1 > 0$, $b_1 b_2 + b_3 > 0$, and $-b_3 > 0$. We will now show that if $\det D(1)$, $\det D(1,2)$, and $\det D$ are each nonzero, then we can successively choose values of a_1 , a_2 and a_3 which make these inequalities true.

We begin by setting a_1 , a_2 and a_3 equal to zero. Since $\det D(1)$ is nonzero, we can choose a value for a_1 such that $-b_1(a_1, 0, 0) > 0$. With a_3 remaining at zero, we next choose a value for a_2 such that $-b_1(a_1, a_2, 0) > 0$

and $b_1(a_1, a_2, 0) \cdot b_2(a_1, a_2, 0) + b_3(a_1, a_2, 0) > 0$. As long as a_3 is equal to zero, $b_3(a_1, a_2, a_3)$ will also be zero, so this second inequality is equivalent to $b_1(a_1, a_2, 0) \cdot b_2(a_1, a_2, 0) > 0$. Since $-b_1(a_1, 0, 0) > 0$, $-b_1(a_1, a_2, 0)$ will also be positive if a_2 is sufficiently small, so our problem is reduced to choosing a small value of a_2 at which $b_2(a_1, a_2, 0)$ is negative. This is easily accomplished, since $b_2(a_1, a_2, 0) = -a_1 a_2 \det D(1, 2)$ and $\det D(1, 2)$ is presumed to be nonzero.

Finally, we will choose a value for a_3 such that $-b_1 > 0$, $b_1 b_2 + b_3 > 0$, and $-b_3 > 0$ will each be true. As long as a_3 is chosen sufficiently close to zero, $-b_1 > 0$ and $b_1 b_2 + b_3 > 0$ will be true by virtue of our earlier choices of a_1 and a_2 , so we must choose a small value of a_3 at which $-b_3(a_1, a_2, a_3)$ will be positive. Since $b_3(a_1, a_2, a_3) = a_1 a_2 a_3 \det D$ and $\det D$ is nonzero by assumption, we only need to be sure to pick a value for a_3 with the correct sign.

Let $\pi(\cdot)$ denote any permutation of the first three positive integers. If $\det D(\pi(1))$, $\det D(\pi(2))$ and $\det D$ are each nonzero, then we can follow the procedure that is outlined above to successively choose $a_{\pi(1)}$, $a_{\pi(2)}$, and $a_{\pi(3)}$ so that each eigenvalue of $A \cdot D$ will have negative real part. This procedure can easily be generalized so that it applies to a $c \times c$ matrix D , where c is any positive integer; it is only necessary that there exist a sequence $D(\pi(1)), D(\pi(1), \pi(2)), \dots, D(\pi(1), \pi(2), \dots, \pi(c)) = D$ of nonsingular principal minors of D .

Given a regular C^1 message process $\dot{m} = G$, our next theorem describes when we can apply this procedure to $D_m G$ over $G^{-1}(0)$, and thereby construct an element $A \cdot G$ of $\text{EMP}(G)$ which is stable on M . Because the entries of $D_m G$

are functions, there are additional hypotheses which must be satisfied by $D_m G$ beyond those that are needed in the case of a constant matrix D . The need for these additional hypotheses will be apparent in our proof of this theorem.

THEOREM 6. Let $\dot{m} = G$ be a regular C^1 message process, and let $\pi(\cdot)$ be a permutation of the first c positive integers such that the following conditions are satisfied by G and $\pi(\cdot)$:

$$i) D_m G \text{ is nonsingular on } G^{-1}(0); \quad (14)$$

$$ii) \text{ each entry of } D_m G \text{ is a bounded function on } G^{-1}(0); \quad (15)$$

$$iii) \text{ for } 1 \leq p \leq c, \det D_m G(\pi(1), \pi(2), \dots, \pi(p)) \text{ does not change sign on } G^{-1}(0), \text{ and for } 1 \leq p \leq c-1, |\det D_m G(\pi(1), \pi(2), \dots, \pi(p))| > K \text{ at any point in this set, where } K \text{ is some fixed positive number.} \quad (16)$$

Then there exists an element $A \cdot G$ of $EMP(G)$ which has the following properties:

$$i) A \text{ is a } c \times c \text{ diagonal matrix whose diagonal entries are nonzero constants;}$$

$$ii) \text{ at any point in } G^{-1}(0), \text{ each eigenvalue of } D_m A \cdot G \text{ has negative real part.} \quad (17)$$

The message process $\dot{m} = A \cdot G$ is therefore stable on M .

Before we prove this theorem, we will first briefly discuss its meaning. Even though a given message process $\dot{m} = G$ may not satisfy (14) - (16) over the entire set $G^{-1}(0)$, Theorem 6 can still be useful in the analysis of this message process. Conditions (14) - (16) can be used to define open (and probably bounded) subsets of $\prod_{i=1}^n E^i \times M$. Over any one of these sets, it is

possible to apply this theorem to find an element of $EMP(G)$ which is locally stable at every equilibrium pair (e', m') in that set. Theorem 6 may prescribe the use of different adjustment rules over these different regions in order to achieve local stability. It shows, however, that local stability can be attained in a neighborhood of any generic point (e', m') of the regular message process which has been chosen to represent the given equilibrium message correspondences.

This result is a counterpoint to Theorems 3 and 5, wherein it is shown that the presence of certain kinds of singular points can prevent any element of $EMP(G)$ from being locally stable near those points. Roughly speaking, these results jointly show that it is relatively easy to find an element of $EMP(G)$ which is locally stable over a "suitable" range of equilibria, but it may be very difficult to find a single message process in $EMP(G)$ which is locally stable at all the equilibria of $\dot{m} = G$. We stop short of trying to make a precise description of what is true for a generic message process, since we have not made the sort of assumptions about the set of message processes and the space $\prod_{i=1}^n E^i \times M$ which we would need in order to decide whether or not a generic message process must have these kinds of singularities.

Finally, the reader should also note that when (14) - (16) are satisfied, the matrix (A_i) which satisfies (17) will not be unique. For the moment, let Ω denote the space of constant $c \times c$ block-diagonal matrices whose i^{th} diagonal block is of size $c_i \times c_i$. The set of elements A in Ω which satisfy (17) form an open subset of Ω . Theorem 6 states conditions on $D_m G$ under which this set must be nonempty. The fact that there is an open subset of Ω whose elements satisfy (17) increases the likelihood that in a given economic setting, it will be possible to find a locally stable message process which can be meaningfully interpreted.

PROOF OF THEOREM 6. We will begin by making the additional hypothesis that $|\det D_m G|$ is also greater than some positive number K on $G^{-1}(0)$. We make this assumption now for the sake of an induction argument, but later we will show that it can be avoided.

In this proof, we will use the notation and the results that were discussed in Section 5: $\psi_p(\cdot)$ denotes the mapping from $L(\mathbb{R}^p, \mathbb{R}^p)$ into $\mathbb{R}^{p+1}$ which assigns to each matrix the coefficients of its characteristic polynomial, and Δ_q denotes the q^{th} Hurwitz determinant. Using the hypothesis on $\det D_m G$ which we added above, we will first prove by an induction argument the existence of a $c \times c$ diagonal matrix A which has the following properties:

- i) each of the diagonal entries of A is a nonzero constant;
- ii) for each $1 \leq p \leq c$, there is a positive constant k_p such that $(-1)^{cp} \Delta_p \circ \psi_c(A \cdot D_m G) \geq k_p$ at every element of $G^{-1}(0)$.

The case of $c = 1$ is trivial, so we will assume that the result is true for all $p < c$, where $c > 1$. For convenience, we can assume that the permutation $\pi(\cdot)$ is the identity permutation. By the induction hypothesis, there exists a $(c-1) \times (c-1)$ diagonal matrix A^* with nonzero constants on its diagonal and positive number $k_1^*, k_2^*, \dots, k_{c-1}^*$ such that the following inequality is true on $G^{-1}(0)$ for each $1 \leq p \leq c-1$:

$$(-1)^{(c-1)p} \cdot \Delta_p \circ \psi_{c-1}(A^* \cdot D_m G(1, 2, \dots, c-1)) > k_p^*.$$

For each $1 \leq i \leq c-1$, we define the (i, i) entry of A to equal the (i, i) entry of A^* . To complete the induction argument, we need to make an appropriate choice of the (c, c) entry of A .

We regard this entry as a variable quantity x , and we will write $A(x)$ when x is placed in the (c,c) entry of A . When x is set equal to zero, the characteristic polynomial of $A(0) \cdot D_m G$ is equal to $-\lambda$ times the characteristic polynomial of $A^* \cdot D_m G(1, 2, \dots, c-1)$; therefore, $(-1)^{pc} \Delta_p \circ \psi_c(A(0) \cdot D_m G) > k_p^*$ on $G^{-1}(0)$ for $1 \leq p \leq c-1$. For this range of p , $(-1)^{pc} \Delta_p \circ \psi_c(A(x) \cdot D_m G)$ can be regarded as a polynomial in x whose constant term (obtained by setting $x = 0$) is $(-1)^{pc} \Delta_p \circ \psi_c(A(0) \cdot D_m G)$, and whose other coefficients are polynomials in the entries of $A(1) \cdot D_m G$. Since each entry of $D_m G$ is bounded on $G^{-1}(0)$, each of these coefficients must also be bounded on $G^{-1}(0)$. This guarantees the existence of a positive number ϵ which has the following property: if $|x| < \epsilon$, then the following inequality will be true on $G^{-1}(0)$ for $1 \leq p \leq c-1$:

$$(-1)^{pc} \Delta_p \circ \psi_c(A(x) \cdot D_m G) > k_p^*/2.$$

For each $1 \leq p \leq c-1$, we define k_p by $k_p \equiv k_p^*/2$.

We will now choose a value of x in the interval $(-\epsilon, \epsilon)$ such that $(-1)^{c^2} \Delta_c \circ \psi_c(A(x) \cdot D_m G) > k_c$ on $G^{-1}(0)$ for some positive number k_c . Using the identity (3) from Section 5, we have the following equations:

$$\begin{aligned} (-1)^{c^2} \Delta_c \circ \psi_c(A(x) \cdot D_m G) &= (-1)^{c^2} \Delta_{c-1} \circ \psi_c(A(x) \cdot D_m G) \det(A(x) \cdot D_m G) \\ &= (-1)^{c^2} \Delta_{c-1} \circ \psi_c(A(x) \cdot D_m G) \cdot x \cdot \det A^* \cdot \det D_m G. \end{aligned}$$

It is now obvious that a value of x can be chosen so that $(-1)^{c^2} \Delta_c \circ \psi_c(A(x) \cdot D_m G)$ will be positive on $G^{-1}(0)$. Since both $\Delta_{c-1} \circ \psi_c(A(x) \cdot D_m G)$ and $\det D_m G$ are bounded away from 0 on $G^{-1}(0)$, $(-1)^{c^2} \Delta_c \circ \psi_c(A(x) \cdot D_m G)$ will also have this property. This concludes the proof of the induction step.

The hypothesis that $|\det D_m G| > K$ is used in the induction argument only to show that $(-1)^{c^2} \Delta_C \circ \psi_C(A(x) \cdot D_m G)$ is bounded away from zero on $G^{-1}(0)$. This is formally needed to complete the induction argument, but it is not needed to insure that each eigenvalue of $A(x) \cdot D_m G$ has negative real part: the conclusion of the theorem holds with the weaker assumption that $\det D_m G$ is non-zero on $G^{-1}(0)$. This can be proven by first applying the result that we just proved by induction to $D_m G(1, 2, \dots, c-1)$, and then again making an appropriate choice of the (c, c) entry of A . Since this argument is virtually identical to the argument which was used in the proof of the induction step, the details will not be included. Q.E.D.

An obvious criticism of Theorem 6 is that it fails to make full use of all the allowable manipulations of the message process $\dot{m} = G$ to construct an element of $\text{EMP}(G)$ which is locally stable on M : here, we only manipulate G by multiplying it by a diagonal matrix with constant entries, while our discussion in Section 4 allows us much more freedom in how we can vary G without leaving $\text{EMP}(G)$. Perhaps the hypotheses of this theorem could be weakened. Our next section is a response to this criticism, for it consists of an example which illustrates the difficulties that are encountered when the hypotheses of Theorem 6 are not satisfied.

10. INSTABILITY CAUSED BY THE GLOBAL BEHAVIOR OF $D_m G$

We will now discuss a message process $\dot{m} = G$ which satisfies hypothesis (14) and (16) of Theorem 6, and yet because several entries of $D_m G$ are unbounded on $G^{-1}(0)$, each element of $\text{EMP}(G)$ will be unstable at every element of m' of M . In this example, there are three agents. The i^{th} agent's

environment is given by $e_i = (x_i, y_i, z_i)$, where $x_i > 3$, $1/2 < y_i < 1$, while z_i can be any real number. M will be $\mathbb{R}^3$. The message process $\dot{m} = G = (g_i)$ is given by the following equations:

$$\dot{m}_1 = z_1 - y_1 m_1 - x_1 m_2 + m_3$$

$$\dot{m}_2 = z_2 + m_1 - y_2 m_2 + x_2 m_3$$

$$\dot{m}_3 = z_3 + x_3 m_1 + m_2 + y_3 m_3.$$

By a direct computation, it can be shown that $|\det D_m G(1)|$, $|\det D_m G(1,2)|$, and $|\det D_m G|$ are bounded below by some positive number on $G^{-1}(0)$. Note, however, that several entries of $D_m G$ are unbounded on $G^{-1}(0)$, so Theorem 6 cannot be applied.

Let m' be any element of M . We will now show that no element of $\text{EMP}(G)$ is locally stable at m' . Since the value of z_i which satisfies $g_i((x_i, y_i, z_i), m') = 0$ is uniquely determined by the values of x_i , y_i , and m' , we can identify $E_{m'}(G)$ with the subset of $\mathbb{R}^6$ given by $x_i > 3$ and $1/2 < y_i < 1$ for $1 \leq i \leq 3$. With this identification, the restriction of a_i to $E_{m'}(G)$, can be regarded as a function of x_i and y_i alone. We will again be analyzing $D_m G$ over $E_{m'}(G)$, and since the entries of $D_m G$ depend only on x_i and y_i for $1 \leq i \leq 3$, the variables z_1 , z_2 and z_3 will not play a role in the rest of this example.

We now assume that $A \cdot G \in \text{EMP}(G)$ is locally stable at m' , where $A = (a_i)$, and a_i is a function of e_i and m . As was discussed in Section 5, the coefficient b_p of λ^{3-p} in the characteristic polynomial of $A \cdot D_m G$ must satisfy $(-1)^3 b_p \geq 0$ at each point in $E_{m'}(G)$. This implies that the following set of inequalities must be satisfied for all $x_i > 3$, $1/2 < y_i < 1$, and $1 \leq i \leq 3$:

$$y_1 a_1 + y_2 a_2 - y_3 a_3 \geq 0 \quad (18)$$

$$(y_1 y_2 + x_1) a_1 a_2 - (y_2 y_3 + x_2) a_2 a_3 - (y_1 y_3 + x_3) a_1 a_3 \geq 0 \quad (19)$$

$$- (\det D_m G) a_1 a_2 a_3 \geq 0. \quad (20)$$

By Theorem 4, each a_i cannot be both positive and negative on $E_m(G)$. Since $\det D_m G$ is negative on $E_m(G)$, (20) implies that the product $a_1 a_2 a_3$ could not be negative at any point in $E_m(G)$, and (18) implies that $a_1 < 0$, $a_2 < 0$, and $a_3 > 0$ also could not occur in $E_m(G)$. Only three possible sign configurations for a_1 , a_2 and a_3 remain: $a_1 \geq 0$, $a_2 \geq 0$, $a_3 \geq 0$; $a_1 \geq 0$, $a_2 \leq 0$, $a_3 \leq 0$; and $a_1 \leq 0$, $a_2 \geq 0$, $a_3 \leq 0$. The functions a_1 , a_2 , and a_3 must satisfy exactly one of these three sets of inequalities at all points in $E_m(G)$.

We will achieve a contradiction to the stability of $A \cdot G$ at m' by showing that a_1 , a_2 , and a_3 could not satisfy any one of these sets of inequalities throughout $E_m(G)$. This will be accomplished by observing (19) as either x_1 , x_2 or x_3 is allowed to go to infinity. We will give a detailed argument only when $a_1 \geq 0$, $a_2 \geq 0$, and $a_3 \geq 0$; the other two remaining cases can be eliminated by similar arguments.

First, note that $y_1 y_2 + x_1$, $y_2 y_3 + x_2$, and $y_1 y_3 + x_3$ are positive at all points in $E_m(G)$. When a_1 , a_2 and a_3 are greater than or equal to zero, we have $a_1 a_2 \geq 0$, $a_2 a_3 \geq 0$, and $a_1 a_3 \geq 0$. Since $A \cdot G$ is effective, there exists a point in $E_m(G)$ that is given by x_1^1 , y_1^1 , x_2^1 , y_2^1 , x_3^1 , y_3^1 at which each a_i is nonzero. The variables x_1 , y_1 , x_3 , y_3 , and y_2 will remain fixed at the values x_1^1 , y_1^1 , x_3^1 , y_3^1 and y_2^1 , respectively, and we will now achieve a contradiction by varying x_2 away from x_2^1 .

The inequality (19) can be rewritten in the following manner at these values of x_1, y_1, x_3, y_3 and y_2 :

$$((y_1'y_2' + x_1')a_1 - (y_2'y_3' + x_2')a_3)a_2 - (y_1'y_3' + x_1')a_1a_3 \geq 0. \quad (21)$$

Since a_3 is positive at x_3, y_3 , the quantity $(y_1'y_2' + x_1')a_1 - (y_2'y_3' + x_2')a_3$ goes to negative infinity when x_2 goes to infinity, while $(y_1'y_3' + x_1')a_1a_3$ remains constant; therefore, x_2a_2 must be bounded above by some positive number k for $x_2 \geq x_2'$, else (21) could not be true at all points in $E_{m'}(G)$. We can conclude from this that a_2 approaches 0 as x_2 goes to infinity.

Using this information, the inequality below follows from (21) as x_2 goes to infinity:

$$- (y_1'y_3' + x_1')a_1a_3 \geq 0.$$

But this inequality is false, since a_1, a_3 , and $y_1'y_3' + x_1'$ are all positive.

The case of $a_1 \geq 0, a_2 \leq 0, a_3 \leq 0$ can be eliminated by first moving to a point in $E_{m'}(G)$ at which each of these functions is nonzero, and then analyzing (19) as x_1 is sent to infinity; the case of $a_1 \leq 0, a_2 \geq 0$, and $a_3 \leq 0$ is handled in a similar manner, except x_3 is the variable which is allowed to go to infinity. This concludes the proof that each element of $EMP(G)$ is unstable at m' .

As we discussed in Section 8, the coefficients of the characteristic polynomial of $A \cdot D_m G$ are themselves polynomials in the variables a_1, a_2 , and a_3 . In our earlier examples, we used the fact that these coefficients change sign on $E_{m'}(G)$ to show that no element of $EMP(G)$ could be stable at m' ; Theorem 5, in fact, formalized this approach. It is worth noting that in this example, none of the coefficients of these polynomials in a_1, a_2, a_3 changes sign on $E_{m'}(G)$.

The message process $\dot{m} = G$ which we just analyzed can be used to construct regular message process $\dot{m} = F$ which satisfies hypotheses (14) and (15) of Theorem 6, and yet no element of $EMP(F)$ is locally stable at any m' in M because F fails to satisfy hypothesis (16). We set $F = B \cdot G$, where B is 3×3 diagonal matrix whose (i,i) entry is $1/x_i$, for $1 \leq i \leq 3$. Note that $EMP(F) = EMP(G)$, so there is nothing that we need to verify in this example.

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$$\Delta_1 = |b_1|, \Delta_2 = \begin{vmatrix} b_1 & b_3 \\ b_0 & b_2 \end{vmatrix}, \dots, \Delta_i = \begin{vmatrix} b_1 & b_3 & b_5 \\ b_0 & b_2 & b_4 \\ 0 & b_1 & b_3 \\ 0 & b_0 & b_2 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$$

Figure 1. Examples of Hurwitz determinants.

Figure 2. The solution sets in M of $g_1(e_1, \bullet) = 0$ and $g_2(e_2, \bullet) = 0$ can be made to "cross over" each other by varying (e_1, e_2) near (e', e') .

Figure 3. The graphs in $E_m'(G)$ of $\det D_m G = 0$ and $a_1(x_1)a_2(x_2) = 0$.

Figure 4. Two graphs in $E_m(G)$:

(i) $a_1(y_1') < 0, a_2(y_2') > 0$;

(ii) $a_1(y_1') > 0, a_2(y_2') < 0$