**THE DYNAMICS ASSOCIATED WITH THE CHAOTIC
TANGLES OF TWO DIMENSIONAL QUASIPERIODIC
VECTOR FIELDS: THEORY AND APPLICATIONS**

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THE DYNAMICS ASSOCIATED WITH THE CHAOTIC TANGLES OF TWO DIMENSIONAL QUASIPERIODIC VECTOR FIELDS: THEORY AND APPLICATIONS

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Abstract. We generalize notions of transport in phase space associated with the classical Poincaré map reduction of a periodically forced two dimensional system to apply to a sequence of nonautonomous maps derived from a quasiperiodically forced two dimensional system. We obtain a global picture of the dynamics in homoclinic and heteroclinic tangle regions using a sequence of time-dependent two dimensional lobe structures derived from the invariant global stable and unstable manifolds of one or more normally hyperbolic invariant sets in a Poincaré section of an associated autonomous system's phase space. We discuss transport in the context of a few example systems, with emphasis on advection in fluids. Transport is specified in terms of two dimensional lobes mapping from one to another within the sequence of lobe structures, and we argue that these lobe structures, and the invariant manifolds from which they are derived, are the dominant structures by which to understand chaotic transport in the tangle regions. We present computer simulations of lobes mapping from one to another within the sequence of lobe structures and discuss qualitatively the essential features of transport in the tangle region: entrainment and detrainment with respect to a sequence of time-dependent cores with boundaries formed by segments of stable and unstable manifolds (for concreteness we focus mainly on the prototypical situation of a single core sequence and hence borrow the fluid mechanical terminology of entrainment and detrainment to make the discussion of transport more apparent), and repeated stretching and folding to give chaotic dynamics. These features are studied in a more exact fashion via the lobe structures. Instantaneous and average flux are quantified analytically in the near-integrable case through the use of a generalized Melnikov function, and the computational prescription for determining flux in systems that are not near-integrable is provided; the time average of the flux is proportional to an average of areas over appropriate regions of the Poincaré section of the autonomous system phase space. We explain how answering basic questions about transport in phase space (both in the near-integrable and general case) necessitates a consideration of issues such as the geometry of intersections of lobes with each other, with the core, and with level sets of the unperturbed Hamiltonian, and the variation of lobe areas. We compare transport rates in the single and multiple frequency cases. Though the comparisons depend on choice of normalization, for some reasonable normalizations the average flux of a class of multiple frequency systems is found to be maximal in a certain single frequency limit. The variation of lobe areas in multiple frequency systems, however, gives one the freedom to enhance or diminish aspects of transport for a fixed average flux. The chaotic nature of the dynamics is understood in the framework of a *traveling horseshoe map sequence*. A search through parameter space to determine when a quasiperiodically forced system is chaotic can uncover simple, practical results: for example, a relative phase shift in the forcing can lead to suppression of chaos in a certain class of systems. We explain the extension of the analysis to more general vector field time dependences.

1. Introduction. Some dynamical systems that exhibit chaotic behavior evolve according to sufficiently simple vector fields that one can obtain a fundamental and detailed global picture of the chaotic dynamics. A canonical example is a periodically forced two dimensional system, which can be reduced by time-periodic sampling of trajectories to a two dimensional Poincaré map (for a detailed account see Guckenheimer and Holmes [1983]). Consider the often studied case where the Poincaré map of the unforced system contains a hyperbolic fixed point connected to

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itself by a homoclinic orbit, or a pair of hyperbolic fixed points connected to each other by heteroclinic orbits (see Figure 1.1(i)). Periodic forcing drastically alters the dynamics near the unperturbed homoclinic/heteroclinic orbits (also called separatrices). For small enough perturbations the Poincaré map's fixed points survive, and, if their global stable and unstable manifolds intersect once, they will intersect infinitely many times to produce the boundary of a complicated two dimensional lobe structure (the so-called homoclinic/heteroclinic tangle region) that is invariant under the Poincaré map (see Figure 1.1(ii)).

Figure 1.1 (i) (a) An unperturbed homoclinic orbit, and (b) a pair of unperturbed heteroclinic orbits. (ii) The tangle region of the perturbed system for (a) the homoclinic case, and (b) the heteroclinic case. The dashed and solid lines are stable and unstable manifolds, respectively, of the surviving hyperbolic fixed points.

Lobes map from one to another within the invariant lobe structure with each application of the Poincaré map, and this provides a global picture of the dynamics, i.e. of *transport in phase space*. A variety of physical systems can be studied in this context, a most notable example in recent years being advection in fluids, where the dynamical system phase space is physical space. The idea that relatively innocuous velocity fields can produce chaotic advection has made quite a stir among the fluid mechanics community in the last few years (see for example Ottino [1990]), even

though the observation that simple vector fields can evolve complicated dynamics has been at the heart of dynamical systems analysis since its inception. On the one hand, the fairly detailed understanding that one can obtain (as we shall see) of homoclinic and heteroclinic tangle regions is quite impressive, given the complicated nature of the dynamics. This understanding prevents the need to resort to statistical theories, and provides a framework for computing exact quantities, such as transport in and out of a defined set of regions in phase space (which has relevance for a number of physical situations, such as roll-to-roll transport in a Rayleigh-Bénard flow (see Camassa and Wiggins [1991])). However, for periodically forced systems the vector field that evolves the system is quite simple, and one is studying transport under iterates of a map. One would like to extend the results to vector fields with as complicated time dependences as possible, while still retaining a fairly complete global picture of the dynamics; this will entail extending the study of maps to a study of *sequences of maps*. Our goal is to generalize some notions of transport in phase space associated with iterates of a Poincaré map derived from a time-periodic two dimensional vector field to apply to a bi-infinite sequence of nonautonomous maps derived from a two dimensional quasiperiodic vector field (i.e. a vector field whose time dependence involves ℓ frequencies, $\ell \geq 2$). As we shall see, the concepts involved in the generalization are robust and apply to vector fields with more general time dependences. We thus wish to introduce these concepts in the context of a familiar and appealing class of systems. Since many of the basic tools of dynamical systems theory, including the use of invariant manifolds in providing a global geometrical framework for the study of chaotic motion, are at present not widely used in the study of chaotic advection, and since we view the extension of these tools to more complicated scenarios to be of basic physical interest, we present a rather lengthy exposition of the subject matter, one which does not assume a sophisticated background in dynamical systems theory, and yet which leads the reader through the relevant mathematics. Emphasis will be given to physically relevant results, and we will illustrate our discussion throughout with an example fluid flow.

Let us first introduce some of the features of transport in the time-periodic case. These features are highlighted by a simulation of a typical system with a heteroclinic tangle region, an oscillating vortex pair (OVP) flow (studied by Rom-Kedar et al. [1990]) that is defined along with its quasiperiodic generalization in Appendix 1. The unperturbed system is a pair of equal and opposite point vortices whose streamlines in the comoving frame are shown in Figure 1.2. Open flow passes above and below an inner core of rotating fluid bounded by two heteroclinic orbits. The perturbed system consists of the vortices oscillating periodically in response to a time-periodic external strain-rate field. Figure 1.3 shows, for seven perturbation periods, snapshots of a material curve initially on the upper unperturbed heteroclinic orbit (the perturbed situation is symmetric about $y = 0$). With each snapshot a new lobe of fluid is entrained into the core region. The lobe stretches as it winds around the core, and then it folds across another entrained lobe, part of it escaping to be advected steadily to the left, part of it remaining in the core to repeat the stretching and folding process.

Figure 1.2 Streamlines in the comoving frame of the unperturbed OVP flow. The hyperbolic fixed points are marked by x_h^a, x_h^b ; the heteroclinic orbits are marked by Ψ_u, Ψ_o, Ψ_l .

This process continues ad infinitum to give chaotic advection. The material curve is being essentially “attracted” to the upper unstable manifold of the Poincaré map’s right fixed point. Since the dynamical system is Hamiltonian (the fluid is incompressible), the unstable manifold is of course not an attractor; however, the strong stretching and contraction of the fluid lobes implies that, though area elements do not shrink to zero, they tend to be stretched in one direction and contracted in another such that the unstable manifold dominates the evolution of material curves in the tangle region (see Figure 1.4 for a heuristic portrayal). The stable and unstable manifolds of the Poincaré map’s fixed points are thus understood to be the dominant structure by which transport in the tangle regions should be studied. Recent comparisons of fluid mixing in experiments with numerical simulations of the associated unstable manifold show this to be the case (see Rom-Kedar et al. [1990] for a discussion of these examples).

Our simulation shows the essential features of transport in the tangle region: (i) entrainment and detrainment, i.e. the back and forth motion across boundaries in phase space that were barriers to transport in the unperturbed case, and (ii) repeated stretching and folding to give chaotic dynamics. These features can be examined via the dynamical system’s associated Poincaré map acting on the invariant lobe structure. Computer simulation of a finite number of the lobes, coupled with a knowledge of how lobes map from one to another, provides a direct portrayal of the transport. Further, one can establish that there are regions in phase space on which a sufficiently large number of iterates of the Poincaré map acts as a horseshoe map (this is essentially a consequence of the stretching and folding of lobes as shown in the simulation).

Figure 1.3 Evolution, according to the single frequency OVP flow, of a material curve that initially lies on Ψ_u . The system parameters are given in Appendix 1.

Figure 1.3 (Continued) Evolution, according to the single frequency OVP flow, of a material curve that initially lies on Ψ_u . The system parameters are given in Appendix 1.

Figure 1.4 Three successive time samples showing a material curve being “attracted” to the unstable manifold. The dashed-dotted line is the stable manifold, the dashed line is the unstable manifold, and the solid line is the material curve. Section 2 will make clear the notation involving ϵ . A similar portrayal appears in Rom-Kedar et al. [1990].

One can then use the standard horseshoe map construction to state in a precise way what is meant by chaotic dynamics in the tangle region. Though establishing chaos, this construction provides only a limited understanding of the topology of stretching in chaotic tangles; however, a study of lobe behavior allows one to obtain a global understanding of the stretch properties in chaotic tangles (see Beigie et al. [1991]). In addition, segments of stable and unstable manifolds of the Poincaré map’s fixed points can be used to define the boundary of a homoclinic or heteroclinic core region (approximately equal to the unperturbed core), and the mapping of lobes from one to another with each application of the Poincaré map can be used to define and quantify notions of entrainment and detrainment with respect to this core. The issue of transport in phase space, under iterates of a map, across boundaries formed by segments of stable and unstable manifolds of invariant hyperbolic sets has been studied by various investigators (see MacKay et al. [1984] and Bensimon and Kadanoff [1984]). Rom-Kedar and Wiggins [1990] provide a more detailed study of transport under iterates of a two dimensional map, one which considers the global geometry of the invariant lobe structures that govern the transport, and

Wiggins [1990] generalizes this study to a class of k -degree-of-freedom Hamiltonian systems ($2 \leq k < \infty$) with hyperbolicity in two dimensions. We should stress that much of the analysis of transport across boundaries formed by segments of stable and unstable manifolds applies to more general circumstances than to a single core of a near-integrable map. Indeed, it can apply to an arbitrary number of simply connected regions bounded by segments of stable and unstable manifolds of an arbitrary number of invariant hyperbolic sets of maps that need not be near-integrable. The study of transport in phase space is relevant to a variety of physical situations besides chaotic advection, such as multiphoton ionization of excited atoms (MacKay and Meiss [1988]), and intramolecular relaxation of highly excited molecules (Davis [1985]).

From both a mathematical and physical standpoint, there is obvious motivation to generalize to more complicated vector field time dependences and from maps to sequences of maps. For concreteness, we mention a few recent physical examples. The OVP flow of Rom-Kedar et al. [1990] considers the interaction of two point vortices with a time-periodic irrotational velocity field, and one would like to be able to address more complicated time dependences in the irrotational field. Further, if one changes the sign of one of the vortices, so that one is considering in the unforced situation a pair of identical vortices rotating about one another (see Beigie et al. [1991]), the vortex response to a time-periodic strain rate field is guaranteed to be time-periodic only if the perturbing field rotates along with the vortices; otherwise, the vortices generically respond with quasiperiodic or chaotic oscillations. Of course as one adds more point vortices to the problem, the vortices need not have a perturbing irrotational field to obtain quasiperiodic and chaotic vortex motion (see for example the discussion of the four point vortex problem by Aref and Pomphrey [1982]). Swinney et al. [1990] study mixing in a planetary-type flow by adding secondary modes to an initial stream function, and there is obvious motivation to be able to study the case with several secondary modes. Solomon and Gollub [1988] mention the observance of quasiperiodic velocity fields in Rayleigh-Bénard convection. There are of course several examples outside the field of fluids, such as comparing molecular dissociation by one and two lasers (see Noid and Stine [1979] and Goggin and Milonni [1988a], [1988b]).

Let us now address the extension of the transport features in the time-periodic case to multiple frequency vector fields. The addition of even one more frequency to the vector field's time dependence requires a fundamental departure from the time-periodic analysis: the evolution of trajectories from one time sample to the next (for whatever choice of sampling) must now in general be described by a different map for each time sample. One can thus no longer construct an invariant two dimensional lobe structure upon which to study phase space transport, or invoke the standard horseshoe map construction to establish that there is chaotic motion. How then does one obtain a detailed global picture of the dynamics? To study the dynamics of a nonautonomous system that evolves according to a two dimensional, ℓ frequency vector field, we consider in Section 2 an associated $(\ell + 2)$ dimensional autonomous system in order to construct an invariant $(\ell + 1)$ dimensional lobe structure in a Poincaré section of the enlarged phase space (the lobe boundaries

being formed out of intersecting global stable and unstable manifolds of normally hyperbolic invariant $(\ell - 1)$ -tori in the defined Poincaré section). We derive from this invariant lobe structure a sequence of two dimensional time-dependent lobe structures in the nonautonomous system phase space that will be used to analyze transport under a sequence of nonautonomous maps. The invariant lobe structure will prove instrumental throughout our analysis in understanding transport relative to the sequence of two dimensional time-dependent lobe structures, and we will see how the transport properties of the nonautonomous system are related to transport properties of the associated autonomous system. For concreteness, we will focus on quasiperiodically forced systems whose unforced solution contains in the Poincaré section of the associated autonomous system a single normally hyperbolic invariant $(\ell - 1)$ -torus connected to itself by a homoclinic orbit, or a pair of such tori connected by two or more heteroclinic orbits. Appendix 1 discusses two such systems (the OVP flow mentioned earlier and a Duffing oscillator) that will be used throughout the paper to illustrate our discussion. Though it will be clear as we proceed how many of the arguments extend to the case of an arbitrary number of normally hyperbolic invariant $(\ell - 1)$ -tori of a map that is not necessarily near-integrable, we will be able to give a more complete global picture of transport in the two prototypical situations we have introduced. For example, a generalized Melnikov function will repeatedly help us understand in the near-integrable case the nature of the invariant $(\ell + 1)$ dimensional lobe structure, which has more complicated geometrical considerations than in the single frequency case. In particular, in Section 2 we study the zero sets of the Melnikov function to determine the geometry of intersections of stable and unstable manifolds, and hence the geometry of the invariant lobes. Section 3 describes how the lobes of one time sample of the nonautonomous system map to the lobes of the next time sample; we define a sequence of time-dependent cores and turnstile lobes, explain the procedure for determining the number of turnstile lobes for each time sample, and illustrate this procedure with some two frequency examples. The turnstile lobes are the mechanism for entrainment and detrainment relative to the core sequence (since for simplicity we are dealing with a single core sequence, we borrow the fluid mechanical terminology of entrainment and detrainment to make the discussion of transport in and out of the core sequence, respectively, more apparent). The main extension from the previously mentioned studies of transport across boundaries is the notion of a sequence of time-dependent cores, and we discuss how such a notion is reasonable, and how the stable and unstable manifolds in the Poincaré section of the autonomous system phase space provide the natural definition of a sequence of time-dependent cores. Indeed, we discuss how for near-integrable systems whose unperturbed separatrix divides bounded and unbounded motion, it is segments of stable and unstable manifolds of the perturbed problem, and *not* the unperturbed separatrix, which form the “last frontier” between bounded and unbounded motion in the perturbed problem. We conclude Section 3 by presenting computer simulations of some sequences of lobe structures and of lobes mapping from one to another within these sequences, and discussing qualitatively the essential features of transport in the time-dependent tangle region. The sampling method used for

the simulations, which we refer to as a *double phase slice method*, contrasts with a previous suggestion by Moon and Holmes [1985] for sampling quasiperiodically forced systems to find underlying structure, and is explained in Appendix 4. The next three sections examine more closely the features discussed qualitatively in Section 3. Section 4 focuses on entrainment and detrainment relative to a sequence of two dimensional time-dependent cores in the nonautonomous system phase space. We calculate analytically instantaneous and average flux with respect to the core sequence in near-integrable systems, and present the computational prescription for calculating these quantities when the system is not near-integrable. We compare average flux in the single and multiple frequency cases: though the comparisons depend on the normalization of perturbation amplitudes, we shall see how, for reasonable normalizations, the average flux for a class of multiple frequency systems is maximal in a particular single frequency limit. We explain how answering basic questions about transport in phase space (both in the near-integrable and general case) necessitates a consideration of more than just flux, and then study such issues as the geometry of intersections of lobes with each other, with the core, and with level sets of the unperturbed Hamiltonian. For example, we will see that even though average flux may decrease in quasiperiodic systems, other factors can outweigh this effect to make it easier for inner level sets of the unperturbed Hamiltonian to escape the core. More generally, *the variation of lobe areas gives one added freedom to enhance or diminish transport properties over finite times for a fixed average flux*. Section 5 deals with the description of chaos associated with stretching and folding of lobes in the framework of a *traveling horseshoe map sequence*. The chaotic dynamics of the nonautonomous system is understood relative to a time-dependent Cantor set of points. Other studies, with different methods, of the chaotic nature of the dynamics under perturbations with time dependences more complicated than periodic, or, equivalently, under a sequence of nonautonomous maps, are given by Scheurle [1986], Stoffer [1988a], [1988b], and Meyer and Sell [1989]. We will see here how the notion of a traveling horseshoe map sequence and of chaotic dynamics relative to a time-dependent Cantor set of points follows quite naturally from previous work by Wiggins (see Wiggins [1987] and [1988]). The generalized Melnikov function allows one to search in parameter space of the quasiperiodic problem to determine when one can and cannot have chaos. Such a search can have simple, practical consequences: for example, we shall see how a relative phase shift in the forcing can lead to suppression of chaos in a class of multiple frequency vector fields. Section 6 returns to a transport problem discussed in Section 4 and treats the problem in a more general way with a formalism that focuses on lobe intersections (for simplicity we consider in Section 4 transport between only two regions, but the framework easily extends to an arbitrary number of regions (such as found in the previously mentioned Rayleigh-Bénard flow), and this extension is part of what is considered in Section 6). We conclude in Section 7 with an explanation of how the analysis extends to vector fields with more general time dependences than quasiperiodic. We will emphasize throughout the paper the two frequency case as a workable and illuminating example of transport in phase space under a bi-infinite sequence of nonrepeating maps.

2. Defining the lobe structures and studying their geometry. We consider two dimensional nonautonomous systems of the form

$$(2.1) \quad \dot{x} = JDH(x) + \varepsilon g(x, t; \mu, \varepsilon),$$

where $x \in \mathbb{R}^2$, $H : X \mapsto \mathbb{R}^1$ is C^{r+1} ($r \geq 2$) on some open set $X \subset \mathbb{R}^2$, $g : X \times \mathbb{R}^1 \times V \times \mathbb{R}^1 \mapsto \mathbb{R}^2$ is C^r ($r \geq 2$) with parameter space $V \subset \mathbb{R}^p$ an open set, $\varepsilon \in \mathbb{R}^1$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$DH(x) = \begin{pmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{pmatrix}.$$

Equation (2.1) represents a class of perturbed single degree-of-freedom Hamiltonian systems (note that the perturbation $\varepsilon g(x, t; \mu, \varepsilon)$ may or may not be Hamiltonian). We restrict our interest to perturbations periodic or quasiperiodic in time for fixed $x \in X, \mu \in V, \varepsilon \in \mathbb{R}^1$, and hence we can rewrite the governing equation as

$$(2.2) \quad \dot{x} = JDH(x) + \varepsilon g(x, \theta_1(t), \theta_2(t), \dots, \theta_\ell(t); \mu, \varepsilon),$$

where $\theta_i(t) = \omega_i t + \theta_{i_0}$ for $i = 1$ to ℓ (some positive integer ≥ 1), and g is 2π periodic in each of $\theta_i(t)$. Though we focus on quasiperiodic systems ($\ell \geq 2$), it should be clear as we proceed how our $\ell \geq 2$ analysis applies to the familiar time-periodic ($\ell = 1$) case, and we will refer from time to time to the single frequency case.

Let $x(t) = \phi(t, t_0, x_0)$ denote the solution to (2.2) which satisfies the initial condition $\phi(t_0, t_0, x_0) = x_0$. We shall also refer to $\phi(t, t_0, x_0)$ as the *trajectory of (2.2) through x_0 at time t_0* . To study the dynamics of (2.2), it will be advantageous to sample the trajectories at discrete time intervals, the interval being one of the perturbation periods, say $2\pi/\omega_\ell$. For time-periodic vector fields, it is well understood that periodic sampling filters out redundant dynamical information to obtain a simpler underlying structure with which to study the dynamics; for quasiperiodic vector fields, periodic sampling will in a similar way simplify the underlying structure with which the dynamics is studied. The evolution of system (2.2) from time $t = t_0 + \frac{2\pi}{\omega_\ell} n$ ($n \in \mathbb{Z}$) to time $t = t_0 + \frac{2\pi}{\omega_\ell} (n + 1)$ defines a map on X :

$$T_\varepsilon(\cdot; n) : X \mapsto X$$

$$x \mapsto \phi\left(\frac{2\pi}{\omega_\ell}(n + 1), \frac{2\pi}{\omega_\ell}n, x\right),$$

where for simplicity we have set $t_0 = 0$, which is not restrictive since the θ_{i_0} are arbitrary). Our goal then is to study the dynamics in X under the bi-infinite sequence of nonautonomous maps $\{T_\varepsilon(\cdot; n); n \in \mathbb{Z}\}$. To do this, we employ a standard technique for the study of nonautonomous systems (see Guckenheimer and Holmes [1983] or Wiggins [1988]), that is to examine an associated autonomous system in order to construct invariant manifolds that are fixed in the enlarged phase space. These manifolds form barriers which constrain the motion in the autonomous

system phase space and hence provide a global picture of the dynamics. Having obtained these manifolds, we will use them to define for each time $t = \frac{2\pi}{\omega_\ell}n$ a set of curves in X that will be used in later sections to study the dynamics in X under the bi-infinite sequence of maps $\{T_\varepsilon(\cdot; n); n \in \mathbb{Z}\}$.

(2.1) *Constructing an invariant lobe structure and studying its geometry*

We rewrite equation (2.2) in the autonomous form

$$\begin{aligned} \dot{x} &= JDH(x) + \varepsilon g(x, \theta_1, \theta_2, \dots, \theta_\ell; \mu, \varepsilon) \\ \dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2 \\ &\vdots \\ \dot{\theta}_\ell &= \omega_\ell. \end{aligned} \tag{2.3}$$

The autonomous system phase space is $\mathbb{R}^2 \times T^\ell$ (where T^ℓ is the ℓ -torus). The expression

$$(x(t), \theta_1(t), \dots, \theta_\ell(t)) = (\phi(t, t_0 = 0, x_0), \omega_1 t + \theta_{1_0}, \dots, \omega_\ell t + \theta_{\ell_0}) \tag{2.4}$$

solves equation (2.3). Let us denote this solution by $\Phi(t, t_0 = 0, x_0, \theta_{1_0}, \theta_{2_0}, \dots, \theta_{\ell_0})$. We shall also refer to this as the *trajectory of (2.3) through $(x_0, \theta_{1_0}, \theta_{2_0}, \dots, \theta_{\ell_0})$ at time $t = 0$* . From equation (2.4), the trajectory of (2.2) through x_0 at time $t = 0$ is the x component of the trajectory of (2.3) through $(x_0, \theta_{1_0}, \theta_{2_0}, \dots, \theta_{\ell_0})$ at time $t = 0$, and hence the study of trajectories of the nonautonomous system is related to the study of the appropriate trajectories of the autonomous system.

We study cases in which the unperturbed system has in X :

- (a) a hyperbolic fixed point x_h which is connected to itself by a homoclinic trajectory $x_h(t)$ (i.e., $\lim_{t \rightarrow \pm\infty} x_h(t) = x_h$), or
- (b) a pair of hyperbolic fixed points x_h^a, x_h^b connected to each other by heteroclinic trajectories $x_h^i(t)$ (i labels different trajectories); for each i either $\{\lim_{t \rightarrow -\infty} x_h^i(t) = x_h^a$ and $\lim_{t \rightarrow \infty} x_h^i(t) = x_h^b\}$ or $\{\lim_{t \rightarrow -\infty} x_h^i(t) = x_h^b$ and $\lim_{t \rightarrow \infty} x_h^i(t) = x_h^a\}$.

From these basic structures, the results will generalize to an arbitrary number of hyperbolic fixed points, each connected to itself and/or other fixed points by homoclinic/heteroclinic trajectories. In the unperturbed homoclinic case, the autonomous system phase space thus contains a normally hyperbolic ℓ -torus of the form

$$T_0 = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | x = x_h\},$$

whose $(\ell + 1)$ dimensional stable and unstable manifolds, denoted $W^s(T_0)$ and $W^u(T_0)$, coincide along the $(\ell + 1)$ dimensional homoclinic manifold given by

$$W^s(T_0) \cap W^u(T_0) = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | x = x_h(t), t \in \mathbb{R}\}.$$

(a)

(b)

Figure 2.1 (a) A homoclinic manifold and (b) two heteroclinic manifolds in the autonomous system phase space of the ℓ frequency case.

Note that “normal hyperbolicity” refers to the fact that expansion and contraction rates normal to T_0 dominate those tangent to it, as to be expected since the θ_i motion is only linear in time (see Wiggins [1988] for further discussion of normal hyperbolicity). The unperturbed heteroclinic case has two normally hyperbolic ℓ -tori of the same form as the homoclinic case, with the $(\ell + 1)$ dimensional stable manifold of one torus and the $(\ell + 1)$ dimensional unstable manifold of the other torus coinciding along an $(\ell + 1)$ dimensional heteroclinic manifold

$$W^s(T_0^a) \cap W^u(T_0^b) = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | x = x_h^i(t), t \in \mathbb{R}\}.$$

Figure 2.1 shows the manifolds in $X \times T^\ell$ (in the forthcoming diagrams, we will not keep presenting both the homoclinic and heteroclinic case, but rather show one or the other interchangeably). The parameter s specifies the location in X along the invariant unperturbed homoclinic or heteroclinic manifold: it represents the time it takes for the point $x(-s)$ on the unperturbed manifold in X to move according to the flow (2.3) to the point $x(s = 0)$ (which we define to be the spatial midpoint of the unperturbed manifold in X).

We define a global cross section of the autonomous system phase space

$$\Sigma^{\theta_{\ell_0}} = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | \theta_\ell = \theta_{\ell_0}\},$$

and the associated Poincaré map generated by the flow of (2.3) is given by

$$(2.5) \quad P_\varepsilon : \Sigma^{\theta_{\ell_0}} \mapsto \Sigma^{\theta_{\ell_0}} \\ \left(x, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n\right) \mapsto \left(\phi\left(\frac{2\pi}{\omega_\ell}(n+1), \frac{2\pi}{\omega_\ell} n, x\right), \theta_0 + 2\pi \frac{\omega}{\omega_\ell}(n+1)\right),$$

where $\theta_0 \equiv (\theta_{1_0}, \theta_{2_0}, \dots, \theta_{(\ell-1)_0})$ and $\omega \equiv (\omega_1, \omega_2, \dots, \omega_{\ell-1})$. Note that the map is independent of n since it is derived from an autonomous system. Studying the flow of (2.3) via this Poincaré map is equivalent to sampling the trajectories of (2.3) at time intervals equal to $2\pi/\omega_\ell$. The Poincaré map of the unperturbed homoclinic system has a normally hyperbolic invariant $(\ell-1)$ -torus of the form

$$\tau_0 = T_0 \cap \Sigma^{\theta_{\ell_0}} = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | x = x_h, \theta_\ell = \theta_{\ell_0}\},$$

with an ℓ dimensional homoclinic manifold

$$W^s(\tau_0) \cap W^u(\tau_0) = \{W^s(T_0) \cap W^u(T_0)\} \cap \Sigma^{\theta_{\ell_0}} \\ = \{(x, \theta_1, \theta_2, \dots, \theta_\ell) | x = x_h(t), \theta_\ell = \theta_{\ell_0}, t \in \mathbb{R}\}.$$

The heteroclinic case is understood similarly. See Figure 2.2 for a portrayal of homoclinic manifolds in the full phase space and in the Poincaré section $\Sigma^{\theta_{\ell_0}}$ for the two frequency case (which will be studied in particular examples in later sections). Note how, in the second portrayal of $W^s(\tau_0) \cap W^u(\tau_0)$ in Figure 2.2(b), the θ_2 specifier is dropped. In the diagrams and notation of the geometrical constructs that clearly lie in the Poincaré section $\Sigma^{\theta_{\ell_0}}$, we will leave out the cumbersome θ_{ℓ_0} specifier; one should remember, however, that these constructs are in $X \times T^\ell$ and correspond to $\theta_\ell = \theta_{\ell_0}$.

To establish for the perturbed system the existence of a lobe structure in $\Sigma^{\theta_{\ell_0}}$, we invoke the following persistence result for small perturbations.

PROPOSITION 2.1. *If $P_{\varepsilon=0}$ contains a normally hyperbolic invariant $(\ell-1)$ -torus τ_0 , then for ε sufficiently small, P_ε possesses a C^r normally hyperbolic invariant $(\ell-1)$ -torus, τ_ε , whose local ℓ dimensional C^r stable and unstable manifolds, $W_{\text{loc}}^s(\tau_\varepsilon)$ and $W_{\text{loc}}^u(\tau_\varepsilon)$, are $C^r\varepsilon$ -close to $W_{\text{loc}}^s(\tau_0)$ and $W_{\text{loc}}^u(\tau_0)$, respectively.*

Proof. This is an immediate consequence of the persistence theory for normally hyperbolic invariant manifolds. See Fenichel [1971], Hirsch et al. [1977], or Wiggins [1988]. $\square$

(a)

or

(b)

Figure 2.2 The $\ell = 2$ homoclinic manifold in (a) the full phase space and (b) the Poincaré section $\Sigma^{\theta_{20}}$ (the second portrayal shows a cut-away view).

In the usual way we can define global stable and unstable manifolds, $W^s(\tau_\epsilon)$ and $W^u(\tau_\epsilon)$, from the local ones:

$$W^s(\tau_\epsilon) = \bigcup_{n \leq 0} P_\epsilon^n(W_{\text{loc}}^s(\tau_\epsilon))$$

$$W^u(\tau_\epsilon) = \bigcup_{n \geq 0} P_\epsilon^n(W_{\text{loc}}^u(\tau_\epsilon)),$$

and these manifolds will soon be seen to provide an invariant lobe structure.

At this point we introduce a generalized Melnikov function (Wiggins [1988]), which provides a criterion for existence of the lobe structure and a tool for studying

the geometry of the lobe structure in the near-integrable case:

(2.6)

$$M(s, \theta_1, \theta_2, \dots, \theta_\ell; \nu) = \int_{-\infty}^{\infty} DH(x_h(t)) \cdot g(x_h(t), \omega_1 t + (\omega_1 s + \theta_1), \omega_2 t + (\omega_2 s + \theta_2), \dots, \omega_\ell t + (\omega_\ell s + \theta_\ell); \mu, \varepsilon = 0) dt,$$

where $(s, \theta_1, \theta_2, \dots, \theta_\ell)$ specify a point on $W^s(T_0^a) \cap W^u(T_0^b)$ (and are thus constant in the integral expression), and $\nu \equiv (\mu, \omega_1, \omega_2, \dots, \omega_\ell)$. Note that we leave out the i superscript on $x_h(t)$; in the heteroclinic case, it is understood that $x_h(t)$ represents one orbit out of a set of possible heteroclinic orbits. Further, whenever we write T_0^a and T_0^b , or τ_0^a and τ_0^b , it is understood that in the homoclinic case $a = b$. Wiggins [1988] gives a full discussion of this function; further, it is explained there how redundancy due to uniqueness of solutions allows one to eliminate one of the parameters when studying global bifurcations, and the $s = 0$ case is thus examined. Here we choose to let the s parameter vary, but set $\theta_\ell = \theta_{\ell_0}$. In the Poincaré section $\Sigma^{\theta_{\ell_0}}$, the first order in ε approximation to the signed distance between the global stable and unstable manifolds is

$$(2.7) \quad d(s, \theta_1, \theta_2, \dots, \theta_{(\ell-1)}; \nu, \varepsilon) = \frac{\varepsilon M(s, \theta_1, \theta_2, \dots, \theta_{(\ell-1)}, \theta_{\ell_0}; \nu)}{\|DH(x_h(-s))\|} + O(\varepsilon^2)$$

(see Wiggins [1988]). More specifically, d is the signed distance, measured along the normal to the unperturbed homoclinic or heteroclinic orbit, between the “first” intersection of $W^u(\tau_\varepsilon^b)$ with the normal and the “first” intersection of $W^s(\tau_\varepsilon^a)$ with the normal (see Figure 2.3).

Figure 2.3 The signed distance function between the stable and unstable manifolds in $\Sigma^{\theta_{\ell_0}}$ for the perturbed two frequency homoclinic system (refer to Wiggins [1988] for determining the sign of the Melnikov function).

“First” is understood relative to moving along $W^u(\tau_\varepsilon^b)$ normally away from τ_ε^b and along $W^s(\tau_\varepsilon^a)$ normally away from τ_ε^a . Note that $\|DH(x_h(-s))\| \rightarrow 0$ as $s \rightarrow \pm\infty$, so the distance function blows up with $|s|$, making the first order in ε approximation in (2.7) a poor one for sufficiently large $|s|$. The approximation

will be valid however over a limited domain: the manifolds $W^s(\tau_\varepsilon^a)$, $W^u(\tau_\varepsilon^b)$ can be approximated uniformly on semi-infinite time intervals (see Guckenheimer and Holmes [1983] and Wiggins [1988]), and on these time intervals the manifolds move only an $O(\varepsilon)$ amount from the unperturbed manifolds. Hence for sufficiently small perturbations the manifold separations will be well approximated by the $O(\varepsilon)$ term over a finite domain. The approximation will prove quite useful in later sections, even though it is valid over a limited domain.

For illustration purposes we consider systems with perturbations of the form

$$(2.8) \quad g(x, t; \mu, \varepsilon) = F_0(x; \mu) + \mathcal{F}(x; \mu) \sum_{i=1}^{\ell} F_i(\mu, \omega_i) \sin(\omega_i t + \theta_{i_0}) + O(\varepsilon).$$

It will be important to keep in mind that the θ_{i_0} specify the initial phases of the perturbation. The generalized Melnikov function associated with these perturbations has the form

$$(2.9) \quad \begin{aligned} M(s, \theta_1, \dots, \theta_{\ell-1}, \theta_{\ell_0}; \nu) = & A_0(\mu) + \sum_{i=1}^{\ell-1} A_i(\mu, \omega_i) \sin(\omega_i s + \theta_i) + A_\ell(\mu, \omega_\ell) \sin(\omega_\ell s + \theta_{\ell_0}) \\ & + \sum_{i=1}^{\ell-1} B_i(\mu, \omega_i) \cos(\omega_i s + \theta_i) + B_\ell(\mu, \omega_\ell) \cos(\omega_\ell s + \theta_{\ell_0}). \end{aligned}$$

In particular, Appendix 1 gives the equations of motion for the two frequency OVP flow mentioned in the introduction and a two frequency Duffing oscillator, and derives their generalized Melnikov functions. In both these examples, as is often the case, the unperturbed homoclinic/heteroclinic orbits have a symmetry, and hence only some of the amplitudes in equation (2.9) are non-zero. (Due to the symmetry in the unperturbed Duffing oscillator, the perturbation is written in terms of cosines, which of course satisfies (2.8) with the proper phase shift.) We shall refer to the ratio of the Melnikov function amplitude, $A_i(\mu, \omega_i)$ or $B_i(\mu, \omega_i)$, to the corresponding relative perturbation amplitude, $F_i(\mu, \omega_i)$, as a *relative scaling factor*:

$$\frac{A_i(\mu, \omega_i)}{F_i(\mu, \omega_i)} \text{ or } \frac{B_i(\mu, \omega_i)}{F_i(\mu, \omega_i)} \equiv \text{relative scaling factor associated with frequency } \omega_i.$$

As seen in the examples in Appendix 1, the relative scaling factors are in general *frequency dependent*, which will be relevant to the study of how transport rates depend on single and multiple frequency forcing. As illustrated in Appendix 1, all scaling factors are determined by two *relative scaling functions* (one for the A 's, one for the B 's) which are derived from the generalized Melnikov function (in fact all the B 's vanish for the examples in Appendix 1, so there is only a single scaling function for these examples). We should point out that if the perturbation amplitudes in (2.8) each had a different spatial dependence, the corresponding Melnikov function would still be expressed in the form (2.9), since the spatial dependences are integrated out. We choose for concreteness, however, to refer to the class of perturbations (2.8) for illustration (whose attractive features include being able to define relative scaling factors).

To define and study an invariant lobe structure necessitates a consideration of the geometry of intersection manifolds of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$, and the following theorem allows one to use the Melnikov function for this consideration.

THEOREM 2.2. *Suppose there exists a point $(\tilde{s}, \tilde{\theta}_1, \dots, \tilde{\theta}_{(\ell-1)})$ for some $\tilde{\nu}$ such that*

- 1) $M(\tilde{s}, \tilde{\theta}_1, \dots, \tilde{\theta}_{(\ell-1)}, \theta_{\ell_0}; \tilde{\nu}) = 0$,
- 2) $D_{(s, \theta_1, \dots, \theta_{(\ell-1)})} M(\tilde{s}, \tilde{\theta}_1, \dots, \tilde{\theta}_{(\ell-1)}, \theta_{\ell_0}; \tilde{\nu})$ is of rank one.

Then for ε sufficiently small $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ intersect transversely near (within $O(\varepsilon)$) this point, and this intersection can be continued to an $(\ell - 1)$ dimensional manifold in $\Sigma^{\theta_{\ell_0}}$.

Proof. This is a straightforward consequence of the implicit function theorem (see Wiggins [1988]). $\square$

Hence within $O(\varepsilon)$ of the zero sets of the Melnikov function which satisfy the above derivative criterion are intersection manifolds of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$. Note that, though the $M = 0$ sets can imply nearby intersection sets, not all intersection manifolds need be near the $M = 0$ sets, as we will describe later. Further note that we are making no assumptions on the geometry of the $(\ell - 1)$ dimensional intersection manifolds; in particular, unlike Wiggins [1988] and Ide and Wiggins [1989], we do not assume them to be $(\ell - 1)$ -tori. We approximately portray (i.e. to $O(\varepsilon)$) some intersection manifolds by plotting the zero sets of a two frequency Melnikov function of the form

$$(2.10) \quad M(s, \theta_1, \theta_{2_0} = 0; \nu) = A_0(\mu) + A_1(\mu, \omega_1) \sin(\omega_1 s + \theta_1) + A_2(\mu, \omega_2) \sin(\omega_2 s).$$

(The systems described in Appendix 1 have Melnikov functions of this form.) To do so, we first flatten out the (s, θ_1) coordinates onto a plane, as shown in Figure 2.4.

Figure 2.4 Flattening the coordinates of an unperturbed homoclinic orbit onto a plane.

Figure 2.5 Some zero sets of the Melnikov function in equation (2.10), for a range of parameter values ($\omega_2/\omega_1 = g^{-1}$, where g is the golden mean, and $\omega_1 \equiv \omega$). The dashed and dashed-dotted lines represent two choices of PIM's according to Definition 2.1.

Figure 2.5 (Continued) Some zero sets of the Melnikov function in equation (2.10), for a range of parameter values (for (h) $\omega_2/\omega_1 = 2$ instead of g^{-1}). The dashed and dashed-dotted lines represent two choices of PIM's according to Definition 2.1.

Figures 2.5 (a) to (g) show the zero sets near $s = 0$ for a range of A_1/A_2 values and $A_0 = 0$, $\omega_2/\omega_1 = g^{-1}$, where g is the golden mean $((\sqrt{5} - 1)/2)$. For $A_1/A_2 < 1$ the intersection manifolds are non-intersecting 1-tori, for $A_1/A_2 > 1$ they are non-intersecting spirals, and for $A_1/A_2 = 1$ they are intersecting spirals (or equivalently intersecting 1-tori). For other ratios of ω_2/ω_1 (both commensurate and incommensurate), the geometry of the intersection manifolds as A_1/A_2 is varied (keeping $A_0 = 0$) is qualitatively similar, as will become more apparent with some additional figures in Section 3. There is a slightly technical point about the non-generic case of $A_1/A_2 = 1$: that the zero sets cross does not necessarily imply that the intersection manifolds cross; however, we have shown that for each crossing of the $M = 0$ sets the intersection manifolds cross within $0(\varepsilon)$, and for the systems defined in Appendix 1 each such crossing occurs for the same value $A_1/A_2 = 1 + 0(\varepsilon)$ (see Appendix 2). Figure 2.5(h) shows a case with $A_0 = \pm 1.5$, $A_1 = A_2 = \pm 1$, and $\omega_2/\omega_1 = 2$: the intersection manifolds exist in a *subset* of $[0, 2\pi)$. As will be proven in an upcoming theorem, the geometry of intersection manifolds for $A_0 \neq 0$ may be qualitatively different for commensurate and incommensurate frequencies, since it is in the commensurate frequency case alone that for certain parameter values the intersection manifolds exist on a subset of $[0, 2\pi)$. We remark parenthetically that the study of zero sets of generalized Melnikov functions is proving to be useful to a number of problems involving transport within chaotic tangle regions. For example, in k degree-of-freedom systems ($2 \leq k < \infty$) with hyperbolicity in two dimensions (see Wiggins [1990]), the definition of the so-called turnstile lobes is a somewhat delicate issue, and necessitates a consideration of the geometry of the zero sets of the Melnikov function.

With these simple plots as illustration, we define *primary intersection manifolds*, study their geometry, and then define *lobes*. The definitions will be sufficiently general to allow arbitrary geometries of intersection manifolds found in systems that satisfy the conditions of Proposition 2.1 and Theorem 2.2. Suppose that the point $(\tilde{s}, \tilde{\theta}_1, \dots, \tilde{\theta}_{\ell-1})$ of Theorem 2.2 satisfies $\frac{\partial M}{\partial s}(\tilde{s}, \tilde{\theta}_1, \dots, \tilde{\theta}_{\ell-1}, \theta_{\ell_0}; \tilde{\nu}) \neq 0$. Then by the implicit function theorem there exists a C^r function $s(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ on some domain

$$\mathcal{Z} \equiv (\alpha_1, \beta_1) \times \dots \times (\alpha_{\ell-1}, \beta_{\ell-1}) \in T^{\ell-1},$$

where $\tilde{\theta}_i \in (\alpha_i, \beta_i) \subset [0, 2\pi)$, $i = 1, 2, \dots, \ell - 1$, such that

$$M(s(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0}), \theta_1, \dots, \theta_{\ell-1}, \theta_{\ell_0}; \tilde{\nu}) = 0$$

(for simplicity of future discussion the dependence of the above function s on the parameters $\tilde{\nu}$ is not explicitly included in the notation). We denote the closure of $\mathcal{Z}$ by

$$\bar{\mathcal{Z}} \equiv [\alpha_1, \beta_1] \times \dots \times [\alpha_{\ell-1}, \beta_{\ell-1}] \in T^{\ell-1},$$

and if $\beta_i = 2\pi$ for some $i = 1, \dots, \ell - 1$ then we define $[\alpha_i, \beta_i] = [\alpha_i, 2\pi)$.

DEFINITION 2.1. Let $s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ with domain $\mathcal{Z}^i$, $i = 1, \dots, p$, be functions as defined above with $\mathcal{Z}^i \cap \mathcal{Z}^j = \emptyset$ and $\bar{\mathcal{Z}}^i \cap \bar{\mathcal{Z}}^j \equiv \mathcal{Z}^{ij} \forall i, j \ i \neq j$. Then from

Theorem 2.2 the set

$$\{(s, \theta_1, \dots, \theta_{\ell-1}) | s = s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0}), (\theta_1, \dots, \theta_{\ell-1}) \in \bar{\mathcal{Z}}^i, i = 1, \dots, p\}$$

parametrizes an $(\ell - 1)$ dimensional surface contained in $W^s(\tau_\varepsilon^a) \cap W^u(\tau_\varepsilon^b)$. In order for this surface to be a single-valued graph over $\bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p$ we further specify

$$s = s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0}) \text{ on } \mathcal{Z}^{ij} \quad i \neq j$$

(provided $\mathcal{Z}^{ij} \neq \emptyset$). We refer to this surface as a *primary intersection manifold (PIM)*, which we denote by τ .

This definition allows several geometrical possibilities, and we list some fundamental ones below.

- (i) $p = 1$ and $\bar{\mathcal{Z}}^1 = T^{\ell-1}$. In this case τ is either an $(\ell - 1)$ -torus or an $(\ell - 1)$ dimensional segment of a spiral manifold. See Figures 2.5(a),(b),(c) for tori, and Figure 2.5(g) for spirals. Note that one could define spiral PIM's in Figure 2.5(f) as well, but we will use this figure to illustrate another type of definition of PIM's.
- (ii) $p > 1$, $\bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p = T^{\ell-1}$, and $s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0}) = s^j(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ on $\mathcal{Z}^{ij}$, $i \neq j$. In this case τ is again either an $(\ell - 1)$ -torus or an $(\ell - 1)$ dimensional segment of a spiral manifold. What typically happens in this situation is that $s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ and $s^j(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ undergo a bifurcation on $\mathcal{Z}^{ij}$ and τ is formed by piecing together the functions $s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$, $i = 1, \dots, p$, at the surfaces where they bifurcate (note that the dimension of $\mathcal{Z}^{ij}$ is generically $(\ell - 2)$, and it is necessary for $D_{(s, \theta_1, \dots, \theta_{\ell-1})} M(s, \theta_1, \dots, \theta_{\ell-1}, \theta_{\ell_0}; \nu)$ to have rank zero on these surfaces). See Figure 2.5(d) for an example.
- (iii) $p > 1$, $\bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p = T^{\ell-1}$, and $s^i(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0}) \neq s^j(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ on $\mathcal{Z}^{ij}$, $i \neq j$. In this case τ is discontinuous on the $\mathcal{Z}^{ij}$; however, τ is a single-valued graph over $T^{\ell-1}$ from our previous condition on the definition of PIM's. See Figures 2.5(e) and (f) for examples. At present, there may seem to be no motivation for such a choice of PIM's in Figure 2.5(f); however, what may seem geometrically unappealing in $\Sigma^{\theta_{\ell_0}}$ will turn out to be geometrically appealing in the nonautonomous system phase space.
- (iv) $p > 1$ and $\mathcal{Z}^{ij} = \emptyset$. Unlike the previous cases, $\bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p \subset T^{\ell-1}$ and τ contains gaps. We define a PIM on a *subset* of $T^{\ell-1}$ when intersection manifolds do not cover $T^{\ell-1}$, as in Figure 2.5(h).

These four cases are not exhaustive: for example, one may have combinations of (ii), (iii), and (iv). Namely, τ may be nondifferentiable and/or discontinuous and/or possess gaps on $\mathcal{Z}^{ij}$.

As for time-periodic vector fields, general properties of $W^s(\tau_\varepsilon^a) \cap W^u(\tau_\varepsilon^b)$ can be inferred from the Melnikov function, so that we can understand the geometry associated with different types of perturbations. We can describe some of these properties after a definition and two lemmas.

DEFINITION 2.2. For $\bar{\theta} \in T^{\ell-1}$, we refer to the plane $\chi(\bar{\theta}) \equiv \{(x, \theta) | \theta = \bar{\theta}\}$ as a *phase slice* of the Poincaré section $\Sigma^{\theta_{\ell_0}}$.

LEMMA 2.3. If there exists an $(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1})$ for some $\bar{\nu}$ such that

$$\begin{aligned} M(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) &= 0 \\ \frac{\partial M}{\partial s}(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) &\neq 0 \end{aligned}$$

then there exists in the phase slice $\chi(\bar{\theta})$ a countable infinity of transversal intersection points of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$.

Proof. See Appendix 3. $\square$

Note how, as illustrated by the cross points in Figure 2.5(d), there can be, in addition to the countable infinity of transversal intersection points in a phase slice, one or more non-transversal intersection points in that phase slice. For the study of transport it does not matter whether the intersections are transversal or nontransversal, so we henceforth refer to intersection points without distinguishing between what type they are.

LEMMA 2.4. (a) If the frequencies are all mutually incommensurate then a countable infinity of intersection points of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ in one phase slice of $\Sigma^{\theta_{\ell_0}}$ implies a countable infinity of such intersection points in all phase slices of $\Sigma^{\theta_{\ell_0}}$; (b) if one or more pairs of frequencies are commensurate then a countable infinity of intersection points of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ in one phase slice of $\Sigma^{\theta_{\ell_0}}$ implies a countable infinity of such intersection points in phase slices defined either over all of $T^{\ell-1}$ or some subset of $T^{\ell-1}$.

Proof. See Appendix 3. $\square$

These lemmas imply the following theorem.

THEOREM 2.5. Suppose there exists a point $(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1})$ for some $\bar{\nu}$ such that

- 1) $M(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) = 0$,
- 2) $\frac{\partial M}{\partial s}(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) \neq 0$.

If the frequencies are all mutually incommensurate, then $W^s(\tau_\varepsilon^a)$ intersects $W^u(\tau_\varepsilon^b)$ in a countable infinity of $(\ell - 1)$ dimensional surfaces that can be represented as graphs over $T^{\ell-1}$. If one or more pairs of frequencies are commensurate, then $W^s(\tau_\varepsilon^a)$ intersects $W^u(\tau_\varepsilon^b)$ in a countable infinity of $(\ell - 1)$ dimensional surfaces that can be represented as graphs over all $T^{\ell-1}$ or some subset of $T^{\ell-1}$.

Proof. See Appendix 3. $\square$

With a definition of PIM's, and an understanding of their geometry, we can now define and study the invariant lobe structure. To deal with the arbitrary geometries of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$, it will prove convenient to define a *lobe* as an appropriate union over phase slices of a simply connected two dimensional region in each phase slice. The first two of the following definitions thus pertain to phase slices.

DEFINITION 2.3. For $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, the intersection of the PIM τ with the phase slice $\chi(\bar{\theta}) \equiv \{(x, \theta) | \theta = \bar{\theta}\}$ defines a unique point. We refer to this point $p(\bar{\theta}) = \tau \cap \chi(\bar{\theta})$ as a *primary intersection point (PIP)*.

DEFINITION 2.4. Let $p_1(\bar{\theta})$ and $p_2(\bar{\theta})$ denote two PIP's in the phase slice $\chi(\bar{\theta})$. Then let $S[p_1(\bar{\theta}), p_2(\bar{\theta})]$ and $U[p_1(\bar{\theta}), p_2(\bar{\theta})]$ denote the segments of $W^s(\tau_\varepsilon^a) \cap \chi(\bar{\theta})$ and $W^u(\tau_\varepsilon^b) \cap \chi(\bar{\theta})$, respectively, from $p_1(\bar{\theta})$ to $p_2(\bar{\theta})$. We say the two PIP's are *adjacent* if $S[p_1(\bar{\theta}), p_2(\bar{\theta})]$ and $U[p_1(\bar{\theta}), p_2(\bar{\theta})]$ contain no other PIP's than $p_1(\bar{\theta})$ and $p_2(\bar{\theta})$.

DEFINITION 2.5. For all $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, let $p_1(\bar{\theta}), p_2(\bar{\theta})$ denote adjacent PIP's in the phase slice $\chi(\bar{\theta})$. A *lobe* $\mathcal{L}$ is an $(\ell + 1)$ dimensional region in $\Sigma^{\theta_{\ell_0}}$ such that:

- (a) for all $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, $\mathcal{L} \cap \chi(\bar{\theta})$ is the region in $\chi(\bar{\theta})$ bounded by $S[p_1(\bar{\theta}), p_2(\bar{\theta})] \cup U[p_1(\bar{\theta}), p_2(\bar{\theta})]$, and
- (b) the sign of $M(s, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \nu)$ in the interval $s(\bar{\theta}) \in [s_1(\bar{\theta}), s_2(\bar{\theta})]$, where $s_i(\bar{\theta})$ is the s value associated with $p_i(\bar{\theta})$, for $i = 1, 2$, is independent of $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$.

The motivation for condition (b) should be clear from the familiar two dimensional lobes associated with time-periodic vector fields: there are two classes of lobes, as distinguished by the sign of the Melnikov function associated with the lobe. When we form the union of regions over different phase slices, we want to choose regions from the same class.

This general definition of a lobe, as with the definition of a PIM, allows several geometrical possibilities; however, our understanding of the geometry of PIM's translates into an understanding of the geometry of lobes. Indeed, for the examples in Figure 2.5, let us choose, for all $\bar{\theta}_1 \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, $p_1(\bar{\theta}_1) = \tau_1 \cap \chi(\bar{\theta}_1)$ and $p_2(\bar{\theta}_1) = \tau_2 \cap \chi(\bar{\theta}_1)$. From the PIM's in Figure 2.5 one can visualize a lobe by remembering that its "thickness", i.e. the separation of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ normal to $W^s(\tau_0^a) \cap W^u(\tau_0^b)$, comes out of the page (see Figure 2.6). For the toral PIM's of Figures 2.5(a),(b),(c), the three dimensional lobe is a simply connected region whose boundary divides $\Sigma^{\theta_{2_0}}$ into an inside and outside; for the spiral PIM's of Figure 2.5(g), the lobe has a discontinuity at $\theta_1 = 0$; for the bifurcating PIM's of Figure 2.5(d) the lobe has no discontinuity but "pinches off" to zero volume at an isolated θ_1 value; for the PIM's of Figures 2.5(e) and (f) with one or more discontinuities in $[0, 2\pi)$, the lobe has discontinuities in $[0, 2\pi)$; for the PIM's of Figure 2.5(h) which contain gaps in T^1 , the lobes contain gaps in T^1 .

From the definition of lobes and Theorem 2.5, when the system satisfies the conditions of Theorem 2.5 then $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ form the boundaries of a countable infinity of $(\ell + 1)$ dimensional lobes in $\Sigma^{\theta_{\ell_0}}$, and hence we refer to $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ as the boundary of an *invariant lobe structure* in $\Sigma^{\theta_{\ell_0}}$ (see Figure 2.7).

Figure 2.6 Visualizing lobes by showing the suppressed dimension.

Figure 2.7 The boundaries of a three dimensional invariant lobe structure in $\Sigma^{\theta_2_0}$ for a two frequency heteroclinic case.

(2.2) Deriving a sequence of two dimensional time-dependent lobe structures from the invariant $(\ell + 1)$ dimensional lobe structure.

(2.2)(a) The autonomous system

Our eventual goal is to study the dynamics of the nonautonomous system (2.2) under the sequence of maps $\{T_\varepsilon(\cdot; n); n \in \mathbb{Z}\}$. From expression (2.4), and the comments that followed, the study of trajectories of (2.2) through x_0 at time $t = 0$ for all x_0 is equivalent to the study of the x component of the trajectories of the autonomous system (2.3) through $(x_0, \theta_{1_0}, \dots, \theta_{\ell_0})$ at time $t = 0$ for all x_0 . The coordinates of the solution $\Phi(t, t_0 = 0, x_0, \theta_{1_0}, \dots, \theta_{\ell_0})$ at time $t = \frac{2\pi}{\omega_\ell} n$ ($n \in \mathbb{Z}$) for all x_0 lie in the plane

$$\left\{ (x, \theta_1, \dots, \theta_\ell) \mid \theta_i = \theta_{i_0} + 2\pi \frac{\omega_i}{\omega_\ell} n; i = 1, \dots, \ell \right\}.$$

We shall refer to this plane as the n^{th} time slice of the phase space of the autonomous system, which is of course identical to a phase slice. To study the dynamics of (2.2), we are interested in the sequence of time slices of the autonomous system phase space, and the intersection of the invariant lobe structure with these time slices defines a sequence of two dimensional time-dependent lobe structures. For the single frequency case, each time slice is identical to the plane defined by the Poincaré section $\Sigma^{\theta_{1_0}}$. Hence a two dimensional lobe structure in this plane is immediately provided by the invariant lobe boundaries $W^s(\tau_\varepsilon^a) \cup W^u(\tau_\varepsilon^b)$ in $\Sigma^{\theta_{1_0}}$ (here the τ 's refer to points). For the two frequency case, the n^{th} time slice is a plane in the three dimensional Poincaré section $\Sigma^{\theta_{2_0}}$ defined by

$$\chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \right) = \left\{ (x, \theta_1) \mid \theta_1 = \theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \right\}$$

(see Figure 2.8). For $\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \in \bar{\mathbb{Z}}^1 \times \dots \times \bar{\mathbb{Z}}^p$ the intersection of this plane with

Figure 2.8 The n^{th} and $(n + 1)^{\text{th}}$ time slice of $\Sigma^{\theta_{2_0}}$.

the invariant lobe boundaries $W^s(\tau_\epsilon^a) \cup W^u(\tau_\epsilon^b)$ in $\Sigma^{\theta_{2_0}}$ defines the boundary of a two dimensional lobe structure:

$$\Gamma\left(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n\right) = \{W^s(\tau_\epsilon^a) \cup W^u(\tau_\epsilon^b)\} \cap \chi\left(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n\right)$$

(we henceforth assume that $\theta_{1_0} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, i.e. that the initial time slice contains a lobe structure, and hence, by invariance under P_ϵ , all time slices have a lobe structure). The intersection of the set of PIM's with the plane $\chi(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n)$ defines a countable infinity of PIP's, and a countable infinity of two dimensional lobes is defined by regions in $\chi(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n)$ bounded by segments of stable and unstable manifolds in that plane that connect adjacent PIP's. The geometry of the two dimensional lobe structure in each time slice is thus the same as the familiar invariant lobe structure of the time-periodic case. The two frequency case easily extends to the ℓ frequency case by replacing θ_1 , ω_1 , and θ_{1_0} with $\theta \equiv (\theta_1, \theta_2, \dots, \theta_{(\ell-1)})$, $\omega \equiv (\omega_1, \omega_2, \dots, \omega_{(\ell-1)})$, and $\theta_0 \equiv (\theta_{1_0}, \theta_{2_0}, \dots, \theta_{(\ell-1)_0})$, respectively. Hence the two dimensional lobe structure in $\Sigma^{\theta_{\ell_0}}$ for the ℓ frequency problem is defined for time $t = \frac{2\pi}{\omega_\ell}n$ by the boundary

$$\Gamma\left(\theta_0 + 2\pi\frac{\omega}{\omega_\ell}n\right) = \{W^s(\tau_\epsilon^a) \cup W^u(\tau_\epsilon^b)\} \cap \chi\left(\theta_0 + 2\pi\frac{\omega}{\omega_\ell}n\right),$$

where

$$\chi\left(\theta_0 + 2\pi\frac{\omega}{\omega_\ell}n\right) = \left\{(x, \theta) \mid \theta = \theta_0 + 2\pi\frac{\omega}{\omega_\ell}n\right\}.$$

We will frequently refer to the expression $\theta_0 + 2\pi\frac{\omega}{\omega_\ell}n$, so it will be useful to remember that it specifies along with θ_{ℓ_0} the phases of the perturbation at the n^{th} time slice.

(2.2) (b) The nonautonomous system

The sequence of two dimensional lobe structures defined in the autonomous system phase space will suffice to study the trajectories of (2.2). Indeed some might prefer to remain in the autonomous system phase space since, after all, the invariant manifolds that constrain the motion are in this space. However, when we study the motion of a two dimensional nonautonomous physical system of the form (2.2), there is conceptual appeal in returning to the physical space, X , rather than dealing with time slices of an $(\ell+2)$ dimensional phase space. This is a simple matter of projecting the coordinates of the autonomous solution $\Phi(t, t_0 = 0, x_0, \theta_{1_0}, \theta_{2_0}, \dots, \theta_{\ell_0})$ onto X , since the x component of this solution solves the nonautonomous system (2.2). Let $\hat{X}$ denote the projection operator onto X :

$$\begin{aligned}\hat{X} : X \times T^\ell &\mapsto X \\ (x, \theta_1, \theta_2, \dots, \theta_\ell) &\mapsto x.\end{aligned}$$

Then the boundaries of the two dimensional lobe structure in X at time $t = \frac{2\pi}{\omega_\ell}n$ are given by

$$\gamma(n) = \hat{X}\left(\Gamma\left(\theta_0 + 2\pi\frac{\omega}{\omega_\ell}n\right)\right),$$

as shown in Figure 2.9. We stress at the outset that for the sake of simple notation, when describing the projected quantities here and elsewhere, the dependence on initial phases (θ_0, θ_{t_0}) and frequencies (ω, ω_t) is to be understood implicitly. Some PIP's and two dimensional lobes are shown explicitly in Figure 2.10. Earlier (following Theorem 2.2) we stated that not all intersection sets need be continued from the $M = 0$ sets, and in Figure 2.10 we explicitly see the two types of intersection sets: PIP's, marked by crosses, which are continued from, and hence within $O(\varepsilon)$ of, the zero sets of the Melnikov function, and SIP's (secondary intersection points), marked by solid dots, which need not be within $O(\varepsilon)$ of the zero sets. Similarly, in $\Sigma^{\theta_{t_0}}$ one has, in addition to PIM's, secondary intersection manifolds (SIM's). We refer to the time-dependent lobe structure in X as a *homoclinic or heteroclinic tangle region*. We proceed to study the dynamics in this region by using the invariance under P_ε of the higher dimensional lobe structure in $\Sigma^{\theta_{t_0}}$. Note that, since in the single frequency case the time slices and hence the lobe boundaries in each slice are the same for all values of n , the projected lobe structure on X recovers the familiar invariant lobe structure of the time-periodic case.

Figure 2.9 Projecting onto X to obtain the time-dependent two dimensional lobe structures in the nonautonomous system phase space.

Figure 2.10 Showing explicitly some intersection points and two dimensional lobes in X for a given time sample. The PIP's are marked by crosses, and the SIP's are marked by dots. Each lobe is filled in differently.

3. Describing transport in the tangle regions. We proceed to describe the dynamics within the homoclinic and heteroclinic tangle regions of (2.2). The global stable and unstable manifolds in Σ^{θ_0} are by definition invariant under P_ε . Hence, by (2.5):

$$(3.1) \quad P_\varepsilon \left(\{W^s(\tau_\varepsilon^a) \cup W^u(\tau_\varepsilon^b)\} \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right) = \{W^s(\tau_\varepsilon^a) \cup W^u(\tau_\varepsilon^b)\} \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} (n+1) \right),$$

where $\theta_0 \equiv (\theta_{1_0}, \theta_{2_0}, \dots, \theta_{(\ell-1)_0})$ and $\omega \equiv (\omega_1, \omega_2, \dots, \omega_{\ell-1})$ (henceforth, when we use θ, ω , or θ_0 they are understood to represent $(\ell-1)$ dimensional vectors as defined here). The n^{th} time slice lobe boundaries are thus mapped under P_ε to the $(n+1)^{th}$ time slice lobe boundaries:

$$(3.2) \quad P_\varepsilon \left(\Gamma \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right) = \Gamma \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} (n+1) \right).$$

Projecting (3.2) onto X and using that

$$(3.3) \quad \hat{X} \circ P_\varepsilon \left(\chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right) = T_\varepsilon \left(\hat{X} \left(\chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right); n \right)$$

gives

$$(3.4) \quad T_\varepsilon(\gamma(n); n) = \gamma(n+1).$$

So the lobe boundaries in X at time $t = \frac{2\pi}{\omega_\ell} n$ map under $T_\varepsilon(\cdot; n)$ to the lobe boundaries in X at time $t = \frac{2\pi}{\omega_\ell} (n+1)$. By the invariance of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$

under P_ε , if an intersection point in the n^{th} time slice satisfies the criterion for a PIP, its image will satisfy the criterion. Hence the n^{th} time slice PIP's map to the $(n+1)^{\text{th}}$ time slice PIP's. Further, the n^{th} time slice segments of the stable and unstable manifolds that connect adjacent PIP's map to the $(n+1)^{\text{th}}$ time slice segments of stable and unstable manifolds connecting adjacent PIP's. Thus, each lobe in the n^{th} time slice maps under P_ε to a lobe in the $(n+1)^{\text{th}}$ time slice. Equivalently, projecting onto X and using (3.3), each lobe in X at time $t = \frac{2\pi}{\omega_t}n$ maps under $T_\varepsilon(\cdot; n)$ to a lobe in X at time $t = \frac{2\pi}{\omega_t}(n+1)$ (see Figure 3.1). The mapping of one lobe to another under successive time samples provides a global picture of the dynamics, i.e. of *transport in phase space*, in the tangle region (see Figure 3.2). We concentrate in this section on describing transport in terms of two dimensional lobes mapping from one to another. We facilitate this discussion with the following definition.

Figure 3.1 A lobe mapping between successive time slices of $\Sigma^{\theta_{20}}$, or, equivalently, between successive time samples of X .

Figure 3.2 A lobe mapping within the sequence of lobe structures in X for an ℓ frequency homoclinic case.

DEFINITION 3.1. We refer to the adjacent PIP's that are used to define a two dimensional lobe in a phase slice as the *bounding PIP's* of that lobe. Further, we say a lobe is *between* two arbitrary PIP's in a phase slice if its bounding PIP's lie on or between these two PIP's (relative to the curve defined by either the stable or unstable manifold in that phase slice).

(3.1) Transport in the time-periodic case

Before discussing transport in the $\ell \geq 2$ case, let us summarize the results of the single frequency case, illustrated by the OVP flow mentioned in the introduction and described more fully in Appendix 1. Figure 3.3 shows the invariant two dimensional lobe structure in the $x_2 \geq 0$ region of $\Sigma^{\theta_{10}=0}$. An upper core boundary, shown by the dashed-dotted line, is defined by

$$C = U[q_\epsilon^b, p_c] \cup S[p_c, q_\epsilon^a] \cup x_1[q_\epsilon^a, q_\epsilon^b],$$

where p_c denotes the PIP at $x_1 = 0$ and $x_1[q_\epsilon^a, q_\epsilon^b]$ denotes the segment along the x_1 axis from the left fixed point q_ϵ^a to the right fixed point q_ϵ^b (the meaning of $U[q_\epsilon^b, p_c]$

Figure 3.3 The invariant lobe structure in $\Sigma^{\theta_{10}=0}$ for the time-periodic OVP flow.

and $S[p_c, q_\varepsilon^a]$ should be clear from Section 2). One can formally choose any PIP to play the role of p_c in defining the core; however, p_c is chosen here to give a core that best approximates the unperturbed core. Regions in $\Sigma^{\theta_{10}=0}$ are *entrained into/detrained from* the core if they are mapped from outside/inside the core to inside/outside the core. The lobes between $P_\varepsilon^{-m}(p_c)$ and $P_\varepsilon^{-(m+1)}(p_c)$ are mapped under P_ε to the lobes between $P_\varepsilon^{-(m-1)}(p_c)$ and $P_\varepsilon^{-m}(p_c)$ such that they preserve orientation and relative ordering (this preservation is a consequence of the fact that the Poincaré map is obtained by discrete time sampling of trajectories of ODE's, as discussed in Wiggins [1988]). By the definition of the core boundary, the lobes between p_c and $P_\varepsilon^{-1}(p_c)$ reverse their orientation relative to the core boundary under application of P_ε ; all other lobes preserve under P_ε their orientation relative to the core boundary. Borrowing the terminology from MacKay et al. [1984], we refer to the lobes between p_c and $P_\varepsilon^{-1}(p_c)$ as *turnstile lobes*. We assume that each turnstile lobe and its image is entirely inside or entirely outside the core (which is usually the case), with the understanding that if any one of the lobes has pieces both inside and outside the core, one can redefine the lobes to be the pieces of the old lobes contained in the appropriate region (inside or outside the core). Each turnstile lobe is either entrained into or detrained from the core under P_ε ; these lobes are the *only* mechanism for transport inside and outside the core under P_ε . This motivates the following definition of *sets* of lobes: for $m \in \mathbb{Z}$, $m > 0$, $E(m)$ and $D(m)$, respectively, denote the set of lobes that are entrained into/detrained from the core upon the m^{th} iterate of P_ε ; for $m \leq 0$ they denote the set of lobes that are detrained from/entrained into the core upon the $(|m| + 1)^{\text{th}}$ iterate of P_ε^{-1} . We refer to $E(m)$ and $D(m)$ as sets of *entraining* and *detraining* lobes, respectively. From previous remarks, $E(m) \cup D(m)$ is the set of lobes between $P_\varepsilon^{-(m-1)}(p_c)$ and $P_\varepsilon^{-m}(p_c)$. For the single frequency OVP flow studied by Rom-Kedar et al. [1990], $E(m)$ and $D(m)$ each contain one lobe for all $m \in \mathbb{Z}$. For single frequency forcing the number of lobes is (generically) independent of m and of whether the lobes are entraining or detraining. In near-integrable systems, the Melnikov function can be used to determine the number of lobes in each set $E(m)$ and $D(m)$; otherwise one can numerically compute the lobe structure. We will discuss these procedures in the multiple frequency case, which will also cover the single frequency case. By their definition the sets of lobes map under P_ε according to

$$\begin{aligned}
 (3.5) \quad & P_\varepsilon(E(m)) = E(m-1) \\
 & \forall m \in \mathbb{Z}. \\
 & P_\varepsilon(D(m)) = D(m-1)
 \end{aligned}$$

Since the sets of lobes in this case are identical to individual lobes, equation (3.5) specifies the dynamics of individual lobes. In our discussion of the multiple frequency case we will explain the more complicated scenario when the sets contain more than one lobe. Since the Poincaré section is two dimensional, it is a trivial matter to project onto the “physical” space X , so we will not bother to do it here.

(3.2) Transport in the quasiperiodic case

We describe transport in a manner that is step by step analogous to the approach in the time-periodic case. We illustrate our discussion with two frequency near-integrable systems with a Melnikov function of the form

$$(3.6) \quad M(s, \theta_1, \theta_{2_0} = 0; \nu) = A_1(\mu, \omega_1) \sin(\omega_1 s + \theta_1) + A_2(\mu, \omega_2) \sin(\omega_2 s)$$

(refer to Section 2 and Appendix 1). We stress, though, that the description of transport does not rely on near-integrability. To illustrate transport using a three dimensional portrayal of the lobe structure in $\Sigma^{\theta_{2_0}=0}$ would certainly be cumbersome; we thus suppress the dimension normal to the unperturbed homoclinic/heteroclinic manifold, as done when we plotted the zero sets of the Melnikov function in the previous section. As described in that section, the set of $M = 0$ curves portray to $O(\varepsilon)$ the set of PIM's, and hence portray the dividing lines between invariant three dimensional lobes, whose thickness comes out of the page (refer back to Figure 2.6). Of course, the $M = 0$ curves give no information on the thickness of the lobes normal to the unperturbed manifolds, and they only approximate the PIM's location, but they are ideally suited for our purposes in this section, that is, to illustrate globally (i.e. for all θ_1) how lobes map from one time slice to the next by allowing one to visualize how the PIM's map from one to another with each iterate of P_ε . Consider some simple choices of A_i and ω_i shown in Figures 3.4(a)-(g). The shading and labeling of these plots will be explained as our notation is developed. We include commensurate frequencies along with incommensurate ones: even though the commensurate cases are really periodic, they are still illuminating for our two frequency prescription. Indeed, for many commensurate frequency cases, the common period of the two frequencies is much longer than the individual periods, so there is motivation to sample at one of the two periods and hence use the two frequency formalism.

We first define an invariant core in $\Sigma^{\theta_{2_0}}$, which necessitates the definition of a manifold that plays the role that p_c does in the time-periodic case. Let τ_c be an $(\ell - 1)$ dimensional PIM.

DEFINITION 3.2. A *homoclinic core boundary* $\mathcal{C}$ in $\Sigma^{\theta_{2_0}}$ is the ℓ dimensional surface $\{U[p_\varepsilon(\bar{\theta}), p_c(\bar{\theta})] \cup S[p_c(\bar{\theta}), p_\varepsilon(\bar{\theta})] | \bar{\theta} \in \bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}\}$, where $p_\varepsilon(\bar{\theta}) = \tau_\varepsilon \cap \chi(\bar{\theta})$ and $p_c(\bar{\theta}) = \tau_c \cap \chi(\bar{\theta})$.

The heteroclinic core boundary involves two heteroclinic orbits, and hence has more than one possible form. We will worry about only one of the heteroclinic orbits and hence make the following definition.

DEFINITION 3.3. A *heteroclinic core boundary* $\mathcal{C}$ in $\Sigma^{\theta_{2_0}}$ is the ℓ dimensional surface $\{U[p_\varepsilon^b(\bar{\theta}), p_c(\bar{\theta})] \cup S[p_c(\bar{\theta}), p_\varepsilon^a(\bar{\theta})] \cup O[p_\varepsilon^a(\bar{\theta}), p_\varepsilon^b(\bar{\theta})] | \bar{\theta} \in \bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}\}$, where $O[p_\varepsilon^a(\bar{\theta}), p_\varepsilon^b(\bar{\theta})]$ denotes the segment of the core boundary in the phase slice $\chi(\bar{\theta})$ due to the other heteroclinic orbit, which may be the same as the unperturbed orbit (as in the OVP flow), or may involve another heteroclinic tangle.

Figure 3.4 Portrayal and labeling, in (s, θ_1) space, of the invariant three dimensional lobes, PIM's, τ_c and $P_\epsilon^{\pm 1}(\tau_c)$ for systems with Melnikov function of the form (3.6).

Figure 3.4 (Continued) Portrayal and labeling, in (s, θ_1) space, of the invariant three dimensional lobes, PIM's, τ_c and $P_\varepsilon^{\pm 1}(\tau_c)$ for systems with Melnikov function of the form (3.6).

As done in the time-periodic case with p_c , we specialize our concern to a specific choice of τ_c , and hence of the core:

$$\tau_c = \{(x, \theta) | x = \zeta(\theta), \theta \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}\},$$

where $\zeta(\theta = \bar{\theta})$ is the x value of the PIP in the phase slice $\chi(\bar{\theta})$ such that $\mathcal{C} \cap \chi(\bar{\theta})$ *best approximates the unperturbed core in the phase slice $\chi(\bar{\theta})$* . For example, the unperturbed heteroclinic and homoclinic orbits in X of the OVP flow and Duffing equation are symmetric about their spatial midpoint ($s = 0$), and a natural choice for τ_c is for $\zeta(\theta = \bar{\theta})$ to be the x value of the PIP in the $\chi(\bar{\theta})$ plane whose s value is closest to zero. Figure 3.4 shows this choice of τ_c for the two frequency examples: for 3.4(c),(d), and (g), τ_c is a 1-torus, while in 3.4(a),(b),(e) and (f) it is a piecewise continuous set of segments of spirals. Note that, for visual clarity, the piecewise continuous sets are joined by verticle dashed lines at the points of discontinuity: it should be understood that τ_c is really single-valued everywhere. The invariant core for this choice of τ_c immediately defines a two dimensional core in any phase slice, $\chi(\bar{\theta}), \bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}$, and equivalently in X for any time sample:

$$(3.7) \quad \begin{aligned} C(\bar{\theta}) &= \mathcal{C} \cap \chi(\bar{\theta}) \\ c(n) &= \hat{X} \left(C \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right). \end{aligned}$$

We stress two fundamental points regarding the invariant core and the resulting sequence of cores. First, since the invariant core boundary in general may be piecewise continuous in θ , it does not necessarily divide the Poincaré section $\Sigma^{\theta_{\ell_0}}$ into an inside and outside. This would present a problem for more general systems, such as the k degree-of-freedom systems discussed in Wiggins [1990]. *However, due to the trivial nature of the dynamics in θ (everything maps from phase slice to phase slice), all that is needed of the invariant core boundary is that it divides each phase slice into an inside and an outside, which our piecewise continuous core boundary indeed does.* By their definitions, for each n , $C(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n)$ divides its two dimensional time slice into an inside and an outside, and $c(n)$ divides X into an inside and outside. Second, though τ_c and hence the invariant core boundary may in certain cases seem complicated and geometrically unappealing, by its very definition it ensures that for each n the core $c(n)$ most closely resembles the unperturbed core boundary. The aim then in the definition of τ_c is geometric appeal in each phase slice.

We now define entrainment and detrainment with respect to the core sequence.

DEFINITION 3.4. The regions in the n^{th} time slice $\chi(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n)$ of $\Sigma^{\theta_{\ell_0}}$ that are *entrained into/detrained from the core under P_ϵ* are the regions in that slice that are outside/inside $C(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n)$ and are mapped to inside/outside $C(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} (n+1))$ in the $(n+1)^{th}$ time slice. Equivalently, the regions in X that are *entrained into/detrained from the core under $T_\epsilon(\cdot; n)$* are the regions in X that are outside/inside $c(n)$ and are mapped to inside/outside $c(n+1)$.

We have defined entrainment and detrainment with respect to a core that changes shape and area from one time sample to the next, which is markedly different from the time-periodic case, where the core corresponds to a fixed spatial region.

Thus, before continuing we should motivate such a definition. Imagine for example a two dimensional material blob that pulsates in some quasiperiodic fashion, as shown in the sequence of snapshots in Figure 3.5(a). We do not want to say there is any entrainment or detrainment here. Though we could define a fixed spatial core and monitor the back and forth motion across this core due to the pulsation, we prefer to define a time-dependent core boundary as the material blob boundary, and say that there is no entrainment/detrainment with respect to this pulsating core. Now consider a material blob which, on top of a net pulsation, shows lobe formations that one would like to associate with entrainment/detrainment, as shown in Figure 3.5(b). Again, here our preferred way to monitor entrainment/detrainment is with respect to a pulsating core boundary, as shown by the dashed line in Figure 3.5(b), since this prevents us from counting net pulsation as entrainment/detrainment.

Figure 3.5 (a) A pulsating material blob and (b) a pulsating and lobe-forming material blob. A defined core boundary is shown by the dashed line.

The obvious complication is: how do we distinguish between “overall pulsation” and “lobe formation”? When a material blob deforms from one time sample to the next, to distinguish between the “reversible” pulsation deformations and the “irreversible” lobe-like deformations, we need to know the long-time behavior of the material (see Figure 3.6(a)). An essential feature of our lobe structure comes into play here: the invariance under P_ε of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ implies that these manifolds contain the long term information needed to provide a natural way of distinguishing between net pulsation and lobe formation for system (2.2). For example, returning to the OVP flow, in the region in X at time $t = \frac{2\pi}{\omega_2}(n+1)$ where $T_\varepsilon(c(n); n)$ shows lobe-like deformations, the curve $\hat{X} \left(W^s(\tau_\varepsilon^a) \cap \chi \left(\theta_{10} + 2\pi \frac{\omega_1}{\omega_2}(n+1) \right) \right)$ pro-

vides the natural dividing line between “inside” and “outside” since by definition the points on $W^s(\tau_\varepsilon^a)$ asymptote to τ_ε^a (see Figure 3.6(b)); fluid particles immediately to the left of the highlighted $\hat{X} \left(W^s(\tau_\varepsilon^a) \cap \chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} (n+1) \right) \right)$ segment are destined to move further and further in the $-x_1$ direction (their x_1 component eventually going to $-\infty$), while fluid particles immediately to the right are destined to move further and further in the $+x_1$ direction until they are rotated around the core to be either detrained or rotated around the core again, and so on.

Figure 3.6 (a) A material blob deforms; the dashed lines show possible dividing lines between “inside” and “outside”. (b) The stable manifold (the dashed line) provides a natural dividing line where the lobes form between “inside” and “outside”.

The invariant manifolds $W^s(\tau_\varepsilon^a), W^u(\tau_\varepsilon^b)$ thus provide a natural way to define a time-dependent core in X with which to monitor entrainment/detrainment for the sequence of nonautonomous maps $\{T_\varepsilon(\cdot; n); n \in \mathbb{Z}\}$. Though for concreteness we have phrased our motivation in the context of a fluid, we do not wish the reader to lose sight of the significance of a time-dependent core in relation to the fundamental notion of bounded and unbounded motion in non-integrable systems. Indeed points within the core may escape, and hence their motion may not be bounded; however,

we shall soon see that they can escape only via the detraining turnstile lobes, and hence, when detrainment leads to unbounded motion (as is the case in the OVP flow, but not in the Duffing oscillator), it is meaningful to regard the time-dependent core boundary as the *last frontier* between bounded and unbounded motion and to view the points within the core as bounded until they escape via the detraining turnstile lobes to unbounded motion.

With a motivated definition of a core sequence, we address the global dynamics associated with lobes mapping from one time slice to the next. The two dimensional lobes in $\chi(\bar{\theta})$, $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, between $P_\epsilon^{-m}(\tau_c) \cap \chi(\bar{\theta})$ and $P_\epsilon^{-(m+1)}(\tau_c) \cap \chi(\bar{\theta})$ are mapped under P_ϵ to the lobes in $\chi\left(\bar{\theta} + 2\pi \frac{\omega}{\omega_t}\right)$ between $P_\epsilon^{-(m-1)}(\tau_c) \cap \chi\left(\bar{\theta} + 2\pi \frac{\omega}{\omega_t}\right)$ and $P_\epsilon^{-m}(\tau_c) \cap \chi\left(\bar{\theta} + 2\pi \frac{\omega}{\omega_t}\right)$ such that they preserve their orientation and relative ordering (this preservation is again a consequence of the fact that P_ϵ is obtained by discrete time sampling of the trajectories of ODE's). By the definition of the invariant core boundary, the lobes in $\chi(\bar{\theta})$ between $\tau_c \cap \chi(\bar{\theta})$ and $P_\epsilon^{-1}(\tau_c) \cap \chi(\bar{\theta})$ reverse their orientation relative to the core boundary under the application of P_ϵ ; all other lobes in that phase slice preserve under P_ϵ their orientation relative to the core boundary. Hence we refer to the lobes in $\chi(\bar{\theta})$ between $\tau_c \cap \chi(\bar{\theta})$ and $P_\epsilon^{-1}(\tau_c) \cap \chi(\bar{\theta})$ as the *turnstile lobes* of that phase slice. Again we assume that each two dimensional turnstile lobe and its image is entirely inside or outside the core, with the understanding that we can redefine the lobes as in the time-periodic case if the need arises. Hence the turnstile lobes of each phase slice are either entrained into or detrained from the core under P_ϵ ; again we stress these lobes are the *only* mechanism for entrainment and detraining relative to the sequence of cores. Previous investigators have studied escape from bounded to unbounded motion in low dimensional chaotic systems by focusing on such notions as overlap of neighboring resonance zones (see for example Goggin and Milonni [1988a] and [1988b]) and cantori as partial barriers to transport (see MacKay and Meiss [1988]). These methods provide only an approximate framework for escape to unbounded motion, in contrast to a theory dealing with the lobes of the homoclinic/heteroclinic tangles, which provides a framework for exact computation of escape from the core, and which, as we shall see, can offer simple explanations of transport phenomena that heretofore have had insufficient explanation. The behavior of the turnstile lobes motivates the following definition of *sets* of two dimensional lobes in a given phase slice, and hence invariant $(\ell + 1)$ dimensional lobes in $\Sigma^{\theta_{\ell_0}}$.

DEFINITION 3.5. For $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}$, $m \in \mathbb{Z}$, $m > 0$, $E(m, \bar{\theta})$ and $D(m, \bar{\theta})$ denote, respectively, the set of lobes in $\chi(\bar{\theta})$ that are entrained into/detrained from the core upon the m^{th} iterate of P_ϵ ; for $m < 0$ they denote the set of lobes in $\chi(\bar{\theta})$ that are detrained from/entrained into the core upon the $(|m| + 1)^{\text{th}}$ iterate of P_ϵ^{-1} . We refer to $E(m, \bar{\theta})$ and $D(m, \bar{\theta})$ as sets of *entraining* and *detraining* lobes, respectively. Invariant $(\ell + 1)$ dimensional lobes in $\Sigma^{\theta_{\ell_0}}$ are defined by:

$$\begin{aligned} E(m) &= \{E(m, \bar{\theta}) | \bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}\} \\ D(m) &= \{D(m, \bar{\theta}) | \bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}\}. \end{aligned}$$

From previous remarks, $E(m, \bar{\theta}) \cup D(m, \bar{\theta})$ denotes the set of lobes in $\chi(\bar{\theta})$ between $P_\varepsilon^{-(m-1)}(\tau_c) \cap \chi(\bar{\theta})$ and $P_\varepsilon^{-m}(\tau_c) \cap \chi(\bar{\theta})$. In the multiple frequency case the number of lobes in $E(m, \bar{\theta})$ and $D(m, \bar{\theta})$, denoted $\mathcal{N}_E(m, \bar{\theta})$ and $\mathcal{N}_D(m, \bar{\theta})$, respectively, in general depend on $m, \bar{\theta}$, and whether the set is entraining or detraining. At times we will want to refer to individual lobes, which we denote by $E(m, \bar{\theta}; i)$, $D(m, \bar{\theta}; i)$. Hence for all $m \in \mathbb{Z}$, $\bar{\theta} \in \bar{\mathbb{Z}}^1 \times \cdots \times \bar{\mathbb{Z}}^p \subseteq T^{\ell-1}$:

$$\begin{aligned} E(m, \bar{\theta}) &= E(m, \bar{\theta}; 1) \cup \cdots \cup E(m, \bar{\theta}; \mathcal{N}_E(m, \bar{\theta})) \\ D(m, \bar{\theta}) &= D(m, \bar{\theta}; 1) \cup \cdots \cup D(m, \bar{\theta}; \mathcal{N}_D(m, \bar{\theta})). \end{aligned}$$

From previous comments on the dynamics:

$$(3.8) \quad \begin{aligned} \mathcal{N}_E(m, \bar{\theta}) &= \mathcal{N}_E \left(1, \bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} (m-1) \right) \\ \mathcal{N}_D(m, \bar{\theta}) &= \mathcal{N}_D \left(1, \bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} (m-1) \right). \end{aligned}$$

Hence, determining the number of turnstile lobes in each phase slice determines the number of lobes in each set of lobes for each phase slice; further it tells us how many lobes in each phase slice are entrained into/detrained from the core under P_ε . This number can be determined visually in each phase slice for the examples in Figure 3.4 by the portrayals of τ_c and $P_\varepsilon^{-1}(\tau_c)$; however, one need not determine these quantities from plots of PIM's. In the near-integrable case, the Melnikov function provides a computational tool for determining these quantities. If $\mathcal{N}(\bar{\theta})$ is the number of zeros of the Melnikov function in $\chi(\bar{\theta})$ between (and not counting) τ_c and $P_\varepsilon^{-1}(\tau_c)$, then generically

$$\mathcal{N}_E(\bar{\theta}) + \mathcal{N}_D(\bar{\theta}) = \mathcal{N}(\bar{\theta}) + 1,$$

where for simplicity of notation we have written $\mathcal{N}_E(1, \bar{\theta}) \equiv \mathcal{N}_E(\bar{\theta})$ and $\mathcal{N}_D(1, \bar{\theta}) \equiv \mathcal{N}_D(\bar{\theta})$. If $\mathcal{N}(\bar{\theta}) + 1$ is even then $\mathcal{N}_E(\bar{\theta}) = \mathcal{N}_D(\bar{\theta}) = (\mathcal{N}(\bar{\theta}) + 1)/2$. If $\mathcal{N}(\bar{\theta}) + 1$ is odd then one of $\mathcal{N}_E(\bar{\theta}), \mathcal{N}_D(\bar{\theta})$ equals $\mathcal{N}(\bar{\theta})/2$, the other $\mathcal{N}(\bar{\theta})/2 + 1$; which one equals which can be determined by the sign of the Melnikov function between the zeros. If $s_c(\bar{\theta})$ denotes the s value of $\tau_c \cap \chi(\bar{\theta})$, then $\left(s_c \left(\bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} \right) + \frac{2\pi}{\omega_\ell} \right)$ is the s value of $P_\varepsilon^{-1}(\tau_c) \cap \chi(\bar{\theta})$, and hence one determines $\mathcal{N}(\bar{\theta})$ by computing the number of zeros of $M(s, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \nu)$ between $s_c(\bar{\theta})$ and $\left(s_c \left(\bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} \right) + \frac{2\pi}{\omega_\ell} \right)$. When systems are not near-integrable, one can resort to explicit simulation of two dimensional lobe structures in a sequence of time slices. We discuss and present computer simulations of lobe structures in Section (3.3) and explain the sampling methods for simulation in Appendix 4. Once one chooses a PIP $\in \tau_c$ for the n^{th} time slice, $\tau_c \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$, one easily determines the PIP $\in P_\varepsilon^{-1}(\tau_c)$ by applying P_ε^{-1} to the PIP $\in \tau_c$ in the $(n+1)^{\text{th}}$ time slice, $\tau_c \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} (n+1) \right)$.

The sets of lobes map according to

$$(3.9) \quad \begin{aligned} P_\varepsilon(E(m, \bar{\theta})) &= E \left(m-1, \bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} \right) \\ P_\varepsilon(D(m, \bar{\theta})) &= D \left(m-1, \bar{\theta} + 2\pi \frac{\omega}{\omega_\ell} \right). \end{aligned}$$

The action of P_ϵ on m follows by the definition of the lobes, and is hence not very informative; this is because we have chosen a lobe labeling that incorporates the dynamics, which will facilitate later discussions of transport. Note though that a determination of the number of turnstile lobes in each phase slice in principle specifies the number of lobes in each set of lobes, and thus along with equation (3.9) specifies how individual lobes map from one to another; in particular, it specifies how many lobes are entrained and detrained with each iterate of P_ϵ . For all useful transport expressions, however, we will be interested in how *sets* of lobes map from one to another.

Projecting onto X immediately labels individual lobes and sets of lobes in X for any time sample:

$$(3.10) \quad \begin{aligned} e(m, n; i) &= \widehat{X} \left(E \left(m, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n; i \right) \right) \\ d(m, n; i) &= \widehat{X} \left(D \left(m, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n; i \right) \right) \end{aligned}$$

and

$$\begin{aligned} e(m, n) &= e(m, n; 1) \cup \dots \cup e(m, n; \mathcal{N}_e(m, n)) \\ d(m, n) &= d(m, n; 1) \cup \dots \cup d(m, n; \mathcal{N}_d(m, n)), \end{aligned}$$

where $\mathcal{N}_e(m, n) = \mathcal{N}_E \left(m, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$, $\mathcal{N}_d(m, n) = \mathcal{N}_D \left(m, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$ (see Figure 3.7).

Figure 3.7 Portraying the core and labeling the sets of lobes in X for the n^{th} time sample of the two frequency OVP flow $(x_c(n) = \widehat{X}(\tau_c \cap \chi(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n))$, $x_{h,\epsilon}^i(n) = \widehat{X}(\tau_\epsilon^i \cap \chi(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n))$ for $i = a, b$, and $T_\epsilon^{-1}(\cdot; n)$ maps from $t = \frac{2\pi}{\omega_2}(n+1)$ to $t = \frac{2\pi}{\omega_2}n$).

The projection further specifies how sets of lobes map from one to another under the sequence of maps $\{T_\epsilon(\cdot; n) | n \in \mathbb{Z}\}$:

$$(3.11) \quad \begin{aligned} T_\epsilon(e(m, n); n) &= e(m-1, n+1) \\ T_\epsilon(d(m, n); n) &= d(m-1, n+1). \end{aligned}$$

Note that, for a general bi-infinite sequence of nonrepeating maps, to specify the number of turnstile lobes at each time sample necessitates determining a countable infinity of numbers. However, multiple frequency perturbations offer a fundamental simplification due to the compactness of $T^{\ell-1}$: if one determines the number of turnstile lobes in each phase slice $\chi(\bar{\theta})$ for all $\bar{\theta} \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}$, then by (3.10) this determines the number of turnstile lobes for each map in the nonrepeating sequence.

Figure 3.4 illustrates our description of transport. The defined τ_c and a few iterates of $P_\epsilon^{\pm 1}$ on τ_c are portrayed; the number of turnstile lobes for each phase slice is immediately deduced by looking at the number of PIP's between τ_c and $P_\epsilon^{-1}(\tau_c)$ in that phase slice. The $\omega_1 = \omega_2 \equiv \omega$ ($A_1 = A_2 = 1$ until otherwise stated) case in Figure 3.4(a) shows a single pair of entraining/detraining turnstile lobes for all $\theta_1 \in [0, 2\pi)$ (except for the isolated value $\theta_1 = \pi$), and the sets of lobes map according to

$$\begin{aligned} P_\epsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + 2\pi) = E(m-1, \theta_1) \\ &\quad \forall \theta_1 \in [0, 2\pi). \\ P_\epsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + 2\pi) = D(m-1, \theta_1) \end{aligned}$$

One can easily visualize the lobes mapping under P_ϵ in Figure 3.4(a) by recognizing that $P_\epsilon^{-1}(\tau_c)$ maps to τ_c ; the mapping is portrayed in all of Figures 3.4 (a) to (g) by showing successive images of a single two dimensional lobe, represented in the (s, θ_1) plane by a dot between the lobe's two bounding PIP's (everything in the (s, θ_1) plane moves rigidly with this dot). The two dimensional lobes of Figure 3.4(a) remain in the same phase slice and thus map within an invariant two dimensional lobe structure. Hence for $\theta_{1_0} = 0$ we recover the time-periodic result given earlier for the OVP flow. Exact cancellation of two sine functions out of phase by π gives the vertical zero set at $\theta_1 = \pi$, and, as we shall see in Section 4, lobe areas vary smoothly and monotonically from zero at $\theta_1 = \pi$ to maxima at $\theta_1 = 0$ and 2π .

The $\omega_1 = 2\omega_2 \equiv 2\omega$ case of Figure 3.4(b) has two pairs of entraining/detraining turnstile lobes for all θ_1 (since we are sampling at the slower frequency), except for the isolated values where the PIM's intersect at $\theta_1 = \pi/2, 3\pi/2$ (we henceforth do not keep pointing out the isolated values). The sets of lobes map according to:

$$\begin{aligned} P_\epsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + 2\pi \cdot 2) = E(m-1, \theta_1) \\ &\quad \forall \theta_1 \in [0, 2\pi). \\ P_\epsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + 2\pi \cdot 2) = D(m-1, \theta_1) \end{aligned}$$

As in the previous figure, P_ϵ acts on the indice m , but leaves (θ_1, θ_2) unchanged, so that the two dimensional lobes remain in the same phase slice. This is an example

of a general property of the commensurate frequency case: if $\omega_1/\omega_2 = i/j$, $i \in Z$, $j \in Z$, then there exists an $N \in Z$ (for example $N = i \cdot j$) such that P_ε^N leaves (θ_1, θ_2) unchanged. This operator P_ε^N thus acts on the two dimensional lobes the same way as does the familiar Poincaré map derived from time-periodic vector fields. The price of this correspondence is that the time between iterates of P_ε^N , $\Delta t = N2\pi/\omega_2$, can be extremely long. The $2\omega_1 = \omega_2 \equiv 2\omega$ case of Figure 3.4(c) is the same problem as Figure 3.4(b) sampled at the faster frequency (we purposely consider both samplings for comparison). Now there is only one pair of entraining/detraining turnstile lobes for all $\theta_1 \in [0, 2\pi)$, and the transport equations are given by

$$\begin{aligned} P_\varepsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + \pi) \\ &\quad \forall \theta_1 \in [0, 2\pi). \\ P_\varepsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + \pi) \end{aligned}$$

The lobes now map back and forth between two phase slices separated by $\Delta\theta_1 = \pi$. In general for an arbitrary sampling of a commensurate frequency case, the lobes will visit a finite number of θ_1 values in $[0, 2\pi)$.

Figure 3.4(d) shows an incommensurate frequency case $\omega_1 = g\omega_2 \equiv \omega$, where g is the golden mean, so the sequence of nonautonomous maps is now truly nonrepeating. The PIM's appear qualitatively quite similar to the previous commensurate example. In fact, for all θ_1 values there is one pair of entraining and detraining turnstile lobes; hence a nonrepeating map sequence can have underlying PIM's with such simple geometry that the lobes map from one to another in a very simple manner. The lobe sets map according to

$$\begin{aligned} P_\varepsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + 2\pi g) \\ &\quad \forall \theta_1 \in [0, 2\pi). \\ P_\varepsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + 2\pi g) \end{aligned}$$

The essential difference in transport compared with the previous example is that the time slices will now visit $\theta_1 \in [0, 2\pi)$ densely, as will always be the case with incommensurate frequencies. Figure 3.4(e) shows the case $g\omega_1 = \omega_2 \equiv \omega$, where the number of turnstile lobes depends on θ_1 :

$$\begin{aligned} N_E(\theta_1) &= \begin{cases} 2 & \theta_1 \in \left[0, \frac{\pi(1+g)}{2}\right) \\ 1 & \theta_1 \in \left[\frac{\pi(1+g)}{2}, \frac{\pi(5-3g)}{2}\right) \\ 2 & \theta_1 \in \left[\pi\left(\frac{5-3g}{2}\right), 2\pi\right) \end{cases} \\ N_D(\theta_1) &= \begin{cases} 2 & \theta_1 \in \left[0, \frac{\pi(3-g)}{2}\right) \\ 1 & \theta_1 \in \left[\frac{\pi(3-g)}{2}, \frac{\pi(7-5g)}{2}\right) \\ 2 & \theta_1 \in \left[\pi\left(\frac{7-5g}{2}\right), 2\pi\right) \end{cases} \end{aligned}$$

The lobe sets map according to

$$\begin{aligned} P_\varepsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + 2\pi/g) \\ P_\varepsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + 2\pi/g) \end{aligned} \quad \forall \theta_1 \in [0, 2\pi).$$

The last two examples deal with simpler and yet generic examples of unequal Melnikov function amplitudes, and hence PIM's that are non-intersecting tori or spirals. Figure 3.4(f) shows a large amplitude, slow frequency term modulated by a small amplitude, fast frequency term ($A_1 = (2\pi A_2) = 1, \omega_1 = \frac{\omega_2}{2\pi} \equiv \omega$). The lobe structure is essentially determined by the large amplitude component, and we sample here at the faster frequency, so only a small set of θ_1 values actually have turnstile lobes. Here we have

$$\begin{aligned} \mathcal{N}_E(\theta_1) &= \begin{cases} 1 & \theta_1 \in [0.5, 1.5) \\ 0 & \text{otherwise} \end{cases} \\ \mathcal{N}_D(\theta_1) &= \begin{cases} 1 & \theta_1 \in [2\pi - 2.5, 2\pi - 1.5) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The lobe sets map according to

$$\begin{aligned} P_\varepsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + 1) \\ P_\varepsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + 1) \end{aligned} \quad \forall \theta_1 \in [0, 2\pi).$$

Figure 3.4(g) is the previous system sampled at the slower frequency (i.e. $(2\pi A_1) = A_2 = 1, \frac{\omega_1}{2\pi} = \omega_2 \equiv \omega$). Now the PIM's are 1-tori, rather than segments of spirals, and there is again a single pair of entraining/detraining turnstile lobes for all $\theta_1 \in [0, 2\pi)$. The lobe sets map according to:

$$\begin{aligned} P_\varepsilon(E(m, \theta_1)) &= E(m-1, \theta_1 + (2\pi)^2) \\ P_\varepsilon(D(m, \theta_1)) &= D(m-1, \theta_1 + (2\pi)^2) \end{aligned} \quad \forall \theta_1 \in [0, 2\pi).$$

For all the examples $\mathcal{N}_E(\bar{\theta}_1), \mathcal{N}_D(\bar{\theta}_1)$ are determined for all $\bar{\theta}_1 \in [0, 2\pi)$, and this directly determines $\mathcal{N}_E\left(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n\right), \mathcal{N}_D\left(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n\right)$, and hence $\mathcal{N}_e(n), \mathcal{N}_d(n)$, for all $n \in \mathbb{Z}$. Hence the number of entraining and detraining turnstile lobes is determined for *all* time samples, due to the compactness of $\bar{\mathcal{Z}}^1 \times \dots \times \bar{\mathcal{Z}}^p \subseteq T^{\ell-1}$.

In summary then, the transport in the tangle region of the nonautonomous system (2.2) is understood in terms of the two dimensional lobes of the n^{th} time slice of $\Sigma^{\theta_{1_0}}$ mapping under P_ε to the lobes of the $(n+1)^{th}$ time slice, and we now know how to describe such a map.

(3.9) Computer simulation of lobe boundaries and qualitative discussion of lobe transport

As in the time-periodic case, there is for quasiperiodic vector fields a generic picture of transport associated with lobes in phase space mapping from one to another in the manner just described. Some essential features of the picture can be described in a precise way, and in the next three sections we will address some of these features. A good prelude to such a study is the computer simulation of the lobe structure. This provides a direct portrayal of transport in the tangle region, and it will make obvious some features of transport that will be studied in later sections. Our sampling method used for computer simulation, which we refer to as a *double phase slice method*, contrasts with a previous suggestion by Moon and Holmes [1985] for analyzing the dynamics under quasiperiodic vector fields, a double Poincaré map method which essentially attempts to treat the system as periodic. We defer a discussion of this procedure to Appendix 4. We present simulations of two cases of the two frequency OVP flow, one with a 1:2 frequency ratio in the forcing term (the oscillating strain rate field),

$$(3.12) \quad \psi_{forcing} = 0.12x_1x_2\{2 \cdot 0.4 \sin(2t) + 4 \cdot 1.05 \sin(4t)\},$$

and one with a $1 : g^{-1}$ frequency ratio,

$$(3.13) \quad \psi_{forcing} = 0.12x_1x_2\{2 \cdot 0.4 \sin(2t + \theta_{1_0}) + 2g^{-1} \cdot 0.95 \sin(2g^{-1}t)\},$$

where $\theta_{1_0} = 2\pi[8g - 4] \approx 5.933$ ($\psi_{forcing}$ is the stream function of the forcing term for the *nondimensional* equations A1.1 in Appendix 1). For each forcing term, the arguments of the first and second sine functions will be referred to as θ_1 and θ_2 , with frequencies ω_1 and ω_2 , respectively. We focus on the heteroclinic example here because it has a larger core region with which to portray lobes. Further, we include a commensurate frequency example with a simple ratio because it contains many of the essential features found in the case of incommensurate frequencies, and its simplicity highlights these features. For comparison with the two frequency case, Figure 3.8 shows some lobes of two single frequency cases: (a) shows the lobes in X at $t = n\pi$ for the case $\psi_{forcing} = 0.12x_1x_2\{2 \cdot 0.4 \sin(2t)\}$, and (b) shows the lobes in X at $t = n\frac{\pi}{2}$ for the case $\psi_{forcing} = 0.12x_1x_2\{4 \cdot 1.05 \sin(4t)\}$. Figure 3.9 shows for the perturbation (3.12) lobes in two phase slices of $\Sigma^{\theta_{2_0}=0}, \chi(\bar{\theta}_1 = 0)$ and $\chi(\bar{\theta}_1 = \pi)$. The stable and unstable manifold separations in the phase slice $\chi(\bar{\theta}_1 = 0)$ are seen to be essentially a superposition of the manifold separations of the two lobe structures in Figure 3.8, as predicted by the linear Melnikov approximation of manifold separations in equation (2.7), valid for sufficiently small perturbations. Figures 3.10 and 3.11 show for the perturbations in equation (3.12) and (3.13), respectively, sequences of four time samples and how the lobes of fluid map within these sequences.

The time-dependent lobe structure in X for any time $t = \frac{2\pi}{\omega_2}n$ has many similarities with the invariant two-dimensional lobe structure of the time-periodic case. There is a reasonable looking core region and a bi-infinite sequence of lobes that

Figure 3.8 Some lobes from the OVP lobe structure in X at times (a) $t = n\pi$ for $\psi_{forcing} = 0.12x_1x_2\{2 \cdot 0.4 \sin(2t)\}$ and (b) $t = n\frac{\pi}{2}$ for $\psi_{forcing} = 0.12x_1x_2\{4 \cdot 1.05 \sin(4t)\}$. The entraining lobes are shaded.

Figure 3.9 Some lobes from the OVP lobe structure in the phase slices $\chi(\bar{\theta}_1 = 0)$ and $\chi(\bar{\theta}_1 = \pi)$ of $\Sigma^{\theta_{20}=0}$ for $\psi_{forcing}$ given by equation (3.12). The entraining lobes are shaded.

Figure 3.10 A sequence of four time samples of the OVP flow forced according to equation (3.12). Four material lobes (entraining lobes) are shaded so that we can monitor their transport. The core boundaries are marked by dashed lines.

extend from the inside and outside of the core boundary, each of which gets thinner and longer as its base (i.e. where it extends from the core) gets closer to the points that lie on the normally hyperbolic invariant 1-tori. The thinner and longer the entraining/detraining lobes that extend from inside the core boundary are, the more they wind about the core and wrap around other entraining/detraining lobes and intersect other detraining/entraining lobes (this happens since $W^s(\tau_\varepsilon^a), W^u(\tau_\varepsilon^b)$ cannot intersect themselves, but can intersect each other). That the time-dependent lobe structure in X has the same geometrical constraints as the invariant lobe structure of the time-periodic case will prove instrumental in easily extending some lobe intersection analysis to be discussed in Sections 4 and 6. A notable difference from the single frequency OVP flow, however, is the variation of lobe areas, and, as we shall see, this variation can have a significant effect on transport, for it gives one added freedom to enhance or diminish features of transport over finite time scales.

Figure 3.11 A sequence of four time samples of the OVP flow forced according to equation (3.13). Three material lobes (de-training lobes) are shaded so that we can monitor their transport. The core boundaries are marked by dashed lines.

A sequence of snapshots of the lobe structure in X shows notable differences from the time-periodic case. Of course, the lobe structure now varies with each sample time $t = \frac{2\pi}{\omega_2}n$. The core region varies in shape and area with each sample, as do lobe areas relative to their ordering with respect to the PIP $\in \tau_c$ (which does not contradict that lobes of fluid conserve area as they map from one lobe structure to the next). In the 1 : 2 frequency ratio case (sampled at the faster frequency), the lobe structure oscillates with successive time samples between two forms: a “tall, skinny” one ($\theta_1 = 0$) and a “short, fat” one ($\theta_1 = \pi$). In the 1 : g^{-1} frequency ratio case (sampled at the faster frequency) the lobe structure varies in a nonrepeating fashion with successive time samples. As should be clear from the previous sections, the key to understanding the time-dependent structure is to recognize that it is the intersection of a time slice with an invariant structure in a higher dimensional Poincaré section; for example, we stress how there are no fixed

points in the two dimensional structure, but rather points on a normally hyperbolic invariant 1-torus in $\Sigma^{\theta_{20}}$.

As lobes of fluid map in X from one lobe structure to the next, their behavior is qualitatively similar to that found in the time-periodic case. They stretch in one direction and contract in another to produce the two essential features of lobe transport found in the time-periodic case: entrainment and detrainment (back and forth motion across boundaries that were barriers to transport in the unperturbed case) and repeated stretching and folding to produce chaotic advection. In Figure 3.12 we explicitly show the entrainment and detrainment process for the time-dependent core associated with the system described by equation (3.12). These features need to be understood in the framework of the sequence of nonautonomous maps on X , which will be the focus of the next three sections. An essential aspect of the dynamics to recognize right away, however, is that, because the repeated stretching and folding of lobes carries over to the quasiperiodic case (see Section 5), material curves in the tangle region tend to get “attracted” to $W^u(\tau_\epsilon^b)$, just like in the time-periodic case (refer back to Section 1). Since $\hat{X}\left(W^u(\tau_\epsilon^b) \cap \chi\left(\theta_{1_0} + 2\pi\frac{\omega_1}{\omega_2}n\right)\right)$ varies with n , the “attracting” structure in X is time-dependent. The time-dependent lobe structure is thus the dominant structure by which to understand motion in the tangle region, and it will allow us to embrace rather than avoid the time-dependent nature of the more complicated transport issues under quasiperiodic perturbations.

Figure 3.12 Showing explicitly entrainment and detrainment between the $t = 2\pi/\omega_2$ sample and the $t = 2 \cdot 2\pi/\omega_2$ sample of the OVP flow with forcing given by equation (3.12). The dashed line is $c(1)$ in the $t = 2\pi/\omega_2$ sample and $T(c(1); 1)$ in the $t = 2 \cdot 2\pi/\omega_2$ sample. The dashed-dotted line is $c(2)$.

We wish to stress the notion of a time-dependent attractor. Often a physical system is a point in an autonomous system phase space, and so there is little motivation to conceptualize anything but the invariant attractor in the autonomous system phase space. In chaotic advection, however, where physical space at sample times is a time slice of the autonomous system phase space, the time-dependent “attractor” associated with each time slice does have physical meaning.

4. Entrainment and detrainment.

(4.1) *Instantaneous and average flux*

Several features of entrainment and detrainment can be studied via the lobe structures. We start with lobe areas, which provide a measure of *flux* across the core boundaries. For simplicity of discussion, let us first consider Hamiltonian perturbations, i.e. sequences of area-preserving maps on X .

DEFINITION 4.1(a). The *instantaneous flux* associated with P_ε acting on the phase slice $\chi(\bar{\theta})$, denoted by $\phi_E(\bar{\theta})$ and $\phi_D(\bar{\theta})$ for entrainment and detrainment, respectively, is the area of the regions in the phase slice that are entrained into/detrained from the core under P_ε , divided by the sampling period.

Hence expressions for instantaneous flux associated with P_ε acting on the n^{th} time slice are given by

$$(4.1) \quad \begin{aligned} \phi_E \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) &= \frac{\omega_\ell}{2\pi} \mu \left(E \left(1, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right) \\ \phi_D \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) &= \frac{\omega_\ell}{2\pi} \mu \left(D \left(1, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right), \end{aligned}$$

where $\mu(\cdot)$ denotes the area of the set of lobes within the parenthesis. Recall that Section 3 explained how to identify the turnstile lobes in each time slice.

DEFINITION 4.1(b). The *instantaneous flux* associated with $T_\varepsilon(\cdot; n)$ acting on X , denoted by $\phi_e(n)$ and $\phi_d(n)$ for entrainment and detrainment, respectively, is the area of the regions in X that are entrained into/detrained from the core under $T_\varepsilon(\cdot; n)$, divided by the sampling period.

Hence the expressions for instantaneous flux associated with $T_\varepsilon(\cdot; n)$ acting on X are given by

$$(4.2) \quad \begin{aligned} \phi_e(n) &= \frac{\omega_\ell}{2\pi} \mu(e(1, n)) \\ \phi_d(n) &= \frac{\omega_\ell}{2\pi} \mu(d(1, n)). \end{aligned}$$

Since $e(i, n) = \widehat{X} \left(E \left(i, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right)$ and $d(i, n) = \widehat{X} \left(D \left(i, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right)$, $\phi_e(n) = \phi_E \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$ and $\phi_d(n) = \phi_D \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$, so for the remainder of this section we deal just with slices of the autonomous system phase space, with the understanding of how the flux results apply directly to the nonautonomous system. Note that, in contrast to the single frequency case, instantaneous flux will in general vary from one time slice to the next, and will be different for entrainment and detrainment.

In addition, we will also be concerned with *average flux*.

DEFINITION 4.2. The *average flux* associated with P_ε acting on the bi-infinite sequence of time slices, or equivalently with the bi-infinite sequence of maps $\{T_\varepsilon(\cdot; n) \mid n \in \mathbb{Z}\}$ acting on X , is the average of the instantaneous flux over all visited time slices.

Since P_ε preserves area as it acts on a time slice, the average flux is identical for entrainment and detrainment, and is thus denoted by a single symbol Φ :

$$(4.3) \quad \Phi(\theta_0) = \frac{1}{2} \left\{ \langle \phi_E \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \rangle_n + \langle \phi_D \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \rangle_n \right\},$$

where $\langle \cdot \rangle_n$ denotes the average over $n \in \mathbb{Z}$ (when one or more frequencies are mutually commensurate this average will depend on θ_0 , and when the frequencies are all mutually incommensurate it will be independent of θ_0).

For sufficiently small perturbations, the Melnikov approximation of manifold separation in equation (2.7) provides, for any phase slice $\chi(\bar{\theta})$ of $\Sigma^{\theta_{\ell_0}}$, $\bar{\theta} \in \bar{\mathbb{Z}}^1 \times \cdots \times \bar{\mathbb{Z}}^p$, a first order in ε approximation of areas of the individual turnstile lobes. From the comments in Section 2 on uniform approximation of $W^s(\tau_\varepsilon^a)$ and $W^u(\tau_\varepsilon^b)$ on semi-infinite time intervals, for sufficiently small perturbations the boundaries of the turnstile lobes are $O(\varepsilon)$ close to the unperturbed manifold. The area element of one of these lobes, as shown in Figure 4.1, has an area

$$\begin{aligned} dA &= \left[\frac{\varepsilon |M(s, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \nu)|}{\|DH(x_h(-s))\|} + O(\varepsilon^2) \right] [d\lambda(1 + O(\varepsilon))] \\ &= \frac{\varepsilon |M(s, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \nu)|}{\|DH(x_h(-s))\|} d\lambda + O(\varepsilon^2), \end{aligned}$$

where $d\lambda$ is an element of arc length, centered at $x_h(-s)$, along the unperturbed homoclinic or heteroclinic manifold in $\chi(\bar{\theta})$. Using that

$$d\lambda = \frac{d\lambda}{ds} ds = \|DH(x_h(-s))\| ds$$

gives

$$(4.4) \quad \mu(\text{turnstile lobe } (\bar{\theta})) = \varepsilon \int_{s_a(\bar{\theta})}^{s_b(\bar{\theta})} |M(s, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \nu)| ds + O(\varepsilon^2),$$

where $s_a(\bar{\theta})$ and $s_b(\bar{\theta})$ are the s values of the bounding PIP's of the individual turnstile lobe in $\chi(\bar{\theta})$. Equations (4.1) and (4.4) provide an analytical prescription for determining instantaneous flux, and we do so shortly for the examples given in Section 3.

If all the frequencies are mutually commensurate, then $\theta \left(t = \frac{2\pi}{\omega_\ell} n \right) = \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n$ will visit a finite number of values, N , in $T^{\ell-1}$, and the average flux is given by

$$(4.5) \quad \Phi(\theta_0) = \frac{\omega_\ell}{2\pi} \frac{\varepsilon}{2N} \sum_{n=0}^{N-1} \left\{ \int_{s_0(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n)}^{s_{-1}(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n)} |M \left(s, \theta_{1_0} + 2\pi \frac{\omega_1}{\omega_\ell} n, \dots, \right. \right. \\ \left. \left. \theta_{(\ell-1)_0} + 2\pi \frac{\omega_{\ell-1}}{\omega_\ell} n, \theta_{\ell_0}; \nu \right) | ds + O(\varepsilon) \right\},$$

where

$$s_0 \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) = s \text{ value of } \tau_c \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right)$$

$$s_{-1} \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) = s \text{ value of } P_\epsilon^{-1}(\tau_c) \cap \chi \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right).$$

Figure 4.1 An area element of a two dimensional turnstile lobe in a given phase slice for an ℓ frequency homoclinic tangle.

Given a choice of sampling phase θ_{ℓ_0} , the average flux is in general a function of the initial phases θ_0 . If all the frequencies are mutually incommensurate, then $\theta \left(t = \frac{2\pi}{\omega_\ell} n \right) = \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n$ will visit $T^{\ell-1}$ densely and uniformly, and hence the average flux is given by

$$(4.6) \quad \Phi = \frac{\omega_\ell}{(2\pi)^\ell} \frac{\epsilon}{2} \int_{T^{\ell-1}} \int_{s_0(\bar{\theta})}^{s_{-1}(\bar{\theta})} |M(s, \bar{\theta}_1, \dots, \bar{\theta}_{(\ell-1)}, \theta_{\ell_0}; \nu)| ds d\bar{\theta} + O(\epsilon^2),$$

where we have left out the θ_0 argument in Φ since the average is now independent of the initial phases. The invariant lobe structure in $\Sigma^{\theta_{\ell_0}}$ is thus seen to provide direct long term information about the nonautonomous system (2.2), since average flux is proportional to volumes of the invariant turnstile lobes in $\Sigma^{\theta_{\ell_0}}$.

For the two frequency case there is only one frequency ratio, either commensurate or incommensurate, and so expressions (4.5) and (4.6) exhaust all possibilities. Figures 4.2(a) to (i) show, for the two frequency examples of Section 3, plots of individual turnstile lobe areas and of total areas of entrainment and detrainment for all phase slices (often the total areas are just individual lobe areas), as well as the averages of these areas over the visited time slices. These plots specify (by dividing the area in the appropriate time slices by the sampling period) the instantaneous flux for all time slices and the average flux.

Figure 4.2 Turnstile lobe areas for the examples of Section 3 (the verticle scale is per unit ε and $\langle \rangle_n$ denotes the average over all n). For the labeling of individual lobes, $E(m, \bar{\theta}_1; i)$ and $D(m, \bar{\theta}_1; i)$, we choose i to increase in the same direction as m . Note that the labeling applies to *types* of lines; for example, in (b) $\mu(E(1, \theta_{10}; 1))$ applies to the solid line, which is discontinuous.

(d)

(e)

(f)

(g)

Figure 4.2 (Continued) Turnstile lobe areas for the examples of Section 3 (the verticle scale is per unit ϵ).

Figure 4.2 (Continued) Turnstile lobe areas for the examples of Section 3 (the vertical scale is per unit ε).

We should remark that in addition to the average of flux over infinite time, another relevant characteristic is how long a system takes for its average over finite time to converge to the average over infinite time. This depends on how quickly $\theta_{n+1} = \theta_n + 2\pi \frac{\omega}{\omega_\ell}$ “spreads out” across $T^{\ell-1}$, as well as how the turnstile lobe areas vary along $T^{\ell-1}$. For example, in the two frequency case, one can choose an incommensurate frequency ratio arbitrarily close to $1 : 1$, so that, though the average over infinite time would be the average over θ_1 of the plot in Figure 4.2(a), it takes an arbitrarily long time for the finite time average to converge to this mean. In contrast, the case where the frequency ratio equals $1 : g^{-1}$ spreads out quite quickly along the unit circle, and hence the finite time average converges quickly to the infinite time average (see Figure 4.3).

For more than two frequencies some of the frequency ratios may be commensurate, some incommensurate, so in general the expressions for flux will be “in between” (4.5) and (4.6), involving a combination of sums and integrals. Since the Melnikov approximation of manifold separation is valid only over a limited domain of s , one might think that we will have to content ourselves with summing and/or integrating over θ to find average flux, which would be cumbersome. However, using that by its definition the Melnikov function satisfies for all $m \in \mathbb{Z}$

$$(4.7) \quad M(s, \theta_1, \dots, \theta_{\ell-1}, \theta_{\ell_0}; \nu) = M\left(s - m \frac{2\pi}{\omega_\ell}, \theta_1 + m \frac{2\pi\omega_1}{\omega_\ell}, \dots, \theta_{\ell-1} + m \frac{2\pi\omega_{\ell-1}}{\omega_\ell}, \theta_{\ell_0}; \nu\right)$$

and employing a simple change of variables, one can rewrite the sum and/or integral over θ as an integral over s in a constant phase slice.

Figure 4.3 Finite-time average areas for the fourth example of Section 3 ($\theta_{1_0} = 0$). The average over the first N time samples is denoted by $\langle \rangle_N$. Again the verticle scale is per unit ε .

When all frequencies are mutually commensurate one obtains (see Appendix 5):

$$(4.8) \quad \Phi(\theta_0) = \frac{\omega_\ell}{2\pi} \frac{\varepsilon}{2N} \int_{s_0(\theta_0)}^{s_{-N}(\theta_0)} |M(s, \theta_{1_0}, \dots, \theta_{(\ell-1)_0}, \theta_{\ell_0}; \nu)| ds + O(\varepsilon^2),$$

where $s_{-N}(\theta_0)$ is the s value of $P_\varepsilon^{-N}(\tau_c) \cap \chi(\theta_0)$ and N is a positive integer such that $\theta_0 + 2\pi \frac{\omega}{\omega_\ell} N = \theta_0$. Recall from Section 3.2 that P_ε^N maps the two dimensional lobes within the same phase slice, and hence acts on these lobes the same way as does the familiar Poincaré map derived from time-periodic vector fields. Hence the integral runs from a chosen PIP $p_0(\theta_0)$ (with s value $s_0(\theta_0)$) to $(P_\varepsilon^N)^{-1}(p_0(\theta_0))$; it should thus be apparent that the commensurate frequency expression (4.8) recovers the standard result for time-periodic vector fields. A more elegant expression is obtained by letting $N \rightarrow \infty$:

$$(4.9) \quad \Phi(\theta_0) = \lim_{T \rightarrow \infty} \frac{\varepsilon}{2T} \int_0^T |M(s, \theta_{1_0}, \dots, \theta_{(\ell-1)_0}, \theta_{\ell_0}; \nu)| ds + O(\varepsilon^2).$$

(That the lower limit is 0 and not $s_0(\theta_0)$ introduces an error that goes to zero as $T \rightarrow \infty$). The case of one or more incommensurate frequencies transforms to (4.9) using the same method as with commensurate frequencies (equation (4.7) and a

simple change of variables as described in Appendix 5). *We should stress that the transformations do not depend on the choice of τ_c , so (4.9) gives the unique value of average flux for any choice of τ_c .* We also point out that for the heteroclinic core there can of course be multiple sets of turnstiles (potentially one set associated with each unperturbed heteroclinic orbit) and the above flux expressions apply to only one set of turnstiles.

With this more convenient expression, one can use the generalized Melnikov function to perform a detailed study of average flux as a function of system parameters, especially to compare single and multiple frequency forcing. There are three ingredients to understanding such a comparison:

- 1) the relative scaling factors are frequency dependent;
- 2) a sum of sinusoidal functions interfere;
- 3) one must choose a normalization of perturbation amplitudes in order to compare single and multiple frequency systems.

Let us compare single and two frequency forcing in the following way (from our two frequency comparisons, the extension to ℓ frequency comparisons should be clear). Consider the perturbation in equation (2.8) with $F_0(x; \mu) = 0$ (which the systems defined in Appendix 1 satisfy with $F_i(\mu, \omega_i) = \omega_i f_i$ for the OVP flow and $F_i(\mu) = f_i$ for the nondissipative Duffing oscillator). For any frequency pair (ω_1, ω_2) , study the dependence of Φ on F_1 , with F_2 chosen by some normalization condition. For example, suppose $F_1 + F_2 = 1$. For equal relative scaling factors, both single frequency cases ($F_1 = 0$ and $F_1 = 1$) will correspond to absolute maxima, due to interference effects (see the solid line in Figure 4.4(a)). For unequal relative scaling factors (say $A_1/F_1 > A_2/F_2$ without loss of generality), the single frequency case associated with the larger relative scaling factor (the case $F_1 = 1$) will correspond to an absolute maximum of average flux (see the dashed line in Figure 4.4(a)). This holds for arbitrary relative scaling factors (see Appendix 6). *The interpretation for this normalization is then this: for a two frequency perturbation, in general one of the frequencies has a greater relative scaling factor and hence a larger contribution to manifold separation and lobe area for a given forcing amplitude; to include the other frequency in the perturbation only detracts from average flux - the best one can do is the single frequency case with the larger relative scaling factor.* Of course, the comparisons depend on the choice of normalization, and it is possible to choose a normalization such that the perturbation amplitudes are large enough at mid-range values of $F_1 \in [0, 1]$ that the effects due to interference and different relative scaling factors are outweighed, and then the multiple frequency case has maximal rates. Preferably the dynamical system equations describe a physical situation which motivates a normalization. For example, as shown in Rom-Kedar et al. [1990], the OVP flow can be thought of as a vortex pair moving in a wavy-wall channel. Suppose we choose to keep the root-mean-square wall amplitude constant for all $f_1 \in [0, 1]$ for any given (ω_1, ω_2) . For the long wavelength limit of the wall amplitude oscillations, the wall amplitudes are simply related to the perturbation amplitudes, and the normalization becomes $(F_1/\omega_1)^2 + (F_2/\omega_2)^2 = f_1^2 + f_2^2 = 1$. Figure 4.4(b) shows plots of average flux for two choices of (ω_1, ω_2) . The profiles

are similar to those in Figure 4.4(a), except that the interference dips are not quite so pronounced as with the previous normalization ($F_1 + F_2 = 1$). Here too then average flux is maximal in a single frequency limit. Note that for perturbations (2.8) the single frequency (ω_i) flux is proportional to A_i , and hence to $F_i \mathcal{RSF}(\omega_i)$, where $\mathcal{RSF}(\omega_i)$ denotes the relative scaling factor. Thus, whereas for the first normalization the maximal single frequency limit corresponds to the frequency ω_i with greater value of $\mathcal{RSF}(\omega_i)$, for the second normalization it corresponds to the frequency ω_i with greater value of $\omega_i \mathcal{RSF}(\omega_i)$.

Figure 4.4 (a) Average flux as a function of F_1 for $F_1 + F_2 = 1$, with $(\omega_1, \omega_2) = (1.48, 0.78)$, $(A_1/F_1, A_2/F_2) = (1.80, 1.80)$ for the solid line and $(\omega_1, \omega_2) = (1.48, 0.66)$, $(A_1/F_1, A_2/F_2) = (1.80, 1.00)$ for the dashed-dotted line (A_i/F_i is taken from the OVP flow: see Appendix 1). (b) Average flux for the OVP flow as a function of f_1 for $f_1^2 + f_2^2 = 1$, with $(\omega_1, \omega_2) = (1.00, 1.94)$, $(A_1/f_1, A_2/f_2) = (2.22, 2.22)$ for the solid line and $(\omega_1, \omega_2) = (1.00, 0.75)$, $(A_1/f_1, A_2/f_2) = (2.22, 1.22)$ for the dashed-dotted line. In both plots the vertical scale is per unit ε and $\theta_{1_0} = \theta_{2_0} = 0$.

Perhaps more important than making a comparison between the single and multiple frequency case, however, is the ability to search through the parameter space of the multiple frequency problem. *Hence, just as one can use the generalized Melnikov function to study where in parameter space chaos can occur (see Ide and Wiggins [1989], and Section 5), one can in addition search these chaotic regions to study the magnitude of the average flux.* As should be clear after Section 5, the amount of material in phase space that is repeatedly stretched and folded is related to the average flux (i.e. to average lobe areas), and so this magnitude can be viewed as a basic measure of how “chaotic” a system is. One could of course produce endless plots like those in Figure 4.4 to sweep through parameter space, but it is not our

goal here to do a detailed numerical study. Before one performs such a study, however, we cannot overemphasize the usefulness of the relative scaling function. *A plot of this function, along with a chosen normalization, immediately indicates the effectiveness of each frequency at producing manifold separations, which allows one to deduce such practical results as which frequencies have the greatest contribution to average flux, and when one frequency component will overwhelm the other, so that the two frequency forcing is essentially single frequency forcing.* For example, in the Duffing oscillator (refer to Appendix 1, specifically to Figure A1.4), the relative scaling function indicates that if one of the frequencies is of order one, then an additional forcing term with the same amplitude and $\omega > 5$ is negligible.

We emphasize that the analytical expressions we have provided are $O(\epsilon)$ approximations and hence are good approximations only in the near-integrable case. An exact determination of lobe areas, and hence flux, necessitates straightforward, yet laborious, computation. One needs to determine the turnstile lobe boundaries in each time slice, and this can be done by the procedure described in Appendix 4. Then it is a simple matter of computing the areas between boundaries in a plane. Other investigators (for example MacKay and Meiss [1988]) point out in the context of time-periodic perturbations that the Melnikov function provides an *exact* expression for flux through the *unperturbed separatrix*. However, as ϵ grows, the core boundary formed by segments of stable and unstable manifolds can move further from the unperturbed separatrix, and hence the difference between the flux associated with the unperturbed separatrix and the flux associated with the core sequence can grow (the former becoming smaller and smaller compared to the latter). The larger flux associated with the core sequence would be a truer measure of the magnitude of the back and forth motion in the tangle region; further, from previous comments in Section 3 it is the perturbed manifolds which should provide the criterion for entrainment and detrainment and act as a “last frontier” between bounded and unbounded motion.

For non-Hamiltonian perturbations there is the additional complication that the two dimensional turnstile lobes do not conserve area as they are entrained into or detrained from the core under P_ϵ . The definitions and analysis for flux extend to non-Hamiltonian perturbations if we replace everywhere the area of the turnstile lobes by the area of the *images* of the turnstile lobes. This replacement introduces *no* changes to the near-integrable $O(\epsilon)$ approximations for flux (see Remark 2 of Appendix 5). When the system is not near-integrable, one needs to resort to numerical simulation of the boundaries of the *images* of the turnstile lobes.

(4.2) Addressing basic transport issues by considering the geometry of lobes and their variation of areas

The average flux provides a measure of the back and forth motion across a reasonable boundary in the tangle region. However, to answer basic questions about transport in phase space necessitates a consideration of lobe geometry and the variation of lobe areas. For example, in the case of a forced system whose unperturbed separatrix divides phase space into bounded and unbounded motion, a fundamental concern (with a variety of physical applications) is the probability of escape from

the core after n time samples. This necessitates a consideration of *lobe content*, rather than just lobe areas, which can be determined by addressing the intersection of lobes with each other and with the core. An additional concern (motivated for example by the semi-classical treatment of molecular dissociation) is the probability of escape for points that lie on a particular level set of the unperturbed Hamiltonian. This necessitates a consideration of the intersection of lobes with the level sets of the unperturbed Hamiltonian. Both these transport problems are affected not only by the average of lobe areas, but by the *variation* of lobe areas.

Let us consider these transport problems. As mentioned in Section (3.3), entraining and detraining lobes can intersect each other; from these intersections we obtain a more detailed knowledge of transport in the tangle region. For example, the $t = 0$ sample of Figure 3.11, which we present again here in Figure 4.5 with labels on some lobes, shows lobe $d(3,0)$ intersecting with lobe $e(1,0)$; the fluid in $d(3,0) \cap e(1,0)$ will not only be entrained into the core in the next time sample, it will then be detrained from the core between the $n = 2$ and $n = 3$ time sample. Similarly $e(-2,0)$ intersects $d(1,0)$: the fluid in $e(-2,0) \cap d(1,0)$ will be detrained from the core in the next time sample and was entrained into the core between the $n = -3$ and $n = -2$ time sample.

Figure 4.5 The lobe structure of the $t = 0$ sample of Figure 3.11, shown again with the necessary labeling.

These concerns relate to the *content* of those lobes about to be entrained/detrained in the next time sample: examining the intersections of these lobes with other lobes provides an understanding of the history of the fluid particles, i.e. when they might have been previously entrained/detrained or in the future when they may be entrained/detrained. An example of such a concern is our first question about escape from the core. We begin by examining this question in the context of the multiple frequency OVP flow, i.e. an area-preserving system whose unperturbed solution contains a pair of heteroclinic orbits that divide bounded and unbounded motion (since the system is symmetric about $x_2 = 0$, again we consider just the $x_2 > 0$ half). We can phrase the transport problem in the following way: suppose at $t = 0$ fluid of one type (A) is in the core, and fluid of another type (B) is outside the core, then at any time $t = \frac{2\pi}{\omega_t}n$ how much of fluid A and B is inside and outside the core, i.e. how much of fluid A and B have been *transported* out of and into the core, respectively? For the OVP system, the $e(m,n)$ lobes for $m \geq 1$ and the $d(m,n)$ lobes for $m \leq 0$ are contained entirely outside the core. The remaining lobes will in general lie both inside and outside the core, except for the turnstile lobe $d(1,n)$ by our previous assumption (see Figure 4.6).

Figure 4.6 Schematic diagram of some lobes of the OVP flow in the $t = 0$ sample. The $d(3,0)$ lobe is contained partially inside and partially outside the $t = 0$ core. The portion of the lobe with type A particles is white, and the portion with type B particles is shaded.

The particle content of a lobe of fluid is of course determined by its intersection with the core at time $t = 0$. If $\mu_A(\cdot)$, $\mu_B(\cdot)$ denote the areas of the pieces of the lobe in the parenthesis containing fluid A and fluid B , respectively, then

$$\begin{aligned}
(4.10) \quad & \mu_A(e(m, 0)) = \mu(e(m, 0) \cap c(0)) \\
& \mu_B(e(m, 0)) = \mu(e(m, 0)) - \mu(e(m, 0) \cap c(0)) \\
& \mu_A(d(m, 0)) = \mu(d(m, 0) \cap c(0)) \\
& \mu_B(d(m, 0)) = \mu(d(m, 0)) - \mu(d(m, 0) \cap c(0)).
\end{aligned}$$

These expressions can be rewritten in terms of lobe intersections alone. For $m \geq 1$:

$$\begin{aligned}
(4.11) \quad & \mu_A(e(m, 0)) = 0 \\
& \mu_B(e(m, 0)) = \mu(e(m, 0)) \\
& \mu_A(d(m, 0)) = \mu(d(m, 0)) - \sum_{i=1}^{m-1} \mu(d(m, 0) \cap e(i, 0)) \\
& \mu_B(d(m, 0)) = \sum_{i=1}^{m-1} \mu(d(m, 0) \cap e(i, 0)),
\end{aligned}$$

where for $m = 1$ we replace the two sums by zero. The last two equations follow from the fact that the stable manifold cannot intersect itself, so the only way for $d(m, 0)$ ($m \geq 1$) to be outside the core is for it to be in the $e(i, 0)$ lobes for $i \geq 1$ (see Figure 4.6). The $i \geq m$ intersections are disallowed because of our assumption that the detraining turnstile lobes are completely inside the core (which is valid for the OVP flow up to quite large perturbations).

Now we wish to specify the change of core content between any two time samples. If $\Delta A_{\text{core}}(n)$ and $\Delta B_{\text{core}}(n)$ are the area changes under $T_\varepsilon(\cdot; n)$ of fluid type A and B , respectively, in the core, then

$$\begin{aligned}
(4.12) \quad & \Delta A_{\text{core}}(n) = \mu_A(e(1, n)) - \mu_A(d(1, n)) \\
& \Delta B_{\text{core}}(n) = \mu_B(e(1, n)) - \mu_B(d(1, n)).
\end{aligned}$$

From previous comments, for all positive $i \in Z$ and all $n \in Z$

$$\begin{aligned}
(4.13) \quad & \mu_A(e(i, n)) = 0 \\
& \mu_B(e(i, n)) = \mu(e(i, n)).
\end{aligned}$$

To deal with $d(i, n)$, recall that $T_\varepsilon(\cdot; n)$ is area-preserving for all $n \in Z$, so that

$$\begin{aligned}
(4.14) \quad & \mu_A(\cdot) = \mu_A(T_\varepsilon(\cdot; n)) \\
& \mu_B(\cdot) = \mu_B(T_\varepsilon(\cdot; n)).
\end{aligned}$$

Using (4.14) and (4.10) gives expressions for the $d(i, n)$ lobes which, when plugged into (4.12) along with (4.13), give

$$\begin{aligned}
(4.15) \quad & \Delta A_{\text{core}}(n) = -\mu(d(1+n, 0) \cap c(0)) \\
& \Delta B_{\text{core}}(n) = \mu(e(1, n)) - \mu(d(1, n)) + \mu(d(1+n, 0) \cap c(0)).
\end{aligned}$$

This specifies the area changes in terms of areas of lobes and of intersections of lobes with the core. Alternatively, one could use (4.11) instead of (4.10) to obtain

$$(4.16) \quad \begin{aligned} \Delta A_{\text{core}}(n) &= -\mu(d(1, n)) + \sum_{j=0}^{n-1} \mu(d(1, n) \cap e(-j, n)) \\ \Delta B_{\text{core}}(n) &= \mu(e(1, n)) - \sum_{j=0}^{n-1} \mu(d(1, n) \cap e(-j, n)) \end{aligned}$$

(for $n = 0$ replace the sums over j by zero). This specifies the area changes in terms of areas of turnstile lobes and intersections of turnstile lobes with other lobes. *The right hand side of (4.16) has the same form as in the time-periodic case with the second lobe argument “ n ” simply going along for the ride. This is due to the fact that, as mentioned in Section 3, the geometrical constraints of the time-dependent lobe structures in X for the quasiperiodic case are the same as those of the invariant lobe structure of the time-periodic case.* Summing the first expression of (4.15) or (4.16) over the first N time samples and dividing by the initial core area gives the percentage of material initially in the core that escapes after N time samples.

It is particularly easy to discuss lobe content in the OVP example since there is no re-entrainment (once fluid escapes the core, it never re-enters) and the sequence of maps is area-preserving. For Hamiltonian systems which involve re-entrainment, such as the non-dissipative Duffing oscillator, to specify area changes in terms of intersections of lobes with other lobes is more difficult. However, to specify area changes in terms of intersections of lobes with the core is no more difficult: one still maps lobes back to the $t = 0$ sample and sees how much of the lobe is inside and outside the $t = 0$ core. Since the lobes are no more fundamental an entity than the core, to specify area changes in terms of intersections of lobes with the core is no less fundamental than specification in terms of intersections of lobes with other lobes. Further, if one wishes to compute a lobe’s content, it is easier to deal with a single intersection of the lobe mapped back to $t = 0$ with the core at $t = 0$ than with expressions involving a sum of intersections of the lobe with other lobes. Nevertheless, there is still interest in understanding the geometry of intersections of lobes, and we will discuss this point further in Section 6. The additional complexity of non-Hamiltonian perturbations is of course that the two dimensional lobes do not preserve area as they map from one time sample to the next, and we also defer this point to Section 6.

The main difference between single and multiple frequency systems in regard to this transport problem is the *variation* of lobe areas: *given a fixed average flux, the variation of lobe areas gives one the freedom to enhance or diminish transport in a given direction over a finite time.* For example, if one wishes to enhance immediate detraining over the first few time samples, one can choose a perturbation time dependence such that the first few detraining lobes are large (relative to the mean) and the first few entraining lobes are small (which forces the first few detraining lobes to start off mainly in the core and hence take more of the initial core material with it). Of course the fact that average flux is typically worse in multiple frequency

systems than in a corresponding single frequency system (as explained earlier) tends to diminish the transport in both directions, but over a finite time interval this can be outweighed by the above effect due to variation of lobe areas.

The second transport problem we introduced in this section deals with the probability of escape for points that lie on a *particular level set of the unperturbed Hamiltonian*. For example, Noid and Stine [1979] and Goggin and Milonni [1988a] and [1988b] study dissociation of diatomic molecules due to forcing by one and two lasers, as modeled by the Morse oscillator; a semi-classical approach to this problem is concerned with the probability that an ensemble of points at a particular level set of the unperturbed Hamiltonian (for example the one corresponding to the quantum mechanical ground state) escape from bounded to unbounded motion. They find in their numerical investigations that the two laser system tends to *enhance* dissociation (i.e. escape from a particular level set to unbounded motion). This contrasts with our earlier result that two frequency systems tend to diminish the average flux, which one might think would *hinder* dissociation. However, average flux is only one of many factors in this transport problem; one must also consider the geometry of lobes and the variation of lobe areas. For example, though the absolute value of the two frequency Melnikov function may have a lower average over s than in a corresponding single frequency case, for some normalizations it can have a larger maximum, and hence the lobes may better penetrate inner level sets of the unperturbed Hamiltonian. Further, as mentioned briefly in Noid and Stine [1979] and elaborated upon by Goggin and Milonni [1988b], additional forcing frequencies entail additional resonance bands within the core. Though no mention is made of the homoclinic tangle, it is clear that more efficient destruction of KAM tori within the core can allow the lobes of the homoclinic tangle to better penetrate inner level sets. Additionally, for penetration of a fixed unperturbed level set, the previously discussed pulsation and “breathing” of the lobe structure can be significant. We see then that though average flux may decrease, there are other factors that can affect the ability of the lobes to penetrate a particular inner level set in a multiple frequency problem. As a simple example, consider Figure 4.7, which shows again in (a) some lobes for the $n = 2$ time slice of the system portrayed in Figure 3.11, and in (b) some lobes of a corresponding single frequency system ($f_1^2 + f_2^2$ is the same for both systems). Though the two frequency system has a smaller average lobe area, its first detraining lobe is large relative to the mean and intersects with more inner level sets than does the first detraining lobe of the single frequency case. Of course, this does not imply that the first N detraining lobes of this two frequency system intersect with more of the level set shown than do the lobes of the single frequency system; we are focusing for visual simplicity on the first detraining lobe. We should point out that the studies of Noid and Stine [1979] and Goggin and Milonni [1988b] consider a fairly specific situation: they focus on the *threshold intensity* needed for the ground state level set to achieve any dissociation in the long time limit. Escape over *all* time scales (including small and medium time scales) for a forcing amplitude that does not happen to be right at threshold for the concerned level set is a more complicated problem for which conclusions are less easily drawn and a study in terms of lobe dynamics is motivated. In particular, one would expect the

variation of lobe areas to affect any studies on finite time scales.

The framework in terms of lobe dynamics is easily written down. One can write for the OVP flow (or equivalently the Morse oscillator, or any system with similar geometry) an expression for probability of escape for this second transport problem that is quite similar to the first expression of equation (4.15). If $\mathcal{L}_0$ denotes the level set of interest of the unperturbed Hamiltonian, and $\Delta\mathcal{L}_0(n)$ denotes the number of points initially on the level set that escape under $T_\varepsilon(\cdot; n)$, where the points are initialized on the level set according to some density, then

$$(4.17) \quad \Delta\mathcal{L}_0(n) = \mu(d(1+n, 0) \cap \mathcal{L}_0),$$

where $\mu(\cdot)$ denotes the number of points on the curve within the parenthesis. Summing $\Delta\mathcal{L}_0(n)$ over the first N time samples and dividing by $\mu(\mathcal{L}_0)$ gives the percentage of points on the level set that have escaped after N time samples. Equations (4.15) to (4.17) thus provide a framework for computing answers to the two transport problems. *Once one recognizes the mechanism for, and topology of, transport, there are rich possibilities for determining answers to transport questions by explicit simulation of a finite number of lobes in the homoclinic/heteroclinic tangle.*

We conclude this section by remarking that a more detailed study of this second transport problem is given in Beigie and Wiggins [1991].

Figure 4.7 Some lobes in the $n = 2$ time sample of (a) the system portrayed in Figure 3.11 and (b) a corresponding single frequency system ($\omega = 2$).

5. Chaos.

(5.1) The periodic case

The chaotic nature of the dynamics in homoclinic and heteroclinic tangle regions under iterates of a two dimensional Poincaré map derived from a time-periodic vector field is well established (see Guckenheimer and Holmes [1983] or Wiggins [1988]). Recall the snapshot sequence, shown in Figure 1.3, of the time-periodic OVP flow. As mentioned in the introduction, with each time sample a new lobe of fluid is entrained into the core; the lobe stretches as it winds around the core, and at the fifth time sample the first entrained lobe wraps around the fifth entrained lobe. The wrapping of lobes around one another in the invariant lobe structure in the two dimensional Poincaré section for time-periodic vector fields entails horseshoe maps, as shown heuristically in Figure 5.1, where $P_\epsilon^3(D)$ intersects D in the shape of a horseshoe. Using techniques such as those found in Wiggins [1988], one can rigorously establish the existence of horseshoe maps in the tangle region of time-periodic vector fields (one does this by considering a region sufficiently close to, or containing, the hyperbolic fixed point).

Figure 5.1 A horseshoe map in the heteroclinic tangle region of $\Sigma^{\theta_{10}=0}$ in the time-periodic case.

One can associate with this map the usual symbolic dynamics formalism and establish the existence of a Cantor set $\Lambda \in \Sigma^{\theta_1}$ on which P_ϵ^k (for large enough k) is topologically conjugate to a full shift on the bi-infinite sequence of two symbols, as shown in the commuting diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{P_\epsilon^k} & \Lambda \\ \downarrow \phi & & \downarrow \phi \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

where σ is the shift map acting on the set of bi-infinite sequences of two symbols Σ , and ϕ is a homeomorphism that takes each point in Λ to a sequence in Σ . This directly entails some standard features of chaotic dynamics:

- 1) a countable infinity of periodic points of all possible periods;
- 2) an uncountable infinity of nonperiodic points;
- 3) a dense orbit.

(5.2) The quasiperiodic case

For quasiperiodic systems (2.2), the map of X from time $t = \frac{2\pi}{\omega_t}n$ to time $t = \frac{2\pi}{\omega_t}(n+1)$, $T_\epsilon(\cdot; n)$, depends on n . So we of course cannot develop the usual two dimensional horseshoe map construction. *So in what sense is the dynamics in X chaotic?* Again we use the autonomous system (2.3) to construct an invariant structure with which to understand the dynamics, and then we project the sequence of time slices onto X to obtain a sequence of time-dependent structures from the invariant one. As a preview of what is to come, one can imagine, rather than a single horseshoe map, a bi-infinite sequence S_H of different “horseshoe maps” $H(\cdot; j) : X \rightarrow X$,

$$S_H = \{\dots, H(\cdot; -j), \dots, H(\cdot; -1), H(\cdot; 0), H(\cdot; 1), \dots, H(\cdot; j), \dots\},$$

and a bi-infinite sequence S_D of different domains $D(j) \in X$,

$$S_D = \{\dots, D(-j), \dots, D(-1), D(0), D(1), \dots, D(j), \dots\},$$

such that $H(D(j); j)$ intersects $D(j+1)$ in the shape of a horseshoe (see Figure 5.2). There is thus a sequence of formed horseshoes landing on different regions of X ; each time the horseshoe happens to land on the region that will next form a horseshoe, and land in such a way that the stretched direction “aligns” with the direction about to be stretched. We refer to this as a *traveling horseshoe map sequence*. It is clear that this sequence of maps retains the essential ingredient of chaos – repeated stretching and folding, and hence sensitive dependence on initial conditions. Though our discussion here is heuristic, Figure 5.2 should make apparent what we mean.

Figure 5.2 A traveling horseshoe map sequence.

A rigorous construction of a traveling horseshoe map sequence can be made for systems of the form (2.2) which possess an invariant lobe structure. For simplicity of discussion, let us consider the two frequency homoclinic case. In Section 3.3 we discussed how the two dimensional lobes in any phase slice of $\Sigma^{\theta_{20}}$ fold and wrap violently around one another just as in the invariant lobe structure of the time-periodic case. These lobes exist in all phase slices $\chi(\bar{\theta}_1)$, $\bar{\theta}_1 \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, to give in $\Sigma^{\theta_{20}}$ a three dimensional lobe structure that folds and wraps violently around itself in the direction “normal” to τ_ϵ . The Poincaré section $\Sigma^{\theta_{20}}$ thus contains a three dimensional region $\mathcal{R}$ whose image under $P_\epsilon^k(\mathcal{R})$ (for k sufficiently large) intersects $\mathcal{R}$ such that, for any $\bar{\theta}_1 \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$, $P_\epsilon^k(\mathcal{R}) \cap \chi(\bar{\theta}_1)$ intersects $\mathcal{R} \cap \chi(\bar{\theta}_1)$ in the shape of a horseshoe (see Figure 5.3). Using techniques such as those found in Wiggins [1988], one can rigorously establish the existence of such a region $\mathcal{R}$ (in a manner similar to the periodic case, one needs to consider a region sufficiently close to or containing the normally hyperbolic invariant 1-torus). The geometry of $\mathcal{R} \cap P_\epsilon^k(\mathcal{R})$, and of the resulting Cantor set, is determined by the geometry of the PIM’s (just as was the geometry of the lobes in Section 2). When the PIM’s are non-intersecting 1-tori (this case is dealt with rigorously in Wiggins [1988]), $\mathcal{R} \cap P_\epsilon^k(\mathcal{R})$ is a simply connected region whose boundaries divide $\Sigma^{\theta_{20}}$ into an inside and outside, and there is a Cantor set of 1-tori, $\tilde{\Lambda}$, on which P_ϵ^k is topologically conjugate to a full shift on the bi-infinite sequence of two symbols:

$$(5.1) \quad \begin{array}{ccc} \tilde{\Lambda} & \xrightarrow{P_\epsilon^k} & \tilde{\Lambda} \\ \downarrow \tilde{\Phi} & & \downarrow \tilde{\Phi} \\ \Sigma & \xrightarrow{\sigma} & \Sigma, \end{array}$$

where $\tilde{\Phi}$ is a homeomorphism that takes each torus in $\tilde{\Lambda}$ to a sequence in Σ . Note how by a “Cantor set of 1-tori” we mean a set of 1-tori whose intersection with $\chi(\bar{\theta}_1)$ for any $\bar{\theta}_1 \in \bar{\mathcal{Z}}^1 \times \cdots \times \bar{\mathcal{Z}}^p$ defines a Cantor set of points. The role of points in the time-periodic case thus applies to 1-tori in the two frequency case (see Figure 5.4), and the dynamics on the Cantor set of 1-tori is thus understood to be chaotic, with:

- 1) a countable infinity of periodic 1-tori of all possible periods;
- 2) an uncountable infinity of nonperiodic 1-tori;
- 3) a 1-torus whose orbit under P_ϵ^k is dense in $\tilde{\Lambda}$.

(a)

(b)

Figure 5.3 $P_\epsilon^k(\mathcal{R})$ intersects $\mathcal{R}$ in a horseshoe fashion. Note how the initial discontinuity of $\mathcal{R}$ at $\theta_1 = 0$ is mapped to another θ_1 value.

X

(a)

(b)

X

(c)

Figure 5.4 (a) A period two point in X for the time-periodic case; (b) a period two 1-torus in $\Sigma^{\theta_{20}}$ for the two frequency case; (c) motion in X under $\{T_\epsilon(\cdot; n); n \in \mathbb{Z}\}$ of a point that initially lies on the period two 1-torus of (b).

Heuristically, then, points which lie on this set of 1-tori move chaotically normal to τ_ϵ as they move in a regular manner “along” τ_ϵ (i.e. in the θ_1 direction). Just as one can establish chaos for time-periodic vector fields in which $W^s(\tau_\epsilon^a)$ and $W^u(\tau_\epsilon^b)$ intersect non-transversally (see Guckenheimer and Holmes [1983]), one can construct an invariant Cantor set in $\Sigma^{\theta_{20}}$ when the toral PIM’s meet at isolated points to give non-transversal intersections at that point. When the PIM’s are segments of spirals (either intersecting or non-intersecting), then $P_\epsilon^k(\mathcal{R}) \cap \mathcal{R}$ will in general consist of piecewise continuous segments of “spiral” volumes. Figure 5.3 shows the case where $\mathcal{R}$ is a segment of a “spiral” volume from 0 to 2π ; hence $\mathcal{R}$ is discontinuous at $\theta_1 = 0$. Applying P_ϵ^k to $\mathcal{R}$ sends this discontinuity to a new θ_1 value, and the intersection of $P_\epsilon^k(\mathcal{R})$ with $\mathcal{R}$ creates another discontinuity at $\theta_1 = 0$. Explicit construction of a Cantor set by repeated application of P_ϵ^k and P_ϵ^{-k} introduces a new discontinuity in θ_1 with each application, so that the resulting Cantor set $\tilde{\Lambda}$ in $\Sigma^{\theta_{20}}$ will consist of a countable infinity of piecewise continuous segments of spirals that intersect each phase slice, $\chi(\bar{\theta}_1)$, $\bar{\theta}_1 \in \tilde{\mathcal{Z}}^1 \times \cdots \times \tilde{\mathcal{Z}}^p$, in a Cantor set of points. Note that for those perturbations for which the lobe structure exists on only a subset of T^1 , the Cantor set will exist on a subset of T^1 (further note that this occurs only in the commensurate frequency case, which from previous discussion can be described by the time-periodic formalism, and hence by the standard horseshoe map construction). For example, in Figure 3.4(a), where the lobe structure vanishes in the plane $\chi(\bar{\theta}_1 = \pi)$, we have to exclude from our consideration a finite but arbitrarily small window in θ_1 around $\theta_1 = \pi$ to obtain a finite k in P_ϵ^k . More significantly, in Figure 2.5(h) there are gaps in θ_1 for which there are no intersection manifolds and hence no lobe structures, and there are thus gaps in the resulting Cantor set.

Regardless of the geometry of $\mathcal{R} \cap P_\epsilon^k(\mathcal{R})$ in the Poincaré section $\Sigma^{\theta_{20}}$, the derivation of a traveling horseshoe map sequence is straightforward. From the three dimensional region $\mathcal{R}$ we can define a two dimensional region in any phase slice $\chi(\bar{\theta}_1)$, $\bar{\theta}_1 \in \tilde{\mathcal{Z}}^1 \times \cdots \times \tilde{\mathcal{Z}}^p$, by $R(\bar{\theta}_1) = \mathcal{R} \cap \chi(\bar{\theta}_1)$. That $P_\epsilon^k(\mathcal{R})$ intersects $\mathcal{R}$ in any of the above phase slices in the shape of a horseshoe directly implies that $P_\epsilon^k(R(\bar{\theta}_1))$ intersects $R(\bar{\theta}_1 + 2\pi \frac{\omega_1}{\omega_2} k)$ in the shape of a horseshoe. Projecting onto X ,

$$r(n) = \hat{X} \left(R \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \right) \right),$$

and using (3.3) gives that $T_\epsilon(\cdot; n+k-1) \circ \cdots \circ T_\epsilon(\cdot; n+1) \circ T_\epsilon(r(n); n)$ intersects $r(n+k)$ in the shape of a horseshoe (see Figure 5.5). We thus have our defined sequence of traveling horseshoe maps, with

$$\begin{aligned} D(j) &\equiv r(j \cdot k) \\ H(\cdot; j) &\equiv T_\epsilon(\cdot; (j+1)k-1) \circ \cdots \circ T_\epsilon(\cdot; j \cdot k+1) \circ T_\epsilon(\cdot; j \cdot k). \end{aligned}$$

Of course nothing magical is happening here: it is just a matter of images of two dimensional lobes still folding and wrapping around one another ad infinitum even though the entire lobe structure is varying from one time sample to the next. In fact, if one thinks of, say, a mixing fluid, the case where fluid lobes always wrap

around each other with each time sample in the exact same way seems more of an anomaly than the case we have here.

Figure 5.5 Obtaining a traveling horseshoe map sequence from $\mathcal{R} \cap P_\epsilon^k(\mathcal{R})$.

We can also use the invariant Cantor set in $\Sigma^{\theta_{20}}$ to define for each time slice a Cantor set of points (when the lobe structure and hence the Cantor set has gaps, recall that we assume we start in a time slice with a lobe structure). We define the Cantor set of points in the phase slice $\chi(\bar{\theta}_1)$ to be $\Lambda(\bar{\theta}_1) = \tilde{\Lambda} \cap \chi(\bar{\theta}_1)$. From the commuting diagram (5.1), we directly obtain

$$(5.2) \quad \begin{array}{ccc} \Lambda(\bar{\theta}_1) & \xrightarrow{P_\epsilon^k} & \Lambda\left(\bar{\theta}_1 + 2\pi \frac{\omega_1}{\omega_2} k\right) \\ \downarrow \Phi & & \downarrow \Phi \\ (\Sigma, \bar{\theta}_1) & \xrightarrow{\hat{\sigma}} & \left(\Sigma, \bar{\theta}_1 + 2\pi \frac{\omega_1}{\omega_2} k\right), \end{array}$$

where

$$\hat{\sigma}(\{\cdots s_{-n} \cdots s_{-1} \cdot s_0 s_1 \cdots s_n \cdots\} \equiv s, \bar{\theta}_1) = \left(\{\cdots s_{-n} \cdots s_{-1} s_0 \cdot s_1 \cdots s_n \cdots\}, \bar{\theta}_1 + 2\pi \frac{\omega_1}{\omega_2} k\right)$$

and

$$\Phi^{-1}(s, \bar{\theta}_1) = \tilde{\Phi}^{-1}(s) \cap \chi(\bar{\theta}_1)$$

(see Guckenheimer and Holmes [1983] or Wiggins [1988] for a discussion of the standard notation for the symbolic dynamics; further note how the s variable in this context is of course different from the parameter for the unperturbed homoclinic/heteroclinic manifold seen elsewhere). The operator $\hat{\sigma}$ is similar to the “extended shift map” of Stoffer [1988a,b]. Projecting (5.2) onto X , we have:

$$(5.3) \quad \begin{array}{ccc} \lambda(j) & \xrightarrow{T_\varepsilon(\cdot; (j+1)k-1) \circ \dots \circ T_\varepsilon(\cdot; j \cdot k)} & \lambda(j+1) \\ \downarrow \phi(\cdot; j) & & \downarrow \phi(\cdot; j+1) \\ \Sigma & \xrightarrow{\sigma} & \Sigma \end{array}$$

where

$$\begin{aligned} \lambda(j) &= \hat{X} \left(\Lambda \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} (j \cdot k) \right) \right) \\ \phi^{-1}(s; j) &= \hat{X} \left(\tilde{\Phi}^{-1}(s) \cap \chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} (j \cdot k) \right) \right). \end{aligned}$$

Some may feel more comfortable with topological conjugacy to an “extended shift map”, shown in (5.2), than with the diagram (5.3), but both relations say the same thing: the dynamics in X from one time sample to the next can be described by a shift map relative to a time-dependent Cantor set of points. Properties such as periodic points of all periods or dense orbits are thus to be understood relative to this time-dependent set, rather than fixed spatial coordinates. Note how, though for simplicity we have discussed the two frequency case throughout this section, the results hold as well for the general ℓ frequency problem, where the Cantor set in $\Sigma^{\theta_{1_0}}$ consists of $(\ell - 1)$ dimensional objects (for example $(\ell - 1)$ -tori or $(\ell - 1)$ dimensional segments of spirals).

Having discussed chaos, we reiterate our previous comments on searching parameter space to determine when one can and cannot have chaos. From previous remarks, such a search is equivalent to a search in parameter space to determine whether a lobe structure exists, which in the near-integrable case is equivalent to determining whether zero sets of the Melnikov function exist. We stress that such a search can have simple, practical consequences. For example, in commensurate two frequency systems the lobe structures, and hence Cantor sets, can have gaps in θ_1 (refer back to Figure 2.5(h)): hence the choice of θ_{1_0} affects whether the system has chaotic dynamics (if θ_{1_0} lies in a gap there will be no chaos). *Thus a shift in the relative initial phase $\theta_{1_0} - \theta_{2_0}$ can suppress chaos.*

6. A transport formalism that focuses on lobe intersections. In the last part of Section 4, we addressed the question for the two frequency OVP flow: if at time $t = 0$ there is fluid of type A in the core and type B outside the core, then how much of A and B is inside and outside the core at any later time $t = \frac{2\pi}{\omega_2} n$? We answered this question two ways: in terms of (i) intersections of lobes with the core, and (ii) intersections of lobes with each other. As was mentioned, the first approach is easily extended to more complicated systems such as the Duffing oscillator, where re-entrainment can occur. There is still interest, however, in

understanding the geometry of lobe intersections with other lobes, and how this applies to transport questions such as the one above. Rom-Kedar and Wiggins [1990] give a detailed study of transport under two dimensional maps by focusing on intersections of lobes with other lobes. The classical Poincaré map reduction of periodically forced systems can thus be described by this analysis. The analysis applies to an arbitrary number of regions of phase space bounded by pieces of stable and unstable manifolds of hyperbolic fixed points, and does not require the map to be area-preserving or near-integrable. The generalization to a sequence of maps derived from a quasiperiodic vector field is fairly easy, as we explain in this section. Our goal here is certainly not to go through the same detailed and laborious analysis given for the time-periodic case; we wish to merely summarize some main results of Rom-Kedar and Wiggins [1990] and explain how the formulas extend to the quasiperiodic case with little change.

(6.1) The periodic case

Consider a $C^r, r \geq 1$, orientation preserving map $T : X \mapsto X$. Let $p_i, i = 1, 2, \dots, N$, denote N hyperbolic fixed points of T , with stable and unstable manifolds denoted by $W^s(p_i)$ and $W^u(p_i)$, respectively. As done when we defined in X the core region of the single frequency OVP flow, we can divide X into *regions*, i.e. simply connected domains of phase space with boundaries consisting of boundaries of phase space and segments of stable and unstable manifolds starting at hyperbolic fixed points and ending at either PIP's or at the boundary of phase space (which can be infinity). Suppose one divides X into N_R disjoint regions, denoted $R_i, i = 1, 2, \dots, N_R$, and that initially particles of species S_i are uniformly distributed throughout region $R_i, i = 1, 2, \dots, N_R$. Then the goal is to compute the area occupied by species S_i in region R_j (for any i and j) after an arbitrary number of iterations of the map. As has been explained, the turnstile lobes are the only mechanism for transport across the boundaries; one can identify these lobes and from lobe intersection analysis specify the particle content of these lobes. For $m \in \mathbb{Z}$ and $m \geq 1$, let $L_{k,j}(m)$ denote the set of lobes that will leave R_k and enter R_j upon the m^{th} iterate of T ; for $m \leq 0$ let $L_{k,j}(m)$ denote the set of lobes that leave R_j and enter R_k upon the $(|m| + 1)^{th}$ iterate of T^{-1} . Further, let $L_{k,j}^i(m)$ denote the portion of these lobes that contain species S_i (i.e., at $t = 0, L_{k,j}(m) \cap R_i$). (Note how, as described briefly in Section 3 and more fully in Rom-Kedar and Wiggins [1990], typically the turnstile lobes $L_{k,j}(1)$ are contained entirely in R_k ; we assume this to be the case, being aware that for cases where $L_{k,j}(1)$ is contained in regions other than R_k we can redefine the lobes to be the pieces of $L_{k,j}(1)$ that are in R_k .) Then for area-preserving maps (we mention later the case where area is not preserved), the area occupied by species S_i in region R_j immediately after the n^{th} iterate of T ($n \geq 1$) is given by

$$(6.1) \quad T_{i,j}(n) = \delta_{ij}\mu(R_j) + \sum_{k=1}^{N_R} \sum_{m=1}^n [\mu(L_{k,j}^i(m)) - \mu(L_{j,k}^i(m))],$$

where δ_{ij} is the Kronecker delta and

$$(6.2) \quad \mu(L_{k,j}^i(m)) = \sum_{s=1}^{N_R} \sum_{p=1}^m \mu(L_{k,j}(m) \cap L_{i,s}(p)) - \sum_{s=1}^{N_R} \sum_{p=1}^{m-1} \mu(L_{k,j}(m) \cap L_{s,i}(p)).$$

Note how (6.2) is purely a geometrical result. The area of the portion of a given lobe that contains particle species S_i is simply the area of the intersection of region R_i with the lobe mapped back to time $t = 0$. This is rewritten in terms of intersections of lobes with other lobes based on the geometrical constraints of the lobe structure: for example, that the stable and unstable manifold can never intersect themselves, but can intersect each other, and that $L_{k,j}(m)$ (by our assumption) will after $(m-1)$ iterations of T be contained entirely in region R_k (this determines the truncation of the sum in (6.2)). Using that $T^q(L_{k,j}(m)) = L_{k,j}(m-q)$ and that T is area-preserving, (6.2) can be rewritten as

$$(6.3) \quad \begin{aligned} \mu(L_{k,j}^i(m)) &= \sum_{s=1}^{N_R} \sum_{p=0}^{m-1} \mu(L_{k,j}(1) \cap L_{i,s}(1-p)) \\ &\quad - \sum_{s=1}^{N_R} \sum_{p=1}^{m-1} \mu(L_{k,j}(1) \cap L_{s,i}(1-p)). \end{aligned}$$

This rewrites areas in terms of intersections of the turnstile lobes with other lobes.

Note that, though not explicitly stated in Rom-Kedar and Wiggins [1990], these transport equations in general apply to a division of phase space such that there is at most a single set of turnstiles between each pair of regions; one can of course relax this restriction by at the end of the transport calculation considering some particle species types as identical, which is equivalent to grouping together previously distinct regions to form more complicated regions that allow multiple sets of turnstiles between pairs of regions.

(6.2) The quasiperiodic case

Let P_ε now have N normally hyperbolic invariant $(\ell-1)$ -tori in $\Sigma^{\theta_{\ell_0}}$, denoted by τ_ε^i , $i = 1, 2, \dots, N$, with stable and unstable manifolds denoted by $W^s(\tau_\varepsilon^i)$ and $W^u(\tau_\varepsilon^i)$, respectively. Further we relax the condition that P_ε be derived from a near-integrable system (though we retain the same notation). As done when we defined the core in Section 3, we can divide each phase slice $\chi(\bar{\theta})$, $\bar{\theta} \in \bar{Z}^1 \times \dots \times \bar{Z}^p$, of $\Sigma^{\theta_{\ell_0}}$ into N_R two dimensional disjoint regions, denoted $R_i(\bar{\theta})$, $i = 1, 2, \dots, N_R$, that are simply connected domains of the phase slice with boundaries consisting of boundaries of phase space and segments of stable and unstable manifolds in that phase slice starting at normally hyperbolic invariant $(\ell-1)$ -tori and ending at either PIM's or at the boundaries of phase space. Invariant $(\ell+1)$ dimensional regions in $\Sigma^{\theta_{\ell_0}}$ are then defined by

$$\mathcal{R}_i = \{R_i(\bar{\theta}) \mid \bar{\theta} \in \bar{Z}^1 \times \dots \times \bar{Z}^p\}.$$

Projecting the n^{th} time slice onto X , we have regions in X for the nonautonomous system (2.2) at time $t = \frac{2\pi}{\omega_\ell}n$ defined by

$$r_i(n) = \hat{X} \left(R_i \left(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right)$$

(assuming that the initial time slice is a phase slice with a set of lobe structures). Note how in the single core case we defined a core region in Σ^{θ_0} (by choosing a specific τ_c) that was fairly easy to deal with and, we argued, that was a meaningful region with which to monitor transport. By our above general definition of a region one can of course construct invariant regions in Σ^{θ_0} whose piecewise continuous variation in θ behaves wildly or whose shape in the phase slice $\chi(\bar{\theta})$ is quite contorted, so that, though one could still monitor what goes in and out of these regions, the analysis would be cumbersome and the results not very useful. The goal of course is to use the definition to construct reasonable regions in the spirit of Section 3. For $m \in \mathbb{Z}$, $m \geq 1$, we denote by $L_{k,j}(m, \bar{\theta})$ the set of two dimensional lobes in $\chi(\bar{\theta})$ that leave $R_k(\bar{\theta} + 2\pi \frac{\omega}{\omega_\ell}(m-1))$ and enter $R_j(\bar{\theta} + 2\pi \frac{\omega}{\omega_\ell}m)$ (or, equivalently, leave $\mathcal{R}_k$ and enter $\mathcal{R}_j$) upon the m^{th} iterate of P_ε ; for $m \leq 0$, $L_{k,j}(m, \bar{\theta})$ denotes the set of lobes in $\chi(\bar{\theta})$ that leave $R_j(\bar{\theta} - 2\pi \frac{\omega}{\omega_\ell}|m|)$ and enter $R_k(\bar{\theta} - 2\pi \frac{\omega}{\omega_\ell}(|m|+1))$ upon the $(|m|+1)^{\text{th}}$ iterate of P_ε^{-1} . Projecting onto X , we obtain for the n^{th} time sample of the nonautonomous system (2.2) a labeling of lobes $\ell_{k,j}(m, n)$:

$$\ell_{k,j}(m, n) = \hat{X} \left(L_{k,j} \left(m, \theta_0 + 2\pi \frac{\omega}{\omega_\ell} n \right) \right).$$

As in the periodic case, the area of the portion of a given lobe that contains particles of species S_i is the area of the intersection of the region $r_i(0)$ with the lobe mapped back to $t = 0$. The geometrical constraints of the lobes in X for any time sample are identical to those of the time-periodic case: the two dimensional lobes are bounded by stable and unstable manifolds which cannot intersect themselves, but which intersect each other, with the added assumption of the turnstile lobes $\ell_{k,j}(1, n)$ being contained totally in $r_k(n)$. So the same arguments that give (6.1) and (6.2) for the time-periodic case give for the quasiperiodic case

$$(6.4) \quad T_{i,j}(n) = \delta_{ij} \mu(r_j(0)) + \sum_{k=1}^{N_R} \sum_{m=1}^n [\mu(\ell_{k,j}^i(m, 0)) - \mu(\ell_{j,k}^i(m, 0))],$$

where $T_{i,j}(n)$ is the area occupied by species S_i in region $r_j(n)$ at the n^{th} time sample, and

$$(6.5) \quad \begin{aligned} \mu(\ell_{k,j}^i(m, 0)) &= \sum_{s=1}^{N_R} \sum_{p=1}^m \mu(\ell_{k,j}(m, 0) \cap \ell_{i,s}(p, 0)) \\ &\quad - \sum_{s=1}^{N_R} \sum_{p=1}^{m-1} \mu(\ell_{k,j}(m, 0) \cap \ell_{s,i}(p, 0)). \end{aligned}$$

Applying $T_\varepsilon(\cdot; m-2) \circ \cdots \circ T_\varepsilon(\cdot; 0)$ to the right hand side of (6.5) and using that $T_\varepsilon(\cdot; n)$ is area-preserving for all n (we are still considering the area-preserving case), we obtain

$$(6.6) \quad \begin{aligned} \mu(\ell_{k,j}^i(m, 0)) &= \sum_{s=1}^{N_R} \sum_{p=0}^{m-1} \mu(\ell_{k,j}(1, m-1) \cap \ell_{i,s}(1-p, m-1)) \\ &\quad - \sum_{s=1}^{N_R} \sum_{p=1}^{m-1} \mu(\ell_{k,j}(1, m-1) \cap \ell_{s,i}(1-p, m-1)). \end{aligned}$$

This rewrites areas in terms of intersections of the turnstile lobes in the $(m - 1)^{th}$ time sample with the other lobes in the $(m - 1)^{th}$ time sample. *Equation (6.6) is identical in form to equation (6.3), with the $(m - 1)$ indice going along for the ride.*

The non-Hamiltonian case is extended from time-periodic to quasiperiodic vector fields in a similar manner, but we will not go through the formulas here: we refer the reader to Rom-Kedar and Wiggins [1990] for the time-periodic case (the additional complexity is that one must now of course keep track of the variation of a lobe's area as it maps from one time slice to the next).

7. Conclusions and explanation of extending the analysis to more general time dependences. On the one hand, as is the case with turbulent fluid flows, the behavior of nonlinear physical systems is often so complex that one can obtain only a very qualitative or partial understanding of the motion. On the other hand, there do exist nonlinear systems which display complicated dynamics, and yet with which one can obtain a fundamental and detailed global picture of the dynamics. At present, there is a wide gap between these two types of systems, and one would of course like to bridge this gap by extending some of the initial paradigms of low dimensional dynamical systems analysis to more complicated scenarios. It is in this spirit that we have extended some notions of transport in phase space associated with the classical Poincaré map reduction of two dimensional periodically forced systems to apply to a bi-infinite sequence of nonautonomous maps derived from a quasiperiodic two dimensional vector field. In a certain sense, this is a mild extension, since it still deals with a highly limited class of perturbation time dependences. We have deliberately focused on a simple extension (especially the two frequency case) since any nontrivial advance past the ubiquitous time-periodic case immediately entails a fundamental departure in the analysis (going from maps to sequences of maps) and the new concepts in the simple two frequency extension are for the most part robust (as we shall describe momentarily), applicable to more general time dependences. Our goal then has been to discuss these new concepts in the context of a simple class of systems.

Suppose we wish to consider more complicated perturbation time dependences and remove the restriction that $g(x, t; \mu, \varepsilon)$ takes on the form in (2.2). We can still always introduce the associated autonomous system:

$$(7.1) \quad \begin{aligned} \dot{x} &= JDH(x) + \varepsilon g(x, \theta; \mu, \varepsilon) \\ \dot{\theta} &= 1. \end{aligned}$$

The phase space is now $X \times \mathbb{R}$ (the θ phase space is no longer compact), and we write $\theta = t + \theta_0$. An unperturbed homoclinic orbit in the autonomous system phase space is now as shown in Figure 7.1(a). If one can establish for one's perturbation, as done in Section 2 for quasiperiodic perturbations, persistence of one or more invariant normally hyperbolic sets, and the existence of infinitely many intersections of global stable and unstable manifolds in a given time slice $\chi(\bar{\theta}) = \{(x, \theta) | \theta = \bar{\theta}\}$ (and hence trivially in all time slices), then one has an invariant lobe structure upon which to define transport (see Figure 7.1(b)). One can define an invariant core boundary and label the lobes in $X \times \mathbb{R}$ just as done in the quasiperiodic case (things will thus be

piecewise continuous in θ). A sequence of monotonically increasing real numbers defines a sequence of sample times, and the evolution under (7.1) from one time slice to the next defines a sequence of maps. For any given time slice a two dimensional lobe structure, a core boundary, and lobe labeling is defined by the intersection of the time slice with the invariant objects in $X \times \mathbf{R}$. Using the invariance of the lobe structure in the autonomous system phase space, one can monitor how the lobes map from one time slice to the next, identify turnstile lobes, and from this define and quantify entrainment and detrainment with respect to the time-dependent core, and so forth. One can simulate the lobe structure in any time slice using a method just like the double phase slice method described in Appendix 4 (more appropriately called a *double time slice method* here).

Figure 7.1 (a) An unperturbed homoclinic manifold in $X \times \mathbf{R}$;
(b) an invariant lobe structure in $X \times \mathbf{R}$ for the perturbed case.

Meyer and Sell [1989] establish persistence of invariant normally hyperbolic manifolds for all perturbations that are bounded and uniformly continuous in time, and then construct a Melnikov functional. Stoffer [1988a], [1988b] defines a Melnikov

function for nonautonomous maps and shows that the sequence of maps admits a hyperbolic set if the Melnikov function has infinitely many simple zeros with derivatives bounded away from zero. Their Melnikov analysis applying to (7.1) can be thought of as providing the following function:

$$(7.2) \quad M(s, \theta; \mu) = \int_{-\infty}^{\infty} DH(x_h(t)) \cdot g(x_h(t), t + s + \theta; \mu, \varepsilon = 0) dt$$

(see Figure 7.1(a) for s and θ). The same arguments that lead to the expression for average flux (4.9) with the quasiperiodic Melnikov function (2.6) gives expression (4.9) with the Melnikov function (7.2) substituted:

$$(7.3) \quad \begin{aligned} \Phi &= \lim_{T \rightarrow \infty} \frac{\varepsilon}{2T} \int_0^T |M(s, \theta; \mu)| ds + O(\varepsilon^2) \\ &= \lim_{T \rightarrow \infty} \frac{\varepsilon}{2T} \int_0^T |M(s, \theta; \mu)| d\theta + O(\varepsilon^2), \end{aligned}$$

(instead of using (4.7), one uses $M(s, \theta; \mu) = M(s - \tau, \theta + \tau; \mu)$, $\tau \in \mathbb{R}$, and then again employs a simple change of variables). Thus the expression for average flux extends trivially. Note that, from (7.2), the first and second expressions of (7.3) will be independent of θ and s , respectively. In addition, the geometrical constraints of the two dimensional lobes in any time slice are identical to those of the time-periodic case (refer back to Section 6), so the transport equations of Sections 4 and 6 are identical under more general time dependences. As we see, then, the ideas for quasiperiodic perturbations are robust and extend easily to more general time dependences. For the more general perturbations, however, the global picture of the dynamics may be less apparent than in the quasiperiodic case, especially the two frequency case, since the θ variable is no longer compact. For instance, to find the turnstile lobes in each time slice one will truly have to deal with an infinite domain $\theta \in \mathbb{R}$, and one will not be able to give a global portrayal of the choice of τ_c and the invariant core boundary. Thus, because of their simplicity and geometric appeal, we wish to stress two frequency systems as workable and illuminating examples of chaotic transport under a bi-infinite sequence of nonrepeating maps.

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Appendix 1. A quasiperiodically oscillating vortex pair flow and a quasiperiodic Duffing oscillator.

We define two dynamical systems that are referred to throughout the paper, and derive their generalized Melnikov functions and relative scaling functions.

(A1.1) *An oscillating vortex pair (OVP) flow*

Consider the quasiperiodic generalization of the oscillating vortex pair flow studied by Rom-Kedar et al. [1990]. As mentioned in the introduction, this two dimensional fluid flow consists of a pair of point vortices of equal and opposite strength $\pm\Gamma$ in the presence of an oscillating strain-rate field which perturbs the vortex motion and which shall be referred to as the forcing term (even though it is understood that the net perturbation of the fluid flow is a sum of this forcing term and the effects of the vortex response). The stream function of the quasiperiodic forcing case is, in the comoving frame,

$$\begin{aligned} \psi(x_1, x_2, t) = & \frac{-\Gamma}{4\pi} \log \left[\frac{(x_1 - x_1^v(t))^2 + (x_2 - x_2^v(t))^2}{(x_1 - x_1^v(t))^2 + (x_2 + x_2^v(t))^2} \right] \\ & - V_v x_2 + \varepsilon x_1 x_2 \left(\sum_{i=1}^{\ell} \omega_i f_i \sin(\omega_i t + \theta_{i_0}) \right), \end{aligned}$$

where $(x_1^v(t), \pm x_2^v(t))$ are the vortex positions, V_v is the average velocity of the vortex pair in the lab frame, and $\varepsilon \omega_i f_i$ is the strain rate amplitude associated with the i^{th} frequency (εf_i is non-dimensional). For $\varepsilon = 0$, $(x_1^v, x_2^v) = (0, d)$ and $V_v = \Gamma/4\pi d$. Let us specialize to the two frequency case and use the dimensionless variables $x_1/d \rightarrow x_1$, $x_2/d \rightarrow x_2$, $\Gamma t/2\pi d^2 \rightarrow t$, $V_v 2\pi d/\Gamma \rightarrow v_v$, $2\pi \omega_i d^2/\Gamma \rightarrow \omega_i$ ($i = 1, 2$). The flow is then specified by

(A1.1)

$$\begin{aligned} \dot{x}_1 = & - \left[\frac{x_2 - x_2^v(t)}{(x_1 - x_1^v(t))^2 + (x_2 - x_2^v(t))^2} - \frac{x_2 + x_2^v(t)}{(x_1 - x_1^v(t))^2 + (x_2 + x_2^v(t))^2} \right] \\ & - v_v + \varepsilon x_1 \{ \omega_1 f_1 \sin(\omega_1 t + \theta_{1_0}) + \omega_2 f_2 \sin(\omega_2 t + \theta_{2_0}) \} \\ \dot{x}_2 = & (x_1 - x_1^v(t)) \left[\frac{1}{(x_1 - x_1^v(t))^2 + (x_2 - x_2^v(t))^2} - \frac{1}{(x_1 - x_1^v(t))^2 + (x_2 + x_2^v(t))^2} \right] \\ & - \varepsilon x_2 \{ \omega_1 f_1 \sin(\omega_1 t + \theta_{1_0}) + \omega_2 f_2 \sin(\omega_2 t + \theta_{2_0}) \}, \end{aligned}$$

where

$$\begin{aligned} x_1^v(t) = & e^{-\varepsilon(f_1 \cos(\omega_1 t + \theta_{1_0}) + f_2 \cos(\omega_2 t + \theta_{2_0}) - f)} \\ & \cdot \int \left(\frac{1}{2} - v_v e^{\varepsilon(f_1 \cos(\omega_1 t + \theta_{1_0}) + f_2 \cos(\omega_2 t + \theta_{2_0}) - f)} \right) dt \\ x_2^v(t) = & e^{\varepsilon(f_1 \cos(\omega_1 t + \theta_{1_0}) + f_2 \cos(\omega_2 t + \theta_{2_0}) - f)}, \end{aligned}$$

and v_v is chosen so that

$$\lim_{T \rightarrow \infty} \int_0^T \left(\frac{1}{2} - v_v e^{\varepsilon(f_1 \cos(\omega_1 t + \theta_{1_0}) + f_2 \cos(\omega_2 t + \theta_{2_0}) - f)} \right) dt = 0.$$

The vortex solutions $(x_1^v(t), \pm x_2^v(t))$ are found using that each vortex is advected by the flow due to the other vortex and the oscillating strain-rate field, so the vortices satisfy

$$\begin{aligned} \frac{dx_1^v}{dt} = & \frac{1}{2x_2^v} - v_v + \varepsilon x_1^v \{ \omega_1 f_1 \sin(\omega_1 t + \theta_{1_0}) + \omega_2 f_2 \sin(\omega_2 t + \theta_{2_0}) \} \\ \frac{dx_2^v}{dt} = & -\varepsilon x_2^v \{ \omega_1 f_1 \sin(\omega_1 t + \theta_{1_0}) + \omega_2 f_2 \sin(\omega_2 t + \theta_{2_0}) \}. \end{aligned}$$

Rom-Kedar et al. [1990] choose, for the single frequency analysis with $\theta_{1_0} = 0$, the initial conditions $(x_1^v(0), x_2^v(0)) = (0, 1)$, which guarantees a vortex response symmetric about $x_1 = 0$ to first order in ε . A simple quasiperiodic generalization will need to keep this symmetry, which is done by choosing the constant of integration in the x_1^v expression to be zero. The x_2^v behavior is determined uniquely by the choice of f : a choice of $f = f_1 \cos \theta_{1_0} + f_2 \cos \theta_{2_0}$ or $f = f_1 + f_2$ follows the spirit of Rom-Kedar et al. [1990]; alternatively, one could choose $f = 0$ to obtain a perturbation of mean zero in the x_2 direction. We choose $f = f_1 + f_2$ for our simulations. The periodic forcing simulation shown in the introduction corresponds to equations (A1.1) with $f_2 = 0$, $\varepsilon f_1 = 0.1$, $\omega_1 = 2.5$, $\theta_{1_0} = 0$, $\varepsilon f = 0.1$.

The equations for x_1 and x_2 in (A1.1) define a two dimensional nonautonomous dynamical system, and the motion of (x_1, x_2) for a given $\varepsilon, f_1, f_2, \omega_1, \omega_2, \theta_{1_0}, \theta_{2_0}$ and choice of initial vortex conditions is the fluid flow whose transport we address. The net perturbation is a sum of the forcing term, linear in ε , and the vortex response, nonlinear in ε ; by Taylor expanding the vortex term about $\varepsilon = 0$, the governing equations in the single frequency case can be put in the autonomous form

$$(A1.2) \quad \begin{aligned} \dot{x} &= JDH(x) + \varepsilon g^{per}(x, \theta_1; \omega_1) + 0(\varepsilon^2) \\ \dot{\theta}_1 &= \omega_1, \end{aligned}$$

(see Rom-Kedar et al. [1990]), and in the two frequency case they will have the form

$$(A1.3) \quad \begin{aligned} \dot{x} &= JDH(x) + \varepsilon g^{qp}(x, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2) + 0(\varepsilon^2) \\ \dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2, \end{aligned}$$

where $x = (x_1, x_2)$. The quasiperiodic forcing term, and thus the first order expansion term in the vortex response, is a superposition of two periodic forcing terms. Hence

$$(A1.4) \quad g^{qp}(x, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2) = f_1 g^{per}(x, \theta_1; \omega_1) + f_2 g^{per}(x, \theta_2; \omega_2).$$

Figure A1.1(a) shows some level sets in $\Sigma^{\theta_{2_0}}$ of the unperturbed ($\varepsilon = 0$) autonomous system (A1.3).

The Melnikov function for the single frequency case (A1.2),

$$(A1.5) \quad M^{per}(s, \theta_1; \omega_1) \equiv \int_{-\infty}^{\infty} DH(x_h(t)) \cdot g^{per}(x_h(t), \omega_1 t + (\omega_1 s + \theta_1); \omega_1) dt,$$

is, for $x_h(t)$ equal to the upper unperturbed heteroclinic orbit in X ,

$$(A1.6) \quad M^{per}(s, \theta_1; \omega_1) = \omega_1 F(\omega_1^{-1}) \sin(\omega_1 s + \theta_1),$$

where $F(\omega_1^{-1})$ is shown in Figure A1.2 (see Rom-Kedar et al. [1990]). From Section 2, the two frequency Melnikov function is

$$\begin{aligned} M^{qp}(s, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2) \\ \equiv \int_{-\infty}^{\infty} DH(x_h(t)) \cdot g^{qp}(x_h(t), \omega_1 t + \omega_1 s + \theta_1, \omega_2 t + \omega_2 s + \theta_2; f_1, f_2, \omega_1, \omega_2) dt, \end{aligned}$$

which by (A1.4) and (A1.5) satisfies for the same $x_h(t)$ as above

$$M^{qp}(s, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2) = f_1 M^{per}(s, \theta_1; \omega_1) + f_2 M^{per}(s, \theta_2; \omega_2).$$

Figure A1.1 (a) Some level sets in $\Sigma^{\theta_{20}}$ of the unperturbed OVP flow. (b) The lobe structure in $\Sigma^{\theta_{20}}$ of the perturbed OVP flow.

Figure A1.2 The plot of $F(\omega_1^{-1})$ in equation (A1.6). Taken from Rom-Kedar et al. [1990].

Hence by (A1.6)

$$(A1.7) \quad \begin{aligned} M^{qp}(s, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2) \\ = f_1 \omega_1 F(\omega_1^{-1}) \sin(\omega_1 s + \theta_1) + f_2 \omega_2 F(\omega_2^{-1}) \sin(\omega_2 s + \theta_2). \end{aligned}$$

From Section 2 we refer to the ratio of each Melnikov function amplitude $f_i \omega_i F(\omega_i^{-1})$ to the corresponding relative perturbation amplitude $\omega_i f_i$ as the *relative scaling factor* associated with frequency ω_i . The fact that the relative scaling factors $F(\omega_i^{-1})$ are frequency dependent is pertinent to the study of transport rates, in particular to a comparison of average flux between single and multiple frequency forcing. Note how all scaling factors are determined by the single *relative scaling function* $F(\omega^{-1})$. M^{qp} is easily seen to have zero sets in all phase slices of $\Sigma^{\theta_{2_0}}$, and hence the upper and lower stable and unstable manifolds in $\Sigma^{\theta_{2_0}}$ of the perturbed flow form the boundary of the defined three dimensional heteroclinic lobe structure (see Figure A1.1(b)) and intersect each phase slice to give the boundary of a two dimensional lobe structure. The physical situation can be pictured as the two dimensional fluid mapping from one time slice to the next in $\Sigma^{\theta_{2_0}}$ with each successive time sample. The lobe structure in $\Sigma^{\theta_{2_0}}$ thus constrains the motion of the fluid in the tangle region, as described in Section 3.

(A1.2) The Duffing oscillator

In contrast to the OVP flow, consider the quasiperiodically forced Duffing oscillator:

$$(A1.8) \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \beta x_1 - x_1^3 + \varepsilon \left[\sum_{i=1}^{\ell} f_i \cos(\omega_i t + \theta_{i_0}) - \gamma x_2 \right]. \end{aligned}$$

Consider again the two frequency case $\ell = 2$. The physical system is now a point in X , rather than all of X (as in the fluid example). However, the chaotic nature of the oscillator's homoclinic tangle region (see Section 5) encourages that we still develop a global picture of the dynamics in X . Though the previous example involved an incompressible fluid and hence a Hamiltonian (i.e. area-preserving) perturbation, there is now a dissipative term ($-\varepsilon \gamma x_2$) in the perturbing vector field. The lobe structure of Section 2 did not assume Hamiltonian perturbations, so we can have a lobe structure upon which to discuss transport even in non-Hamiltonian systems.

The two frequency autonomous system (with $\beta = 1$) is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 (1 - x_1^2) + \varepsilon [f_1 \cos \theta_1 + f_2 \cos \theta_2 - \gamma x_2] \\ \dot{\theta}_1 &= \omega_1 \\ \dot{\theta}_2 &= \omega_2. \end{aligned}$$

The generalized Melnikov function is

$$\begin{aligned} M(s, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2, \gamma) = \\ \int_{-\infty}^{\infty} DH(x_h(t)) \cdot g^{qp}(x_h(t), \omega_1 t + (\omega_1 s + \theta_1), \omega_2 t + (\omega_2 s + \theta_2); f_1, f_2, \gamma) dt, \end{aligned}$$

where

$$H(x_1, x_2) = \frac{x_2^2}{2} - \frac{x_1^2}{2} + \frac{x_1^4}{4}$$

and

$$g^{qp}(x_1, x_2, \theta_1, \theta_2; f_1, f_2, \gamma) \equiv (g_1, g_2) = (0, f_1 \cos \theta_1 + f_2 \cos \theta_2 - \gamma x_2).$$

Figure A1.3(a) shows in $\Sigma^{\theta_2_0}$ some level sets of the unperturbed system. Using that the x component of the unperturbed homoclinic orbits is

$$x_h(t) = (x_{1_h}(t), x_{2_h}(t)) = \pm \sqrt{2} \operatorname{sech}(t)(1, -\tanh(t)),$$

the Melnikov function is

(A1.9)

$$\begin{aligned} M(s, \theta_1, \theta_2; f_1, f_2, \omega_1, \omega_2, \gamma) = & -\frac{4\gamma}{3} \pm \sqrt{2} \pi f_1 \omega_1 \operatorname{sech}\left(\frac{\pi \omega_1}{2}\right) \sin(\omega_1 s + \theta_1) \\ & \pm \sqrt{2} \pi f_2 \omega_2 \operatorname{sech}\left(\frac{\pi \omega_2}{2}\right) \sin(\omega_2 s + \theta_2). \end{aligned}$$

Figure A1.3 (a) Some level sets in $\Sigma^{\theta_2_0}$ of the unperturbed Duffing oscillator. (b) The lobe structure in $\Sigma^{\theta_2_0}$ of the perturbed nondissipative Duffing oscillator.

Figure A1.4 shows a plot of the relative scaling function $\sqrt{2}\pi\omega\text{sech}(\pi\omega/2)$ versus ω . Zero sets of the Melnikov function are not guaranteed here for $\varepsilon > 0$ because of the presence of the dissipative term; Ide and Wiggins [1989] study where in parameter space the zero sets do occur. For $\gamma = 0$ (the Hamiltonian case), zero sets are guaranteed for all phase slices in $\Sigma^{\theta_{2_0}}$ and hence the homoclinic lobe structure exists in this case in each phase slice (see Figure A1.3(b)).

Figure A1.4 The relative scaling function for the Duffing oscillator.

The OVP fluid and the nondissipative Duffing oscillator both have Melnikov functions in $\Sigma^{\theta_{2_0}=0}$ of the form

$$(A1.10) \quad M(s, \theta_1, \theta_{2_0} = 0; \nu) = A_1(\mu, \omega_1) \sin(\omega_1 s + \theta_1) + A_2(\mu, \omega_2) \sin(\omega_2 s),$$

where $A_1(\mu, \omega_1), A_2(\mu, \omega_2)$ are independent amplitudes. Since the Poincaré maps defined for any two cross sections $\Sigma^{\theta_{2_0}}, \Sigma^{\theta'_{2_0}}$ are topologically conjugate to one another, we choose for simplicity to set $\theta_{2_0} = 0$ in our transport examples.

Appendix 2. The persistence of zero set crossings in parametrized families.

Let

$$\tilde{d}(s, \theta_1; A_1, \varepsilon) = \frac{M(s, \theta_1, \theta_{2_0} = 0; A_1, \omega_1, \omega_2)}{\|DH(x_h(-s))\|} + O(\varepsilon),$$

where

$$M(s, \theta_1, \theta_{2_0} = 0; A_1, \omega_1, \omega_2) = A_1 \sin(\omega_1 s + \theta_1) + (2 - A_1) \sin(\omega_2 s)$$

and the denominator is defined in Section 2. The conditions for the zero sets of the Melnikov function crossing at $(\bar{s}, \bar{\theta}_1; \bar{A}_1)$ are

$$(A2.1) \quad \begin{aligned} \tilde{d}(\bar{s}, \bar{\theta}_1; \bar{A}_1, 0) &= 0 \\ \frac{\partial \tilde{d}}{\partial s}(\bar{s}, \bar{\theta}_1; \bar{A}_1, 0) &= 0 \\ \frac{\partial \tilde{d}}{\partial \theta_1}(\bar{s}, \bar{\theta}_1; \bar{A}_1, 0) &= 0. \end{aligned}$$

Since $\det\{D_{(s, \theta_1, A_1)} f(\bar{s}, \bar{\theta}_1; \bar{A}_1, 0)\} \neq 0$, where f denotes the three component vector that is the left hand side of equation (A2.1), the implicit function theorem implies that equations (A2.1) can be continued to non-zero ε . Hence the intersection manifolds cross within $0(\varepsilon)$ of $(\bar{s}, \bar{\theta}_1)$ for some $A_1 = \bar{A}_1 + 0(\varepsilon)$.

Further, with the systems defined in Appendix 1, for each crossing of the $M = 0$ sets, the intersection manifolds will cross for the same value of A_1 . Using the symmetries of the full equations, one can easily establish that if the intersection manifolds cross within $0(\varepsilon)$ of $(\bar{s}, \bar{\theta}_1)$ at some value of A_1 , they will also cross within $0(\varepsilon)$ of $(-\bar{s}, -\bar{\theta}_1)$ at the same value of A_1 . Applying $P_\varepsilon^{\pm n}$ to these cross points implies that for each crossing of the $M = 0$ sets, the intersection manifolds will cross for the same value of A_1 .

Appendix 3. Proofs for Section 2.

Lemma 2.3. Suppose there exists an $(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1})$ for some $\bar{\nu}$ such that

$$(A3.1) \quad \begin{aligned} M(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) &= 0 \\ \frac{\partial M}{\partial s}(\bar{s}, \bar{\theta}_1, \dots, \bar{\theta}_{\ell-1}, \theta_{\ell_0}; \bar{\nu}) &\neq 0. \end{aligned}$$

Theorem 2.2 implies that there exists a $\Delta\theta \in T^{\ell-1}$ such that defined on the interval $(\bar{\theta} - \Delta\theta, \bar{\theta} + \Delta\theta)$ there is a C^r function $s(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ that parametrizes a segment of an $(\ell - 1)$ dimensional PIM. From the definition of the generalized Melnikov function (2.6) it immediately follows that

$$(A3.2) \quad \begin{aligned} M(\bar{s} + \frac{2\pi}{\omega_\ell} n, \bar{\theta}_1 - 2\pi \frac{\omega_1}{\omega_\ell} n, \dots, \bar{\theta}_{\ell-1} - 2\pi \frac{\omega_{\ell-1}}{\omega_\ell} n, \theta_{\ell_0}; \bar{\nu}) &= 0 \\ \frac{\partial M}{\partial s}(\bar{s} + \frac{2\pi}{\omega_\ell} n, \bar{\theta}_1 - 2\pi \frac{\omega_1}{\omega_\ell} n, \dots, \bar{\theta}_{\ell-1} - 2\pi \frac{\omega_{\ell-1}}{\omega_\ell} n, \theta_{\ell_0}; \bar{\nu}) &\neq 0 \end{aligned} \quad \forall n \in \mathbb{Z}.$$

When all the frequencies are mutually commensurate, $\bar{\theta} - 2\pi \frac{\omega}{\omega_t} n$, $n \in Z$, visits $\bar{\theta}$ identically a countable infinity of times, and hence by (A3.2) there are an infinity of s values among $\{\bar{s} + \frac{2\pi}{\omega_t} n; n \in Z\}$ that satisfy (A3.1). Hence, by Theorem 2.2 there exists a countable infinity of transversal intersection points of $W^s(\tau_\epsilon^a)$ and $W^u(\tau_\epsilon^b)$ in the phase slice $\chi(\bar{\theta})$. When one or more pairs of frequencies are incommensurate, it is still true that for $\Delta\theta' \in T^{\ell-1}$ *arbitrarily small* there are a countable infinity of $n \in Z$ such that $\bar{\theta} + 2\pi \frac{\omega}{\omega_t} n \in [\bar{\theta} - \Delta\theta', \bar{\theta} + \Delta\theta']$. By (A3.2) and the comments that precede it, for each such n there exists a C^r function $s_n(\theta_1, \dots, \theta_{\ell-1}; \theta_{\ell_0})$ defined on the interval $[\bar{\theta} + 2\pi \frac{\omega}{\omega_t} n - \Delta\theta, \bar{\theta} + 2\pi \frac{\omega}{\omega_t} n + \Delta\theta]$ that parametrizes a segment of an $(\ell - 1)$ dimensional PIM. As long as we choose $\Delta\theta' < \Delta\theta$ (by this we mean each component of $\Delta\theta'$ is smaller than the corresponding component of $\Delta\theta$), the interval $[\bar{\theta} + 2\pi \frac{\omega}{\omega_t} n - \Delta\theta, \bar{\theta} + 2\pi \frac{\omega}{\omega_t} n + \Delta\theta]$ contains $\bar{\theta}$, and hence $\chi(\bar{\theta})$ contains a countable infinity of transversal intersection points of $W^s(\tau_\epsilon^a)$ and $W^u(\tau_\epsilon^b)$ in the phase slice $\chi(\bar{\theta})$.

Lemma 2.4. This follows immediately from the invariance of $W^s(\tau_\epsilon^a)$ and $W^u(\tau_\epsilon^b)$ under P_ϵ . An intersection of $W^s(\tau_\epsilon^a)$ and $W^u(\tau_\epsilon^b)$ remains an intersection under $P_\epsilon^{\pm 1}$. Hence, if there exists a countable infinity of intersection points in the phase slice $\chi(\bar{\theta})$, there will be a countable infinity of intersection points in the phase slice $\chi(\bar{\theta} + 2\pi \frac{\omega}{\omega_t} n)$ for all $n \in Z$. If all frequencies are mutually incommensurate, then $\{\bar{\theta} + 2\pi \frac{\omega}{\omega_t} n; n \in Z\}$ is dense in $T^{\ell-1}$, which implies that there is a countable infinity of intersection points in all phase slices of $\Sigma^{\theta_{\ell_0}}$. If one or more pairs of frequencies are commensurate then $\bar{\theta} - 2\pi \frac{\omega}{\omega_t} n$ does *not* in general visit $T^{\ell-1}$ densely, and hence we can only conclude that there exist a countable infinity of intersection points in all phase slices defined either over all of $T^{\ell-1}$ or some subset of $T^{\ell-1}$.

Theorem 2.5. This follows immediately from Lemmas 2.3 and 2.4.

Appendix 4. Computing lobe structures using a *double phase slice sampling* method.

Suppose we wish to portray lobe boundaries in X at time $t = \frac{2\pi}{\omega_t} n$, or, equivalently, the invariant lobe boundaries in the time slice $\chi(\theta_0 + 2\pi \frac{\omega}{\omega_t} n)$. For simplicity of discussion, let us consider the two frequency homoclinic case. Evolving the curve

$$W_{\text{loc}}^u(\tau_\epsilon) \cap \chi\left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} (n - i)\right)$$

forward in time according to (2.3) for i sample periods (where i is some positive integer) gives a curve which extends from

$$\tau_\epsilon \cap \chi\left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n\right)$$

along a finite length of

$$W^u(\tau_\epsilon) \cap \chi\left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n\right)$$

(the greater i is, of course, the greater the length will be). Similarly, evolving the curve

$$W_{\text{loc}}^s(\tau_\varepsilon) \cap \chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} (n+i) \right)$$

backwards in time according to (2.3) for i sample periods gives a curve which extends from

$$\tau_\varepsilon \cap \chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \right)$$

along a finite length of

$$W^s(\tau_\varepsilon) \cap \chi \left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n \right)$$

(see Figure A4.1).

Figure A4.1 Simulating a finite length of (a) $W^u(\tau_\varepsilon)$ and (b) $W^s(\tau_\varepsilon)$ in the n^{th} time slice for a two frequency homoclinic case.

The resulting two curves form the boundary of a finite number of lobes of the two dimensional lobe structure in the time slice $\chi\left(\theta_{1_0} + 2\pi \frac{\omega_1}{\omega_2} n\right)$.

We still need to be able to find $W_{\text{loc}}^u(\tau_\varepsilon)$ and $W_{\text{loc}}^s(\tau_\varepsilon)$ in the appropriate slices. The procedure for doing this is a straightforward generalization of the standard trial and error procedure used in the time-periodic case, so our discussion here will only be heuristic. Suppose for some arbitrary phase slice $\chi(\bar{\theta}_1)$ of $\Sigma^{\theta_{2_0}}$ we wish to find $\tau_\varepsilon \cap \chi(\bar{\theta}_1)$ and one or both of $W_{\text{loc}}^u(\tau_\varepsilon) \cap \chi(\bar{\theta}_1)$ and $W_{\text{loc}}^s(\tau_\varepsilon) \cap \chi(\bar{\theta}_1)$. Let $C_\pm$ be a closed curve in the phase slice $\chi\left(\bar{\theta}_1 \pm 2\pi \frac{\omega_1}{\omega_2} j\right)$ that contains the point $\tau_\varepsilon \cap \chi\left(\bar{\theta}_1 \pm 2\pi \frac{\omega_1}{\omega_2} j\right)$ and is pierced by $W_{\text{loc}}^u(\tau_\varepsilon)$ and $W_{\text{loc}}^s(\tau_\varepsilon)$, where j is some positive integer (see Figure A4.2).

Figure A4.2 Evolving $C_\pm$ to the phase slice $\chi(\bar{\theta}_1)$ in $\Sigma^{\theta_{2_0}}$ to obtain τ_ε and its local stable and unstable manifolds in that plane.

Due to the normal hyperbolicity of τ_ε , $P_\varepsilon^j(C_-)$ will be stretched along $W^u(\tau_\varepsilon) \cap \chi(\bar{\theta}_1)$ and $P_\varepsilon^{-j}(C_+)$ will be stretched along $W^s(\tau_\varepsilon) \cap \chi(\bar{\theta}_1)$. The region in $\chi(\bar{\theta}_1)$ bounded by $P_\varepsilon^j(C_-)$ will intersect with the region in $\chi(\bar{\theta}_1)$ bounded by $P_\varepsilon^{-j}(C_+)$ in one or more disjoint regions, one of which, $\Omega(\bar{\theta}_1)$, will contain $\tau_\varepsilon \cap \chi(\bar{\theta}_1)$ and shrink to zero area as $j \rightarrow \infty$ (again see Figure A4.2). Thus, the trial and error procedure,

similar in spirit to the time-periodic case, is to make reasonable tries at $C_{\pm}$ and evolve them to $\chi(\bar{\theta}_1)$ as described above to pinpoint $\tau_{\epsilon} \cap \chi(\bar{\theta}_1)$ and its local stable and unstable manifolds by watching how the two curves stretch and intersect. By choosing $C_{\pm}$ sufficiently small, we can obtain an arbitrarily good approximation of $W_{\text{loc}}^u(\tau_{\epsilon}) \cap \chi(\bar{\theta}_1)$ and $W_{\text{loc}}^s(\tau_{\epsilon}) \cap \chi(\bar{\theta}_1)$. When the system is near-integrable, one can make a good initial estimate of $C_{\pm}$ due to the knowledge of the location of the fixed point of the unperturbed system; otherwise one must employ a more arduous trial-and-error effort, which would be the case under time-periodic vector fields as well.

The procedure for simulating global stable and unstable manifolds in the n^{th} time slice of the autonomous system phase space is thus quite similar to the procedure for simulating the invariant lobe structure in the time-periodic case, except that one has to take into account that curves are, with each application of P_{ϵ} , mapped around T^1 in an enlarged phase space. Hence one finds the intersection of $W_{\text{loc}}^u(\tau_{\epsilon})$ and $W_{\text{loc}}^s(\tau_{\epsilon})$ with an appropriate pair of phase slices, which in turn are used to simulate a finite length of $W^u(\tau_{\epsilon})$ and $W^s(\tau_{\epsilon})$ in the desired phase slice. From our discussion of the two frequency homoclinic case, the procedure for more frequencies and for the heteroclinic case should be clear. Note how the described procedure contrasts with a previous suggestion by Moon and Holmes [1985] for analyzing the dynamics under quasiperiodic vector fields, a double Poincaré map method which essentially attempts to treat the system as periodic. Samples of an equation like (2.2) are taken only when both $\theta_2 = \bar{\theta}_2$ and $\theta_1 \in [\bar{\theta}_1 - \beta, \bar{\theta}_1 + \beta]$ (for some choice of $\bar{\theta}_1, \bar{\theta}_2$ and $\beta \ll 2\pi$), and the results are summed. The time between samples can be much longer with this approach, and there is a “fuzziness” of the resulting structure due to the finite width of the sampling window, 2β . In contrast to this method, we shall refer to our approach as a *double phase slice method*, since to simulate the lobe structure in any phase slice one evolves two curves, each originating in a different phase slice.

Appendix 5. Derivation of equation (4.8) from equations (4.5) and (4.7).

From equation (4.5):

$$(A5.1) \quad \Phi(\theta_0) = \frac{\omega_{\ell}}{2\pi} \frac{\epsilon}{2N} \sum_{n=0}^{N-1} I_n + O(\epsilon^2)$$

where

$$I_n = \int_{s_0(\theta_0 + 2\pi \frac{\omega}{\omega_{\ell}} n)}^{s_{-1}(\theta_0 + 2\pi \frac{\omega}{\omega_{\ell}} n)} \left| M \left(s, \theta_{1_0} + 2\pi \frac{\omega_1}{\omega_{\ell}} n, \dots, \theta_{(\ell-1)_0} + 2\pi \frac{\omega_{\ell-1}}{\omega_{\ell}} n, \theta_{\ell_0}; \nu \right) \right| ds.$$

From equation (4.7)

$$I_n = \int_{s_0(\theta_0 + 2\pi \frac{\omega}{\omega_{\ell}} n)}^{s_{-1}(\theta_0 + 2\pi \frac{\omega}{\omega_{\ell}} n)} \left| M \left(s + \frac{2\pi}{\omega_{\ell}} n, \theta_{1_0}, \dots, \theta_{(\ell-1)_0}, \theta_{\ell_0}; \nu \right) \right| ds.$$

Letting $s' = s + n \frac{2\pi}{\omega_\ell}$,

$$I_n = \int_{s_0(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n) + \frac{2\pi}{\omega_\ell} n}^{s_{-1}(\theta_0 + 2\pi \frac{\omega}{\omega_\ell} n) + \frac{2\pi}{\omega_\ell} n} |M(s', \theta_{1_0}, \dots, \theta_{(\ell-1)_0}, \theta_{\ell_0}; \nu)| ds',$$

which gives

$$(A5.2) \quad I_n = \int_{s_{-n}(\theta_0)}^{s_{-(n+1)}(\theta_0)} |M(s', \theta_{1_0}, \dots, \theta_{(\ell-1)_0}, \theta_{\ell_0}; \nu)| ds',$$

where

$$s_{-j}(\theta_0) = s \text{ value of } P_\epsilon^{-j}(\tau_c) \cap \chi(\theta_0) \quad \forall j \in \mathbb{Z}.$$

Substituting (A5.2) into (A5.1) gives (4.8).

Remark 1. The proof of equation (4.9) for one or more incommensurate frequency ratios is the same as the above proof with $N \rightarrow \infty$.

Remark 2. Note that (A5.2) with $n = -1$ implies that for non-Hamiltonian perturbations the turnstile lobes and their images have the same area through $O(\epsilon)$.

Appendix 6. Maximal average flux in a single frequency limit.

Consider the perturbation in (2.8) with $F_0(x; \mu) = 0$, $F_1 + F_2 = 1$, and the larger relative scaling factor at $F_1 = 1$. The average flux is given by

$$(A6.1) \quad \Phi(\theta_{1_0}) = \frac{\epsilon}{2} < |a_1 F_1 \sin(\omega_1 s + \theta_{1_0}) + a_2 F_2 \sin(\omega_2 s)| >_s,$$

where a_1 and a_2 denote the relative scaling factors associated with ω_1 and ω_2 , respectively, and $< >_s$ denotes the average over s . We rewrite this as

$$(A6.2) \quad \Phi(\theta_{1_0}) = \frac{\alpha}{2} < |\beta_1 \sin(\omega_1 s + \theta_{1_0}) + \beta_2 \sin(\omega_2 s)| >_s,$$

where

$$\begin{aligned} \alpha &= \epsilon(a_1 F_1 + a_2 F_2) \\ \beta_i &= \frac{a_i F_i}{a_1 F_1 + a_2 F_2} \quad \text{for } i = 1, 2. \end{aligned}$$

The new amplitudes β_i satisfy $\beta_1 + \beta_2 = 1$ and $(\beta_1(F_1 = 0), \beta_2(F_1 = 0)) = (0, 1)$, $(\beta_1(F_1 = 1), \beta_2(F_1 = 1)) = (1, 0)$; hence the expression $< | \cdot | >_s$ in (A6.2) has the profile of the equal relative scaling factor case, but skewed in F_1 ; it will thus still have absolute maxima at $F_1 = 0$ and 1 . The term α is monotonic increasing in F_1 , hence $\Phi(\theta_{1_0})$ will have an absolute maximum at $F_1 = 1$. Thus the single frequency case with the larger relative scaling factor corresponds to an absolute maximum of average flux for the above class of systems.

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