**FINITE DIMENSIONAL EXPONENTIAL ATTRACTOR
FOR THE PHASE FIELD MODEL**

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1. Introduction.

The phase field equations are a parabolic system of equations which describe the temperature u and an “order parameter” or phase field φ . The dimensionless temperature u is scaled so that $u = 0$ is the ordinary planar equilibrium melting temperature and φ is scaled so that φ near $+1$ is associated with the liquid phase and φ near -1 is the solid. The interface is defined as the set of points for which φ vanishes. We refer to Caginalp [3] for more details and references.

In their simplest form (neglecting anisotropy, etc.) the phase field equations can be written as [2]

$$\tau\varphi_t = \xi^2\Delta\varphi - g(\varphi) + 2u \quad \text{in } \Omega \times \mathbb{R}^+ \quad (1.1)$$

$$u_t + \frac{\ell}{2}\varphi_t = K\Delta u \quad \text{in } \Omega \times \mathbb{R}^+ \quad (1.2)$$

where Ω is an open bounded set of $\mathbb{R}^N$ ($N \geq 1$) with smooth boundary $\partial\Omega$, $g(s) = \frac{1}{2}(s^3 - s)$, and where ℓ and K are positive constants representing the dimensionless latent heat and diffusivity, respectively, while the positive parameters τ and ξ represent a time and a length scale.

In this paper we assume that the function g has the slightly more general form

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j \quad \text{with} \quad b_{2p-1} > 0, \quad p \geq 2. \quad (1.3)$$

As for the boundary conditions we consider three cases:

(i) the Dirichlet conditions

$$u = \varphi = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+; \quad (1.4)$$

(ii) the Neumann conditions

$$\frac{\partial u}{\partial n} = \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+ \quad (1.5)$$

where n denotes the unit outward normal on $\partial\Omega$; and

(iii) the periodic conditions

$$\begin{aligned} \varphi|_{x_i=0} &= \varphi|_{x_i=L_i}, & \varphi_{x_i}|_{x_i=0} &= \varphi_{x_i}|_{x_i=L_i}, & i &= 1, \dots, N, \\ u|_{x_i=0} &= u|_{x_i=L_i}, & u_{x_i}|_{x_i=0} &= u_{x_i}|_{x_i=L_i}, & i &= 1, \dots, N \end{aligned} \quad (1.6)$$

where we assume that $\Omega = \prod_{i=1}^N (0, L_i)$, $L_i > 0$.

When an initial value problem is considered we supplement the equations with the initial conditions

$$\varphi(x, 0) = \varphi_0(x), \quad u(x, 0) = u_0(x), \quad x \in \Omega. \quad (1.7)$$

The proofs will be given in the case of the Neumann boundary conditions (1.5); they would be quite similar for the periodic boundary conditions (1.6) and somewhat simpler for the Dirichlet boundary conditions (1.4). Either choice of the boundary conditions allows to work with the unknowns

$$(\varphi, v) := \left(\varphi, u + \frac{\ell}{2} \varphi \right)$$

so that, in the case of Neumann boundary conditions, we are led to the problem

$$P \quad \begin{cases} \tau \varphi_t = \xi^2 \Delta \varphi - g(\varphi) - \ell \varphi + 2v & \text{in } \Omega \times \mathbb{R}^+, \\ v_t = K \Delta v - \frac{K\ell}{2} \Delta \varphi & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial \varphi}{\partial n} = \frac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ \varphi(x, 0) = \varphi_0(x), \quad v(x, 0) = v_0(x) & x \in \Omega \end{cases}$$

where $v_0 = u_0 + \frac{\ell}{2} \varphi_0$.

For the mathematical setting we introduce the spaces

$$H = (L^2(\Omega))^2 \quad \text{and} \quad V = (H^1(\Omega))^2.$$

Note that the space integral of the function v in Problem P is conserved in time, namely

$$\int_{\Omega} v(x, t) dx = \int_{\Omega} v_0(x) dx \quad \text{for all } t > 0.$$

It is convenient to introduce the function spaces

$$H_{\beta} = \left\{ (\varphi, v) \in H; \frac{1}{|\Omega|} \int_{\Omega} v(x) dx = \beta \right\} \quad \text{and} \quad \mathcal{H}_{\alpha} = \bigcup_{|\beta| \leq \alpha} H_{\beta}.$$

In Section 2, we shall show that if $(\varphi_0, v_0) \in H$, there exists a unique solution $(\varphi(t), v(t)) = S(t)(\varphi_0, v_0)$ of Problem P such that $S(t)$ satisfies the usual semi-group properties and for each $t > 0$, $S(t)$ is a Lipschitz continuous operator from H into itself.

The semi-group $S(t)$ maps $\mathcal{H}_\alpha$ into itself for each $\alpha \geq 0$. In Section 3 we show that there exist absorbing sets in $\mathcal{H}_\alpha$ and in $\mathcal{H}_\alpha \cap V$; this implies in particular that the semi-group $S(t)$ is uniformly compact in $\mathcal{H}_\alpha$ so that the semi-group possesses a maximal attractor $\mathcal{A}_\alpha$ that is compact and connected. In fact, it turns out that there exists an absorbing set in $(C^m(\bar{\Omega}))^2$ for each $m \in \mathbb{N}^+$ so that $\mathcal{A}_\alpha$ is bounded in all the spaces $(C^m(\bar{\Omega}))^2$.

In Section 4, we prove the existence of an inertial set, namely a compact set which contains the attractor, is positively invariant by the semi-group, has a finite fractal dimension and attracts all solutions at an exponential rate. Our methods of proof strongly rely on those of Temam [10], Eden, Foias, Nicolaenko and Temam [4] and Eden, Milani and Nicolaenko [5].

Note Added. After this paper was completed we received a preprint by Bates and Zheng [1] who considered the same problem with $g(s) = \frac{1}{2}(s^3 - s)$ and $N \leq 3$. They showed (i) the existence of a universal attractor, (ii) the existence of inertial manifolds in the one space dimension and in the case of a square domain in two dimensions, and (iii) the existence of inertial sets for $N \leq 3$. Our proofs are based on working with $H = (L^2(\Omega))^2$ and are slightly simpler.

2. Existence of the Semi-Group.

The following well-posedness result extends that of Elliott and Zheng [6].

Theorem 2.1.

(i) *For any $(\varphi_0, v_0) \in H_\beta$, Problem P has a unique solution (φ, v) which satisfies*

$$(\varphi, v) \in L^\infty(0, T; H_\beta) \cap L^2(0, T; V), \quad \varphi \in L^{2p}(Q_T)$$

for all $T > 0$ where $Q_T := \Omega \times (0, T)$, and

$$(\varphi, v) \in C(\mathbb{R}^+; H_\beta).$$

(ii) *If $(\varphi_0, v_0) \in V \cap H_\beta$ and $\varphi_0 \in L^{2p}(\Omega)$, then*

$$(\varphi, v) \in L^\infty(0, T; V) \cap L^2(0, T; (H^2(\Omega))^2),$$

$$\varphi \in L^\infty(0, T; L^{2p}(\Omega)) \quad \text{and} \quad (\varphi_t, v_t) \in L^2(Q_T).$$

Moreover, the mapping

$$S(t) : (\varphi_0, v_0) \mapsto (\varphi(t), v(t))$$

is Lipschitz continuous on H for all $t > 0$ and $\{S(t)\}_{t \geq 0}$ is a semi-group on $\mathcal{H}_\alpha$.

Remark 2.1. Similar theorems were proved by Elliott and Zheng [6] when $\varphi_0 \in H^2(\Omega) \cap L^{2p}(\Omega)$ ($p = 2$) and $u_0 \in H^2(\Omega)$ and by Bates and Zheng [1] when $\varphi_0 \in H^1(\Omega) \cap L^{2p}(\Omega)$ ($p = 2$) and $u_0 \in L^2(\Omega)$.

Remark 2.2 A consequence of Theorem 2.1 is that for any $(\varphi_0, v_0) \in H_\beta$,

$$(\varphi, v) \in L^\infty(\delta, T; V) \cap L^2(\delta, T; (H^2(\Omega))^2)$$

and

$$\begin{aligned} \varphi &\in L^\infty(\delta, T; L^{2p}(\Omega)), \\ \text{and } (\varphi_t, v_t) &\in (L^2(\Omega \times (\delta, T)))^2 \end{aligned}$$

for all $T > 0$ and all $\delta \in (0, T)$.

Before proving Theorem 2.1 we recall some properties of the polynomial function g which will be useful in what follows, namely

$$\frac{3}{4b_{2p-1}^{q-1}} |g(s)|^q - C_q \leq g(s)|s|^{(2p-1)(q-1)-1} s \leq \frac{5}{4b_{2p-1}^{q-1}} |g(s)|^q + C_q \quad \forall s \in \mathbb{R}, \quad q \geq 1, \quad (2.1)$$

$$\frac{3}{4} b_{2p-1}^q |s|^{(2p-1)q} - C_q \leq |g(s)|^q \leq \frac{5}{4} b_{2p-1}^q |s|^{(2p-1)q} + C_q \quad \forall s \in \mathbb{R}, \quad q \geq 1, \quad (2.2)$$

$$g'(s) \geq -C \quad \forall s \in \mathbb{R} \quad (2.3)$$

for some positive constants C_q and C .

Proof of Theorem 2.1. The proof relies on the Galerkin method and we shall only give its main steps. Denote by

$$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_k \leq \dots$$

the eigenvalues of the operator $-\Delta$ with homogeneous Neumann boundary conditions and by w_k , $k = 1, \dots$ the corresponding unit eigenfunctions. Note that they constitute an orthogonal basis of $L^2(\Omega)$ and that they are smooth functions.

For each integer m we look for an approximate solution (φ_m, v_m) of the form

$$\varphi_m(t) = \sum_{i=1}^m \varphi_{im}(t) w_i, \quad v_m(t) = \sum_{i=1}^m v_{im}(t) w_i$$

satisfying

$$\tau \int_{\Omega} \varphi_{mt} w_j + \xi^2 \int_{\Omega} \nabla \varphi_m \nabla w_j = \int_{\Omega} (-g(\varphi_m) - \ell \varphi_m + 2v_m) w_j, \quad (2.4)$$

$$\int_{\Omega} v_{mt} w_j + K \int_{\Omega} \nabla v_m \nabla w_j = \frac{K\ell}{2} \int_{\Omega} \nabla \varphi_m \nabla w_j \quad (2.5)$$

for $j = 1, \dots, m$ and

$$\begin{aligned} \varphi_m(0) &= \varphi_{0m} \rightarrow \varphi_0 \text{ in } L^2(\Omega) \text{ as } m \rightarrow \infty \\ v_m(0) &= v_{0m} \rightarrow v_0 \text{ in } L^2(\Omega) \text{ as } m \rightarrow \infty. \end{aligned} \quad (2.6)$$

Problem (2.4)–(2.6) is an initial value problem for a system of $2m$ ordinary differential equations, so that it has a unique solution (φ_m, v_m) on some interval $(0, T_m)$, $T_m > 0$; in fact the a priori estimates below show that $T_m = +\infty$.

We multiply (2.4) by $\varphi_{jm}(t)$ and sum on $j = 1, \dots, m$ to obtain

$$\begin{aligned} \frac{\tau}{2} \frac{d}{dt} \int_{\Omega} \varphi_m^2 + \xi^2 \int_{\Omega} |\nabla \varphi_m|^2 &= \int_{\Omega} (-g(\varphi_m) - \ell \varphi_m + 2v_m) \varphi_m \\ &\leq \int_{\Omega} \left(-\frac{b_{2p-1}}{4} \varphi_m^{2p} - \ell \varphi_m^2 + \varepsilon v_m^2 + C_{\varepsilon} \right) \end{aligned} \quad (2.7)$$

by using (2.1), (2.2) (with $q = 1$), and the inequality $|ab| \leq \varepsilon a^2 + \varepsilon b^{2p} + C_{\varepsilon}$.

Multiplying (2.5) by $v_{jm}(t)$ and summing on $j = 1, \dots, m$, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} v_m^2 + K \int_{\Omega} |\nabla v_m|^2 = \frac{K\ell}{2} \int_{\Omega} \nabla \varphi_m \nabla v_m \leq \int_{\Omega} \frac{K}{2} |\nabla v_m|^2 + \frac{K\ell^2}{8} |\nabla \varphi_m|^2. \quad (2.8)$$

Next we multiply (2.8) by $\frac{4\xi^2}{K\ell^2}$ and add it up to (2.7). This gives

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \varphi_m^2 + \frac{2\xi^2}{K\ell^2} v_m^2 \right) + \frac{\xi^2}{2} \int_{\Omega} |\nabla \varphi_m|^2 + \frac{2\xi^2}{\ell^2} \int_{\Omega} |\nabla v_m|^2 \\ + \frac{b_{2p-1}}{4} \int_{\Omega} \varphi_m^{2p} + \ell \int_{\Omega} \varphi_m^2 \leq \varepsilon \int_{\Omega} v_m^2 + C_{\varepsilon}. \end{aligned} \quad (2.9)$$

Note that w_1 is a constant function, so that by taking $j = 1$ in (2.5), one gets $\frac{d}{dt} \int_{\Omega} v_m(x, t) dx = 0$; that is,

$$\frac{1}{|\Omega|} \int_{\Omega} v_m(x, t) dx = \frac{1}{|\Omega|} \int_{\Omega} v_0(x) dx = \beta. \quad (2.10)$$

By Poincaré's inequality [10, p. 50], there exists a positive constant $C(\Omega)$ such that

$$\int_{\Omega} (v_m - \beta)^2 \leq C(\Omega) \int_{\Omega} |\nabla v_m|^2$$

or equivalently

$$\int_{\Omega} v_m^2 \leq \beta^2 |\Omega| + C(\Omega) \int_{\Omega} |\nabla v_m|^2. \quad (2.11)$$

Substituting (2.11) into (2.9) and letting $\varepsilon = \frac{\xi^2}{\ell^2 C(\Omega)}$, we deduce that

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \varphi_m^2 + \frac{2\xi^2}{K\ell^2} v_m^2 \right) + \frac{\xi^2}{2} \int_{\Omega} |\nabla \varphi_m|^2 + \frac{\xi^2}{2\ell^2} \int_{\Omega} |\nabla v_m|^2 \\ + \frac{b_{2p-1}}{4} \int_{\Omega} \varphi_m^{2p} + \ell \int_{\Omega} \varphi_m^2 + \frac{\xi^2}{2\ell^2 C(\Omega)} \int_{\Omega} v_m^2 \leq C(1 + \beta^2). \end{aligned} \quad (2.12)$$

Integrating this inequality from 0 to T (with $T > 0$ arbitrary), we conclude that there exists a constant C depending on β such that

$$\|(\varphi_m, v_m)\|_{L^\infty(0,T;H)}, \|(\varphi_m, v_m)\|_{L^2(0,T;V)}, \|\varphi_m\|_{L^{2p}(Q_T)}^{2p} \leq C(\|\varphi_0, v_0\|_H + T).$$

These estimates will be necessary for the proof of (i). Next we give the estimates necessary for the proof of (ii). We now suppose that $\varphi_{0m} \rightarrow \varphi_0$ in $H^1(\Omega) \cap L^{2p}(\Omega)$ and $v_{0m} \rightarrow v_0$ in $H^1(\Omega)$. We multiply the equations (2.4) and (2.5) by $-\lambda_j \varphi_{jm}(t)$ and $-\lambda_j v_{jm}(t)$ respectively and sum on $j = 1, \dots, m$ to obtain, using also (2.3),

$$\begin{aligned} \frac{\tau}{2} \frac{d}{dt} \int_{\Omega} (\text{grad } \varphi_m)^2 + \xi^2 \int_{\Omega} (\Delta \varphi_m)^2 + \ell \int_{\Omega} (\text{grad } \varphi_m)^2 \\ \leq C_2 \int_{\Omega} (\text{grad } \varphi_m)^2 - 2 \int_{\Omega} v_m \Delta \varphi_m \\ = -C_2 \int_{\Omega} \varphi_m \Delta \varphi_m - 2 \int_{\Omega} v_m \Delta \varphi_m. \end{aligned}$$

This gives

$$\begin{aligned} \frac{\tau}{2} \frac{d}{dt} \int_{\Omega} (\text{grad } \varphi_m)^2 + \frac{\xi^2}{2} \int_{\Omega} (\Delta \varphi_m)^2 + \ell \int_{\Omega} (\text{grad } \varphi_m)^2 \\ \leq C \left(\int_{\Omega} \varphi_m^2 + \int_{\Omega} v_m^2 \right) \end{aligned} \quad (2.13)$$

For v_m we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} (\text{grad } v_m)^2 + \frac{K}{2} \int_{\Omega} (\Delta v_m)^2 \\ \leq \frac{K\ell^2}{8} \int_{\Omega} (\Delta \varphi_m)^2 \end{aligned} \quad (2.14)$$

Next we multiply (2.14) by $\frac{2\xi^2}{K\ell^2}$ and add it to (2.13). This yields

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} (\text{grad } \varphi_m)^2 + \frac{\xi^2}{K\ell^2} (\text{grad } v_m)^2 \right) \\ + \frac{\xi^2}{4} \int_{\Omega} (\Delta \varphi_m)^2 + \frac{\xi^2}{\ell^2} \int_{\Omega} (\Delta v_m)^2 \leq C(\|\varphi_0, v_0\|_H + T) \end{aligned} \quad (2.15)$$

where we have used the estimates for (i). Integrating this inequality from 0 to T ($T > 0$ is arbitrary), we deduce that there exists a positive constant C such that

$$\begin{aligned} \|\varphi_m\|_{L^\infty(0,T;H^1(\Omega))}, \|v_m\|_{L^\infty(0,T;H^1(\Omega))} \leq C \\ \|\varphi_m\|_{L^2(0,T;H^2(\Omega))}, \|v_m\|_{L^2(0,T;H^2(\Omega))} \leq C. \end{aligned}$$

Finally we multiply the j^{th} equation in (2.4) by $\varphi'_{jm}(t)$ and sum on $j = 1, \dots, m$. This gives

$$\begin{aligned} \tau \int_{\Omega} \varphi_{mt}^2 + \xi^2 \int_{\Omega} \text{grad } \varphi_m \text{ grad } \varphi_{mt} + \int_{\Omega} g(\varphi_m) \varphi_{mt} \\ = -\ell \int_{\Omega} \varphi_m \varphi_{mt} + 2 \int_{\Omega} v_m \varphi_{mt} \end{aligned}$$

so that for all $t \in (0, T]$

$$\begin{aligned} & \tau \int \int_{Q_t} \varphi_{mt}^2 + \frac{\xi^2}{2} \int_{\Omega} (\text{grad } \varphi_m(t))^2 + \int_{\Omega} G(\varphi_m(t)) \\ & \leq \frac{\xi^2}{2} \int_{\Omega} (\text{grad } \varphi_{0m})^2 + \int_{\Omega} G(\varphi_{0m}) + \frac{\tau}{2} \int \int_{Q_t} \varphi_{mt}^2 \\ & \quad + C \left(\int \int_{Q_t} \varphi_m^2 + \int \int_{Q_t} v_m^2 \right) \end{aligned}$$

where $Q_t := \Omega \times (0, t)$ and where

$$G(s) = \int_0^s g(\tau) d\tau$$

is the primitive of g . We deduce from the properties of g and the previous estimates that

$$\tau \int \int_{Q_t} \varphi_{mt}^2 + \frac{3b_{2p-1}}{4p} \int_{\Omega} |\varphi_m(t)|^{2p} \leq C.$$

Thus

$$\|\varphi_{mt}\|_{L^2(Q_T)} \leq C$$

and

$$\|\varphi_m\|_{L^\infty(0, T; L^{2p}(\Omega))} \leq C.$$

Next we complete the proof of existence of a solution of Problem P , with only supposing that $(\varphi_0, v_0) \in H_\beta$. From the uniformly bounded sequence of the functions $\{(\varphi_m, v_m)\}_{m=1}^\infty$, we can extract a subsequence, which we denote again by (φ_m, v_m) , such that as $m \rightarrow \infty$

$$\begin{aligned} (\varphi_m, v_m) & \rightharpoonup (\varphi, v) \text{ weakly in } L^2(0, T; V) \quad \text{and} \quad \text{weak}^* \text{ in } L^\infty(0, T; H), \\ g(\varphi_m) & \rightharpoonup \chi \quad \text{weakly in } L^{\hat{p}}(Q_T) \quad (\hat{p} = 2p/(2p-1)) \end{aligned}$$

for some function $\chi \in L^{\hat{p}}(Q_T)$ and

$$(\varphi, v) \in L^\infty(0, T, H_\beta) \cap L^2(0, T, V) \text{ with } \varphi \in L^{2p}(Q_T).$$

Passing to the limit in (2.4)–(2.6), we find

$$\begin{aligned} \tau \frac{d}{dt} \int_{\Omega} \varphi w + \xi^2 \int_{\Omega} \nabla \varphi \nabla w &= \int_{\Omega} (-\chi - \ell \varphi + 2v) w \quad \forall w \in V \cap L^{2p}(\Omega), \\ \frac{d}{dt} \int_{\Omega} v z + K \int_{\Omega} \nabla v \nabla z &= \frac{K\ell}{2} \int_{\Omega} \nabla \varphi \nabla z \quad \forall z \in V, \\ \varphi(0) &= \varphi_0, \quad v(0) = v_0. \end{aligned}$$

This shows that

$$\begin{cases} \tau\varphi_t = \xi^2 \Delta\varphi - \chi - \ell\varphi + 2\tilde{v} + 2\beta & \text{in } L^2(0, T; V') + L^p(Q_T) \\ \tilde{v}_t = K\Delta\tilde{v} - \frac{K\ell}{2}\Delta\varphi & \text{in } L^2(0, T; V'). \end{cases}$$

With classical arguments [7] one sees that $(\varphi, v) \in C([0, T]; H_\beta)$. It remains to check that $\chi = g(\varphi)$. This is done by means of a standard monotonicity argument; we refer to Marion [9] for more details.

We have thus proven the existence part of Theorem 2.1. The uniqueness of the solution of Problem P and the Lipschitz continuity of $S(t)$ on H are a consequence of the following result.

Theorem 2.2. *Let (φ_1, v_1) and (φ_2, v_2) be two solutions of Problem P with initial functions (φ_{01}, v_{01}) and $(\varphi_{02}, v_{02}) \in H$. Set*

$$y(t) = \int_{\Omega} \left(\frac{\tau}{2}(\varphi_1 - \varphi_2)^2 + \frac{2\xi^2}{K\ell^2}(v_1 - v_2)^2 \right).$$

Then there exists a positive constant D depending only on τ, ξ, K, ℓ and g such that

$$y(t) \leq y(0)e^{Dt} \text{ for all } t > 0.$$

Proof. We multiply the difference of the equations for φ_1 and φ_2 by $\varphi_1 - \varphi_2$, the difference of the equations for v_1 and v_2 by $\frac{4\xi^2}{K\ell^2}(v_1 - v_2)$ and add up the resulting equations. This gives, after integrating by parts,

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2}(\varphi_1 - \varphi_2)^2 + \frac{2\xi^2}{K\ell^2}(v_1 - v_2)^2 \right) \\ &= -\xi^2 \int_{\Omega} |\nabla(\varphi_1 - \varphi_2)|^2 - \frac{4\xi^2}{\ell^2} \int_{\Omega} |\nabla(v_1 - v_2)|^2 + \frac{2\xi^2}{\ell} \int_{\Omega} \nabla(\varphi_1 - \varphi_2) \nabla(v_1 - v_2) \\ & \quad + \int_{\Omega} -(g(\varphi_1) - g(\varphi_2))(\varphi_1 - \varphi_2) - \ell(\varphi_1 - \varphi_2)^2 + 2(v_1 - v_2)(\varphi_1 - \varphi_2) \\ &\leq D \int_{\Omega} \left(\frac{\tau}{2}(\varphi_1 - \varphi_2)^2 + \frac{2\xi^2}{K\ell^2}(v_1 - v_2)^2 \right) \quad (\text{by (2.3)}) \end{aligned}$$

for some positive constant D . The assertion of Theorem 2.2 thus follows from Gronwall's inequality.

3. The Maximal Attractor.

In this section we shall show the existence of absorbing sets in $\mathcal{H}_\alpha$ and in $\mathcal{H}_\alpha \cap V$ for the semi-group $S(t)$ associated with Problem P and conclude that $S(t)$ possesses a

maximal attractor in $\mathcal{H}_\alpha$. We then show that the attractor is bounded in $(C^m(\bar{\Omega}))^2$ for all $m \in \mathbb{N}^+$.

In the sequel, we shall always assume that $(\varphi_0, v_0) \in H_\beta$ with $\beta \in [-\alpha, \alpha]$, $\alpha > 0$.

First we show the existence of an absorbing set in $\mathcal{H}_\alpha$.

Multiplying the equation for φ by φ and the equation for v by $\frac{4\xi^2}{K\ell^2}v$, integrating by parts and adding up the resulting equations, we get (cf. (2.12))

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \varphi^2 + \frac{2\xi^2}{K\ell^2} v^2 \right) + \frac{\xi^2}{2} \int_{\Omega} |\nabla \varphi|^2 + \frac{\xi^2}{2\ell^2} \int_{\Omega} |\nabla v|^2 \\ + \frac{b_{2p-1}}{4} \int_{\Omega} \varphi^{2p} + \ell \int_{\Omega} \varphi^2 + \frac{\xi^2}{2\ell^2 C(\Omega)} \int_{\Omega} v^2 \leq C(1 + \beta^2) \end{aligned} \quad (3.1)$$

from which we deduce that there exists a positive constant d such that

$$\frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \varphi^2 + \frac{2\xi^2}{K\ell^2} v^2 \right) + d \int_{\Omega} \left(\frac{\tau}{2} \varphi^2 + \frac{2\xi^2}{K\ell^2} v^2 \right) \leq C(1 + \beta^2).$$

Applying Gronwall's lemma we get

$$\int_{\Omega} \left(\frac{\tau}{2} \varphi^2(t) + \frac{2\xi^2}{K\ell^2} v^2(t) \right) \leq \left\{ \int_{\Omega} \left(\frac{\tau}{2} \varphi_0^2 + \frac{2\xi^2}{K\ell^2} v_0^2 \right) \right\} e^{-dt} + \frac{C(1 + \beta^2)}{d},$$

which in turn implies that there exist positive constants C_1 and C_2 such that

$$\int_{\Omega} \{ \varphi^2(t) + v^2(t) \} \leq C_1 \left(\int_{\Omega} \{ \varphi_0^2 + v_0^2 \} \right) e^{-dt} + C_2(1 + \beta^2). \quad (3.2)$$

Thus, we have proven the following:

Theorem 3.1. *For any $\alpha > 0$, there exist a constant $\mathcal{R} = \mathcal{R}(\alpha) > 0$ and a monotonic function $\mathcal{T}(\cdot)$ such that*

$$\|S(t)(\varphi_0, v_0)\|_H \leq \mathcal{R}(\alpha) \quad \text{if } t \geq \mathcal{T}(\|(\varphi_0, v_0)\|_H). \quad (3.3)$$

Proof. By taking $\mathcal{R}(\alpha) = \sqrt{1 + C_2(1 + \alpha^2)}$ and $\mathcal{T}(r) = \frac{1}{d} \ln(r^2 C_1)$, the assertion of the theorem follows immediately from (3.2).

We now proceed to show that the absorbing set is in $\mathcal{H}_\alpha \cap V$.

Multiplying the equation for φ by $-\Delta \varphi$ and the equation for v by $-\frac{2\xi^2}{K\ell^2} \Delta v$, and adding up the resulting equations, we get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} |\nabla \varphi|^2 + \frac{\xi^2}{K\ell^2} |\nabla v|^2 \right) + \int_{\Omega} \left(\xi^2 (\Delta \varphi)^2 + \frac{2\xi^2}{\ell^2} (\Delta v)^2 \right) \\ = \int_{\Omega} \left(g(\varphi) \Delta \varphi + \ell \varphi \Delta \varphi - 2v \Delta \varphi + \frac{\xi^2}{\ell} \Delta \varphi \Delta v \right). \end{aligned}$$

Using the estimate

$$\int_{\Omega} g(\varphi) \Delta \varphi = - \int_{\Omega} g'(\varphi) |\nabla \varphi|^2 \leq C \int_{\Omega} |\nabla \varphi|^2 = -C \int_{\Omega} \varphi \Delta \varphi$$

and the Cauchy inequality $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$, we get

$$\frac{d}{dt} \int_{\Omega} \left\{ \frac{\tau}{2} |\nabla \varphi|^2 + \frac{\xi^2}{K \ell^2} |\nabla v|^2 \right\} \leq C \left\{ \int_{\Omega} \varphi^2 + \int_{\Omega} v^2 \right\} \leq C \mathcal{R}^2(\alpha) \quad (3.4)$$

if $t \geq \mathcal{T}(R)$ where $R := \|(\varphi_0, v_0)\|_H$.

In order to be able to apply the uniform Gronwall lemma [10, Lemma 1.1, p. 89] we have to show that

$$\int_t^{t+r} \int_{\Omega} |\nabla \varphi|^2 \leq a_1, \quad \int_t^{t+r} \int_{\Omega} |\nabla v|^2 \leq a_2 \quad (3.5)$$

for some positive constants r, a_1, a_2 and all $t \geq \mathcal{T}(R)$. To that purpose we integrate (3.1) from t to $t+r$ to obtain

$$\frac{\xi^2}{2} \int_t^{t+r} \int_{\Omega} \left\{ |\nabla \varphi|^2 + \frac{1}{\ell^2} |\nabla v|^2 \right\} \leq C(1 + \alpha^2)r + C \mathcal{R}^2(\alpha) \quad \text{if } t \geq \mathcal{T}(R),$$

so that (3.5) is satisfied. Therefore we can apply the uniform Gronwall lemma [10, Lemma 1.1, p. 89] to (3.4) and (3.5), to get

$$\int_{\Omega} \left\{ \frac{\tau}{2} |\nabla \varphi(t)|^2 + \frac{\xi^2}{K \ell^2} |\nabla v(t)|^2 \right\} \leq C(\alpha)$$

for all $t \geq \mathcal{T}(R) + r$. This establishes the existence of an absorbing set in $\mathcal{H}_{\alpha} \cap V$.

It then follows from the existence of an absorbing set in $\mathcal{H}_{\alpha} \cap V$ that the semi-group $S(t)$ is uniformly compact in $\mathcal{H}_{\alpha}$.

We are now in a position to deduce from Temam [10, Theorem 1.1, p. 23] the existence of a universal attractor.

Theorem 3.2. *For every $\alpha \geq 0$ the semi-group $S(t)$ associated with Problem P maps $\mathcal{H}_{\alpha}$ into itself. It possesses in $\mathcal{H}_{\alpha}$ a maximal attractor $\mathcal{A}_{\alpha}$ that is compact and connected.*

In what follows we shall show that the solution enters an absorbing set of $(C^m(\bar{\Omega}))^2$ for each $m \in \mathbb{N}^+$. A consequence is that the attractor $\mathcal{A}_{\alpha}$ is also in $(C^m(\bar{\Omega}))^2$ for all $m \in \mathbb{N}^+$. In addition, this regularity result will allow us to show the existence of inertial sets for arbitrary space dimension in section 4.

We first derive some a priori estimates which we then use in a bootstrap argument.

Lemma 3.3. *Let $\delta \in (0, 1)$ and $q > 1$ be any two constants and assume that $\varphi_0 \in L^2(\Omega)$ and $v \in L^q(Q_1)$. Then there exists a positive constant $C = C(\delta, q)$ such that*

$$\|g(\varphi)\|_{L^q(Q_\delta^1)} \leq C(1 + \|v\|_{L^q(Q_1)}) \quad (3.6)$$

where $Q_\delta^1 := \Omega \times (\delta, 1)$ and $Q_1 := \Omega \times (0, 1)$.

Remark 3.1. The estimate (3.6) is independent of the initial value φ_0 . This can be true since g is super-linear. The condition $\varphi_0 \in L^2$ is needed only for the existence of solutions.

Proof. We first give a formal argument. Set $\gamma = \frac{(2p-1)q}{2p-2}$. Multiplying the equation for φ by $t^\gamma |\varphi|^{(2p-1)(q-1)-1} \varphi$ and integrating by parts, we get

$$\begin{aligned} & \frac{1}{(2p-1)(q-1)+1} \frac{d}{dt} \int_{\Omega} t^\gamma |\varphi|^{(2p-1)(q-1)+1} \\ & \leq \int_{\Omega} (-g(\varphi) - \ell\varphi + 2v) t^\gamma |\varphi|^{(2p-1)(q-1)-1} \varphi \\ & \quad + \frac{\gamma}{(2p-1)(q-1)+1} \int_{\Omega} t^{\gamma-1} |\varphi|^{(2p-1)(q-1)+1} \\ & \leq -\varepsilon \int_{\Omega} t^\gamma |g(\varphi)|^q + C(\varepsilon, p, q) \int_{\Omega} (t^\gamma |v|^q + 1 + t^\gamma) \quad \text{for some } \varepsilon > 0 \end{aligned} \quad (3.7)$$

by using (2.1), (2.2), the inequality $|v||g|^{q-1} \leq \varepsilon |g|^q + C_\varepsilon |v|^q$, and

$$t^{\gamma-1} |\varphi|^{(2p-1)(q-1)+1} = (t^\gamma |\varphi|^{(2p-1)q})^{\frac{(2p-1)(q-1)+1}{(2p-1)q}} \leq \varepsilon t^\gamma |\varphi|^{(2p-1)q} + C_\varepsilon.$$

Integrating the last inequality in (3.7) from 0 to 1, the assertion of the lemma follows.

In order to make this proof rigorous, one should first approximate v and φ_0 by smooth functions, derive (3.7) and (3.6) for this approximate solution and finally pass to the limit.

Next we recall some technical lemmas which will be useful in what follows.

Lemma 3.4. *Let $u_0 \in L^2(\Omega)$, $f \in L^q(Q_1)$ for some $q \in (1, \infty)$ and let u be the solution of the time evolution problem*

$$L \quad \begin{cases} u_t - \Delta u = f & \text{in } Q_1 \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \times (0, 1), \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases}$$

Then for each $0 < \delta < 1$, there exists a positive constant $C = C(\delta, \Omega)$ such that

$$\|u\|_{W_q^{2,1}(Q_\delta^1)} \leq C \left(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^q(Q_1)} \right). \quad (3.8)$$

Lemma 3.5. *One has the following embedding*

$$W_q^{2,1}(Q_\delta^1) \subset L^{q^*}(Q_\delta^1)$$

where

$$q^* = \begin{cases} \frac{(N+2)q}{N+2-2q} & \text{if } q < \frac{N+2}{2} \\ \text{any } \tilde{q} \text{ finite} & \text{if } q = \frac{N+2}{2} \end{cases}$$

and

$$W_q^{2,1}(Q_\delta^1) \subset C^{\lambda,\lambda/2}(\bar{Q}_\delta^1) \text{ with } \lambda = 2 - \frac{N+2}{q} \text{ if } q > \frac{N+2}{2} \text{ and } q \neq N+2. \quad (3.9)$$

Lemma 3.4 follows from the standard potential analysis [7, Chapter 4, Section 3] whereas Lemma 3.5 follows from the embedding theorems in [7, Chapter 2, Section 3].

We are now in a position to prove the following result.

Lemma 3.6. *Let $\lambda, \delta \in (0, 1)$ be any two constants and assume that $(\varphi_0, v_0) \in H$. Then there exists a positive constant $C = C(\lambda, \delta)$ such that*

$$\|\varphi\|_{C^{\lambda,\lambda/2}(Q_\delta^1)} + \|v\|_{C^{\lambda,\lambda/2}(Q_\delta^1)} \leq C(1 + \|\varphi_0, v_0\|_H). \quad (3.10)$$

Proof. Let $\delta_1 (\ll \delta)$ be a small positive constant to be determined. Applying Lemma 3.4 to the solution φ for the parabolic equation for φ in Problem P, one gets

$$\begin{aligned} \|\varphi\|_{W_q^{2,1}(Q_{2\delta_1}^1)} &\leq C(\|g(\varphi)\|_{L^q(Q_{\delta_1}^1)} + \|\varphi\|_{L^q(Q_{\delta_1}^1)} + \|v\|_{L^q(Q_{\delta_1}^1)} + \|\varphi(\delta_1)\|_{L^2(\Omega)}) \\ &\leq C(1 + \|v\|_{L^q(Q_1)} + \|(\varphi_0, v_0)\|_H) \end{aligned} \quad (3.11)$$

by Lemma 3.3 and the estimate (3.2) (Note that $\beta \leq C\|v_0\|_{L^2(\Omega)}$).

Similarly, we can deduce from the equation for v that

$$\begin{aligned} \|v\|_{W_q^{2,1}(Q_{3\delta_1}^1)} &\leq C(\|\Delta\varphi\|_{L^q(Q_{2\delta_1}^1)} + \|v(2\delta_1)\|_{L^2(\Omega)}) \\ &\leq C(1 + \|v\|_{L^q(Q_1)} + \|(\varphi_0, v_0)\|_H) \end{aligned} \quad (3.12)$$

by using (3.11).

Now we can use a bootstrap argument to complete the proof of Lemma 3.6. Let $q^0 = 2$ and iteratively define q^j by

$$q^j = (q^{j-1})^*, \quad j = 1, 2, \dots, J$$

until $q^J > (N+2)/2$. Choosing $\delta_1 = \delta/3(J+1)$, we can successively apply Lemma 3.5 and the estimate (3.12) on the domain $Q_{3J\delta_1}^1$ to deduce that $v \in L^{q^J}(Q_{3J\delta_1}^1)$, and finally, apply the embedding (3.9) in Lemma 3.5 to deduce the assertion of Lemma 3.6.

Theorem 3.7. *Let $m \in \mathbb{N}^+$ be arbitrary, let $\lambda, \delta \in (0, 1)$ be any two constants and assume that $(\varphi_0, v_0) \in H$. Then there exists a constant $C = C(m, \lambda, \delta)$ such that*

$$\|\varphi\|_{C^{2m+\lambda, m+\lambda/2}(Q_\delta^\infty)} + \|v\|_{C^{2m+\lambda, m+\lambda/2}(Q_\delta^\infty)} \leq C(1 + \|\varphi_0, v_0\|_H).$$

Proof. For each $k \geq 0$, applying Lemma 3.6 on the domain Q_k^{k+1} and using the estimate (3.2), we can deduce that

$$\|\varphi\|_{C^{\lambda, \lambda/2}(Q_{k+\delta}^{k+1})} + \|v\|_{C^{\lambda, \lambda/2}(Q_{k+\delta}^{k+1})} \leq C(1 + \|\varphi(k), v(k)\|_H) \leq \bar{C}(1 + \|\varphi_0, v_0\|_H).$$

Therefore, we have

$$\|\varphi\|_{C^{\lambda, \lambda/2}(Q_\delta^\infty)} + \|v\|_{C^{\lambda, \lambda/2}(Q_\delta^\infty)} \leq \bar{C}(1 + \|\varphi_0, v_0\|_H).$$

The assertion of Theorem 3.7 thus follows from the standard bootstrap argument [7, Theorem 10.1, p. 351].

Corollary 3.8. *Let $m \in \mathbb{N}^+$ be arbitrary. Then the pair $(\varphi(t), v(t)) = S(t)(\varphi_0, v_0)$ enters an absorbing set of $\mathcal{H}_\alpha \cap (C^m(\bar{\Omega}))^2$.*

Proof. Apply Theorem 3.7 with choosing a large enough initial time.

Now we can conclude the following.

Theorem 3.9. *The attractor $\mathcal{A}_\alpha$ is included in $(C^\infty(\Omega))^2$ and bounded in all the spaces $(C^m(\bar{\Omega}))^2$, $m \in \mathbb{N}^+$.*

4. Existence of Inertial Sets.

From now on we shall show the existence of inertial sets for the semi-group $S(t)$ corresponding to the Problem P on $\mathcal{H}_\alpha$ where α is any given fixed positive constant.

We first recall the definition of inertial sets [4, 5]. Let $\mathcal{B}_\alpha$ be the absorbing ball in $\mathcal{H}_\alpha$ given by Theorem 3.1 and define the invariant set

$$\mathcal{X}_\alpha := \overline{\cup_{t \geq t_0} S(t, \mathcal{B}_\alpha)}, \quad t_0 = 1 + \mathcal{T}\left(\max_{(\varphi_0, v_0) \in \mathcal{B}_\alpha} \|(\varphi_0, v_0)\|_H\right) \quad (4.1)$$

From Theorem 3.7, we know that $\mathcal{X}_\alpha$ is a bounded set in $(C^m(\bar{\Omega}))^2$ for any $m \in \mathbb{N}^+$.

Definition 4.1. *A set $\mathcal{M}_\alpha$ is called an inertial set for $(\{S(t)\}_{t \geq 0}, \mathcal{X}_\alpha)$ if*

- (i) $\mathcal{A}_\alpha \subset \mathcal{M}_\alpha \subset \mathcal{X}_\alpha$,
- (ii) $S(t)\mathcal{M}_\alpha \subset \mathcal{M}_\alpha$ for every $t \geq 0$,
- (iii) there exist positive constants C_1 and C_2 such that $\text{dist}_H(S(t)\mathcal{X}_\alpha, \mathcal{M}_\alpha) \leq C_1 e^{-C_2 t}$ for all $t \geq 0$ where $\text{dist}_H(A, B)$ is the Hausdorff distance of two sets defined by $\text{dist}_H(A, B) = \max_{x \in A} \min_{y \in B} \|x - y\|_H$, and

(iv) $\mathcal{M}_\alpha$ has finite fractal dimension.

In order to show the existence of an inertial set we prove below that the semi-group $S(t)$ corresponding to Problem P satisfies the so called **squeezing property**: *there exists a time $t^* > 0$, an integer $N_0 \geq 1$, and an orthogonal projection P of rank N_0 such that for every $U_1 = (\varphi_1, v_1)$ and $U_2 = (\varphi_2, v_2)$ in X_α satisfying*

$$\|P(S(t^*)(U_1) - S(t^*)(U_2))\|_H \leq \|(I - P)(S(t^*)(U_1) - S(t^*)(U_2))\|_H, \quad (4.2)$$

one has the inequality

$$\|S(t^*)(U_1) - S(t^*)(U_2)\|_H \leq \frac{1}{8} \|U_1 - U_2\|_H. \quad (4.3)$$

The squeezing property and the Lipschitz continuity of $S(t)$ (for each $t > 0$) will imply the existence of an inertial set $\mathcal{M}_\alpha$ with fractal dimension bounded by a constant times N_0 (see, for example, [5]).

Let $\{w_1, w_2, \dots\}$ be the eigenfunctions defined in section 2 and let

$$H_N = \text{span} \{w_1, \dots, w_N\}.$$

We define p_N as the orthogonal projection from $L^2(\Omega)$ onto H_N and $q_N = I - p_N$. One can easily show that

$$\|u\|_{L^2(\Omega)}^2 \leq \frac{1}{\lambda_{N+1}} \|\nabla u\|_{L^2(\Omega)}^2 \quad \text{for all } u \in q_N H^1(\Omega). \quad (4.4)$$

Next we define the corresponding product projections P_N and Q_N on H by

$$P_N(\varphi, v) := (p_N \varphi, p_N v) \quad \text{and} \quad Q_N = I - P_N. \quad (4.5)$$

Before proving the squeezing property we shall show the following:

Lemma 4.1. *There exists a positive constant C such that for any two solutions (φ_1, v_1) and (φ_2, v_2) to Problem P with initial functions in $\mathcal{X}_\alpha$, the pair $(\phi, V) := Q_N(\varphi_1 - \varphi_2, v_1 - v_2)$ satisfies*

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \phi^2 + \frac{2\xi^2}{K\ell^2} V^2 \right) + \frac{\xi^2}{4} \int_{\Omega} |\nabla \phi|^2 + \frac{\xi^2}{\ell^2} \int_{\Omega} |\nabla V|^2 \\ & + \left(\ell + \frac{\xi^2}{4} \lambda_{N+1} - \frac{3}{2} \right) \int_{\Omega} \phi^2 + \left(\frac{\xi^2}{\ell^2} \lambda_{N+1} - 1 \right) \int_{\Omega} V^2 \leq C \int_{\Omega} (\varphi_1 - \varphi_2)^2. \end{aligned} \quad (4.6)$$

Proof. Applying the operator Q_N to the differences of the equations for (φ_1, v_1) and (φ_2, v_2) , we obtain

$$\begin{cases} \tau \phi_t = \xi^2 \Delta \phi - q_N(g(\varphi_1) - g(\varphi_2)) - \ell \phi + 2V, \\ V_t = K \Delta V - \frac{K\ell}{2} \Delta \phi. \end{cases}$$

Multiplying the first equation by ϕ and the second equation by $\frac{4\xi^2}{K\ell^2}V$, adding up the resulting equations, and integrating by parts, we get

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \phi^2 + \frac{2\xi^2}{K\ell^2} V^2 \right) + \xi^2 \int_{\Omega} |\nabla \phi|^2 + \frac{4\xi^2}{\ell^2} \int_{\Omega} |\nabla V|^2 + \ell \int_{\Omega} \phi^2 \\ &= 2 \int_{\Omega} V \phi - \int_{\Omega} \phi q_N(g(\varphi_1) - g(\varphi_2)) + \frac{2\xi^2}{\ell} \int_{\Omega} \nabla \phi \nabla V. \end{aligned} \quad (4.7)$$

To estimate the second term on the right-hand side, notice that (φ_1, v_1) and (φ_2, v_2) remain in X_{α} which is a bounded set in $(C^0(\bar{\Omega}))^2$, so that there exists a positive constant R such that $\|\varphi_1, \varphi_2\|_{C(\bar{\Omega} \times \mathbb{R}^+)} \leq R$. Consequently

$$\begin{aligned} \left| \int_{\Omega} \phi q_N(g(\varphi_1) - g(\varphi_2)) \right| &= \left| \int_{\Omega} \phi (g(\varphi_1) - g(\varphi_2)) \right| \\ &\leq C(R) \int_{\Omega} |\phi| |\varphi_1 - \varphi_2| \\ &\leq \frac{(C(R))^2}{2} \int_{\Omega} (\varphi_1 - \varphi_2)^2 + \frac{1}{2} \int_{\Omega} \phi^2. \end{aligned}$$

Substituting it into (4.7) and using Cauchy's inequality for the first and last terms on the right-hand side of (4.7), we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \phi^2 + \frac{2\xi^2}{K\ell^2} V^2 \right) + \frac{\xi^2}{2} \int_{\Omega} |\nabla \phi|^2 + \frac{2\xi^2}{\ell^2} \int_{\Omega} |\nabla V|^2 \\ &+ \left(\ell - \frac{3}{2} \right) \int_{\Omega} \phi^2 - \int_{\Omega} V^2 \leq \frac{(C(R))^2}{2} \int_{\Omega} (\varphi_1 - \varphi_2)^2 \end{aligned}$$

which, together with (4.4), yields (4.6).

Now we prove the squeezing property.

Theorem 4.2. *The semi-group $S(t)$ corresponding to Problem P satisfies the squeezing property. More precisely, for any $t^* > 0$, there exists an $N_0 = N_0(t^*)$ such that for any $U_1 = (\varphi_1, v_1)$ and $U_2 = (\varphi_2, v_2)$ in X_{α} satisfying (4.2) for $P := P_{N_0}$ defined in (4.5), U_1 and U_2 satisfy (4.3) also.*

Proof. Let t^* be any fixed positive constant and let $(\varphi_{01}, v_{01}), (\varphi_{02}, v_{02}) \in X_{\alpha}$. Set for $t \geq 0$

$$(\psi(t), W(t)) = (\varphi_1(t) - \varphi_2(t), v_1(t) - v_2(t)).$$

By Lemma 4.1 and Theorem 2.2 we have

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \left(\frac{\tau}{2} \phi^2 + \frac{2\xi^2}{K\ell^2} V^2 \right) + \left(\ell + \frac{\xi^2}{4} \lambda_{N+1} - \frac{3}{2} \right) \int_{\Omega} \phi^2 + \left(\frac{\xi^2}{\ell^2} \lambda_{N+1} - 1 \right) \int_{\Omega} V^2 \\ &\leq C \int_{\Omega} (\varphi_1 - \varphi_2)^2 \leq \frac{2C}{\tau} \left(\int_{\Omega} \left\{ \frac{\tau}{2} (\varphi_{01} - \varphi_{02})^2 + \frac{2\xi^2}{K\ell^2} (v_{01} - v_{02})^2 \right\} \right) e^{Dt} \end{aligned}$$

where $(\phi, V) = Q_N(\psi, W)$. It follows that there exists a function f satisfying $f(s) \rightarrow +\infty$ as $s \rightarrow +\infty$ such that when N is large enough,

$$\frac{d}{dt} \|(\phi, \mu V)\|_H^2 + f(\lambda_{N+1}) \|(\phi, \mu V)\|_H^2 \leq \tilde{C} \|(\varphi_{01} - \varphi_{02}, v_{01} - v_{02})\|_H^2 e^{Dt}$$

where $\mu = \frac{2\xi}{\sqrt{\tau K} \ell}$. By Gronwall's lemma this implies that

$$\begin{aligned} \|(\phi, \mu V)(t)\|_H^2 &\leq \|(\phi(0), \mu V(0))\|_H^2 e^{-f(\lambda_{N+1})t} + \frac{\tilde{C} e^{Dt}}{D + f(\lambda_{N+1})} \|(\psi(0), W(0))\|_H^2 \\ &\leq \|(\psi(0), \mu W(0))\|_H^2 \left(e^{-f(\lambda_{N+1})t} + \frac{\tilde{C} e^{Dt}}{D + f(\lambda_{N+1})} \right). \end{aligned}$$

Now assume that (4.2) holds; i.e., the inequality

$$\|P_{N_0}(\psi(t^*), W(t^*))\|_H \leq \|Q_{N_0}(\psi(t^*), W(t^*))\|_H$$

holds. Then,

$$\begin{aligned} \|(\psi, W)(t^*)\|_H^2 &= \|Q_{N_0}(\psi, W)(t^*)\|_H^2 + \|P_{N_0}(\psi, W)(t^*)\|_H^2 \\ &\leq 2 \|Q_{N_0}(\psi, W)(t^*)\|_H^2 \leq 2 \left(1 + \frac{1}{\mu} \right) \|(\phi, \mu V)(t^*)\|_H^2 \\ &\leq 2 \|(\psi, \mu W)(0)\|_H^2 \left(e^{-f(\lambda_{N_0+1})t^*} + \frac{\tilde{C} e^{Dt^*}}{D + f(\lambda_{N_0+1})} \right) \left(1 + \frac{1}{\mu} \right) \\ &\leq \frac{1}{8} \|(\psi, W)(0)\|_H^2 \end{aligned}$$

if we choose $N_0 = N_0(t_*)$ large enough. This completes the proof of Theorem 4.2.

Applying the result in [4], we thus have the following:

Theorem 4.3. *Let $\mathcal{X}_\alpha$ be defined as in (4.1). Then there exists an inertial set $\mathcal{M}_\alpha$ for $(S(t)_{t \geq 0}, \mathcal{X}_\alpha)$ which has fractal dimension $\leq \text{constant} \cdot N_0$.*

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