**ON A DISSOLUTION-GROWTH PROBLEM
WITH SURFACE TENSION IN THE NEIGHBORHOOD
OF A STATIONARY SOLUTION**

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F. ABERGEL**, D. HILHORST** AND F. ISSARD-ROCH**

1. Introduction. Consider a system composed of a solid phase of a single compound and an incompressible liquid phase which is a dilute solution of that compound and suppose that the evolution in time of the system is governed by two processes: a diffusion process in a diffusion layer in the fluid and a dissolution-growth process, located at the interface between solid and fluid.

The mathematical problem can be described by three basic equations

$$(1.1) \quad C_t = \Delta C \quad \text{in the diffusion layer}$$

where $C = C(x, y, t)$ is the concentration in the liquid phase,

$$(1.2) \quad \frac{\partial C}{\partial n} \Big|_{\Gamma(t)} = V_n$$

where $\Gamma(t)$ denotes the interface between solid and fluid, n is the normal unit vector to the interface directed towards the fluid and V_n is the normal velocity of $\Gamma(t)$

$$(1.3) \quad V_n = -\alpha e^{\gamma K} + C \Big|_{\Gamma(t)}$$

where α is a positive constant, the positive constant γ is proportional to the surface tension of the interface and K is the curvature of the interface. Let us note that in view of (1.3), the boundary condition (1.2) can be rewritten as

$$\left(C - \frac{\partial C}{\partial n} \right) \Big|_{\Gamma(t)} = \alpha e^{\gamma K}.$$

In this paper, we consider the case where the interface can be parametrized in the form $y = f(x, t)$ and denote by e the width of the diffusion layer. Then f and C satisfy the following (rescaled) problem

$$P_0 \left\{ \begin{array}{ll} C_t = \Delta C \text{ in } Q_f := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, f(x, t) < y < f(x, t) + e\} & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ \left(C - \frac{\partial C}{\partial n} \right) (x, f(x, t), t) = \alpha e^{-\gamma(f_{xx}/(1+f_x^2)^{3/2})(x, t)} & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ \frac{\partial C}{\partial n} (x, f(x, t) + e, t) = 0 & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ C(x + 2L, y, t) = C(x, y, t) & (x, y, t) \in Q_f \\ \frac{f_t}{(1+f_x^2)^{1/2}} = -\alpha e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} + C(x, f(x, t), t) & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ f(x + 2L, t) = f(x, t) & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ f(x, 0) = f_0(x) & x \in \mathbb{R} \\ C(x, y, 0) = C_0(x, y) \quad (x, y) \in \Sigma_f := \{(x, y) \in \mathbb{R}^2, f_0(x) < y < f_0(x) + e\} \end{array} \right.$$

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where f_0 and C_0 satisfy the hypothesis H_0 :

$f_0 \in C_{\text{per}}^{3+\lambda}(\mathbb{R})$, $C_0 \in C_{\text{per}}^{2+\lambda}(\overline{\Sigma}_f)$ for some $\lambda \in (0, 1)$ and C_0 satisfies the compatibility conditions

$$\begin{aligned} \frac{\partial C_0}{\partial n}(x, f_0(x) + e) &= 0 \quad x \in \mathbb{R} \\ \left(C_0 - \frac{\partial C_0}{\partial n}\right)(x, f_0(x)) &= \alpha e^{-\gamma(f_0''/(1+f_0'^2)^{3/2})}(x) \quad x \in \mathbb{R} \end{aligned}$$

(By the subscript per we denote functions periodic in the variable x with period $2L$).

We remark that pairs of the form $(C, f) = (\alpha, \text{constant})$ are stationary solutions of Problem P_0 . The purpose of this paper is to prove the following existence and uniqueness result.

THEOREM 1.1. *There exists a positive constant $\rho_0 = \rho_0(T)$ such that if the initial data satisfies the condition*

$$\|f_0\|_{\mathbb{R}}^{(3+\lambda)} + \|C_0 - \alpha\|_{\Sigma_f}^{(2+\lambda)} \leq \rho_0$$

then Problem P_0 has a unique solution (f, C) in $C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\mathbb{R} \times [0, T]) \times C_{\text{per}}^{2+\lambda, 1+\lambda/2}(\overline{Q}_f^T)$ where $Q_f^T := \{(x, y, t) \in Q_f, t \in (0, T)\}$.

The paper is organized as follows.

In Section 2, we present a physical derivation of the problem which is due to Cournil [5].

In Section 3, we transform Problem P_0 into a problem on a fixed domain.

In Section 4, we recall the definitions of the Hölder spaces that we use and prove some properties which are useful in the following sections.

In Section 5, we consider a related linear problem and prove its well-posedness.

We prove Theorem 1.1 transposed to the coordinates of the fixed domain in the sections 6 and 7. The uniqueness proof, which extends the one of Section 5 and is based on multiplications by suitable test functions, is given in Section 6. The existence proof is given in Section 7. It is done by means of a fixed point method: one iteratively solves the linear problem of Section 5, while considering as known the nonlinear terms. The idea is to show that the iterative map is a strict contraction from a ball of small enough radius $\rho > \rho_0$ into itself.

Let us remark that our proof would carry over to the case of a domain given by

$$\{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, f(x, t) < y < A\}$$

with A a fixed constant. A suitable change of variables for transforming the problem to a fixed domain is then given by $\tilde{y} = (A - y)/(A - f(x, t))$ and the constants ρ_0 and ρ should be chosen small enough so that the interface does not hit the boundary $y = A$.

In a forthcoming paper we shall extend our method to the proof of local existence and uniqueness for arbitrary initial data. For the well-posedness of a very similar problem in the case of spherical symmetry we refer to [4,7].

For local existence and uniqueness results for related problems we refer to a study of Duchon and Robert [6] of a quasistationary problem and to papers by X.Y. Chen [3] and Xinfu Chen [2] about a two phase problem.

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2. The physical derivation. In heterogeneous media, a problem of interest is the creation or the consumption of one or more phases. These phenomena are accompanied by a change in the geometry of the reaction field and, in particular, by an evolution of interfaces. The interfaces are of essential importance: on the one hand, they determine the geometrical configuration of the system and on the other hand they intervene, by means of their curvature, in the equilibrium and stability conditions.

In this paper, we consider a system composed of a solid phase of a single compound and an incompressible liquid phase which is a dilute solution of that compound. The evolution in time of this system induces mass transfer processes: a homogeneous one which consists of a diffusion process in a diffusion layer in the fluid and a heterogeneous one, namely a dissolution-growth process, located at the interface between solid and liquid.

Let $y = f(x, t)$ be a parametrization of the interface, let e denote the width of the diffusion layer and let $C(x, y, t)$ with $f(x, t) < y < f(x, t) + e$, denote the concentration in the liquid phase. The equations of the problem are deduced from the following physical laws:

(i) *Mass balance in the liquid.*

Let v denote the convective velocity of the fluid and J the diffusion flux. Then the mass flow is governed by Fick's law of diffusion

$$(2.1) \quad J = -D \operatorname{grad} C$$

where the diffusion coefficient D is a positive constant, and C satisfies

$$(2.2) \quad C_t = -\operatorname{div}(J + Cv).$$

Further, the incompressibility of the fluid is expressed by

$$(2.3) \quad \operatorname{div} v = 0.$$

Substituting (2.1) and (2.3) in equation (2.2) yields the following parabolic equation for the concentration

$$(2.4) \quad C_t = D\Delta C - v \operatorname{grad} C$$

in $\{(x, y, t), t > 0, x \in \mathbb{R}, f(x, t) < y < f(x, t) + e\}$.

(ii) *Mass balance at the solid-liquid interface.*

Let L and M be arbitrary positive constants, $x \in (-L, L)$ and $\bar{f}$ be an arbitrary function such that

$$-M < f(r, \cdot) < \bar{f}(r) < f(r, \cdot) + e \quad \text{for } r \in (-L, L)$$

on some time interval $(t - \delta, t + \delta)$. Further we define

$$\theta := \{(r, y) \in (-L, x) \times \mathbb{R} \text{ such that } -M \leq y \leq \bar{f}(r)\}$$

The number of moles of the compound contained in θ at time t is given by

$$(2.5) \quad n(t) = \int_{-L}^x dr \int_{f(r,t)}^{\bar{f}(r)} dy C(r, y, t) + \int_{-L}^x dr \frac{f(r, t) + M}{V}$$

where $1/V$ is the concentration in the solid phase. The variation of $n(t)$ is due to exchanges by diffusion through the boundary Γ_t between $\mathcal{O}$ and the liquid phase

$$(2.6) \quad \begin{aligned} \dot{n}(t) &= \int_{\Gamma_t} -(J + Cv) \cdot \nu d\gamma \\ &= - \int_{-L}^x dr \int_{f(r,t)}^{\bar{f}(r)} \operatorname{div}(J + Cv) \\ &\quad - \int_{-L}^x (J + Cv) \cdot n|_{y=f(r,t)} \sqrt{1 + f_x^2(r, t)} dr \end{aligned}$$

where ν denotes the normal to Γ_t , directed towards the outside of $\mathcal{O}$, n denotes the normal to the interface $y = f(r, t)$ directed towards the fluid and where the interface is taken just inside the fluid in the last integral.

Next we differentiate (2.5) with respect to t . This yields

$$\begin{aligned} \dot{n}(t) &= \int_{-L}^x dr \int_{f(r,t)}^{\bar{f}(r)} dy C_t(r, y, t) + \\ &\quad - \int_{-L}^x dr C(r, f(r, t), t) f_t(r, t) + \int_{-L}^x dr \frac{1}{V} f_t(r, t) \end{aligned}$$

which together with (2.6) and (2.2) implies

$$\begin{aligned} & \int_{-L}^x \left(\frac{1}{V} - C(r, f(r, t), t) \right) f_t(r, t) dr \\ &= - \int_{-L}^x (J + Cv) \cdot n|_{y=f(r, t)} \sqrt{1 + f_x^2(r, t)} dr \end{aligned}$$

and after differentiation with respect to x

$$\left(\frac{1}{V} - C \right) |_{y=f(x, t)} f_t = -(J + Cv) \cdot n|_{y=f(x, t)} \sqrt{1 + f_x^2}.$$

Finally, using furthermore the impermeability condition of the interface

$$v \cdot n = \frac{f_t}{\sqrt{1 + f_x^2}}$$

we deduce the equation

$$(2.7) \quad \frac{1}{V} f_t = -J \cdot n|_{y=f(x, t)} \sqrt{1 + f_x^2}.$$

(iii) *Dissolution and growth of the solid.*

We suppose that the kinetics of dissolution and growth follows Nernst's law which derives, either from a rate determining diffusion in a layer of constant thickness, or from a rate determining first order interface reaction [8], namely

$$(2.8) \quad J \cdot n|_{y=f(x, t)} = -h(C, R)$$

where R is the radius of curvature of the interface

$$(2.9) \quad R(x, t) = -\frac{(1 + f_x^2)^{3/2}}{f_{xx}}$$

and the function h is given by

$$(2.10) \quad h(C, R) = K(C - s_0 e^{\gamma/R})$$

where K is a kinetic constant, s_0 is the concentration at saturation of the solution and γ is proportional to the surface tension of the interface.

Using (2.7)–(2.10) we deduce that

$$(2.11) \quad \frac{1}{V} f_t = (1 + f_x^2)^{1/2} K(C(x, f(x, t), t) - s_0 e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}})$$

whereas (2.1) and (2.8)–(2.10) imply the boundary condition

$$(2.12) \quad D \frac{\partial C}{\partial n} = K(C - s_0 e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}})$$

on the interface $y = f(x, t)$, where n is the normal to the interface directed towards the fluid.

(iv) *Other boundary conditions.*

We assume that C and f are periodic in the x -direction, namely

$$\begin{cases} C(x + 2L, y, t) = C(x, y, t) & \text{in } Q_f \\ f(x + 2L, t) = f(x, t) & \text{in } \mathbb{R} \times \mathbb{R}_+ \end{cases}$$

and that C satisfies the Neumann condition

$$(2.13) \quad \frac{\partial C}{\partial n}(x, f(x, t) + e, t) = 0 \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+.$$

In what follows we suppose that v is small enough so that the convection term is negligible with respect to the diffusion term in (2.4). Then C satisfies the diffusion equation

$$(2.14) \quad C_t = D \Delta C.$$

Finally we set

$$\begin{aligned} \tilde{x} &= Kx/D, \tilde{y} = Ky/D, \tilde{t} = K^2 t/D, \tilde{C} = VC, \tilde{\gamma} = K\gamma/D, \\ \tilde{f} &= Kf/D, \alpha = Vs_0, \tilde{L} = KL/D, \tilde{e} = Ke/D \end{aligned}$$

and omit the tilda. The equations (2.14), (2.12), (2.13) and (2.11) transform into

$$\begin{cases} C_t = \Delta C & \text{in } Q_f := \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, f(x, t) < y < f(x, t) + e\} \\ (C - \frac{\partial C}{\partial n})(x, f(x, t), t) = \alpha e^{-\gamma(f_{xx}/(1+f_x^2)^{3/2})(x, t)} & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ \frac{\partial C}{\partial n}(x, f(x, t) + e, t) = 0 & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ f_t = (1 + f_x^2)^{1/2} \left(-\alpha e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} + C(x, f(x, t), t) \right) & (x, t) \in \mathbb{R} \times \mathbb{R}^+ \end{cases}$$

which together with periodic boundary conditions and periodic initial conditions for C and f yield Problem P_0 .

3. Transformation of the problem to a problem on a fixed domain. Let us define new coordinates by

$$\hat{x} = x, \quad \hat{y} = y - f(x, t), \quad \hat{t} = t$$

and set

$$g(x, y, t) = C(x, y, t) - \alpha, \quad \hat{g}(\hat{x}, \hat{y}, \hat{t}) = g(x, y, t), \quad \hat{f}(\hat{x}, \hat{t}) = f(x, t).$$

Then g satisfies the problem

$$\left\{ \begin{array}{l} g_t = \Delta g \quad \text{in } Q_f \\ \left(g - \frac{\partial g}{\partial n}\right)(x, f(x, t), t) = \alpha \left(e^{-\gamma(f_{xx}/(1+f_x^2)^{3/2}}(x, t) - 1\right) \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ \frac{\partial g}{\partial n}(x, f(x, t) + e, t) = 0 \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ g(x + 2L, y, t) = g(x, y, t) \quad \text{in } Q_f \\ g(x, y, 0) = C_0(x, y) - \alpha \quad x \in \mathbb{R}, f(x, 0) \leq y \leq f(x, 0) + e \end{array} \right.$$

and f satisfies the problem

$$\left\{ \begin{array}{l} f_t = -(1 + f_x^2)^{1/2} \left(\alpha e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - g(x, f(x, t), t) - \alpha\right) \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ f(x + 2L, t) = f(x, t) \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \\ f(x, 0) = f_0(x) \quad x \in \mathbb{R} \end{array} \right.$$

where C_0 and f_0 are periodic in x with period $2L$. Some easy computations permit to show that $\hat{g}$ and $\hat{f}$ satisfy

$$P \left\{ \begin{array}{l} \hat{g}_t = \hat{g}_{\hat{x}\hat{x}} + \hat{g}_{\hat{y}\hat{y}} + \hat{f}_{\hat{x}}^2 \hat{g}_{\hat{y}\hat{y}} - (\hat{f}_{\hat{x}\hat{x}} - \hat{f}_{\hat{t}}) \hat{g}_{\hat{y}} - 2\hat{f}_{\hat{x}} \hat{g}_{\hat{x}\hat{y}} \\ \quad \text{in } Q_0 = \{(x, y, t) \in \mathbb{R}^2 \times \mathbb{R}^+, 0 < y < e\} \\ \hat{g}(\hat{x}, 0, \hat{t}) - (1 + \hat{f}_{\hat{x}}^2)^{1/2} \hat{g}_{\hat{y}}(\hat{x}, 0, \hat{t}) + \frac{\hat{f}_{\hat{x}}}{(1 + \hat{f}_{\hat{x}}^2)^{1/2}} \hat{g}_{\hat{x}}(\hat{x}, 0, \hat{t}) \\ \quad = \alpha(e^{-\gamma \hat{f}_{\hat{x}\hat{x}}/(1 + \hat{f}_{\hat{x}}^2)^{3/2}} - 1) \quad (\hat{x}, \hat{t}) \in \mathbb{R} \times \mathbb{R}^+ \\ (1 + \hat{f}_{\hat{x}}^2)^{1/2} \hat{g}_{\hat{y}}(\hat{x}, e, \hat{t}) - \frac{\hat{f}_{\hat{x}}}{(1 + \hat{f}_{\hat{x}}^2)^{1/2}} \hat{g}_{\hat{x}}(\hat{x}, e, \hat{t}) = 0 \quad (\hat{x}, \hat{t}) \in \mathbb{R} \times \mathbb{R}^+ \\ \hat{g}(\hat{x} + 2L, \hat{y}, \hat{t}) = \hat{g}(\hat{x}, \hat{y}, \hat{t}) \quad \text{in } Q_0 \\ \hat{g}(\hat{x}, \hat{y}, 0) = \hat{g}_0(\hat{x}, \hat{y}) \quad (\hat{x}, \hat{y}) \in \mathbb{R} \times (0, e) \\ \text{where} \\ \hat{g}_0(\hat{x}, \hat{y}) = C_0(x, y) - \alpha \quad (\text{with } \hat{y} = y - f_0(x)) \\ \text{and} \\ \hat{f}_{\hat{t}} = -(1 + \hat{f}_{\hat{x}}^2)^{1/2} (\alpha e^{-\gamma \hat{f}_{\hat{x}\hat{x}}/(1 + \hat{f}_{\hat{x}}^2)^{3/2}} - \hat{g}(\hat{x}, 0, \hat{t}) - \alpha) \quad (\hat{x}, \hat{t}) \in \mathbb{R} \times \mathbb{R}^+ \\ \hat{f}(\hat{x} + 2L, \hat{t}) = \hat{f}(\hat{x}, \hat{t}) \quad (\hat{x}, \hat{t}) \in \mathbb{R} \times \mathbb{R}^+ \\ \hat{f}(\hat{x}, 0) = \hat{f}_0(\hat{x}) \quad \hat{x} \in \mathbb{R} \\ \text{where} \\ \hat{f}_0(\hat{x}) = f_0(x). \end{array} \right.$$

Remark 3.1. In what follows, it will be convenient to write

$$(1 + f_x^2)^{1/2} (e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1) = -\gamma f_{xx} \\ + (1 + f_x^2)^{1/2} (e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1 + \gamma f_{xx}/(1 + f_x^2)^{1/2})$$

and

$$(1 + f_x^2)^{1/2} g(x, 0, t) = g(x, 0, t) - (1 - (1 + f_x^2)^{1/2}) g(x, 0, t) :$$

this permits to separate linear from nonlinear terms.

4. A few properties of Hölder spaces. In this paper we work with the Hölder spaces $C^{\ell, \ell/2}(\overline{\Omega} \times [0, T])$ as defined by Ladyženskaja, Solonnikov and Ural'ceva [9]. We indicate below some notations and properties which will be useful in what follows.

Let $\Omega = \mathbb{R} \times \Omega'$ where Ω' is a smooth open set of $\mathbb{R}^{N-1}$ and $Q_T = \Omega \times (0, T)$. We define the space

$$C_{\text{per}}^{\ell, \ell/2}(\overline{Q}_T) = \left\{ f \in C^{\ell, \ell/2}(\overline{Q}_T), \right. \\ \left. f(x_1 + 2L, x', t) = f(x_1, x', t) \quad \text{for all } (x_1, x', t) \in \mathbb{R} \times \Omega' \times (0, T) \right\}$$

of $2L$ -periodic functions in the x_1 -direction and define $C_{\text{per}}^{\ell}(\overline{\Omega})$ in a similar way.

In what follows we often use the notation

$$C^{m+\lambda, (m+\lambda)/2}(\overline{Q}_T); \text{ then } m = [\ell] \text{ and } \lambda = \ell - [\ell].$$

Next we state some technical results which will be useful in the sequel.

LEMMA 4.1. (i) *Let $m \in \mathbb{N}$. There exists a constant $C > 0$ which depends only on m such that*

$$\|fg\|_{Q_T}^{(m+\lambda)} \leq C \|f\|_{Q_T}^{(m+\lambda)} \|g\|_{Q_T}^{(m+\lambda)}$$

for all $f, g \in C^{m+\lambda, (m+\lambda)/2}(\overline{Q}_T)$, $0 < \lambda < 1$.

(ii) *There exists a constant $C > 0$ such that*

$$\left\| \frac{1}{f} \right\|_{Q_T}^{(1+\lambda)} \leq C (\|f\|_{Q_T}^{(1+\lambda)})^2$$

for all $f \in C^{1+\lambda, (1+\lambda)/2}(\overline{Q}_T)$ such that $f \geq 1$.

(iii) *There exists a constant $C > 0$ such that*

$$\|\varphi \circ f\|_{Q_T}^{(1+\lambda)} \leq C \|\varphi\|_{(-M, M)}^{(1+\lambda)} \max\{1, (\|f\|_{Q_T}^{(1+\lambda)})^2\}$$

for all $f \in C^{1+\lambda, (1+\lambda)/2}(\overline{Q}_T)$ such that $-M \leq f \leq M$ and all $\varphi \in C^{1+1}([-M, M])$ (by which we mean the space of continuously differentiable functions with Lipschitz continuous derivatives).

Remark. We will use Lemma 4.1 (i) in cases where $g \in C^{m+\lambda, (m+\lambda)/2}(\overline{Q}_T)$ and $f \in C^{m+\lambda, (m+\lambda)/2}(\overline{\Sigma}_T)$ with $\Sigma_T = \mathbb{R} \times (0, T)$. Then we implicitly work with $\tilde{f}(x, y, t) = f(x, t)$ for all $(x, y, t) \in \overline{Q}_T$ so that $\tilde{f} \in C^{m+\lambda, (m+\lambda)/2}(\overline{Q}_T)$ and $\|\tilde{f}\|_{Q_T}^{(m+\lambda)} = \|f\|_{\Sigma_T}^{(m+\lambda)}$.

In what follows we will have to estimate terms of the form

$$E = \exp\left(-\gamma \frac{f_{xx}}{(1+f_x^2)^{3/2}}\right) - 1 + \frac{\gamma f_{xx}}{(1+f_x^2)^{1/2}}$$

as well as differences of such terms. To that purpose we set

$$\chi(s) = e^{\gamma s} - 1 - \gamma s$$

Then

$$(4.2) \quad E = \chi\left(-\frac{f_{xx}}{(1+f_x^2)^{3/2}}\right) + \gamma f_{xx} \left(\frac{1}{(1+f_x^2)^{1/2}} - \frac{1}{(1+f_x^2)^{3/2}}\right).$$

Next we prove the following result.

LEMMA 4.2. *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and for all $r \in C^{1+\lambda, (1+\lambda)/2}(\overline{\Sigma}_T)$ satisfying $\|r\|_{\Sigma_T}^{(1+\lambda)} \leq \delta$ there holds*

$$\|\chi(r)\|_{\Sigma_T}^{(1+\lambda)} \leq \gamma^2 C (\|r\|_{\Sigma_T}^{(1+\lambda)})^2.$$

where C is the constant introduced in Lemma 4.1 (i).

Proof. We use the Taylor expansion

$$\chi(s) = \sum_{k=2}^{+\infty} \frac{\gamma^k s^k}{k!} = s^2 \sum_{k=0}^{\infty} \frac{\gamma^{k+2} s^k}{(k+2)!};$$

then we deduce from Lemma 4.1 (i) that, with C being the constant introduced there,

$$\begin{aligned} \|\chi(r)\|_{\Sigma_T}^{(1+\lambda)} &\leq C \|r^2\|_{\Sigma_T}^{(1+\lambda)} \sum_{k=0}^{\infty} \frac{\gamma^{k+2} \|r^k\|_{\Sigma_T}^{(1+\lambda)}}{(k+2)!} \\ &\leq C^2 (\|r\|_{\Sigma_T}^{(1+\lambda)})^2 \left\{ \sum_{k=0}^{\infty} \frac{\gamma^{k+2}}{(k+2)!} C^{k-1} (\|r\|_{\Sigma_T}^{1+\lambda})^k \right\} \\ &\leq \frac{1}{C \delta^2} (\|r\|_{\Sigma_T}^{(1+\lambda)})^2 \sum_{k=0}^{\infty} \frac{(\gamma C \delta)^{k+2}}{(k+2)!} \\ &\leq \frac{1}{C \delta^2} (e^{\gamma C \delta} - 1 - \gamma C \delta) (\|r\|_{\Sigma_T}^{(1+\lambda)})^2. \end{aligned}$$

LEMMA 4.3. *There exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and for all $r, r' \in C^{1+\lambda, (1+\lambda)/2}(\bar{\Sigma}_T)$ satisfying $\|r\|_{\Sigma_T}^{(1+\lambda)}, \|r'\|_{\Sigma_T}^{(1+\lambda)} \leq \delta$ there exists $K = K(\delta_0)$ such that*

$$\|\chi(r) - \chi(r')\|_{\Sigma_T}^{(1+\lambda)} \leq K \|r - r'\|_{\Sigma_T}^{(1+\lambda)} \max \left(\|r\|_{\Sigma_T}^{(1+\lambda)}, \|r'\|_{\Sigma_T}^{(1+\lambda)} \right).$$

Proof. We have that

$$\begin{aligned} \chi(s) - \chi(s') &= \sum_{k=2}^{\infty} \frac{\gamma^k}{k!} (s^k - s'^k) \\ &= (s - s') \sum_{k=2}^{\infty} \frac{\gamma^k}{k!} \sum_{p=0}^{k-1} s^p s'^{k-1-p} \end{aligned}$$

which yields

$$\begin{aligned} \|\chi(r) - \chi(r')\|_{\Sigma_T}^{(1+\lambda)} &\leq C \|r - r'\|_{\Sigma_T}^{(1+\lambda)} \sum_{k=2}^{\infty} \frac{\gamma^k}{k!} k C^{k-2} \delta^{k-1} \\ &\leq \|r - r'\|_{\Sigma_T}^{(1+\lambda)} \sum_{j=1}^{\infty} \gamma \frac{\gamma^j C^j \delta^j}{j!} \\ &\leq \gamma (e^{\gamma C \delta} - 1) \|r - r'\|_{\Sigma_T}^{(1+\lambda)} \\ &\leq \tilde{K} \|r - r'\|_{\Sigma_T}^{(1+\lambda)} \max \left(\|r\|_{\Sigma_T}^{(1+\lambda)}, \|r'\|_{\Sigma_T}^{(1+\lambda)} \right) \end{aligned}$$

where

$$\tilde{K} = \gamma \frac{e^{\gamma C \delta} - 1}{\delta} \leq \gamma^2 C e^{\gamma C \delta_0}.$$

5. The linear problem. In this section we prove that there exists a unique solution of the following linear problem

$$\begin{aligned} (5.1) \quad & \begin{cases} g_t - \Delta g = F_1 & \text{in } Q_T := \{(x, y, t) \in \mathbb{R} \times (0, e) \times (0, T)\} \\ (g - g_y)(x, 0, t) + \alpha \gamma f_{xx}(x, t) = F_2(x, t) & (x, t) \in \Sigma_T := \mathbb{R} \times (0, T) \\ g_y(x, e, t) = F_3(x, t) & (x, t) \in \Sigma_T \\ g(x + 2L, y, t) = g(x, y, t) & (x, y, t) \in Q_T \\ g(x, y, 0) = g_0(x, y) & (x, y) \in \mathbb{R} \times (0, e) \end{cases} \\ (5.2) \quad & \\ (5.3) \quad & \\ (5.4) \quad & \\ (5.5) \quad & \\ (5.6) \quad & \begin{cases} f_t - \alpha \gamma f_{xx} - g(x, 0, t) = F_4(x, t) & \text{in } \Sigma_T \\ f(x + 2L, t) = f(x, t) & (x, t) \in \Sigma_T \\ f(x, 0) = f_0(x) & x \in \mathbb{R} \end{cases} \\ (5.7) \quad & \\ (5.8) \quad & \end{aligned}$$

where

$$\begin{aligned}
& T \text{ is a fixed (arbitrary) positive number ,} \\
& F_1 \in C_{\text{per}}^{\lambda, \lambda/2}(\overline{Q}_T), \quad F_i \in C_{\text{per}}^{1+\lambda, (1+\lambda)/2}(\overline{\Sigma}_T), i = 2, \dots, 4, \text{ for some } \lambda \in (0, 1), \\
& f_0 \in C_{\text{per}}^{3+\lambda}(\mathbf{R}), \\
& g_0 \in C_{\text{per}}^{2+\lambda}(\mathbf{R} \times [0, e]) \text{ and satisfies the compatibility conditions:} \\
& g_{0y}(x, e) = F_3(x, 0) \quad x \in \mathbf{R} \\
& (g_0 - g_{0y})(x, 0) + \alpha \gamma f_0''(x) = F_2(x, 0) \quad x \in \mathbf{R}.
\end{aligned}$$

THEOREM 5.1. *Problem (5.1)–(5.8) has a unique solution*

$$(g, f) \in C_{\text{per}}^{2+\lambda, 1+\lambda/2}(\overline{Q}_T) \times C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T).$$

Furthermore there exists a constant $\mathcal{C} = \mathcal{C}(T)$ such that

$$\begin{aligned}
\|g\|_{Q_T}^{(2+\lambda)} + \|f\|_{\Sigma_T}^{(3+\lambda)} &\leq \mathcal{C} \left(\|g_0\|_{\mathbf{R} \times (0, e)}^{(2+\lambda)} + \|f_0\|_{\mathbf{R}}^{(3+\lambda)} \right. \\
&\quad \left. + \|F_1\|_{Q_T}^{(\lambda)} + \|F_2\|_{\Sigma_T}^{(1+\lambda)} + \|F_3\|_{\Sigma_T}^{(1+\lambda)} + \|F_4\|_{\Sigma_T}^{(1+\lambda)} \right).
\end{aligned}$$

Proof. We consider the mapping $\mathcal{L} : f \rightarrow \mathcal{L}(f) = h$ defined as follows. Given $f \in C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ with $f(0) = f_0$ (where we use the notation $f(0) = f|_{t=0}$) we consider Problem (5.1)–(5.5). By [g, Theorem 5.3 p. 320] this problem has a unique solution $g \in C^{2+\lambda, 1+\lambda/2}(\overline{Q}_T)$ and the uniqueness property together with the periodicity of the data implies that $g \in C_{\text{per}}^{2+\lambda, 1+\lambda/2}(\overline{Q}_T)$. One then computes $h = \mathcal{L}f \in C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ as the unique solution of Problem (5.6)–(5.8) [g, Theorem 5.1 p. 320]. The following results on continuous dependence then hold

$$\begin{aligned}
(5.9) \quad \|g\|_{Q_T}^{(2+\lambda)} &\leq \mathcal{C}_1 \left(\|g_0\|_{\mathbf{R} \times (0, e)}^{(2+\lambda)} + \|F_1\|_{Q_T}^{(\lambda)} \right. \\
&\quad \left. + \|F_2\|_{\Sigma_T}^{(1+\lambda)} + \|F_3\|_{\Sigma_T}^{(1+\lambda)} + \|f\|_{\Sigma_T}^{(3+\lambda)} \right)
\end{aligned}$$

$$(5.10) \quad \|h\|_{\Sigma_T}^{(3+\lambda)} \leq \mathcal{C}_2 \left(\|f_0\|_{\mathbf{R}}^{(3+\lambda)} + \|F_4\|_{\Sigma_T}^{(1+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)} \right)$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ only depend on T, λ and on the parameters e, α and γ .

Clearly if f is a fixed point of $\mathcal{L}$, i.e. if $\mathcal{L}(f) = f$ and if g is the solution of the problem (5.1)–(5.5), then (f, g) satisfies the equations (5.1)–(5.8). Moreover $\mathcal{L}$ is an affine mapping from the affine subspace $\{f \in C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T), f(0) = f_0\}$ of $C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ into itself. The linear mapping $\mathcal{M}$ corresponding to $\mathcal{L}$ which is such that

$$(5.11) \quad \mathcal{L}(f) - \mathcal{L}(\bar{f}) = \mathcal{M}(f - \bar{f})$$

is defined on

$$\mathring{C}_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T) := \{f \in C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T), f(0) = 0\}$$

by

$$f \rightarrow \mathcal{M}(f) = k,$$

where k is the solution of

$$(5.12) \quad \begin{cases} k_t - \alpha \gamma k_{xx} = w(x, 0, t) & \text{in } \Sigma_T \\ k(x + 2L, t) = k(x, t) & \text{in } \Sigma_T \\ k(x, 0) = 0 & x \in \mathbb{R}, \end{cases}$$

where w is the solution of

$$(5.13) \quad \begin{cases} w_t - \Delta w = 0 & \text{in } Q_T \\ w - w_y = -\alpha \gamma f_{xx} & \text{on } \Sigma_T \\ w_y(x, e, t) = 0 & (x, t) \in \Sigma_T \\ w(x + 2L, y, t) = w(x, y, t) & \text{in } Q_T \\ w(x, y, 0) = 0 & (x, y) \in \mathbb{R} \times (0, e). \end{cases}$$

In what follows we shall prove that $(Id - \mathcal{M})$ is a continuous invertible mapping from $\mathring{C}_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ into itself; next we show that this property is equivalent to the existence of a unique fixed point of $\mathcal{L}$. Adding $f - \bar{f}$ to both sides of equality (5.11) yields

$$\mathcal{L}(f) - \mathcal{L}(\bar{f}) + f - \bar{f} - \mathcal{M}(f - \bar{f}) = f - \bar{f}$$

so that

$$(Id - \mathcal{M})(f - \bar{f}) = \mathcal{L}(\bar{f}) - \bar{f} - \mathcal{L}(f) + f.$$

Thus solving

$$\mathcal{L}(f) = f$$

is equivalent to solving

$$(Id - \mathcal{M})(f - \bar{f}) = \mathcal{L}(\bar{f}) - \bar{f}$$

which gives

$$(5.14) \quad f = \bar{f} + (Id - \mathcal{M})^{-1}(\mathcal{L}(\bar{f}) - \bar{f}).$$

In a first step we shall prove that $\mathcal{M}$ is a compact operator and in a second step that $\text{Ker}(Id - \mathcal{M}) = \{0\}$. It will then follow from the Fredholm alternative (see for instance Brezis [1, Theorem 6.1]) that the operator $(Id - \mathcal{M})$ is invertible.

(i) $\mathcal{M}$ is a compact operator from $\mathring{C}_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ into itself.

Let f be given in $\mathring{C}_{\text{per}}^{3+\lambda,(3+\lambda)/2}(\overline{\Sigma}_T)$; by [9] the solution w of (5.13) belongs to $\mathring{C}_{\text{per}}^{2+\lambda,(2+\lambda)/2}(\overline{Q}_T)$ so that also by [9] the solution k of (5.12) belongs to $\mathring{C}_{\text{per}}^{4+\lambda,2+\lambda/2}(\overline{\Sigma}_T)$. Thus $\mathcal{M}$ is a bounded operator from $\mathring{C}_{\text{per}}^{3+\lambda,(3+\lambda)/2}(\overline{\Sigma}_T)$ into $\mathring{C}_{\text{per}}^{4+\lambda,2+\lambda/2}(\overline{\Sigma}_T)$. Using furthermore the compactness of the embedding from $\mathring{C}_{\text{per}}^{4+\lambda,2+\lambda/2}(\overline{\Sigma}_T)$ into $\mathring{C}_{\text{per}}^{3+\lambda,(3+\lambda)/2}(\overline{\Sigma}_T)$ we conclude that $\mathcal{M}$ is a compact operator from $\mathring{C}_{\text{per}}^{3+\lambda,(3+\lambda)/2}(\overline{\Sigma}_T)$ into itself.

(ii) $\text{Ker} (Id - \mathcal{M}) = \{0\}$.

Let $f \in \mathring{C}_{\text{per}}^{3+\lambda,(3+\lambda)/2}(\overline{\Sigma}_T)$ be such that $(Id - \mathcal{M})f = 0$, i.e. $\mathcal{M}f = f$. Then if w is the unique solution of (5.13), f satisfies the system:

$$(5.15) \quad \begin{cases} f_t - \alpha\gamma f_{xx} = w(x, 0, t) & \text{in } \Sigma_T \\ f(x + 2L, t) = f(x, t) & \text{in } \Sigma_T \\ f(x, 0) = 0 & x \in \mathbb{R}. \end{cases}$$

Next we multiply the differential equation in (5.15) first by f and then by $-\alpha\gamma f_{xx}$ and integrate by parts on $(-L, L)$. This gives

$$(5.16) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-L}^L f^2(x, t) dx + \alpha\gamma \int_{-L}^L f_x^2(x, t) dx \\ = \int_{-L}^L w(x, 0, t) f(x, t) dx \end{aligned}$$

and

$$(5.17) \quad \begin{aligned} \frac{\alpha\gamma}{2} \frac{d}{dt} \int_{-L}^L f_x^2(x, t) dx + \alpha^2\gamma^2 \int_{-L}^L f_{xx}^2(x, t) dx \\ = -\alpha\gamma \int_{-L}^L w(x, 0, t) f_{xx}(x, t) dx. \end{aligned}$$

Now we multiply the differential equation in (5.13) by w and deduce from the boundary condition that w satisfies

$$(5.18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^\epsilon \int_{-L}^L w^2(x, y, t) dx dy + \int_0^\epsilon \int_{-L}^L (\text{grad } w(x, y, t))^2 dx dy \\ = \int_{-L}^L (w(x, 0, t) + \alpha\gamma f_{xx}(x, t)) w(x, 0, t) dx. \end{aligned}$$

Using Young's inequality we deduce from (5.16) that

$$\begin{aligned}
(5.19) \quad & \frac{1}{2} \frac{d}{dt} \int_{-L}^L f^2(x, t) dx + \alpha \gamma \int_{-L}^L f_x^2(x, t) dx \\
& \leq \int_{-L}^L w^2(x, 0, t) dx + \frac{1}{4} \int_{-L}^L f^2(x, t) dx
\end{aligned}$$

Adding (5.17), (5.18) and (5.19) yields

$$\begin{aligned}
(5.20) \quad & \frac{1}{2} \frac{d}{dt} \int_{-L}^L f^2 dx + \alpha \gamma \int_{-L}^L f_x^2 dx + \frac{\alpha \gamma}{2} \frac{d}{dt} \int_{-L}^L f_x^2 dx + \alpha^2 \gamma^2 \int_{-L}^L f_{xx}^2 dx + \\
& \frac{1}{2} \frac{d}{dt} \int_0^e \int_{-L}^L w^2 dx dy + \int_0^e \int_{-L}^L (\text{grad } w)^2 dx dy \\
& \leq 2 \int_{-L}^L w^2(x, 0, t) dx + \frac{1}{4} \int_{-L}^L f^2 dx.
\end{aligned}$$

Finally we use the following result which we shall prove later.

LEMMA 5.2. *Let Ω be a bounded Lipschitzian domain in $\mathbb{R}^2$; for all $\varepsilon > 0$ there exists a positive constant C_ε such that*

$$\int_{\partial\Omega} w^2(s) ds \leq \varepsilon \iint_{\Omega} (\text{grad } w)^2 dx + C_\varepsilon \iint_{\Omega} w^2 dx$$

for all $w \in H^1(\Omega)$.

We set $\varepsilon = 1/4$ to deduce that

$$\begin{aligned}
(5.21) \quad & \frac{1}{2} \frac{d}{dt} \left(\int_{-L}^L f^2 dx + \alpha \gamma \int_{-L}^L f_x^2 dx + \int_0^e \int_{-L}^L w^2 dx dy \right) \\
& \leq \frac{1}{4} \int_{-L}^L f^2 dx + 2C_{1/4} \int_0^e \int_{-L}^L w^2 dx dy.
\end{aligned}$$

The inequality above is of the form $\frac{dY}{dt} \leq CY$, with

$$Y(t) = \frac{1}{2} \int_{-L}^L f^2 dx + \frac{\alpha \gamma}{2} \int_{-L}^L f_x^2 dx + \frac{1}{2} \int_0^e \int_{-L}^L w^2 dx dy$$

and since $Y(0) = 0$ we conclude that $Y(t) = 0$ for all t . This implies that f and w are identically equal to zero and therefore that $Id - \mathcal{M}$ is an invertible mapping.

Finally the continuous dependence of the fixed point f of $\mathcal{L}$ on the data $F_i, i = 1, 2, 3, 4, f_0$ and g_0 can be seen as follows. Choose $\bar{f}(x, t) = f_0(x)$ in (5.14); then f is given by

$$(5.22) \quad f = f_0 + (Id - \mathcal{M})^{-1}(\mathcal{L}(f_0) - f_0)$$

and we deduce from (5.9), (5.10) that

$$(5.23) \quad \begin{aligned} \|\mathcal{L}(f_0)\|_{\Sigma_T}^{(3+\lambda)} &\leq \mathcal{C}_3 \left(\|f_0\|_{\mathbf{R}}^{(3+\lambda)} + \|F_4\|_{\Sigma_T}^{(1+\lambda)} \right. \\ &\quad \left. + \|g_0\|_{\mathbf{R} \times (0, \epsilon)}^{(2+\lambda)} + \|F_1\|_{Q_T}^{(\lambda)} + \|F_2\|_{\Sigma_T}^{(1+\lambda)} + \|F_3\|_{\Sigma_T}^{(1+\lambda)} \right) \end{aligned}$$

where we have used the fact that $\|f_0\|_{\Sigma_T}^{(3+\lambda)} = \|f_0\|_{\mathbf{R}}^{(3+\lambda)}$. Therefore, since $(Id - \mathcal{M})^{-1}$ is a bounded operator we have that

$$(5.24) \quad \|f\|_{\Sigma_T}^{(3+\lambda)} \leq \mathcal{C}_4 \left(\|f_0\|_{\mathbf{R}}^{(3+\lambda)} + \|\mathcal{L}(f_0)\|_{\Sigma_T}^{(3+\lambda)} \right).$$

The result of Theorem 5.1 then follows from (5.9), (5.23) and (5.24).

Proof of Lemma 5.2. We have that

$$\int_{\partial\Omega} w^2(s) ds = \|w\|_{H^0(\partial\Omega)}^2 \leq C \|w\|_{H^{1/2+\delta}(\Omega)}^2$$

for all $\delta > 0$. Next we use the following result given for instance by [10, p. 47]

$$H^{1/2+\delta}(\Omega) = [H^1(\Omega), H^0(\Omega)]_{1/2-\delta}$$

so that

$$\|w\|_{H^{1/2+\delta}(\Omega)} \leq C(\delta) \|w\|_{H^1(\Omega)}^{1/2+\delta} \|w\|_{L^2(\Omega)}^{1/2-\delta}.$$

Choose for instance $\delta = 1/4$. Then

$$\|w\|_{H^{3/4}(\Omega)} \leq C \|w\|_{H^1(\Omega)}^{3/4} \|w\|_{L^2(\Omega)}^{1/4}$$

that is

$$\|w\|_{H^{3/4}(\Omega)}^2 \leq C (\|w\|_{H^1(\Omega)}^2)^{3/4} (\|w\|_{L^2(\Omega)}^2)^{1/4}.$$

Using the inequality

$$a^{3/4} b^{1/4} = (\varepsilon a)^{3/4} (b/\varepsilon^3)^{1/4} \leq \varepsilon a + b/\varepsilon^3$$

for all $a, b, \varepsilon > 0$, we deduce that

$$\|w\|_{H^{3/4}(\Omega)}^2 \leq C(\tilde{\varepsilon}) \|w\|_{H^1(\Omega)}^2 + \|w\|_{L^2(\Omega)}^2 / \tilde{\varepsilon}^3,$$

which implies the result of Lemma 5.2.

6. Uniqueness of the solution. To begin with, we rewrite Problem P in the form

$$(6.1) \quad g_t - \Delta g = f_x^2 g_{yy} - 2f_x g_{xy} - (f_{xx} - f_t)g_y \quad \text{in } Q_T$$

$$(6.2) \quad g(x, 0, t) - (1 + f_x^2)^{1/2} g_y(x, 0, t) + \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, 0, t) \\ = \alpha(e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1) \quad \text{in } \Sigma_T$$

$$(6.3) \quad (1 + f_x^2)^{1/2} g_y(x, e, t) - \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, e, t) = 0 \quad \text{in } \Sigma_T$$

$$(6.4) \quad g(x + 2L, y, t) = g(x, y, t) \quad \text{in } Q_T$$

$$(6.5) \quad g(x, y, 0) = g_0(x, y) \quad (x, y) \in \mathbb{R} \times (0, e)$$

$$(6.6) \quad f_t = -(1 + f_x^2)^{1/2} (\alpha e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - g(x, 0, t) - \alpha) \quad \text{in } \Sigma_T$$

$$(6.7) \quad f(x + 2L, t) = f(x, t) \quad \text{in } \Sigma_T$$

$$(6.8) \quad f(x, 0) = f_0(x) \quad x \in \mathbb{R}.$$

THEOREM 6.1. *There exists at most one solution (g, f) of Problem P such that $g \in L^\infty(0, T; W^{2,\infty}(\mathbb{R} \times (0, e)))$, $g_t \in L^\infty(Q_T)$ and $f \in L^\infty(0, T; W^{3,\infty}(\mathbb{R}))$, $f_t \in L^\infty(\Sigma_T)$.*

Proof. The method of proof is similar to that in Section 5, when we proved the uniqueness of the solution of the linear problem. To begin with, we multiply the difference of the equations for g_1 and g_2 by $g_1 - g_2$ and integrate by parts. We obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^e \int_{-L}^L (g_1 - g_2)^2 + \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 \\ &= \int_0^e \int_{-L}^L (f_{1x}^2 g_{1yy} - f_{2x}^2 g_{2yy})(g_1 - g_2) \quad E_1 \\ & \quad - 2 \int_0^e \int_{-L}^L (f_{1x} g_{1xy} - f_{2x} g_{2xy})(g_1 - g_2) \quad E_2 \\ & \quad - \int_0^e \int_{-L}^L ((f_{1xx} - f_{1t})g_{1y} - (f_{2xx} - f_{2t})g_{2y})(g_1 - g_2) \quad E_3 \\ & \quad - \int_{-L}^L \left[\frac{\partial}{\partial y} (g_1 - g_2)(g_1 - g_2) \right] (x, 0) dx \quad E_4 \\ & \quad + \int_{-L}^L \left[\frac{\partial}{\partial y} (g_1 - g_2)(g_1 - g_2) \right] (x, e) dx \quad E_5 \end{aligned}$$

Then

$$\begin{aligned}
E_1 &= \int_0^e \int_{-L}^L \{ f_{1x}^2 (g_{1yy} - g_{2yy}) (g_1 - g_2) + g_{2yy} (f_{1x}^2 - f_{2x}^2) (g_1 - g_2) \} \\
&\leq - \int_0^e \int_{-L}^L f_{1x}^2 (g_{1y} - g_{2y})^2
\end{aligned} \tag{E11}$$

$$+ \int_{-L}^L [f_{1x}^2 (g_{1y} - g_{2y}) (g_1 - g_2)] (x, e, t) dx \tag{E12}$$

$$- \int_{-L}^L [f_{1x}^2 (g_{1y} - g_{2y}) (g_1 - g_2)] (x, 0, t) dx \tag{E13}$$

$$+ C \int_{-L}^L (f_1 - f_2)_x^2 + C \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$E_2 = -2 \int_0^e \int_{-L}^L f_{1x} (g_1 - g_2)_{xy} (g_1 - g_2) - 2 \int_0^e \int_{-L}^L g_{2xy} (f_1 - f_2)_x (g_1 - g_2)$$

$$E_2 = 2 \int_0^e \int_{-L}^L f_{1x} (g_1 - g_2)_x (g_1 - g_2)_y \tag{E21}$$

$$- 2 \int_{-L}^L [f_{1x} (g_1 - g_2)_x (g_1 - g_2)] (x, e, t) dx \tag{E22}$$

$$+ 2 \int_{-L}^L [f_{1x} (g_1 - g_2)_x (g_1 - g_2)] (x, 0, t) dx \tag{E23}$$

$$- 2 \int_0^e \int_{-L}^L g_{2xy} (f_1 - f_2)_x (g_1 - g_2) \tag{E24}$$

$$|E_{24}| \leq C \int_{-L}^L (f_1 - f_2)_x^2 + C \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$E_{22} = -f_{1x} (g_1 - g_2)^2 (L, e, t) + f_{1x} (g_1 - g_2)^2 (-L, e, t)$$

$$+ \int_{-L}^L f_{1xx} (g_1 - g_2)^2 (x, e, t) dx$$

so that

$$\begin{aligned}
|E_{22}| &\leq C \int_{-L}^L (g_1 - g_2)^2(x, e, t) dx \\
&\leq \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2
\end{aligned}$$

and a similar estimate holds for E_{23} . Let $\delta \in (0, 1)$ be a constant to be determined. We have that

$$\begin{aligned}
|E_{21}| &\leq (1 - \delta) \int_0^e \int_{-L}^L (g_1 - g_2)_x^2 + \frac{1}{1 - \delta} \int_0^e \int_{-L}^L f_{1x}^2 (g_1 - g_2)_y^2 \\
&\leq (1 - \delta) \int_0^e \int_{-L}^L (g_1 - g_2)_x^2 + \int_0^e \int_{-L}^L f_{1x}^2 (g_1 - g_2)_y^2 \\
&\quad + \left(\frac{1}{1 - \delta} - 1 \right) \int_0^e \int_{-L}^L f_{1x}^2 (g_1 - g_2)_y^2 \tag{E_{212}} \\
|E_{212}| &\leq \|f_{1x}\|_{L^\infty(\Sigma_T)} \frac{\delta}{1 - \delta} \int_0^e \int_{-L}^L (g_1 - g_2)_y^2
\end{aligned}$$

Thus

$$(6.9) \quad |E_{212}| \leq (1 - \delta) \int_0^e \int_{-L}^L (g_1 - g_2)_y^2$$

if

$$\frac{\delta}{1 - \delta} (\|f_{1x}\|_{L^\infty(\Sigma_T)})^2 \leq 1 - \delta$$

and thus, if $\|f_{1x}\|_{L^\infty(\Sigma_T)} \leq M$, if

$$\frac{\delta}{1 - \delta} M^2 \leq 1 - \delta$$

that is

$$\delta^2 - (2 + M^2)\delta + 1 \geq 0.$$

Choose

$$\delta \leq \delta^- := \frac{2 + M^2 - \sqrt{(2 + M^2)^2 - 4}}{2}.$$

Then (6.9) is satisfied so that

$$|E_{21}| + E_{11} \leq (1 - \delta) \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2.$$

$$E_3 = - \int_0^e \int_{-L}^L (f_1 - f_2)_{xx} g_{1y} (g_1 - g_2) \quad E_{31}$$

$$- \int_0^e \int_{-L}^L f_{2xx} (g_1 - g_2)_y (g_1 - g_2) \quad E_{32}$$

$$- \alpha \int_0^e \int_{-L}^L \left((1 + f_{1x}^2)^{1/2} e^{-\gamma f_{1xx}/(1+f_{1x}^2)^{3/2}} - (1 + f_{2x}^2)^{1/2} e^{-\gamma f_{2xx}/(1+f_{2x}^2)^{3/2}} \right) g_{1y} (g_1 - g_2) \quad E_{33}$$

$$- \alpha \int_0^e \int_{-L}^L (1 + f_{2x}^2)^{1/2} e^{-\gamma f_{2xx}/(1+f_{2x}^2)^{3/2}} (g_1 - g_2)_y (g_1 - g_2) \quad E_{34}$$

$$+ \int_0^e \int_{-L}^L \left[(1 + f_{1x}^2)^{1/2} (g_1(x, 0, t) + \alpha) - (1 + f_{2x}^2)^{1/2} (g_2(x, 0, t) + \alpha) \right] g_{1y} (g_1 - g_2) \quad E_{35}$$

$$+ \int_0^e \int_{-L}^L (1 + f_{2x}^2)^{1/2} (g_2(x, 0, t) + \alpha) (g_1 - g_2)_y (g_1 - g_2) \quad E_{36}$$

$$|E_{31}| \leq \varepsilon \int_{-L}^L (f_1 - f_2)_{xx}^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$|E_{32}| \leq \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

and similar estimates hold for $|E_{34}|$ and $|E_{36}|$.

$$E_{35} = \int_0^e \int_{-L}^L (1 + f_{1x}^2)^{1/2} (g_1 - g_2)(x, 0, t) g_{1y} (g_1 - g_2) \quad E_{351}$$

$$+ \int_0^e \int_{-L}^L \left((1 + f_{1x}^2)^{1/2} - (1 + f_{2x}^2)^{1/2} \right) (g_2(x, 0, t) + \alpha) g_{1y} (g_1 - g_2) \quad E_{352}$$

$$|E_{351}| \leq \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$|E_{352}| \leq \int_0^e \int_{-L}^L \left| \int_0^1 d\theta \frac{\theta f_{1x} + (1-\theta)f_{2x}}{(1 + (\theta f_{1x} + (1-\theta)f_{2x})^2)^{1/2}} \right| |(g_2(x, 0, t) + \alpha)| |g_{1y}| |(f_1 - f_2)_x| |g_1 - g_2|$$

so that

$$|E_{352}| \leq C \int_{-L}^L (f_1 - f_2)_x^2 + C \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$E_{33} = -\alpha \int_0^e \int_{-L}^L e^{-\gamma f_{1xx}/(1+f_{1x}^2)^{3/2}} \left((1+f_{1x}^2)^{1/2} - (1+f_{2x}^2)^{1/2} \right) g_{1y}(g_1 - g_2) \quad E_{331}$$

$$- \alpha \int_0^e \int_{-L}^L (1+f_{2x}^2)^{1/2} \left(e^{-\gamma f_{1xx}/(1+f_{1x}^2)^{3/2}} - e^{-\gamma f_{2xx}/(1+f_{2x}^2)^{3/2}} \right) g_{1y}(g_1 - g_2) \quad E_{332}$$

$$|E_{331}| \leq C \int_{-L}^L (f_1 - f_2)_x^2 + C \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$|E_{332}| \leq C \int_0^e \int_{-L}^L \left(\int_0^1 d\theta e^{-\gamma(\theta f_{1xx}/(1+f_{1x}^2)^{3/2} + (1-\theta)f_{2xx}/(1+f_{2x}^2)^{3/2})} \right)$$

$$\left| \frac{f_{1xx}}{(1+f_{1x}^2)^{3/2}} - \frac{f_{2xx}}{(1+f_{2x}^2)^{3/2}} \right| |g_1 - g_2|$$

$$\leq C \int_0^e \int_{-L}^L |(f_1 - f_2)_{xx}| |g_1 - g_2| + C \int_0^e \int_{-L}^L |(f_1 - f_2)_x| |g_1 - g_2|$$

so that finally

$$|E_{332}| \leq \varepsilon \int_0^e \int_{-L}^L (f_1 - f_2)_{xx}^2 + C \int_0^e \int_{-L}^L (f_1 - f_2)_x^2$$

$$+ C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$E_{12} = \int_{-L}^L f_{1x}^2 \left(\frac{f_{1x}}{1+f_{1x}^2} g_{1x}(x, e, t) - \frac{f_{2x}}{1+f_{2x}^2} g_{2x}(x, e, t) \right) (g_1 - g_2)(x, e, t) dx$$

$$E_{12} = \int_{-L}^L \frac{f_{1x}^3}{1+f_{1x}^2} ((g_1 - g_2)_x (g_1 - g_2))(x, e, t) dx \quad E_{121}$$

$$+ \int_{-L}^L f_{1x}^2 \left(\frac{f_{1x}}{1+f_{1x}^2} - \frac{f_{2x}}{1+f_{2x}^2} \right) g_{2x}(x, e, t) (g_1 - g_2)(x, e, t) dx \quad E_{122}$$

$$|E_{122}| \leq C \int_{-L}^L (f_1 - f_2)_x^2 + \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2$$

$$E_{121} = -\frac{1}{2} \int_{-L}^L \left(\frac{f_{1x}^3}{1+f_{1x}^2} \right)_x (g_1 - g_2)^2(x, e, t)$$

so that

$$|E_{121}| \leq \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2.$$

Similar results hold for E_{13} , E_4 and E_5 .

Next we write an equation for $f_1 - f_2$.

$$\begin{aligned} (f_1 - f_2)_t &= \\ &= -\alpha((1+f_{1x}^2)^{1/2} - (1+f_{2x}^2)^{1/2}) e^{-\gamma \frac{f_{1xx}}{(1+f_{1x}^2)^{3/2}}} \\ &\quad - \alpha(1+f_{2x}^2)^{1/2} \left(e^{-\gamma \frac{f_{1xx}}{(1+f_{1x}^2)^{3/2}}} - e^{-\gamma \frac{f_{2xx}}{(1+f_{2x}^2)^{3/2}}} \right) \\ &\quad + (1+f_{1x}^2)^{1/2} (g_1 - g_2)(x, 0, t) \\ &\quad + (g_2(x, 0, t) + \alpha) \left(\int_0^1 d\theta \frac{\theta f_{1x} + (1-\theta) f_{2x}}{(1 + (\theta f_{1x} + (1-\theta) f_{2x})^2)^{1/2}} \right) (f_1 - f_2)_x \end{aligned}$$

from which we deduce that

$$(6.10) \quad (f_1 - f_2)_t = a(x, t)(f_1 - f_2)_{xx} + b(x, t)(f_1 - f_2)_x + c(x, t)(g_1 - g_2)(x, 0, t)$$

with

$$a, b, c \in L^\infty(0, T; W^{1,\infty}(\mathbb{R})) \text{ and } a \geq \chi > 0 \text{ for some constant } \chi.$$

We multiply (6.10) by $f_1 - f_2$ and integrate by parts to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-L}^L (f_1 - f_2)^2 + \chi \int_{-L}^L (f_1 - f_2)_x^2 &= - \int_{-L}^L a_x (f_1 - f_2)_x (f_1 - f_2) \\ &+ \int_{-L}^L b (f_1 - f_2)_x (f_1 - f_2) + \int_{-L}^L c (g_1 - g_2)(x, 0, t) (f_1 - f_2) \end{aligned}$$

from which we deduce that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-L}^L (f_1 - f_2)^2 + \chi \int_{-L}^L (f_1 - f_2)_x^2 \\ \leq \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2 \\ + C \int_{-L}^L (f_1 - f_2)_x^2 + C \int_{-L}^L (f_1 - f_2)^2. \end{aligned}$$

Finally we multiply (6.10) by $-(f_1 - f_2)_{xx}$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-L}^L (f_1 - f_2)_x^2 + \chi \int_{-L}^L (f_1 - f_2)_{xx}^2 \\ \leq \varepsilon \int_{-L}^L (f_1 - f_2)_{xx}^2 + C_\varepsilon \int_{-L}^L (f_1 - f_2)_x^2 \\ + \varepsilon \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + C_\varepsilon \int_0^e \int_{-L}^L (g_1 - g_2)^2. \end{aligned}$$

Adding the inequalities obtained in this section yields

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\int_0^e \int_{-L}^L (g_1 - g_2)^2 + \int_{-L}^L (f_1 - f_2)_x^2 + \int_{-L}^L (f_1 - f_2)^2 \right) \\
& + \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + \chi \int_{-L}^L ((f_1 - f_2)_x^2 + (f_1 - f_2)_{xx}^2) \\
& \leq (1 - \delta + \varepsilon) \int_0^e \int_{-L}^L (\text{grad}(g_1 - g_2))^2 + \varepsilon \int_{-L}^L (f_1 - f_2)_{xx}^2 \\
& + C_\varepsilon \left(\int_0^e \int_{-L}^L (g_1 - g_2)^2 + \int_{-L}^L (f_1 - f_2)_x^2 + \int_{-L}^L (f_1 - f_2)^2 \right).
\end{aligned}$$

Then, choosing ε small enough and setting

$$Y(t) = \int_0^e \int_{-L}^L (g_1 - g_2)^2 + \int_{-L}^L (f_1 - f_2)_x^2 + \int_{-L}^L (f_1 - f_2)^2,$$

we deduce that

$$\frac{dY}{dt} \leq CY \text{ with } Y(0) = 0$$

and uniqueness follows.

7. Existence. In order to prove the existence of a solution we rewrite the nonlinear problem P in the form

$$\begin{aligned}
(7.1) \quad & \begin{cases} g_t - \Delta g = f_x^2 g_{yy} - 2f_x g_{xy} - f_{xx} g_y + f_t g_y & \text{in } Q_T \\ (g - g_y)(x, 0, t) + \alpha \gamma f_{xx}(x, t) = \alpha(e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1 + \gamma f_{xx}) \\ + ((1 + f_x^2)^{1/2} - 1)g_y(x, 0, t) - \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, 0, t) & (x, t) \in \Sigma_T \end{cases} \\
(7.2) \quad & \\
(7.3) \quad & \begin{cases} g_y(x, e, t) = (1 - (1 + f_x^2)^{1/2})g_y(x, e, t) + \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, e, t) & (x, t) \in \Sigma_T \end{cases} \\
(7.4) \quad & \begin{cases} g(x + 2L, y, t) = g(x, y, t) & (x, y, t) \in Q_T \end{cases} \\
(7.5) \quad & \begin{cases} g(x, y, 0) = g_0(x, y) & (x, y) \in \mathbb{R} \times (0, e) \end{cases} \\
(7.6) \quad & \begin{cases} f_t - \alpha \gamma f_{xx} - g(x, 0, t) = -\alpha(1 + f_x^2)^{1/2} \left(e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1 + \gamma \frac{f_{xx}}{(1 + f_x^2)^{1/2}} \right) \\ + ((1 + f_x^2)^{1/2} - 1)g(x, 0, t) & \text{in } \Sigma_T \end{cases} \\
(7.7) \quad & \begin{cases} f(x + 2L, t) = f(x, t) & (x, t) \in \Sigma_T \end{cases} \\
(7.8) \quad & \begin{cases} f(x, 0) = f_0(x) & x \in \mathbb{R}. \end{cases}
\end{aligned}$$

Remark that the left-hand-sides of the equations in the linear problem (5.1)-(5.8) and in Problem (6.1)-(6.8) coincide. Here again we suppose that

$$g_0 \in C_{\text{per}}^{2+\lambda}(\mathbb{R} \times [0, e]), f_0 \in C_{\text{per}}^{3+\lambda}(\mathbb{R})$$

and that g_0 satisfies compatibility conditions, namely equations (7.2) and (7.3) in which f and g are replaced by f_0 and g_0 .

Next we define

$$(7.9) \quad F_1(f, g) = f_x^2 g_{yy} - 2f_x g_{xy} - f_{xx} g_y + f_t g_y,$$

$$(7.10) \quad F_2(f, g) = \alpha(e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1 + \gamma f_{xx}) \\ + ((1 + f_x^2)^{1/2} - 1)g_y(x, 0, t) - \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, 0, t)$$

$$(7.11) \quad F_3(f, g) = (1 - (1 + f_x^2)^{1/2})g_y(x, e, t) + \frac{f_x}{(1 + f_x^2)^{1/2}} g_x(x, e, t)$$

$$(7.12) \quad F_4(f, g) = -\alpha(1 + f_x^2)^{1/2} \left(e^{-\gamma f_{xx}/(1+f_x^2)^{3/2}} - 1 + \gamma \frac{f_{xx}}{(1 + f_x^2)^{1/2}} \right) \\ + ((1 + f_x^2)^{1/2} - 1)g(x, 0, t)$$

and prove the following preliminary lemma.

LEMMA 7.1. *Let $\delta_0 > 0$ be arbitrary.*

(i) *There exists a positive constant $K_1 = K_1(\delta_0)$ such that for each $\delta \in (0, \delta_0)$*

$$\|F_1(f, g)\|_{Q_T}^{(\lambda)} + \sum_{i=2}^4 \|F_i(f, g)\|_{\Sigma_T}^{(1+\lambda)} \\ \leq K_1(\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)})^2$$

for all $f \in C^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$ and $g \in C^{2+\lambda, 1+\lambda/2}(\overline{Q}_T)$ satisfying $\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)} \leq \delta$.

(ii) *There exists a constant $K_2 = K_2(\delta_0)$ such that for each $\delta \in (0, \delta_0)$ and for all $f, h \in C^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T)$, $g, k \in C^{2+\lambda, 1+\lambda/2}(\overline{Q}_T)$ satisfying*

$$\max(\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)}, \|h\|_{\Sigma_T}^{(3+\lambda)} + \|k\|_{Q_T}^{(2+\lambda)}) \leq \delta,$$

there holds

$$\|F_1(f, g) - F_1(h, k)\|_{Q_T}^{(\lambda)} + \sum_{i=2}^4 \|F_i(f, g) - F_i(h, k)\|_{\Sigma_T}^{(1+\lambda)} \\ \leq K_2 \delta (\|f - h\|_{\Sigma_T}^{(3+\lambda)} + \|g - k\|_{Q_T}^{(2+\lambda)}).$$

Proof. (i) For F_i the property directly follows from Lemma 4.1 (i). Next we show the property for F_2 . By Lemma 4.1 (i), (ii)

$$\left\| \frac{f_{xx}}{(1+f_x^2)^{j/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \leq C \|f_{xx}\|_{\Sigma_T}^{(1+\lambda)} \left(\|(1+f_x^2)^{j/2}\|_{\Sigma_T}^{1+\lambda} \right)^2 \quad \text{for } j = 1, 3.$$

Using also Lemma 4.1 (iii) we deduce that if $f \in C^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma_T})$ there exists a constant $K_1 = K_1(\delta_0)$ such that

$$\left\| \frac{f_{xx}}{(1+f_x^2)^{j/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \leq K_1 \|f\|_{\Sigma_T}^{(3+\lambda)} \quad \text{for } j = 1, 3.$$

Similarly one can check that there exists $K_2 = K_2(\delta_0)$ such that

$$\|(1+f_x^2)^{1/2} - 1\|_{\Sigma_T}^{(1+\lambda)} \leq K_2 \|f\|_{\Sigma_T}^{(3+\lambda)}$$

and $K_3 = K_3(\delta_0)$ such that

$$\left\| \frac{1}{(1+f_x^2)^{1/2}} - \frac{1}{(1+f_x^2)^{3/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \leq K_3 (\|f\|_{\Sigma_T}^{(3+\lambda)})^2.$$

Thus using also (4.1) and Lemma 4.2 we deduce that

$$\begin{aligned} \|F_2(f, g)\|_{\Sigma_T}^{(1+\lambda)} &\leq K_4 \left\{ (\|f\|_{\Sigma_T}^{(3+\lambda)})^2 + \|f\|_{\Sigma_T}^{(3+\lambda)} \|g\|_{Q_T}^{(2+\lambda)} \right\} \\ &\leq K_5 (\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)})^2, \end{aligned}$$

and that similar results hold for F_3 and F_4 .

(ii) The property is obvious for F_1 . For instance

$$\begin{aligned} &\|f_x^2 g_{yy} - h_x^2 k_{yy}\|_{Q_T}^{(\lambda)} \\ &\leq \|(f_x^2 - h_x^2) g_{yy}\|_{Q_T}^{(\lambda)} + \|h_x^2 (g_{yy} - k_{yy})\|_{Q_T}^{(\lambda)} \\ &\leq C \left(\|g_{yy}\|_{Q_T}^{(\lambda)} \|f_x + h_x\|_{\Sigma_T}^{(\lambda)} \|f_x - h_x\|_{\Sigma_T}^{(\lambda)} + (\|h_x\|_{\Sigma_T}^{(\lambda)})^2 \|g_{yy} - k_{yy}\|_{Q_T}^{(\lambda)} \right) \\ &\leq C \delta^2 \left(\|f - h\|_{\Sigma_T}^{(3+\lambda)} + \|g - k\|_{Q_T}^{(2+\lambda)} \right). \end{aligned}$$

As for F_2 we first consider the term

$$\begin{aligned} F &= \left\| \frac{f_x}{(1+f_x^2)^{1/2}} g_x(x, 0, t) - \frac{h_x}{(1+h_x^2)^{1/2}} k_x(x, 0, t) \right\|_{\Sigma_T}^{(1+\lambda)} \\ &\leq C \left\| \frac{f_x}{(1+f_x^2)^{1/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \|g_x(x, 0, t) - k_x(x, 0, t)\|_{\Sigma_T}^{(1+\lambda)} \\ &\quad + C \|k_x(x, 0, t)\|_{\Sigma_T}^{(1+\lambda)} \left\| \frac{f_x}{(1+f_x^2)^{1/2}} - \frac{h_x}{(1+h_x^2)^{1/2}} \right\|_{\Sigma_T}^{(1+\lambda)}. \end{aligned}$$

Since

$$\frac{f_x}{(1+f_x^2)^{1/2}} - \frac{h_x}{(1+h_x^2)^{1/2}} = \left(\int_0^1 \frac{d\theta}{(1+(\theta f_x + (1-\theta)h_x)^2)^{3/2}} \right) (f_x - h_x)$$

we deduce that

$$\begin{aligned} & \left\| \frac{f_x}{(1+f_x^2)^{1/2}} - \frac{h_x}{(1+h_x^2)^{1/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \\ & \leq C_1 \left(\int_0^1 d\theta \|(1+(\theta f_x + (1-\theta)h_x)^2)^{3/2}\|_{\Sigma_T}^{(1+\lambda)} \right)^2 \|f_x - h_x\|_{\Sigma_T}^{(1+\lambda)} \\ & \leq C_2 \left(\|(1+f_x^2+h_x^2)^{3/2}\|_{\Sigma_T}^{(1+\lambda)} \right)^2 \|(f-h)_x\|_{\Sigma_T}^{(1+\lambda)} \\ & \leq C_3 \left(\|(1+f_x^2)^{3/2}\|_{\Sigma_T}^{(1+\lambda)} + \|(1+h_x^2)^{3/2}\|_{\Sigma_T}^{(1+\lambda)} \right)^2 \|(f-h)_x\|_{\Sigma_T}^{(1+\lambda)} \\ & \leq C \|(f-h)_x\|_{\Sigma_T}^{(1+\lambda)} \end{aligned}$$

so that

$$\begin{aligned} F & \leq K \left(\|f_x\|_{\Sigma_T}^{(1+\lambda)} \|(g-k)_x(x,0,t)\|_{\Sigma_T}^{(1+\lambda)} \right. \\ & \quad \left. + \|k_x(x,0,t)\|_{\Sigma_T}^{(1+\lambda)} \|(f-h)_x\|_{\Sigma_T}^{(1+\lambda)} \right) \\ & \leq K\delta \left(\|f-h\|_{\Sigma_T}^{(3+\lambda)} + \|g-k\|_{Q_T}^{(2+\lambda)} \right). \end{aligned}$$

The term $((1+f_x^2)^{1/2} - 1)g_y(x,0,t) - ((1+h_x^2)^{1/2} - 1)k_y(x,0,t)$ can be estimated in a similar way. Finally let us consider the term

$$E = \chi \left(\frac{f_{xx}}{(1+f_x^2)^{3/2}} \right) - \frac{f_x^2 f_{xx}}{(1+f_x^2)^{3/2}}$$

in (4.2). Then one can estimate the difference

$$E_2 = \frac{f_x^2 f_{xx}}{(1+f_x^2)^{3/2}} - \frac{h_x^2 h_{xx}}{(1+h_x^2)^{3/2}}$$

as it has been done above with F . Finally we consider

$$E_1 = \chi \left(\frac{f_{xx}}{(1+f_x^2)^{3/2}} \right) - \chi \left(\frac{h_{xx}}{(1+h_x^2)^{3/2}} \right).$$

We have that

$$\begin{aligned} \left\| \frac{f_{xx}}{(1+f_x^2)^{3/2}} \right\|_{\Sigma_T}^{(1+\lambda)} & \leq C \|f_{xx}\|_{\Sigma_T}^{(1+\lambda)} \left(\|(1+f_x^2)^{3/2}\|_{\Sigma_T}^{(1+\lambda)} \right)^2 \\ & \leq K\delta \end{aligned}$$

so that by Lemma 4.3

$$\begin{aligned} E_1 &\leq K' \delta \left\| \frac{f_{xx}}{(1+f_x^2)^{3/2}} - \frac{h_{xx}}{(1+h_x^2)^{3/2}} \right\|_{\Sigma_T}^{(1+\lambda)} \\ &\leq K' \delta \|f - h\|_{\Sigma_T}^{(3+\lambda)}. \end{aligned}$$

A similar proof can be given for F_3 and F_4 .

Next we state the main result of this section.

THEOREM 7.2. *There exists a positive constant $\rho_0 = \rho_0(T)$ such that if the initial data satisfy the condition*

$$(7.13) \quad \|f_0\|_{\mathbf{R}}^{(3+\lambda)} + \|g_0\|_{\mathbf{R} \times (0,e)}^{(2+\lambda)} \leq \rho_0$$

then, there exists a constant $\rho > \rho_0$ such that Problem (7.1)–(7.8) has a unique solution (f, g) in $C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T) \times C_{\text{per}}^{2+\lambda, 1+\lambda/2}(\overline{Q}_T)$ satisfying $\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)} \leq \rho$.

Proof. We consider the mapping $\mathcal{T}$

$$(f, g) \rightarrow \mathcal{T}(f, g) = (f', g')$$

defined as follows: (f', g') is the solution of the linear problem (5.1)–(5.8) with initial conditions f_0 and g_0 and where the right-hand-sides $F_i, i = 1, \dots, 4$ are given by the formulas (7.9)–(7.12).

By Lemma 7.1 (i) and Theorem 5.1 the mapping $\mathcal{T}$ is well-defined from the affine space

$$A = \left\{ (f, g) \in C_{\text{per}}^{3+\lambda, (3+\lambda)/2}(\overline{\Sigma}_T) \times C_{\text{per}}^{2+\lambda, 1+\lambda/2}(\overline{Q}_T), f(0) = f_0, g(0) = g_0 \right\}$$

into itself. Next we define the norm $\|\cdot\|$ on A by

$$\|(f, g)\| = \|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)}.$$

In what follows we prove that if the constant ρ_0 in (7.13) is small enough then the mapping $\mathcal{T}$ has a unique fixed point in a certain ball. It follows from the study of the linear case (Theorem 5.1) that

$$\begin{aligned} \|\mathcal{T}(f, g)\| &= \|f'\|_{\Sigma_T}^{(3+\lambda)} + \|g'\|_{Q_T}^{(2+\lambda)} \\ (7.14) \quad &\leq \mathcal{C} \left(\|f_0\|_{\mathbf{R}}^{(3+\lambda)} + \|g_0\|_{\mathbf{R} \times (0,e)}^{(2+\lambda)} + \|F_1\|_{Q_T}^{(\lambda)} + \sum_{i=2}^4 \|F_i\|_{\Sigma_T}^{(1+\lambda)} \right) \end{aligned}$$

where $\mathcal{C}$ is a constant which depends on T . This implies that

$$(7.15) \quad \begin{aligned} \|\mathcal{T}(f, g) - \mathcal{T}(h, k)\| &\leq \mathcal{C}(\|F_1(f, g) - F_1(h, k)\|_{Q_T}^{(\lambda)} \\ &+ \sum_{i=2}^4 \|F_i(f, g) - F_i(h, k)\|_{\Sigma_T}^{(1+\lambda)}) \end{aligned}$$

Also using Lemma 7.1 (ii), we deduce that if (f, g) and (h, k) satisfy

$$\|(f, g)\| \leq \delta, \|(h, k)\| \leq \delta$$

then

$$\|\mathcal{T}(f, g) - \mathcal{T}(h, k)\| \leq \mathcal{C}K_2\delta\|(f - h, g - k)\|$$

so that if

$$(7.16) \quad \delta < \frac{1}{\mathcal{C}K_2}$$

and if $\mathcal{B}(0, \delta)$ denotes the ball of center 0 and radius δ , $\mathcal{T}$ is a strict contraction from the closed convex set $\mathcal{B}(0, \delta) \cap A$ into A .

Finally we deduce from (7.14) and from Lemma 7.1 (i) that if

$$(7.17) \quad \mathcal{C} \left(\|f_0\|_{\mathbb{R}}^{(3+\lambda)} + \|g_0\|_{\mathbb{R} \times (0, \epsilon)}^{(2+\lambda)} + K_1 \left(\|f\|_{\Sigma_T}^{(3+\lambda)} + \|g\|_{Q_T}^{(2+\lambda)} \right)^2 \right) \leq \delta$$

then $\mathcal{T}$ maps the set $\mathcal{B}(0, \delta) \cap A$ into itself. In turn the inequality (7.17) is implied by the condition

$$(7.18) \quad \tilde{\mathcal{C}}(\rho_0 + K_1\delta^2) \leq \delta$$

with

$$\tilde{\mathcal{C}} = \max(\mathcal{C}, 1).$$

Note that (7.18) implies that $\delta > \rho_0$ so that the set $\mathcal{B}(0, \delta) \cap A$ is non empty. Choose for example

$$\delta = \frac{1}{2\tilde{\mathcal{C}}K} \quad \text{and} \quad \rho_0 = \frac{1}{4\tilde{\mathcal{C}}^2K}$$

with $K = \max(K_1, K_2)$. Then one can check that the conditions (7.16) and (7.18) are satisfied.

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