**DISCONTINUOUS MEDIA AND DIRICHLET FORMS
OF DIFFUSION TYPE**

By

Marco Biroli

and

Umberto Mosco

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MARCO BIROLI* AND UMBERTO MOSCO**

Abstract. We suggest that Dirichlet forms may be used in describing the variational behaviour of possibly highly nonhomogeneous and nonisotropic bodies and prove structural Harnack inequalities and Saint-Venant type decays for local solutions.

We consider a body X with a very irregular internal structure, highly nonhomogeneous and possibly nonisotropic. We suppose that $u : X \rightarrow \mathbb{R}$ describes a physical state of X , whose equilibrium is subjected to a *variational principle* of a suitable nature.

In order to formulate such a principle with very few requirements about the internal structure of X , we shall assume that the *energy functional* minimized by u can be written as a *Dirichlet form* $a(u, v)$ in the Hilbert space

$$H = L^2(X, m),$$

for a suitable choice of a locally compact separable Hausdorff topology on X and of a positive Radon measure m on X , with $\text{supp } m = X$.

We recall that a **Dirichlet form** on H is a bilinear form $a(u, v)$ defined on a dense linear subspace $D[a]$ of H , which can be written as

$$a(u, v) = (\sqrt{L}u, \sqrt{L}v) \text{ for every } u, v \in D[a],$$

where $(\cdot, \cdot)$ denotes the inner product of H and L is a positive semi-definite selfadjoint operator in H with $D\sqrt{L} = D[a]$, and which has in addition the following fundamental *Markovianity property*: $T \circ u \in D[a]$ and $a(T \circ u, T \circ u) \leq a(u, u)$, whenever $u \in D[a]$ and $T : \mathbb{R} \rightarrow \mathbb{R}$, $T(0) = 0$, $|T(x) - T(y)| \leq |x - y|$ for every $x, y \in \mathbb{R}$.

We shall restrict our study to Dirichlet forms of **diffusion type**, that is to forms a that have the following *strong local property*:

$$a(u, v) = 0 \quad \text{for every } u, v \in D[a] \text{ with } v \text{ constant on a neighborhood of } \text{supp } u,$$

where $\text{supp } u$ is taken to be the support of the measure $u \cdot m$ in X .

Furthermore, we shall assume that the form a is **regular** in H , that is, there exists a subset C of $D[E] \cap C_0(X)$ which is both dense in $C_0(X)$ with the uniform norm and dense in $D[E]$ with the *intrinsic norm* $(a(u, u) + (u, u))^{1/2}$. Such a set C , that without

*Department of Mathematics, Politecnico di Milano, P. Leonardo de Vinci, 32 I-20133 MILANO.

**Department of Mathematics, University of Rome "La Sapienza", I-00185 ROMA.

restriction can be assumed to be a subalgebra of $D[E] \cap C_0(X)$, is called a *core* of a in H . The functions that belong to C play the role of *test functions* in our variational theory. By $C_0(X)$ we are denoting the space of continuous functions with compact support in X .

From the physical point of view taken at the beginning of this paper, the choice of the class of regular Dirichlet forms of diffusion type in order to state our variational principle for X is motivated by the fact that any such form can be given the following integral expression:

$$(1) \quad a(u, v) = \int_X \mu(u, v)(dx),$$

for every $u, v \in D[a]$, where μ is a Radon-measure-valued positive-semidefinite symmetric bilinear form on $D[a]$, uniquely associated with a , called the *energy measure* of a .

One of the most important property of μ is its *local* character: the restriction of the measure $\mu(u, v)$ to any open subset A of X only depends on the restrictions of u and v to A . This property entitles us to interpret μ as a measure-valued description of the *physical characteristics* of the body X on which our variational principle relies. Moreover, it enables us to define in a natural way the space of functions u that *belong locally* to the domain of the form on a given open subset A of X . We shall denote this space by $D[a, A]_{\text{loc}}$ and simply by $D[a]_{\text{loc}}$ if $A = X$. We may suppress the explicit reference to the form a in our notation, if we are dealing with forms that have a common domain.

Furthermore, the local character of μ allows us to define a function u to be a *local minimizer* of the *energy functional*

$$E[u] = \frac{1}{2}a(u, u)$$

in a given arbitrary open subset X_0 of X , if u is a function on X that satisfies the minimality condition:

$$(2) \quad u \in D[a, X_0]_{\text{loc}} : \frac{1}{2} \int_{X_0} \mu(u, u)(dx) \leq \frac{1}{2} \int_{X_0} \mu(u + \phi, u + \phi)(dx)$$

for every $\phi \in C$ with $\text{supp } \phi \subset X_0$.

Clearly, u is a solution of (2) if and only if u is a solution of

$$(3) \quad u \in D[a, X_0]_{\text{loc}} : \quad a(u, v) = 0 \text{ for every } v \in D_0[a, X_0],$$

where for any open subset A of X we denote by $D_0[a, A]$ the closure of the $D[E] \cap C_0(A)$ in $D[a]$ with its intrinsic norm. We will refer to any solution of (3) as to a *local solution* in X_0 of the equation formally written as

$$(4) \quad Lu = 0.$$

Before going on, let us point out that our present interpretation of μ as the measure-valued characteristics of the body is further supported by the special coordinate-invariant expression that is taken by the form a , whenever we are ready to introduce in X , or in some open portion of it, the additional structure of a *differentiable manifold*. In this case, in fact, if there exist coordinate functions $x_1, \dots, x_n$, that belong locally to $D[a]$ on their domain of definition, then any differentiable function u on X also belongs locally to $D[a]$ and the energy measure $\mu(u, v)$, for every $u, v \in C^1(X)$, can be written in local coordinates as

$$\mu(u, v) = \sum_{ij=1}^n (\partial u / \partial x_i)(x_1, x_2, \dots, x_n) (\partial v / \partial x_j)(x_1, x_2, \dots, x_n) \nu^{ij},$$

where

$$\nu^{ij} = \mu(x_i, x_j), \quad i, j = 1, \dots, n,$$

defines a positive-semidefinite symmetric tensor v on X . The form a , for every $u, v \in C_0^1(X)$, takes now the following more familiar invariant integral expression:

$$a(u, v) = \int_X \sum_{ij=1}^n (\partial u / \partial x_i)(x_1, x_2, \dots, x_n) (\partial v / \partial x_j)(x_1, x_2, \dots, x_n) \nu^{ij}(dx),$$

where the integral at the right hand side has to be intended as reduced to the coordinate domains by means of a partition of unit in C and the following (degenerate) *ellipticity condition*

$$\sum_{ij=1}^n \xi_i \xi_j \nu^{ij} \geq 0 \text{ for every } \xi \in R^n$$

is satisfied in the sense of measures on X .

We refer for instance to M. Fukushima [F] for a general treatment of the theory of Dirichlet forms and their probabilistic interpretation.

Let us now come back to our general setting and to equation (4).

Our aim is to describe the behaviour of an arbitrary local solution u of (4) in X_0 , in a neighborhood of an arbitrary given point x_0 of X_0 . Moreover, we want our theory and estimates have a *structural* character. By this we mean that the properties we will establish have to hold *uniformly* for a whole family of equivalent Dirichlet forms of diffusion type, in a sense that will be made precise below.

We will suppose then that we are given a whole family of regular Dirichlet forms of diffusion type, a , which have a common domain $D[a] = D$ in $L^2(X, m)$ and are mutually *equivalent* in the following sense:

However we choose a form in the family, say b , there exist two constants $0 < \lambda \leq \Lambda$, depending on b but whose ratio Λ/λ is independent of b , such that any other form a of the family is related to b by the condition

$$(5) \quad \lambda b(u, u) \leq a(u, u) \leq \Lambda b(u, u) \quad \text{for every } u \in D[a] = D[b] = D.$$

We remark that, by a well known *domination principle* for energy measures, condition (5) is equivalent to the condition

$$(6) \quad \lambda\mu_b(u, u) \leq \mu_a(u, u) \leq \Lambda\mu_b(u, u) \text{ in } X, \text{ for every } u \in D[a] = D[b] = D,$$

where μ_a, μ_b are the energy measures of a, b , respectively.

We will develop our theory under the assumption that the set of all test function $\phi \in C$ whose energy measures have a *bounded density* with respect to the measure m , is rich enough to separate the points of X . More precisely, we suppose that there is a form of the family (5), say b , that admits a **m -separating** core, that is, a core C that has the following separating property:

$$(7) \quad \text{For every } x, y \in X, x \neq y, \exists \phi \in C \text{ with } \mu(\phi, \phi) \leq m \text{ on } X, \text{ such that } \phi(x) \neq \phi(y),$$

where $\mu = \mu_b$. Clearly, if a set C is a core of b , then C is also a core for any other form a of the family (5); moreover, in view of (6), if in addition C has the separating property (7) with respect to the given form b , then C has the same property with respect to any other form a of the family.

We are now in a position to introduce the basic notion that is at the heart of our theory, that is, a family of (equivalent) *metrics* induced on the space X by the forms of our family. For related notions we refer to the fundamental papers by A. Nagel, E.M. Stein and S. Wainger [NSW] and by L. Rothschild and E.M. Stein [RS].

Given a form a , we define the *distance function* $d = d_a : X \rightarrow [0, +\infty]$ by setting for every $x, y \in X$:

$$(8) \quad d(x, y) = \sup\{\phi(x) - \phi(y) : \phi \in C, \mu(\phi, \phi) \leq m \text{ on } X\},$$

and we denote by $B = B_a$ the metric balls associated with d :

$$(9) \quad B(x, r) = \{y \in X : d(x, y) < r\},$$

for every $x \in X$ and $r > 0$. Clearly, under our assumptions (6) and (7), for every form a of the family (5) the function d just defined has indeed the properties of a distance. Moreover, the distance functions associated with two arbitrary forms of the family induce equivalent metric structures on X .

We point out that, while the energy measures in (1) are intrinsically defined in terms of the form itself, their densities occurring in (7), hence the metric induced by the form on X , can be highly affected by the initial choice of the measure m . In any case, the metric balls $B_a(x, r)$ of a given form a of the family single out special regions of the space X , on which the form a , as well as its local solutions, should be expected to enjoy special properties that might not hold on other regions of X .

Moreover, when taken up to the metric equivalence pointed out before, the intrinsic balls $B(x, r)$ play a basic role in two main regards. From one side, they allow us to formulate a *compatibility condition*, relating the whole family (5) to the initial topology of X and to the measure m initially chosen on X . This compatibility condition is expressed by **Assumption I** below. From the other side, the system of balls $B(x, r)$ allows us to formulate special *scaling* and *imbedding* properties of the forms and their domains, that also have a structural character for the family (5). These properties, in the form of suitably scaled Poincaré and Sobolev-Poincaré inequalities, are expressed by **Assumption II** below.

Assumption I. The forms (5) admit a m -separating core C in the space $L^2(X, m)$ and the following two properties hold:

- (i) The metric topology induced by the distance (8) on X is *equivalent* to the initial topology of X ;
- (ii) The measure m is *doubling* with respect to the balls (9), that is, there exists a constant c_0 such that $0 < m(B(x, 2r)) \leq c_0 m(B(x, r)) < +\infty$ for every $x \in X$ and $r > 0$.

We remark that under this assumption the space X with the distance d acquires the structure of a *homogeneous space*, according to R. Coifman and G. Weiss [CW], Ch. III, sec. 1.

Assumption II. Given a relatively compact open subset X_0 of X , there exists constants $c_1 > 0$, $c_2 > 0$, $s > 2$ and an integer $k \geq 1$, such that for every $x \in X_0$ and every $r > 0$ with $B(x, r) \subset X_0$, the following inequalities hold:

$$(j) \quad \int_{B(x, r/k)} |u - \bar{u}|^2 m(dx) \leq c_1 r^2 \int_{B(x, r)} \mu(u, u)(dx),$$

for every $u \in D[X_0]_{\text{loc}}$, where $\bar{u} = \frac{1}{m(B(x, r/k))} \int_{B(x, r/k)} u m(dx)$;

$$(jj) \quad \left(\frac{1}{m(B(x, r))} \int_{B(x, r)} |u|^s m(dx) \right)^{1/s} \leq c_2 r \left(\int_{B(x, r)} \mu(u, u)(dx) \right)^{1/2},$$

for every $u \in D[X_0]_{\text{loc}}$ with $\text{supp } u \subset B(x, r)$.

It is easily checked that if Assumptions I and II are satisfied by a given form of the family (5), with some constants c_0 , c_1 , c_2 , k and s , then they are also satisfied by any other form of the family, with the same constant s and with possibly new constants c'_0 , c'_1 , c'_2 , k' depending on the initial c_0 , c_1 , c_2 , k and on the ratio Λ/λ .

We can now state our main results. We denote below by X_0 a connected relatively compact open subset of X and by u an arbitrary solution of (3), where a is any form of a

given family (5) for which Assumptions I and II hold. The *structural constants* c and α in the estimates below, possibly different ones, only depend on the given constants c_0, c_1, c_2, k and s occurring in the Assumptions I and II and on the ratio Λ/λ .

THEOREM 1 (Harnack inequality). *If u is positive, then*

$$\sup_{B(x,r)} u \leq c \inf_{B(x,r)} u$$

for every $B(x, r) \subset X_0$.

A standard consequence of Theorem 1 is the following

COROLLARY. *There exists $\alpha > 0$ such that*

$$\operatorname{osc}_{B(x,r)} u \leq c \left(\frac{r}{R} \right)^\alpha \operatorname{osc}_{B(x,R)} u,$$

for every $0 < r \leq \frac{R}{4} \leq \frac{R_0}{10}$, $B(x, R_0) \subset X_0$.

Therefore, u is Hölder continuous with respect to the intrinsic distance of X , hence u is also continuous with respect to the initial topology of X .

By taking also into account the following L^∞ -estimate:

$$\sup_{B(x,r)} |u| \leq cR^2 m(B(x, R))^{-1/p} \|f\|_{L^p(B(x, R), m)},$$

that holds in $B(x, R) \subset X_0$ with $p > \max \left\{ \frac{s}{s-2}, 2 \right\}$ for every solution u of the equation

$$(10) \quad u \in D_0[a, B(x, R)] : a(u, v) = \int_{B(x, R)} f v m(dx) \text{ for every } v \in D_0[a, B(x, R)],$$

the Corollary enables us to define the *Green function* $G_{B_R}^{x_0}$, for $x_0 \in B(x, R) \subset X_0$, as the unique function $G \in L^{p'}(B(x, R), m) \cap C(B(x, R) \setminus \{x_0\})$ such that

$$u(x_0) = \int_{B(x, R)} G f m(dx),$$

for every $f \in L^p(B(x, R), m)$ and $u = u_f$ solution of (10).

THEOREM 2 (Size of Green's functions). *For every $B(x_0, R) \subset X_0$ and every $0 < r \leq \frac{R}{2}$, the following estimate holds on $\partial B(x_0, r)$*

$$\frac{1}{c\Lambda} \int_r^R \frac{s^2}{m(B(x_0, s))} \frac{ds}{s} \leq G_{B(x_0, R)}^{x_0} \leq \frac{c}{\lambda} \int_r^R \frac{s^2}{m(B(x_0, s))} \frac{ds}{s}.$$

The structural estimates in Theorem 3 and in the Corollary below are the analogue in our present setting of the *Saint-Venant principle* for the energy decay in linear elasticity.

THEOREM 3 (Saint-Venant principle). *There exists $\alpha > 0$ such that*

$$\frac{\int_{B(x_0, r)} G_{B(x_0, q^{-1}r)}^{x_0} \mu(u, u)(dx)}{\int_{B(x_0, R)} G_{B(x_0, q^{-1}R)}^{x_0} \mu(u, u)(dx)} \leq c \left(\frac{r}{R} \right)^\alpha$$

for every $0 < r \leq 2q^2 R \leq 2q^3 R_0$, $B(x, R_0) \subset X_0$, and some fixed $q \in (0, 1/6)$ depending on k .

Furthermore,

$$\int_{B(x_0, R)} G_{B(x_0, q^{-1}R)}^{x_0} \mu(u, u)(dx) \leq c \frac{1}{m(B(x_0, R_0))} \|u\|_{L^2(B(x_0, R_0), m)}^2.$$

From Theorem 2 and Theorem 3, we obtain the more explicit special estimate:

COROLLARY. *For r, R and R_0 as in Theorem 3, we have*

$$\frac{\frac{r^2}{m(B(x_0, r))} \int_{B(x_0, r)} \mu(u, u)(dx)}{\frac{R^2}{m(B(x_0, R))} \int_{B(x_0, R)} \mu(u, u)(dx)} \leq c \left(\frac{r}{R} \right)^\alpha.$$

The proof of Theorem 1 is obtained by adapting De Giorgi's truncation method to the metric structure of the homogeneous space X . The proof of Saint-Venant principle is obtained by a modified "hole-filling" argument that takes into account the relative size of Green's functions on homothetic balls, as given by Theorem 2. Detailed proofs will be given in [BM].

We illustrate the previous results from the point of view of PDE by referring to two important classes of operators in $\mathbb{R}^n$: (a) *Weighted* uniformly elliptic operators in divergence form with measurable coefficients, (b) *Subelliptic* (selfadjoint) second order operators with *bounded measurable* coefficients. Moreover, (c), we shall give an example of an operator L in $\mathbb{R}^2$ whose local solutions are Hölder continuous with respect to the intrinsic distance of L , and only continuous with respect to the euclidean metric of $\mathbb{R}^2$.

(a) We consider the form

$$a(u, v) = \int_{\mathbb{R}^n} \sum_{ij=1}^n \partial u / \partial x_i \partial v / \partial x_j a^{ij}(x) dx,$$

on $C_0^1(\mathbb{R}^n)$, where the measurable coefficients $a^{ij}(x) = a^{ji}(x)$, $i, j = 1, \dots, n$, satisfy the condition

$$\lambda |\xi|^2 w(x) \leq \sum_{ij=1}^n \xi_i \xi_j a^{ij}(x) \leq \Lambda |\xi|^2 w(x) \quad \text{a.e. in } \mathbb{R}^n,$$

for every $\xi \in \mathbf{R}^n$. Here w is a weight in the Muckenhoupt class A_2 , or the weight $w(x) = |\det F'|^{1-2/n}$ associated with a quasi-conformal transformation F in $\mathbf{R}^n$. The domain $D[a]$ of this form is obtained by completion of $C_0^1(\mathbf{R}^n)$ with respect to the norm $(a(u, u) + (u, u))^{1/2}$, where $(\cdot, \cdot)$ is taken to be the inner product of the Hilbert space $H = L^2(\mathbf{R}^n, w(x)dx)$, and $D[a]$ is found to be injected in H . This class of operators has been considered by E. Fabes, C. Kenig and R. Serapioni [FKS] and E. Fabes, D. Jerison and C. Kenig [FJK], who proved Theorems 1 and 2, where now $X = \mathbf{R}^n$, the balls $B(x, r)$ are the usual euclidean balls and $m(dx) = w(x)dx$. The doubling property of our Assumption I is a well known property of the weights considered above and the scaled Poincaré and Sobolev-Poincaré inequalities of Assumption II are given in [FKS].

(b) In the space $H = L^2(\mathbf{R}^n, dx)$, we consider a family of forms (5), where b is a given selfadjoint subelliptic form with smooth coefficients in $\mathbf{R}^n$, that is b is given on $C_0^1(\mathbf{R}^n)$ by

$$b(u, v) = \int_{\mathbf{R}^n} \sum_{ij=1}^n \partial u / \partial x_i \partial v / \partial x_j b^{ij}(x) dx,$$

where $b^{ij}(x) = b^{ji}(x)$, $ij = 1, \dots, n$, are smooth functions that satisfy the (degenerate) ellipticity condition

$$0 \leq \sum_{ij=1}^n \xi_i \xi_j b^{ij}(x) \quad \text{in } \mathbf{R}^n \text{ for every } \xi \in \mathbf{R}^n,$$

and b satisfies in addition the following subellipticity estimate for some $\varepsilon \in (0, 1)$

$$c \|u\|_{H_\varepsilon}^2 \leq b(u, u) + \|u\|_{L^2}^2 \quad \text{for every } u \in C_0^1(\mathbf{R}^n),$$

where H_ε denotes the usual fractional Sobolev space of order ε . The form a is any form of the following type

$$a(u, v) = \int_{\mathbf{R}^n} \sum_{ij=1}^n \partial u / \partial x_i \partial v / \partial x_j a^{ij}(x) dx,$$

with measurable coefficients $a^{ij} = a^{ji}$ that satisfy the *uniform subellipticity condition*

$$\lambda \sum_{ij=1}^n \xi_i \xi_j b^{ij}(x) \leq \sum_{ij=1}^n \xi_i \xi_j a^{ij}(x) \leq \Lambda \sum_{ij=1}^n \xi_i \xi_j b^{ij}(x) \quad \text{a.e. in } \mathbf{R}^n,$$

for some given constants $0 < \lambda \leq \Lambda$.

The distance d on $\mathbf{R}^n$ induced by the form b according to our definition (8) turns out to be equal to the distance d^* associated to the form b according to C.L. Fefferman and D.H. Phong [FP], which is shown by these authors to satisfy the condition

$$\frac{1}{c} |x - y| \leq d^*(x, y) \leq c |x - y|^\varepsilon.$$

By noting that $m(dx) = dx$, this shows that Assumption I above is satisfied on any bounded open domain X of $\mathbb{R}^n$.

The scaled Poincaré inequalities on the intrinsic balls for the form b have been proved by D. Jerison [J] and D. Jerison and A. Sanchez Calle, [JSC] Th. 5.1. These authors also provide examples that show that Poincaré inequality may not hold on regions that do not coincide with an intrinsic ball, what may serve as an illustration of a remark we made before. The Poincaré-Sobolev inequality on X is an immediate consequence of the subellipticity estimate and of Sobolev imbedding theorems. The scaled Poincaré-Sobolev inequalities, as occurring in Assumption II, are then obtained by using the diffeomorphism Φ associated with the form b , given by Theorem 3.1 of [JSC]. For related results see also S. Kusuoka and D. Stroock [KS].

Theorem 1 to 3 above seem to be new in this subelliptic setting. The main example to which they apply is that of an operator of *Hörmander type*

$$L = - \sum_{h,k=1}^m X_k^* \alpha^{hk}(x) X_h, \quad x \in \mathbb{R}^n,$$

where $X_h, h = 1, \dots, m$ are m smooth vector fields in $\mathbb{R}^n$ that satisfy the Hörmander condition and where $\alpha = (\alpha^{hk})$ is any symmetric $m \times m$ matrix of measurable functions on $\mathbb{R}^n$, such that $\lambda|\eta|^2 \leq \alpha\eta \cdot \eta \leq \Lambda|\eta|^2$ for every $\eta \in \mathbb{R}^m$, a.e. on $\mathbb{R}^n$.

(c) In the space $L^2(B, m(dxdy))$, where $B = \left\{ (x, y) \in \mathbb{R}^2 : \frac{1}{(\log|x|)^2} + |y|^2 \leq \frac{1}{4} \right\}$ and $m(dxdy) = \frac{1}{|x|(\log|x|)^2} dxdy$, we consider the form

$$a(u, v) = \int_B \{ x^2 (\log|x|)^4 \partial u / \partial x \partial v / \partial x + \partial u / \partial y \partial v / \partial y \} \frac{1}{|x|(\log|x|)^2} dxdy.$$

Assumptions I and II can be easily checked by suitably rescaling the form on a euclidean ball of $\mathbb{R}^2$. The intrinsic balls $B(o, r)$ shrink to $\{o\}$ as $\exp(-1/r)$ in the x -direction, as $r \rightarrow 0$.

We finally mention that any family of forms satisfying a condition like (5) enjoys special *variational compactness properties*, that can be expressed in terms of Γ -convergence of the functionals $a(u, u)$ now defined on the whole of H by extending them to $+\infty$ outside their domain. In case the domains are uniformly compactly imbedded in H , as for instance in the uniform subelliptic case (b), these convergence properties can be expressed in terms of the *resolvent operators* associated with the forms and are thus related to convergence of spectra and semigroups. The compactness properties play an important role in *homogenization theory* and, more generally, in the asymptotic variational approach to *composite media*, see [M]. The structural estimates presented before describe some properties of local solutions that are kept in the variational limit and this is of particular relevance in the asymptotic theory, due to the fact that the explicit expression of the limit energy form in terms of the *effective characteristics* of the composite body may not always be easily determined.

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