THE PROPAGATION OF TURBULENT BURSTS

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ABSTRACT

We consider the equation of turbulent burst propagation

$$u_t = l(t)\Delta u^m - \frac{ku^m}{l(t)}$$

with $m > 1$ and $k \geq 0$, or in an equivalent formulation

$$u_\tau = \Delta u^m - \frac{k u^m}{r(\tau)^2}.$$

We consider initial data $u(x, 0)$ which are radially symmetric and have compact support. $|x| = l(t) = r(\tau)$ describes the interface of the solution.

We prove existence and uniqueness of radially symmetric solutions with compact support to this problem.

We also establish the asymptotic behaviour as $t \rightarrow \infty$. The equation possesses special self-similar solutions which are nonnegative and have compact support in the space variable. However, the similarity exponents are anomalous. We prove convergence of general solutions to these selfsimilar profiles.

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1. Introduction. In [B] G. I. Barenblatt proposed the following equation as a mathematical model for the propagation of turbulent bursts from a plane source

$$(1.1) \quad u_t = \partial_x(l(t)\sqrt{u}\partial_x u) - \frac{ku\sqrt{u}}{l(t)}.$$

Here u measures the turbulent energy, which is a positive symmetric function in the interval $|x| < l(t)$, and $k \geq 0$ is a constant. Formally the equation looks like a porous medium equation. However, the presence of the support radius $l(t)$ leads to essential differences, in particular the occurrence of anomalous exponents in the description of the large-time behaviour, [BGL]. Also the existence and uniqueness of the solutions is an interesting question.

Mathematically we pose the Cauchy problem for the general equation

$$(1.2) \quad u_t = l(t)\Delta u^m - \frac{ku^m}{l(t)}$$

with $m > 1$ and $k \geq 0$ in the domain $Q = \{(x, t) : x \in \mathbf{R}^N, t > 0\}$, and take initial data

$$(1.3) \quad u(x, 0) = u_0(x)$$

for $x \in \mathbf{R}^N$. We assume that u_0 is continuous, radially symmetric, positive in a ball $B_R(0)$ for some $R > 0$ and zero outside. Introducing the change of time

$$(1.4) \quad d\tau = l(t)dt, \quad \tau(0) = 0$$

we can pass to the equation

$$(1.5) \quad u_\tau = \Delta u^m - \frac{k}{r(\tau)^2}u^m.$$

where $r(\tau) = l(t)$ is the interface expressed in terms of the new time τ . Therefore, we can work with the new equation (1.5) and translate in the end our results to (1.2). In changing back time we use the formula

$$(1.6) \quad dt = \frac{d\tau}{r(\tau)},$$

inverse to (1.4). Let us take as an example the simplest case $k = 0$. Then equation (1.5) becomes the standard Porous Medium Equation, which has a unique solution for every initial data in $L^1(\mathbf{R}^N)$. The unique solution is continuous and nonnegative. If the initial data have compact support, so does the solution for every fixed $\tau > 0$ and the interface $r(\tau)$ is given by a continuous and nondecreasing function of τ . Moreover, the interface function is analytic after the initial waiting time and behaves as $\tau \rightarrow \infty$ like $O(\tau^{1/(m+1)})$.

More precisely, the large-time behaviour of the solutions of the PME is described by the family of self-similar solutions

$$(1.7) \quad U(x, \tau) = \tau^{-\alpha} F(\xi), \quad \xi = x \tau^{-\beta}$$

where the self-similarity exponents are $\alpha = N/(N(m-1)+2)$ and $\beta = \alpha/N$. The function F is explicit in this case, given by $F(\xi; c) = (c - d|\xi|^2)_+^{1/(m-1)}$ with $d = (m-1)\gamma/(2mN)$ and $c > 0$ a free constant. Therefore, the interface is given by

$$(1.8) \quad r(\tau) = (c/d)^{\frac{1}{2}} \tau^\beta.$$

Using now (1.6) we get

$$(1.9) \quad t = \frac{(d/c)^{\frac{1}{2}}}{1-\beta} \tau^{1-\beta}$$

so that U takes on the form

$$(1.10) \quad U(x, t) = t^{-\mu} \tilde{F}(\xi), \quad \xi = x t^{-\nu}$$

with exponents

$$(1.11) \quad \mu = \frac{\alpha}{1-\beta} = \frac{N}{N(m-1)+1} \quad \text{and} \quad \nu = \frac{\beta}{1-\beta} = \frac{\mu}{N}.$$

The corresponding interface is then given by

$$(1.12) \quad l(t) = L t^\nu.$$

Notice the equivalence of the similarity factors

$$(1.13) \quad t^\mu = \lambda_1 \tau^\alpha, \quad t^\nu = \lambda_2 \tau^\beta.$$

The similarity exponents μ and ν can be directly obtained from equation (1.5) (with $k = 0$) by making two claims: (i) U is a solution of the form (1.9), which implies that

$$(1.14) \quad \mu(m-1) + \nu = 1$$

and (ii) the mass $\int U(x, t) dx$ is a conserved quantity, which means that

$$(1.15) \quad \mu = N \nu.$$

Barenblatt and collaborators [BGL] studied the existence of self-similar solutions for equation (1.1) for $k > 0$ and found numerically that there exists such a solution, which is moreover nonnegative and has compact support, but the corresponding similarity exponents are *anomalous*, i.e. they fail to satisfy (1.15) or any other simple relation and have to be obtained instead as eigenvalues of a nonlinear elliptic problem. Of course we always have $\nu = 1 - \mu(m-1)$, so the problem is reduced to determine μ . A rigorous analytical investigation of this solution is done in [KP] for the general equation (1.2) with arbitrary $m > 1$, $N > 1$ and $k > 0$.

Notice finally that once a selfsimilar solution $U(x, t)$ of (1.2) of the form (1.10) is obtained, a whole family can be derived by means of

$$(1.16) \quad U(x, t; c) = c^{\frac{1}{m-1}} U(x/c, t).$$

In the case of equation (1.5) the transformation is similar but the exponent is $2/(m-1)$ instead of $1/(m-1)$.

2. Existence of solutions. After the discussion in the Introduction we may consider the equation

$$(E) \quad u_t = \Delta u^m - \frac{k u^m}{r(t)^2}$$

with $m > 1$ and $k > 0$. We take initial data $u(x, 0) = u_0(x)$. As said above we assume that u_0 is continuous, radially symmetric, positive in a ball $B_R(0)$ for some $R > 0$ and zero outside. We have

THEOREM 2.1. *There exists at least one continuous weak solution of the above Cauchy Problem, which is radially symmetric in the space variable and has compact support in x for every $t > 0$. The interface function is continuous and nondecreasing.*

PROOF: We use an iterative process. We start from the solution u_1 of the equation

$$(2.1) \quad u_t = \Delta u^m - \frac{k u^m}{R^2},$$

where $R > 0$ is the support radius of u_0 . Such a solution exists, is unique, continuous nonnegative and is supported in the region $\{(x, t) : t > 0, |x| < r_1(t)\}$, where the interface function $r_1(t)$ is continuous, nondecreasing and $r_1(0) = R$. Therefore, $r_1(t) \geq R$.

Iteratively, once constructed $u_i(x, t)$ and $r_i(t)$ for $i = 1, 2, \dots, n-1$ with the property that $u_{i+1}(x, t) \geq u_i(x, t)$ and $r_{i+1}(t) \geq r_i(t)$, we define $u_n(x, t)$ as the solution of

$$(2.2) \quad u_t = \Delta u^m - \frac{k u^m}{r_{n-1}^2(t)}.$$

Since $r_{n-1}(t) \geq r_{n-2}(t)$ we easily conclude that $u_n(x, t) \geq u_{n-1}(x, t)$ and $r_n(t) \geq r_{n-1}(t)$, thus justifying the induction hypotheses.

On the other hand the sequence $\{u_n\}$ is bounded above by the solution of the PME with same initial data, which as we know has compact support. We may thus take the limit

$$(2.3) \quad u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t).$$

It is easily verified that this is a solution of (E) with interface

$$(2.4) \quad r(t) = \lim_{n \rightarrow \infty} r_n(t).$$

The continuity of u follows from the compactness results for solutions of general porous media equations, cf. [dB]. The continuity of the interface comes from two facts: (i) as a limit of monotone functions, $r(t)$ is also monotone nondecreasing. (ii) solutions of (E) are weak subsolutions of the PME and it is well-known that solutions of this latter equation have always continuous interfaces. This eliminates the possible jump discontinuities of $r(t)$. ■

3. Uniqueness. We recall that we are dealing with solutions of the equation

$$(3.1) \quad u_t = \Delta u^m - \frac{k u^m}{r(t)^2}$$

with initial data $u(x, 0) = u_0(x)$ as above. We consider continuous weak solutions which are radially symmetric in the space variable and have compact support for very fixed time $t > 0$.

LEMMA 3.1. *Let u and v be any two solutions with admissible initial data f and g and let the interfaces be $r_u(t)$ and $r_v(t)$ respectively. If $r_u(0)$ is strictly smaller than $r_v(0)$ and*

$$(3.2) \quad f < g \quad \text{in the support of } g$$

and , then $u \leq v$ in Q .

PROOF: By continuity of the interfaces there exists an interval $I = \{0 \leq t \leq T\}$ such that the interface $r_u(t)$ is smaller than $r_v(t)$. In this time interval we have

$$(3.3) \quad v_t = \Delta v^m - \frac{k v^m}{r_v(t)^2} \geq \Delta v^m - \frac{k v^m}{r_u(t)^2},$$

i.e. u is a supersolution for the nonlinear parabolic equation

$$(3.4) \quad u_t = \Delta u^m - h(t)u^m, \quad h(t) = \frac{k}{r_u(t)^2},$$

i.e. we have frozen the coefficient $r(t)$ to the value taken by the interface of u . Since there is uniqueness for this latter equation we get $u(x, t) \leq v(x, t)$ for $t \in I$. Moreover, the Strong Maximum Principle implies that u is strictly smaller than v whenever u is positive and $t \in I$.

Let us prove now that T has to be infinite, which clearly implies our result. If T were finite then two possibilities might occur.

(i) First, $u(\cdot, T) \equiv v(\cdot, T)$. This is impossible because of the following energy estimate. In fact, we have with $E(t) = \int (v - u) dx$

$$(3.5) \quad \frac{dE}{dt} = - \int \left(\frac{v^m}{r_v(t)^2} - \frac{u^m}{r_u(t)^2} \right) dx$$

In the interval $0 < t < T$ we have $r_v \geq r_u$ hence

$$(3.6) \quad \frac{dE}{dt} \geq - \frac{k}{r_u(t)^2} \int (v^m - u^m) dx \geq -C \int (v - u) dx = -CE(t),$$

which means that E cannot vanish at $t = T$.

(ii) The second possibility is that $u(x, T) \leq v(x, T)$ but they are not identically equal. Again from the Strong Maximum Principle we conclude that u is strictly smaller than v for all points in the support of u . By the maximality of T we also have

$$(3.7) \quad r_u(T) = r_v(T).$$

We are going to prove that in the present circumstances (3.7) is impossible. If we replace v by its translate

$$(3.8) \quad \tilde{v}(x, t) = v(x + \delta e_1, t)$$

for $e_1 = (1, 0, \dots, 0)$ and $\delta > 0$ is small, then $\tilde{v}$ will still be a supersolution to equation (3.4) and it has larger data on the parabolic boundary of the domain $S = \mathbf{R}^N \times (0, T)$. We conclude that $\tilde{v} \geq u$ in S , which in terms of supports means that the support of u at time T is contained in the ball of center $-\delta e_1$ and radius $r_v(t)$. This of course means that

$$(3.9) \quad r_v(t) - \delta \geq r_u(t) \quad \text{for} \quad t_0 \leq t \leq T.$$

In particular, $r_u(T) < r_v(T)$. This ends the proof. ■

THEOREM 3.2. *Uniqueness and comparison hold.*

PROOF: Suppose that we have two solutions u and v with the same initial data. We introduce the rescaled function $\tilde{v}$ by

$$(3.10) \quad \tilde{v}(x, t) = \left(\frac{\lambda^2}{\sigma}\right)^{\frac{1}{m-1}} v\left(\frac{x}{\lambda}, \frac{t}{\sigma}\right)$$

for λ and $\sigma > 0$. $\tilde{v}$ is again a solution of our problem. Moreover, if we choose $\lambda = 1 + \varepsilon \approx 1$ and then we take a suitable $\sigma < 1$, then $\tilde{v}$ has strictly larger initial data and initial support than u . By Lemma 3.1 we conclude that $u \leq \tilde{v}$ in Q . Letting $\varepsilon \rightarrow 0$ so that also $\sigma \rightarrow 1$ we conclude that $u \leq v$. Reversing the roles of u and v we obtain uniqueness.

The comparison result is now obvious. ■

4. Asymptotic behaviour. In this section we consider a solution $u(x, t)$ of the Cauchy Problem for equation (1.2) with initial data

$$(4.1) \quad u(x, 0) = u_0(x),$$

which are assumed to be continuous, nonnegative, radially symmetric and compactly supported. We want to show that the large-time behaviour of u is precisely described by a certain member of the family of selfsimilar solutions $U(x, t; c)$ mentioned in the Introduction. Actually, we prove

THEOREM 4.1. *Under the above assumptions there exists a constant $c_* > 0$ such that*

$$(4.2) \quad t^\mu |u(x, t) - U(x, t; c_*)| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

uniformly in x . Here $\mu = \mu(m, k)$ is the decay exponent of the selfsimilar solutions $U(x, t; c)$.

PROOF: Following our previous idea we may change the time variable according to formula (1.4) and replace equation (1.2) by equation (1.5) and the exponent μ by the exponent α related to μ by (1.10). Then the change of variables (1.13) allows us to recover the original result.

We therefore consider equation (E) and prove (4.2) with α instead of μ . The proof will be organized into a series of partial results. The basic idea is to adapt a solution $U(x, t; c)$ as closely as possible to our solution u by choosing the appropriate c .

LEMMA 4.2. *There exist constants C and $c > 0$ (depending on u) such that*

$$(4.3) \quad U(x, t; C) \geq u(x, t) \geq U(x, t; c)$$

for every $x \in \mathbf{R}^N$ and every $t \geq 0$.

PROOF: We compare the functions $U(x, t; C)$, $u(x, t)$ and $U(x, t; c)$, with $C > c > 0$. If C is large and c is small enough we will have inequality (4.3) for $t = 0$ and $x \in \mathbf{R}^N$, hence the conclusion by the Maximum Principle. #

We can now define for t the optimal lower bound

$$(4.4) \quad c(t) = \sup\{c > 0 : u(x, t) \geq U(x, t; c) \text{ for } x \in \mathbf{R}^N\}.$$

As a consequence of Lemma 4.2 $c(t)$ is positive and finite, $0 < c(t) \leq C$. It is a simple consequence of the Maximum Principle that $c(t)$ is a nondecreasing function of t . Therefore, there exists the limit

$$(4.5) \quad c_* = \lim_{t \rightarrow \infty} c(t)$$

and this limit is positive and finite.

We now introduce the rescaling operator $\mathcal{T}_k$, with parameter $k > 0$, which acts on solutions of (E) according to the formula

$$(4.6) \quad (\mathcal{T}_k u)(x, t) = k^\alpha u(kx, k^\beta t)$$

to produce a new solution of (E). Moreover U is invariant under such a transformation. Let us abbreviate $\mathcal{T}_k u = u_k$. Given a solution $u(x, t)$ under the assumptions of Theorem 4.1 we consider the family $\{u_k(x, t)\}$ for $k \geq 1$. By known compactness of the solutions

of (E) we may extract from every sequence in this family a convergent subsequence (on compact subsets of Q). Let

$$(4.7) \quad \tilde{u}(x, t) = \lim_{j \rightarrow \infty} u_{k_j}(x, t).$$

We have

$$(4.8) \quad U(x, t; c_*) \leq \tilde{u}(x, t) \leq U(x, t; C) \quad \text{in } Q.$$

If we prove that necessarily $\tilde{u} = U_{c_*}$ the Theorem will follow. Our main lemma is the following.

LEMMA 4.3. *Let $\tilde{u}$ be a sequence limit and assume that for $t = 1$ we have $\tilde{u}(x, 1) \not\equiv U(x, 1; c_*)$. Then for every $t_1 > 1$ there exists $\varepsilon > 0$ and $\tau > 0$ such that*

$$(4.9) \quad \tilde{u}(x, t) \geq U(x, t - \tau; c_* + \varepsilon).$$

for $t \geq t_1$.

PROOF: Let $R(t; c)$ be the interface of $U(x, t; c)$. Since u_k satisfies equation (E) and $r_k(t) \geq R(t; c_* - \delta)$ for some $\delta > 0$ if k is large enough, passing to the limit $k \rightarrow \infty$, so that $\delta \rightarrow 0$, we conclude that the function $\tilde{u}$ satisfies the inequality

$$(4.10) \quad u_t \geq \Delta u^m - \frac{1}{R_*(t)^2} u^m,$$

where $R_*(t) = R(t; c_*)$. It is therefore a supersolution for the equation

$$(4.11) \quad u_t = \Delta u^m - \frac{1}{R_*(t)^2} u^m.$$

which is exactly the equation satisfied by U_{c_*} . Therefore, not only $\tilde{u}(x, t) \geq U(x, t; c_*)$ for $t > 1$ but also, by the Strong Maximum Principle, we have $\tilde{u}(x, t)$ strictly larger than $U(x, t; c_*)$ whenever $\tilde{u}$ is positive. If the interfaces are separated at $t = t_1$ we can safely increase c_* . If both interfaces coincide we use the now well-known trick of introducing a delay $\tau > 0$ on the larger solution so that the supports also separate. This allows us to increase the value of c_* without losing the ordering $\tilde{u} \geq U$. We thus obtain (4.9). We omit further details since they are not essentially different from those of [FK]. ■

LEMMA 4.4. *Under the assumptions of Lemma 4.3 the constant c_* can be improved.*

PROOF: Use the fact that u_{k_j} converges to $\tilde{u}$ on compact sets to arrive at the inequality

$$(4.12) \quad u_k(x, t_1) \geq U(x, t_1 - \tau; c_* + \frac{\varepsilon}{2})$$

if k is large enough. Undoing the transformation this means that

$$(4.13) \quad c(k^\beta t_1) > c_*$$

By the monotonicity of $c(t)$ we get

$$(4.14) \quad \lim_{t \rightarrow \infty} c(t) > c_* \quad \blacksquare$$

By the very definition of c_* this conclusion is impossible. We must therefore have $\tilde{u} \equiv U_{c_*}$. This proves that u_k converges to U_{c_*} along every sequence $k_j \rightarrow \infty$, hence the whole family $\{u_k(x, t)\}$ converges to U_{c_*} as $k \rightarrow \infty$. Undoing the rescaling (4.6) we obtain precisely the conclusion of the Theorem. $\blacksquare$

This proof is similar in spirit to that of [FK], where the porous medium equation was first treated in all space dimensions, and that of [KPV], where the asymptotic behaviour of the solutions of the Barenblatt elasto-plastic filtration model was established. The latter was also a case of anomalous exponents. The main feature where the special form of our equation enters is probably the step in formula (4.10), where we find that $\tilde{u}$ is a supersolution of the equation with frozen coefficient (4.11). Such a property was also essential in the uniqueness proof of Section 3.

5. Final Remarks. We conclude with some interesting observations. Our asymptotic result proves indirectly the uniqueness of the self-similar solution family $U(x, t; c)$ corresponding to given $m > 1$ and $k > 0$. It also proves that the anomalous exponent, α for equation (1.5), μ for (1.2), is uniquely determined. In fact, a way of calculating it consists in taking some admissible initial data u_0 , computing the solution $u(x, t)$ and taking limit as $t \rightarrow \infty$. According to Theorem 4.1 we have for equation (1.2)

$$(5.1) \quad \mu = - \lim_{t \rightarrow \infty} \frac{\log u(0, t)}{\log t}.$$

and the same formula holds for α in equation (1.5). In fact this formula does not need the whole force of the Theorem, but only the bounds of Lemma 4.2. The uniqueness of the self-similar solution does need the result of the Theorem.

Another easy consequence is the fact that α is a nondecreasing function of the parameter k . In fact, solutions of (E) with $k = k_1$ are supersolutions for the same equation with $k = k_2 > k_1$. Since α is the decay rate, it follows that $\alpha(k_1) \leq \alpha(k_2)$.

An interesting observation concerns the asymptotic constant c_* . This constant depends only on the initial data u_0 and the parameters m and k . We can define a functional

$$(5.2) \quad J : u_0 \mapsto c_*(u_0).$$

This mapping represents a certain conservation law expressed in an unusual form. A first example of such functional was considered in [KPV] in the study of the equation of elasto-plastic filtration. We refer to that article for further insight into the subject.

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