**A VARIATIONAL APPROACH TO A FREE BOUNDARY
PROBLEM ARISING IN ELECTROPHOTOGRAPHY**

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Abstract

A least square approach to a free boundary problem modeling the photocopying process posed by A. Friedman and B. Hu [4] is considered. The corresponding variational functional is nonconvex and nonsmooth, so that its generalized gradient is multivalued.

1 Introduction

The following problem arises as the mathematical model for the photocopying process (see [3]).

Given a plane domain $\Omega = \{(x, y); |x| < a; |y| < a\}$ find a continuous curve Γ in $\bar{\Omega}^+$ which is C^1 inside $\Omega^+ = \{(x, y); |x| < a, 0 < y < a\}$ and a function z in $\bar{\Omega}$ such that

$$\Delta z = 1 \text{ in } D \tag{1}$$

$$\Delta z = 0 \text{ in } \Omega \setminus \bar{D} \tag{2}$$

$$z \in C^1(\bar{\Omega} \setminus \bar{I}), \quad I = (-\varepsilon, \varepsilon) \tag{3}$$

$$z_y(x, 0+) - z_y(x, 0-) = -\sigma \quad \forall x \in I \tag{4}$$

$$\frac{\partial z}{\partial \nu} = 0 \text{ on } \Gamma \cap \Omega^+ \quad (5)$$

where $\frac{\partial}{\partial \nu}$ is the outward normal to D .

The function z satisfies also the boundary conditions

$$\begin{aligned} z(x, -a) &= 0 & -a < x < a \\ z(x, a) &= 1 & -a < x < a \\ z_x(\pm a, y) &= 0 & -a < y < a \end{aligned} \quad (6)$$

Here D is the subset of Ω^+ which is enclosed by Γ and parts of the x -axis (See Fig. 1 below). If χ_D is the characteristic function of the domain D

Figure 1.

then it is readily seen that we may write Equations (1) to (6) as

$$\begin{aligned} \Delta z &= \chi_D - \delta & \text{in } \Omega \\ z(x, a) &= 0, & |x| < a \\ z(x, -a) &= 1, & |x| < a \\ z_x(\pm a, y) &= 0, & |y| < a \end{aligned} \quad (7)$$

$$\frac{\partial z}{\partial \nu} = 0 \text{ in } \Gamma \quad (8)$$

where δ is the measure defined by

$$\delta(u) = \sigma \int_{-\varepsilon}^{\varepsilon} \varphi(x, 0) dx \quad \forall \varphi \in H_0^1(\Omega). \quad (9)$$

It will be convenient to approximate δ by the Dirac measure (with different σ)

$$\delta(\varphi) = \sigma \varphi(0, 0) \quad \forall \varphi \in C(\Omega). \quad (10)$$

Problem (1) - (6) has been posed by A. Friedman and B. Hu [4] and studied in some particular cases in the framework of variational inequality theory. Here we shall use a different approach reducing it via least square method to a shape optimization problem. This problem is presented and studied in Section 2. In Section 3 we deduce a maximum principle result and we study the differentiability properties of the corresponding variational functional. In Section 4 we show how to minimize it numerically.

The main difficulty (and beauty) of the problem is that the variational functional is not continuously differentiable (it is also nonconvex). That amounts to the phenomenon that the adjoint equation is a nonstandard elliptic problem with the right hand side in H^{-2} and with discontinuous solution. Discontinuity of the adjoint variable corresponds to the multivaluedness of the generalized gradient of the variational functional.

2 A corresponding shape optimization problem

We shall use the polar coordinates $(r, \theta) \in [0, \infty) \times [0, 2\pi)$ to represent domains and curves Γ of the form arising in problem (1) - (6). Consider the set $U \subset H^4(0, \pi)$,

$$U = \{u \in H^4(0, \pi); 0 < u(\theta) < a, \theta \in [0, \pi]\} \quad (11)$$

and denote by Γ_u the curve $\{r = u(\theta); 0 < \theta < \pi\}$ for every $u \in U$. We shall denote also by D_u the domain enclosed by Γ_u and the x -axis (see Fig. 1).

Consider the following shape optimization problem

$$\begin{aligned} & \text{minimize } \left\{ \int_{\Gamma_u} \left(\frac{\partial z}{\partial \nu} \right)^2 d\sigma + \alpha \|u\|_{H^4(0, \pi)}^2 \right\} \\ & \text{on all } (z, u) \in H_{loc}^2(\Omega \setminus 0) \times U \text{ subject to} \end{aligned} \quad (12)$$

$$\begin{aligned} \Delta z &= \chi_{D_u} - \delta \quad \text{in } \Omega \\ z &= 1 \quad \text{in } \{y = a\}; z = 0 \quad \text{in } \{y = -1\} \\ z_x &= 0 \quad \text{in } \{x = a\} \cup \{x = -a\}. \end{aligned} \quad (13)$$

Here α is a positive constant, and δ is defined in (10).

Note that for every $u \in U$ the boundary value problem (13) has a unique solution $z^u \in W^{1,q}(\Omega)$ where $1 < q < \frac{3}{2}$ (see e.g. [1], p. 495). Moreover by standard elliptic regularity theory we see that $z^u \in W_{loc}^{2,p}(\Omega \setminus \{0\}) \forall p < \infty$ and so in particular it follows that $z^u \in C^1(\bar{\Omega} \setminus \{0\})$. In Section 3 we shall prove more precise regularity result (see Lemma 2, below).

Proposition 1 *For large enough a , problem (12) has at least one solution $u^* \in U$.*

Proof: Let us consider a sequence $\{u_n\} \subset U$ such that

$$\varphi(u_n) = \int_{\Gamma_{u_n}} \left(\frac{\partial z^{u_n}}{\partial \nu} \right)^2 d\sigma + \alpha \|u_n\|_{H^4(0,\pi)}^2 \rightarrow \inf \{\varphi(u); u \in U\}. \quad (14)$$

We may assume that $u_n \rightarrow u^*$ weakly in $H^4(0, \pi)$ and strongly in $C^1([0, \pi])$. Let us show that $u^* \in U$, i.e., $0 < u^*(\theta) < a$ for all $\theta \in [0, \pi]$. The rest can be proved using a standard device.

The following representation holds for $z_n = z^{u_n}$

$$z_n(x, y) = \sigma \log \frac{1}{r} + \tilde{z}_n(x, y) \quad \forall (x, y) \in \Omega \quad (15)$$

where $\tilde{z}_n$ is a solution of $\Delta \tilde{z}_n = \chi_{D_{u_n}}$ with appropriate boundary conditions. By elliptic estimates,

$$\|\tilde{z}_n\|_{C^1(\Omega)} \leq C \quad \forall n \quad (16)$$

Hence

$$\frac{\partial z_n}{\partial \nu} + \frac{c}{r} \leq C \quad \text{in } \Gamma_{u_n} \quad (17)$$

(here we have used the fact that, without loss of generality, we can assume that $\|u_n\|_{H^4(0,\pi)} \leq C$ so that $\vec{v} = \vec{v}_{u_n} = \alpha_n(\theta)\vec{r} + \beta_u(\theta)\vec{\tau}$, $0 < \delta \leq \alpha_n(\theta) \leq 1$, $\vec{r} = (\cos \theta, \sin \theta)$, $\vec{\tau} = (\sin \theta, -\cos \theta)$). This yields

$$\int_{\Gamma_{u_n}} \left(\frac{\partial z_n}{\partial \nu} \right)^2 d\sigma \geq -c + c \int_0^\pi \frac{1}{u_n^2(\theta)} \sqrt{u_n^2(\theta) + (u_n'(\theta))^2} d\theta \geq -c + c \int_0^\pi \frac{d\theta}{u_n(\theta)}. \quad (18)$$

Let us assume now that $u_n(\theta_0) \rightarrow 0$ as $n \rightarrow \infty$ for some $\theta_0 \in [0, \pi]$. Since $\{u_n\}$ is uniformly convergent to u^* along with its first derivative we conclude that $u^*(\theta_0) = (u^*)'(\theta_0) = 0$. Hence $\frac{1}{u^*} \notin L^1(0, \pi)$ and so, by Fatou's lemma,

$$\liminf_{n \rightarrow \infty} \int_{\Gamma_{u_n}} \left(\frac{\partial z_n}{\partial \nu} \right)^2 d\sigma \geq -c + c \liminf_{n \rightarrow \infty} \int_0^\pi \frac{d\theta}{u_n(\theta)} \geq -c + c \int_0^\pi \frac{d\theta}{u^*(\theta)} = +\infty \quad (19)$$

which contradicts (14). Hence $u^* > 0$ in $[0, \pi]$. Now set $u^* = u_a^*$, $\Omega = \Omega_a$ etc., to emphasize the dependence on parameter a (the size of the domain). Then

$$\alpha \|u_a^*\|_{H^1(0,\pi)}^2 \leq \int_{\Gamma_{u_0}} \left(\frac{\partial z_a^0}{\partial \nu}\right)^2 d\sigma + \alpha \|u_0\|_{H^1(0,\pi)}^2 \quad (20)$$

where u_0 is an arbitrary element of U , and $z_a^0 = z_a^{u_0}$. To fix ideas, take $u_0 = 1$, so that z_a^0 is determined by

$$\begin{aligned} \Delta z_a^0 &= \chi_{B_1} - \delta && \text{in } \Omega_a \\ z_a^0 &= 1 \text{ in } \{y = a\}, \quad z_a^0 = 0 && \text{in } \{y = -a\} \\ (z_a^0)_x &= 0 && \text{in } \{x = \pm a\} \end{aligned} \quad (21)$$

We write $z_a^0 = z_1^a + z_2^a$, where

$$\begin{aligned} \Delta z_1^a &= \chi_{B_1} && \text{in } \Omega_a \\ z_1^a &= 0 && \text{in } \{y = \pm a\} \\ (z_1^a)_x &= 0 && \text{in } \{x = \pm a\} \end{aligned} \quad (22)$$

We need an estimate on

$$\int_{\Gamma_{u_0}} \left(\frac{\partial z_1^a}{\partial \nu}\right)^2 d\sigma.$$

Define, by scaling $Y = \frac{X}{a}$ ($X \in \mathbb{R}^2, a > 0$),

$$v(Y) = z_1^a(X). \quad (23)$$

Then

$$\Delta v(Y) = a^2 \Delta z_1^a(X) = a^2 \chi_{B_1}(X) = a^2 \chi_{B_{1/a}}(Y). \quad (24)$$

Hence

$$\begin{aligned} \|v\|_{W^{2,p}} &\leq ca^2 \|\chi_{B_{1/a}}\|_{L^p} \\ &= ca^{\frac{2(p-1)}{p}}. \end{aligned} \quad (25)$$

Hence, if $p = \frac{4}{3}$ and $\tilde{\Gamma} = \frac{1}{a}\Gamma_{u_0}$, by trace theorem,

$$\int_{\tilde{\Gamma}} \left(\frac{\partial v}{\partial \nu}\right)^2 d\tilde{\sigma} \leq \|v\|_{W^{2,p}}^2 \leq ca. \quad (26)$$

Finally

$$\int_{\Gamma_{u_0}} \left(\frac{\partial z_1^a}{\partial \nu} \right)^2 d\sigma = \int_{\tilde{\Gamma}} \frac{1}{a^2} \left(\frac{\partial v}{\partial \nu} \right)^2 a d\tilde{\sigma} \leq c, \quad (27)$$

where c does not depend on a . On the other hand,

$$\begin{aligned} \Delta z_2^a &= -\delta \quad \text{in } \Omega_a \\ z_2^a &= 1 \quad \text{in } \{y = a\}, \\ z_2^a &= 0 \quad \text{in } \{y = -a\} \\ (z_2^a)_x &= 0 \quad \text{in } \{x = \pm a\}. \end{aligned} \quad (28)$$

Then

$$0 \leq z_2^a \leq \sigma \log \frac{a\sqrt{2}}{r} + 1 \quad \text{in } \Omega_a. \quad (29)$$

Hence

$$|\nabla z_2^a| \leq c\sigma \log a + c \quad \text{in } \Gamma_{u_0}. \quad (30)$$

From (27) and (30) we conclude that

$$\int_{\Gamma_{u_0}} \left(\frac{\partial z_a^0}{\partial \nu} \right)^2 d\sigma \leq c(\log a)^2 + c \quad (31)$$

where c does not depend on a . Hence, by (20), $u^* < a$ in $[0, \pi]$ for a large enough.

3 Optimality system and differentiability properties of the variational functional

We associate with problem (12) the dual system

$$\begin{aligned} \Delta p &= \zeta \quad \text{in } \Omega \\ p &= 0 \quad \text{in } \{y = a\} \cup \{y = -a\} \\ p_x &= 0 \quad \text{in } \{x = a\} \cup \{x = -a\} \end{aligned} \quad (32)$$

where $\zeta \in \mathcal{D}'(\Omega)$ is defined by

$$\zeta(\varphi) = \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma \quad \forall \varphi \in \mathcal{D}(\Omega) \quad (33)$$

$u \in U$ and z is a solution to Eq. (7). The boundary value problem is considered of course in the following weak sense

$$\int_{\Omega} p \Delta \varphi dx = \zeta(\varphi) \quad (34)$$

for all $\varphi \in H^2(\Omega)$ such that $\varphi = 0$ in $\{y = a\} \cup \{y = -a\}$, $\varphi_x = 0$ in $\{x = a\} \cup \{x = -a\}$.

Lemma 1 *Problem (32) has a unique solution $p \in L^\infty(\Omega) \cap C^\infty(\bar{\Omega} \setminus \Gamma_u)$. Moreover,*

$$p_e(\bar{x}, \bar{y}) = \lim_{\substack{(x,y) \in \Omega \setminus D_u \\ (x,y) \rightarrow (\bar{x}, \bar{y})}} p(x, y); \quad p_i(\bar{x}, \bar{y}) = \lim_{\substack{(x,y) \in D_u \\ (x,y) \rightarrow (\bar{x}, \bar{y})}} p(x, y) \quad (35)$$

exist for all $(\bar{x}, \bar{y}) \in \Gamma_u$ and

$$p_e(\bar{x}, \bar{y}) - p_i(\bar{x}, \bar{y}) = -\frac{\partial z}{\partial \nu}(\bar{x}, \bar{y}) \quad \forall (\bar{x}, \bar{y}) \in \Gamma_u \quad (36)$$

Proof: The linear functional $\psi \rightarrow \zeta(\varphi)$ where $\Delta \varphi = \psi$ in Ω ; $\varphi = 0$ in $\{y = a\} \cup \{y = -a\}$, $\varphi_x = 0$ in $\{x = a\} \cup \{x = -a\}$ is continuous on $L^2(\Omega)$,

$$|\zeta(\varphi)| \leq \left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\Gamma_u)} \left\| \frac{\partial z}{\partial \nu} \right\|_{L^2(\Gamma_u)} \leq C \|\psi\|_{L^2(\Omega)} \quad \forall \psi \in L^2(\Omega).$$

Hence there is $p \in L^2(\Omega)$ such that

$$\int_{\Omega} p \psi dx = \zeta(\varphi) \quad \forall \psi \in L^2(\Omega).$$

Moreover, by elliptic regularity it follows that $p \in C^\infty(\bar{\Omega} \setminus \Gamma_u)$. We shall prove now that p has lateral limits on Γ_u which satisfy (35). Indeed, using (34), we may represent p as

$$p(x, y) = \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \frac{\partial}{\partial \nu} G(x, y; \bar{x}, \bar{y}) d\sigma_{\bar{x}\bar{y}} + p_0(x, y) \quad \forall (x, y) \in B \setminus \Gamma_u \quad (37)$$

where

$$G(x, y; \bar{x}, \bar{y}) = \frac{1}{2\pi} \log(|x - \bar{x}|^2 + |y - \bar{y}|^2)^{\frac{1}{2}}$$

where p_0 is a smooth function on $\bar{B}$, $\Gamma_u \subset B \subset \bar{B} \subset \Omega$. We see that the function $p(x, y) = \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \frac{\partial}{\partial \nu} G(x, y; \bar{x}, \bar{y}) d\sigma_{\bar{x}\bar{y}}$ is a double layer potential on Γ_u with density $\frac{\partial z}{\partial \nu} \in C(\Gamma_u)$ and since Γ_u is regular we conclude that

$$p_e(\bar{x}, \bar{y}) - p_i(\bar{x}, \bar{y}) = -\frac{\partial z}{\partial \nu}(\bar{x}, \bar{y}) \quad \forall (\bar{x}, \bar{y}) \in \Gamma_u$$

where p_e and p_i are the lateral limits to Γ_u defined by (35).

Let us denote again by p_e and p_i (also, z_e and z_i) the restrictions of p (also, z) to $\Omega \setminus D_u$ and D_u respectively. By definition (34) we see that

$$\int_{\Omega} p \Delta \varphi dx dy = \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma \quad \forall \varphi \in \mathcal{D}(\Omega)$$

and so if p is smooth enough on Γ_u we have

$$\int_{\Gamma_u} (p_e \frac{\partial \varphi}{\partial \nu} - \varphi \frac{\partial p_e}{\partial \nu}) d\sigma - \int_{\Gamma_u} (p_i \frac{\partial \varphi}{\partial \nu} - \varphi \frac{\partial p_i}{\partial \nu}) d\sigma = - \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \frac{\partial \varphi}{\partial \nu} d\sigma.$$

and this yields

$$\frac{\partial p_e}{\partial \nu} - \frac{\partial p_i}{\partial \nu} = 0 \quad \text{in } \Gamma_u$$

in a generalized sense. We may therefore view problem (32) as a generalized formulation of the boundary value problem

$$\begin{aligned} \Delta p &= 0 \quad \text{in } \Omega \setminus \Gamma_u \\ p &= 0 \quad \text{in } \{y = a\} \cup \{y = -a\} \\ p_x &= 0 \quad \text{in } \{x = a\} \cup \{x = -a\} \\ p_e - p_i &= -\frac{\partial z}{\partial \nu} \quad \text{in } \Gamma_u \\ \frac{\partial p_e}{\partial \nu} - \frac{\partial p_i}{\partial \nu} &= 0 \quad \text{in } \Gamma_u. \end{aligned} \tag{38}$$

We also need a sharper regularity result for z^u in the neighborhood of Γ_u . Namely, $z^u \in W^{2,q} \quad \forall q < \infty$ is not enough for our purposes.

Lemma 2 *If $u \in W^{3,q}(0, \pi)$, $q > 2$, then*

$$z_e^u \in C^{2,\alpha}((\Omega \setminus D_u) \cup T) \tag{39}$$

$$z_i^u \in C^{2,\alpha}(D_u \cup T) \tag{40}$$

for any T such that $\bar{T} \subset \Gamma_u$. Here $\alpha = \frac{q-2}{q}$.

Proof: Without loss of generality we can assume that Γ_u is given locally (in new coordinates) as

$$\Gamma_u = \{(x, y) : y = \psi(x), \quad x \in (-\varepsilon, \varepsilon)\} \tag{41}$$

where $\psi \in W^{3,q}$ and $|\psi_x| < \varepsilon$. Define v by

$$v(x, y) = u(x, y + \psi(x)). \quad (42)$$

Then $\Delta u = Lv$ and

$$Lv = \Delta v + v_{yy}(\psi_x)^2 - 2v_{xy}\psi_x - v_y\psi_{xx} = \chi_{\mathbf{R}_-^2} \quad (43)$$

in $B_\varepsilon = \{(x, y) : |(x, y)| < \varepsilon\}$. We can regularize right hand side by $\chi_\eta \in C^\infty$, such that $(\chi_\eta)_x = 0$ and

$$\chi_\eta \rightarrow \chi_{\mathbf{R}_-^2} \text{ in } L^q(B_\varepsilon) \quad \forall q < \infty. \quad (44)$$

as $\eta \rightarrow 0$. Then if v_η is a solution of

$$\begin{aligned} Lv_\eta &= \chi_\eta \text{ in } B_\varepsilon \\ v_\eta &= v \text{ in } \partial B_\varepsilon \end{aligned} \quad (45)$$

then

$$v_\eta \rightarrow v \text{ in } W^{2,q}(B_\varepsilon) \quad \forall q < \infty. \quad (46)$$

Also if we differentiate equation in (45) with respect to x we get

$$L_1(v_\eta)_x = -(v_\eta)_{yy}2\psi_x\psi_{xx} + (v_\eta)_y\psi_{xxx} \quad (47)$$

Hence

$$\| (v_\eta)_x \|_{W^{2,q}(B_{\varepsilon/2})} < c \quad (48)$$

uniformly in η , which implies

$$\| v_x \|_{W^{2,q}(B_{\varepsilon/2})} < c \quad (49)$$

By the Sobolev imbedding theorem, (49) implies

$$\| \nabla v_x \|_{C^\alpha(B_{\varepsilon/2})} < c \quad (50)$$

and hence

$$\| v|_{\{y=0\}} \|_{C^{2,\alpha}(B_{\varepsilon/2} \cap \{y=0\})} < c. \quad (51)$$

By Schauder's estimates (39) and (40) follow.

Now we are ready to formulate a maximum principle type result for problem (12).

Theorem 1 *Let (z, u) be a solution to problem (12). Then u satisfies the equation*

$$\begin{aligned} & \left(\frac{\partial z}{\partial \nu} \frac{\partial z}{\partial \tau} - \frac{1}{2} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{u'}{\sqrt{u^2 + u'^2}} \right)' - \frac{\partial z}{\partial \nu} \frac{\partial z}{\partial r} + \\ & + \frac{1}{2} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{u}{\sqrt{u^2 + u'^2}} - \alpha(u + u^{iiii}) = \eta \end{aligned} \quad (52)$$

where $\eta \in L_{loc}^\infty(0, \pi)$ and

$$(p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}) \sqrt{u^2 + u'^2} \leq \eta \leq (p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}) \sqrt{u^2 + u'^2} \text{ a.e. in } (0, \pi) \quad (53)$$

and $\vec{r} = (\cos \theta, \sin \theta)$, $\vec{\tau} = (\sin \theta, -\cos \theta)$.

Proof: Let (z, u) be optimal in problem (12). We have

$$\int_{\Gamma_{u+\lambda v}} \left(\frac{\partial z^{u+\lambda v}}{\partial \nu} \right)^2 d\sigma - \int_{\Gamma_u} \left(\frac{\partial z}{\partial \nu} \right)^2 d\sigma + \alpha(\|u + \lambda v\|_{H^4(0, \pi)}^2 - \|u\|_{H^4(0, \pi)}^2) \geq 0$$

for all $\lambda > 0$ sufficiently small and $v \in H_0^4(0, \pi)$. We have therefore

$$\begin{aligned} \liminf_{\lambda \downarrow 0} \lambda^{-1} \int_0^\pi \frac{\partial z}{\partial \nu} \frac{\partial}{\partial \nu} (z^{u+\lambda v} - z^u + z^u(u + \lambda v, \theta) - z^u(u, \theta)) \\ (u^2(\theta) + (u')^2(\theta))^{\frac{1}{2}} d\theta \geq \\ -\alpha(Au, v) - \int_0^\pi \left(\frac{\partial z}{\partial \nu} \right)^2 (uv + u'v')(u^2 + (u')^2)^{-\frac{1}{2}} d\theta \\ - \int_0^\pi \frac{\partial z}{\partial \nu} (\nabla z, \nabla \nu_u(v)) (u^2 + (u')^2)^{\frac{1}{2}} d\theta. \end{aligned} \quad (54)$$

where $\nabla \nu_u(v) = \lim_{\lambda \rightarrow 0} (\nu_{u+\lambda v} - \nu_u)$ and ν_u is the normal at Γ_u . We have denoted by $(\cdot, \cdot)$ the pairing between $H^4(0, \pi)$ and its dual space $(H^4(0, \pi))'$. And by $A : H^4(0, \pi) \rightarrow (H^4(0, \pi))'$ the operator $(Au, v) = \int_0^\pi (uv + u^{iv}v^{iv}) d\theta$ $\forall u, v \in H^4(0, \pi)$.

Next by (34) we have

$$\zeta\left(\frac{z^{u+\lambda v} - z}{\lambda}\right) = \lambda^{-1} \int_\Omega p(\chi_{D_{u+\lambda v}} - \chi_{D_u}) dx dy$$

or equivalently

$$\begin{aligned} \lambda^{-1} \int_0^\pi \frac{\partial z}{\partial \nu} \frac{\partial}{\partial \nu} (z^{u+\lambda v} - z) (u^2 + (u')^2)^{\frac{1}{2}} d\theta = \\ = \lambda^{-1} \int_{D_{u+\lambda v} \setminus D_u} p(x, y) dx dy. \end{aligned} \quad (55)$$

On the other hand, we have

$$\lim_{\lambda \downarrow 0} \lambda^{-1} \int_{D_{u+\lambda v} \setminus D_u} p(x, y) dx dy = \int_{\Gamma_u} \xi_v(x, y) d\sigma_{xy} \quad \forall v \in H_0^4(0, \pi) \quad (56)$$

where

$$\xi_v(x, y) = \begin{cases} p_e(x, y)v(x, y) & \text{if } v(x, y) \geq 0 \\ p_i(x, y)v(x, y) & \text{if } v(x, y) \leq 0 \end{cases} \quad \forall (x, y) \in \Gamma_u. \quad (57)$$

In other words,

$$\lim_{\lambda \downarrow 0} \lambda^{-1} \int_{D_{u+\lambda v} \setminus D_u} p(x, y) dx dy = \int_{\Gamma_u} (p_e v^+ - p_i v^-) d\sigma \quad \forall v \in H_0^4(0, \pi). \quad (58)$$

Also, using Lemma 2, we compute

$$\begin{aligned} \lim_{\lambda \downarrow 0} \lambda^{-1} \int_0^\pi \frac{\partial z}{\partial \nu} \frac{\partial}{\partial \nu} (z^u(u + \lambda v, \theta) - z^u(u, \theta)) (u^2(\theta) + (u')^2(\theta))^{1/2} d\theta = \\ \int_{\Gamma_u} \frac{\partial z}{\partial \nu} \left(\frac{\partial(z_e)_r}{\partial \nu} v^+ - \frac{\partial(z_i)_r}{\partial \nu} v^- \right) d\sigma. \end{aligned} \quad (59)$$

Substituting (58) and (59) in (54) and (55) we get

$$\begin{aligned} & -\alpha(Au, v) - \frac{1}{2} \int_0^\pi \left(\frac{\partial z}{\partial \nu} \right)^2 (uv + u'v') (u^2 + (u')^2)^{\frac{1}{2}} d\theta \\ & - \int_0^\pi \frac{\partial z}{\partial \nu} (\nabla z, \nabla \nu_u(v)) (u^2 + (u')^2)^{\frac{1}{2}} d\theta \\ & \leq \int_{\Gamma_u} \left((p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}) v^+ - (p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}) v^- \right) d\sigma \quad \forall v \in H_0^4(0, \pi). \end{aligned} \quad (60)$$

Let us denote by $\eta \in (H^4(0, \pi))'$ the functional defined by the left-hand side of inequality (60). We have therefore

$$(\eta, v) \leq \int_{\Gamma_u} \left((p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}) v^+ - (p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}) v^- \right) d\sigma \quad \forall v \in H_0^4(0, \pi)$$

and this yields

$$(p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}) \sqrt{u^2 + u'^2} \leq \eta \leq (p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}) \sqrt{u^2 + u'^2} \quad (61)$$

in the sense of $(H^4(0, \pi))'$ and in particular in the sense of $\mathcal{D}'(0, \pi)$. Hence $\eta \in L_{loc}^\infty(0, \pi)$ and (53) holds a.e. on $(0, \pi)$.

Now we notice that

$$\begin{aligned}\nu_u &= (u^2 + (u')^2)^{-\frac{1}{2}}(u \cos \theta + u' \sin \theta, u \sin \theta \\ &\quad - u' \cos \theta) = (u^2 + (u')^2)^{-\frac{1}{2}}(u\vec{r} + u'\vec{\tau})\end{aligned}$$

and therefore

$$\begin{aligned}\nabla \nu_u(v) &= \lim_{\lambda \rightarrow 0} \lambda^{-1}(\nu_{u+\lambda v} - \nu_u) \\ &= (u^2 + (u')^2)^{-\frac{1}{2}}(v\vec{r} + v'\vec{\tau}) \\ &\quad - \frac{1}{2}(uv + u'v')(u\vec{r} + u'\vec{\tau})(u^2 + (u')^2)^{-\frac{3}{2}}.\end{aligned}$$

Substituting in (60) we get

$$\eta = -\alpha Au - Bu \tag{62}$$

where $B : H^4(0, \pi) \rightarrow (H^4(0, \pi))'$ is defined by

$$\begin{aligned}(Bu, v) &= \frac{1}{2} \int_0^\pi \left(\frac{\partial z}{\partial \nu}\right)^2 (uv + u'v')(u^2 + (u')^2)^{-\frac{1}{2}} d\theta + \\ &\quad + \int_0^\pi \frac{\partial z}{\partial \nu} (\nabla z, v\vec{r} + v'\vec{\tau}) d\theta - \\ &\quad \int_0^\pi \frac{\partial z}{\partial \nu} (\nabla z, u\vec{r} + u'\vec{\tau}) (uv + u'v')(u^2 + (u')^2)^{-1} d\theta \\ &\quad \forall v \in H_0^4(0, \pi).\end{aligned}$$

Equivalently

$$\begin{aligned}(Bu, v) &= \int_0^\pi \frac{\partial z}{\partial \nu} \left(v \frac{\partial z}{\partial r} + v' \frac{\partial z}{\partial \tau}\right) d\theta - \\ &\quad - \frac{1}{2} \int_0^\pi \left(\frac{\partial z}{\partial \nu}\right)^2 (uv + u'v')(u^2 + (u')^2)^{-\frac{1}{2}} d\theta.\end{aligned}$$

Now by (61) we see that u satisfies in the sense of distributions on $(0, \pi)$ equation (52) thereby completing the proof.

We observe that from (53) it follows that if (z, u) is a solution to problem (12), then

$$\frac{\partial z}{\partial \nu} \left(1 + \frac{\partial(z_i)_r}{\partial \nu} - \frac{\partial(z_e)_r}{\partial \nu}\right) \leq 0 \text{ in } \Gamma_u. \tag{63}$$

In the rest of the paper, instead of studying the functional in (12), we shall concentrate on the principal part of it. Namely, let

$$\varphi(u) = \int_{\Gamma_u} \left(\frac{\partial z}{\partial \nu} \right)^2 d\sigma \quad (64)$$

subject to (13). Also, δ can be defined either by (9) or (10). Then, similarly as in the proof of Theorem 1, for given $u \in U$ (not necessarily a minimizer), we can compute one sided directional derivative of φ in direction $v \in H_0^4(0, \pi)$

$$\begin{aligned} \varphi'(u; v) &= \lim_{\varepsilon \downarrow 0} \frac{\varphi(u + \varepsilon v) - \varphi(u)}{\varepsilon} \\ &= \int_{\Gamma_u} \left((p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}) v^+ - (p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}) v^- \right) d\sigma \\ &\quad - \frac{1}{2} \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \right)^2 \frac{uv + u'v'}{(u^2 + u'^2)^{\frac{1}{2}}} d\theta + \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial r} v + \frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial \tau} v' \right) d\theta \\ &\geq \int_0^\pi (\zeta v^+ - \zeta v^-) \sqrt{u^2 + u'^2} d\theta \\ &\quad - \frac{1}{2} \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \right)^2 \frac{uv + u'v'}{(u^2 + u'^2)^{\frac{1}{2}}} d\theta + \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial r} v + \frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial \tau} v' \right) d\theta \\ &= \int_0^\pi \zeta v \sqrt{u^2 + u'^2} d\theta \\ &\quad - \frac{1}{2} \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \right)^2 \frac{uv + u'v'}{(u^2 + u'^2)^{\frac{1}{2}}} d\theta + \int_0^\pi \left(\frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial r} v + \frac{\partial z^u}{\partial \nu} \frac{\partial z^u}{\partial \tau} v' \right) d\theta \end{aligned} \quad (65)$$

for any ζ such that $p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu} \leq \zeta \leq p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}$.

Such ζ exists *provided*

$$p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu} \leq p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu} \quad (66)$$

i.e., *provided* (63) holds for a given u .

Recall that the Clarke generalized directional derivative is defined as

$$\varphi^0(u; v) = \limsup_{\substack{y \rightarrow u \\ t \downarrow 0}} \frac{\varphi(y + tv) - \varphi(y)}{t} \quad (67)$$

and that the generalized gradient of φ is defined by

$$\partial \varphi(u) = \{ \eta \in (H^4(0, \pi))'; \varphi^0(u; v) \geq (\eta, v) \ \forall v \in H^4(0, \pi) \}. \quad (68)$$

Also, we shall need the following

Definition 1 For any $u \in H^4(0, \pi)$ the restricted generalized gradient $\partial_0 \varphi$ is defined as

$$\partial_0 \varphi(u) = \{\eta \in (H^4(0, \pi))'; \varphi^0(u; v) \geq (\eta, v) \quad \forall v \in H_0^4(0, \pi)\}. \quad (69)$$

Now since

$$\varphi^0(u; v) \geq \varphi'(u; v) \quad (70)$$

we have proved the following

Theorem 2 If in addition (66) holds, as well, then

$$\begin{aligned} \partial_0 \varphi(u) \supset [p_i + \frac{\partial z}{\partial \nu} \frac{\partial(z_i)_r}{\partial \nu}, p_e + \frac{\partial z}{\partial \nu} \frac{\partial(z_e)_r}{\partial \nu}] \sqrt{u^2 + u'^2} + \\ \frac{\partial z}{\partial \nu} \frac{\partial z}{\partial r} - \frac{1}{2} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{u}{\sqrt{u^2 + u'^2}} + \left(\frac{1}{2} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{u'}{\sqrt{u^2 + u'^2}} - \frac{\partial z}{\partial \nu} \frac{\partial z}{\partial \tau} \right)'. \end{aligned} \quad (71)$$

4 Computational issues

We observe that if (66) does not hold, then it is not clear how to compute the generalized gradient. Fortunately, we know (see (63)) that for the *minimizer*, (66) does hold. So, if one attempts to solve the minimization problem in practice using the “steepest descent method”:

$$\begin{aligned} u_0 &\in U \\ u_{n+1} &\in u_n - \rho_n \partial_0 \varphi(u_n), \quad \rho_n > 0 \end{aligned} \quad (72)$$

one should stay in the set $V \subset U$ such that (66) holds for any $u \in V$.

Finally, we discuss how to solve (in practice) equation (32). We observe that existence proof presented above does not offer a way of *solving* the equation. Our method is motivated by the formulation (38). Let f be a given regular function on Γ_u , and so we want to solve

$$\begin{aligned} \Delta z &= 0 \quad \text{in } \Omega \setminus \Gamma_u \\ z &= 0 \quad \text{in } \{y = \pm a\} \\ z_x &= 0 \quad \text{in } \{x = \pm a\} \\ z_e - z_i &= f \quad \text{in } \Gamma_u \\ \frac{\partial z_e}{\partial \nu} - \frac{\partial z_i}{\partial \nu} &= 0 \quad \text{in } \Gamma_u \end{aligned} \quad (73)$$

Let us decompose domain Ω into two nonoverlapping domains:

$$\begin{aligned}\Omega_1 &= (\Omega \cap \{y > 0\}) \setminus \bar{D}_u \\ \Omega_2 &= \text{int}[(\Omega \cap \{y \leq 0\}) \cup D_u].\end{aligned}\quad (74)$$

Then $\Omega = \text{int}(\bar{\Omega}_1 \cup \Omega_2)$ and $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma_u \cup (\partial\Omega_1 \cap \{y = 0\})$.

For $(v, w) \in [L^2(\partial\Omega_1 \cap \partial\Omega_2)]^2$ consider the functional

$$j(v, w) = \frac{1}{2} \int_{\partial\Omega_1 \cap \partial\Omega_2} \left\{ (v - z_2)^2 + \left(w - \frac{\partial z_1}{\partial \nu} \right)^2 \right\} d\sigma \quad (75)$$

where ν is the exterior normal to $\partial\Omega_2$, and

$$\begin{aligned}\Delta z_1 &= 0 \quad \text{in } \Omega_1 \\ z_1 &= v + f \quad \text{in } \Gamma_u \\ z_1 &= v \quad \text{in } \partial\Omega_1 \cap \{y = 0\} \\ z_1 &= 0 \quad \text{in } \{y = a\} \\ (z_1)_x &= 0 \quad \text{in } \{x = \pm a\}\end{aligned}\quad (76)$$

and

$$\begin{aligned}\Delta z_2 &= 0 \quad \text{in } \Omega_2 \\ \frac{\partial z_2}{\partial \nu} &= w \quad \text{in } \partial\Omega_1 \cap \partial\Omega_2 \\ z_2 &= 0 \quad \text{in } \{y = -a\} \\ (z_2)_x &= 0 \quad \text{in } \{x = \pm a\}.\end{aligned}\quad (77)$$

Then, it is not difficult to see that equation (32), (equivalently (38)) is equivalent to

$$\min j(v, w) = 0. \quad (78)$$

So we propose to solve (32) by minimizing (75). To this end we can apply the steepest descent method.

$$\begin{aligned}(v_0, w_0) &\in [L^2(\partial\Omega_1 \cap \partial\Omega_2)]^2 \\ (v_{n+1}, w_{n+1}) &= (v_n, w_n) - \rho_n \nabla j(v_n, w_n), \quad \rho_n > 0\end{aligned}\quad (79)$$

It is not difficult to compute ∇j :

$$\nabla j(v_n, w_n) = \left(\left(\frac{\partial p_2}{\partial \nu} - \frac{\partial p_1}{\partial \nu} \right) |_{\partial\Omega_1 \cap \partial\Omega_2}, (p_2 - p_1) |_{\partial\Omega_1 \cap \partial\Omega_2} \right) \quad (80)$$

where p_1 and p_2 are defined by

$$\begin{aligned} \Delta p_1 &= 0 & \text{in } \Omega_1 \\ p_1 &= \frac{\partial z_1}{\partial \nu} - w_n & \text{in } \partial\Omega_1 \cap \partial\Omega_2 \\ p_1 &= 0 & \text{in } \{y = a\} \\ (p_1)_x &= 0 & \text{in } \{x = \pm a\} \end{aligned} \tag{81}$$

$$\begin{aligned} \Delta p_2 &= 0 & \text{in } \Omega_2 \\ \frac{\partial p_2}{\partial \nu} &= z_2 - v_n & \text{in } \partial\Omega_1 \cap \partial\Omega_2 \\ p_2 &= 0 & \text{in } \{y = -a\} \\ (p_2)_x &= 0 & \text{in } \{x = \pm a\} \end{aligned} \tag{82}$$

and where z_1 and z_2 are defined by (76) and (77) with $v = v_n$ and $w = w_n$ respectively.

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