**ON CLASSICAL SOLUTIONS OF THE TWO-PHASE
STEADY-STATE STEFAN PROBLEM IN STRIPS**

By

Jose-Francisco Rodrigues

and

Boris Zaltzman

IMA Preprint Series # 813

May 1991

ON CLASSICAL SOLUTIONS OF THE TWO-PHASE STEADY-STATE STEFAN PROBLEM IN STRIPS

JOSE-FRANCISCO RODRIGUES* AND BORIS ZALTZMAN**

Abstract. We show that the free boundary of a two-phase Stefan problem of continuous casting, in two dimensions, is given by a smooth curve in both cases of finite or infinite strip. We show then the existence of a classical solution and the continuous dependence of the free boundary with respect to the length L of the strip. We also prove that the difference of the temperature and the free boundaries associated with the finite and the infinite strips, in the L^1 -norm, is of order $1/\sqrt{L}$ as $L \rightarrow \infty$.

1. Introduction. The two-dimensional model for the solidification of a material subjected to a continuous casting process, with constant velocity $\nu > 0$, leads to the study of a two-phase steady-state Stefan problem in domains of the following type

$$(1.1) \quad \Omega_L =]0, \ell[\times] - L, L[, 0 < \ell < \infty, 0 < L \leq \infty.$$

The temperature $\vartheta = \vartheta(x, z)$, with respect to a fixed system of coordinates (x, z) , satisfies in the liquid zone $\{\vartheta > 0\}$ and in the solid one $\{\vartheta < 0\}$ the elliptic equation

$$(1.2) \quad \nu b'(\vartheta) \partial_z \vartheta = \Delta \vartheta \text{ in } \Omega_L \setminus \Phi = \{\vartheta > 0\} \cup \{\vartheta < 0\},$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function, such that $b'(\theta) \geq b_* > 0$ and $b(0) = 0$.

At the liquid-solid interface Φ , consisting of points at the normalized melting temperature $\vartheta = 0$, we have the Stefan condition

$$(1.3) \quad \lambda \nu n_z = [\nabla \vartheta \cdot \vec{n}]_-^+ \text{ on } \Phi = \{\vartheta = 0\}.$$

Here $\vec{n} = (n_x, n_z)$ denotes the normal vector to Φ , $\nabla = (\partial_x, \partial_z)$ denotes the gradient, $[\cdot]_-^+$ the flux jump at the free boundary and $\lambda > 0$ is, essentially, the latent heat.

At the boundary of the domain, we prescribe

$$(1.4) \quad \partial_x \vartheta = 0 \text{ at } x = 0 \text{ and } \partial_x \vartheta = f(z, \vartheta) \text{ at } x = \ell,$$

representing, respectively, a symmetry condition and a lateral flux on $\Gamma_L = \{\ell\} \times] - L, L[$. finally, for the finite and for the infinite strips we shall assume, respectively

$$(1.5) \quad \vartheta = \tau \text{ at } z = -L \text{ and } \vartheta = \mu \text{ at } z = L, \text{ or}$$

$$(1.6) \quad \vartheta = \tau \text{ as } z \rightarrow -\infty \text{ and } \vartheta = \mu \text{ as } z \rightarrow +\infty,$$

*CMAF/INIC and University of Lisbon, Av. Prof. Gama Pinto, 2, 1699 LISBOA CODEX, PORTUGAL.

**ASCU-Blaustein Inst. for Desert Research, Ben-Gurion University of the Negev, Sede Boqer Campus, 84993 ISRAEL.

for some prescribed constants μ, τ :

$$(1.7) \quad \mu < 0 < \tau.$$

For finite strips, introducing the space $V_0 = \{v \in H^1(\Omega_L) : v = 0 \text{ on } z = \pm L\}$ and the maximal monotone graph $\beta(\theta)$, defined by $\beta(\theta) = b(\theta)$ for $\theta < 0$, $\beta(0) = [0, \lambda]$ and $\beta(\theta) = b(\theta) + \lambda$ for $\theta > 0$, the weak formulation of the problem (1.2) – (1.5) consists in finding a pair $(\vartheta, \eta) \in H^1(\Omega_L) \cap L^2(\Omega_L)$

$$(1.8) \quad \eta \in \beta(\vartheta) \text{ a.e. in } \Omega_L, \vartheta \text{ satisfies (1.5) and}$$

$$(1.9) \quad \int_{\Omega_L} \{\nabla \vartheta \cdot \nabla v - \nu \eta \partial_z v\} dx dz = \int_{\Gamma_L} f(\vartheta) v dz, \quad \forall v \in V_0$$

Assuming $f = f(z, \theta) : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is a Caratheodory function, i.e. , continuous in θ for a.e. z and measurable in z for every θ , such that,

$$(1.10)$$

$f(\theta) = f(z, \theta)$ is monotone non-increasing and locally Lipschitz-continuous in θ , for a.e. z

$$(1.11)$$

$f(z, \theta) \leq 0$ for $\theta \geq \tau$, $f(z, \theta) \geq 0$ for $\theta \leq \mu$

$$(1.12)$$

$\sup_{\mu < \theta < \tau} |f(z, \theta)| \in L^p_{\text{loc}}(\mathbf{R})$ for some $p > 1$,

the following result is contained, as a special case, in [RY]:

PROPOSITION 1. *For each $L < +\infty$, there exists a unique weak solution $(\vartheta, \eta) \in [H^1(\Omega_L) \cap C^{0,\alpha}(\overline{\Omega}_L)] \times L^\infty(\Omega_L)$, for some $0 < \alpha < 1$, to the problem (1.8)(1.9), which satisfies*

$$(1.13) \quad \mu \leq \vartheta(x, z) \leq \tau, (x, z) \in \Omega_L.$$

Under certain restrictive conditions on the lateral flux, it was shown in [R2], that the free boundary may be given by a Lipschitz curve. This result has been previously established in [Z1] for the case of an infinite strip with a particular linear flux. The aim of this paper is two fold. In Section 2, we improve and extend the previous results of [R2] for finite strips on the Lipschitz continuity of the free boundary. When $b(\theta)$ is piecewise linear, the theory of regularity developed in [C1,2] and [KNS] applies and the free boundary is a $C^{1,\alpha}$ -curve and even analytic at nondegenerating points. As a consequence, we show that the weak solution is in fact the a.e.-classical one. In Section 3, we use a stability estimate of [RY] in order to show the continuous dependence of the free boundary with respect to the length L of the strip. For the infinite strip, in Section 4 we extend the results of [Z1] to the

case of a nonlinear flux condition. We use a natural procedure by letting $L \rightarrow +\infty$, after obtaining, in Section 3, the appropriate “a priori” estimates. We obtain an error of order $1/\sqrt{L}$ for the difference in the L^1 -norm of the temperatures and of the free boundaries associated with the finite and the infinite strips, respectively.

We note that a free boundary problem arising in fluid mechanics and studied in [AD] has some mathematical analogies in what concerns the condition (1.3). However the different boundary conditions of [AD] lead to another mathematical problem.

For more information on the Stefan problem and its variants we refer to the books [F] and [M] and to the survey papers [D] and [R1].

2. Classical Solutions in Finite Strips. In this section we give sufficient conditions on f such that the weak solution of (1.8) (1.9) is in fact a classical solution in the following sense.

DEFINITION. A solution ϑ is said to be a.e.-classical (resp. classical) if the free boundary $\Phi = \{\vartheta = 0\}$ is given by a Lipschitz-continuous curve (resp. C^1 -curve), and ϑ verifies (1.2), (1.3) and (1.4) in almost every point, with respect to their respective Lebesgue measures (resp. everywhere).

We start by exploiting a natural monotonicity property of the solution under the assumption

$$(2.1) \quad f = f(z, \theta) \text{ is monotone non-increasing in } z, \text{ for each } \theta \in]\mu, \tau[.$$

PROPOSITION 2. *If (2.1) holds, then we have*

$$(2.2) \quad \partial_z \vartheta \leq 0 \text{ in } \Omega_L.$$

Proof. As in [R2], we apply the weak maximum principle to $w = \partial_z \vartheta_\varepsilon$, where ϑ_ε is an approximating solution of a regularized problem with β and f replaced by smooth β_ε and f_ε , respectively. In fact $\partial_z \vartheta_\varepsilon$ solves the elliptic equation

$$(2.3) \quad \partial_{xx}^2 w + \partial_z [\partial_z w - \nu \beta'_\varepsilon(\vartheta_\varepsilon) w] = 0 \text{ in } \Omega_L$$

with the following boundary conditions

$$\begin{aligned} w &\leq 0 \text{ at } z = \pm L, \partial_x w = 0 \text{ at } x = 0 \text{ and} \\ \partial_x w - (\partial_\theta f_\varepsilon) w &= \partial_z f_\varepsilon \leq 0 \text{ at } x = \ell, \end{aligned}$$

and, by the maximum principle, (2.2) follows for ϑ_ε and hence for ϑ . $\square$

By assuming the restrictive sign condition $f \leq 0$ (resp. $f \geq 0$) it is easy to see that we can also conclude, as in [R2], that $\partial_x \vartheta \leq 0$ (resp. $\partial_x \vartheta \geq 0$), which is sufficient to obtain

the Lipschitz continuity of the free boundary. In order to allow simultaneously heating or cooling on the lateral boundary, we introduce the following assumptions on the variation of f , for some fixed small $\delta_0 > 0$:

$$(2.4) \quad \partial_z f \leq -\zeta < 0 \text{ in } F_0 = \{(\theta, z) : |\theta| < \delta_0 \text{ and } f(0, z) > 0\}$$

$$(2.5) \quad |\partial_z f| \leq C \text{ for } \theta \in]-\delta_0, \delta_0[, z \in \mathbb{R}.$$

PROPOSITION 3. *If (2.1), (2.4) and (2.5) hold, then there is a constant $\omega_0 = \omega(\delta_0) > 0$ such that*

$$(2.6) \quad \partial_z \vartheta + \omega_0 \partial_x \vartheta \leq 0 \text{ in } \Lambda_0 \equiv \{(x, z) : |\vartheta(x, z)| < \delta_0\}.$$

Proof. Due to (2.2), we may take z_+ and z_- , such that $\vartheta(\ell, z_+) = \delta$, $\vartheta(\ell, z_-) = -\delta$ and $|\vartheta(\ell, z)| > \delta$ for $z \notin]z_+, z_-[$, where $\delta > 0$ is an arbitrary number smaller than δ_0 . Since $\vartheta \in C^{0,\alpha}(\overline{\Omega})$, we can find smooth curves $S_{\pm} : z = s_{\pm}(x)$, such that, $s_{\pm}(\ell) = z_{\pm}$, $-L < s_-(x) < s_+(x) < L$ and

$$\vartheta(x, s_+(x)) \geq \delta, \vartheta(x, s_-(x)) \leq -\delta.$$

We may consider the neighbourhoods of S_+ and S_- defined by $N_+ = \{(x, z) : \delta/2 < \vartheta(x, z) < 3\delta/2\}$ and $N_- = \{(x, t) : -3\delta/2 < \vartheta(x, z) < \delta/2\}$.

On the other hand, since ϑ satisfies the equation

$$(2.7) \quad \nu \partial_z [b(\vartheta)] = \Delta \vartheta \text{ in } \Omega \setminus \{\vartheta = 0\},$$

using (2.5) and elliptic local estimates, we have $\vartheta \in C^{1,\alpha}(\overline{N}_{\pm})$, and we may take ϑ_{ε} converging to ϑ in $C^{1,\alpha}$ on the compact subsets $\overline{N}_+$ and $\overline{N}_-$.

Our aim is to apply the weak maximum principle to the function $u_0 = \partial_z \vartheta_{\varepsilon} + \omega_0 \partial_x \vartheta_{\varepsilon}$ in the subset $\mathcal{O} = \{(x, z) : 0 < x < \ell, s_+(x) < z < s_-(x)\}$ for some $\omega_0 > 0$. As in Proposition 1, it is enough to verify $u_0 \leq 0$ on $\partial \mathcal{O}$.

On $\partial \mathcal{O} \cap \{x = 0\}$, we have $\partial_x \vartheta_{\varepsilon} = 0$ and $\partial_z \vartheta_{\varepsilon} \leq 0$. On $\overline{S}_+ \cup \overline{S}_-$, from the strong maximum principle, $\partial_z \vartheta < -2\eta < 0$ so that, for all $\varepsilon < \varepsilon(\eta)$, we have $\partial_z \vartheta_{\varepsilon} \leq -\eta < 0$ on $\overline{S}_+ \cup \overline{S}_-$. Since $\partial_x \vartheta_{\varepsilon}$ is bounded in $C^{1+\alpha}(\overline{N}_{\pm})$, we may find $\omega_{\pm} > 0$, such that,

$$\partial_z \vartheta_{\varepsilon} + \omega_{\pm} \partial_x \vartheta_{\varepsilon} \leq 0 \text{ on } S_+ \cup S_-.$$

On $\partial \mathcal{O} \cap \{x = \ell\}$, we may suppose $|\vartheta_{\varepsilon}| < \delta_0$ and take approximations f_{ε} of f verifying also (2.4) and we need only to control the case where $\partial_x \vartheta_{\varepsilon}(\ell, z) = f_{\varepsilon}(\vartheta_{\varepsilon}(z, \ell), z) > 0$ for $z \in]z_+, z_-[$. Suppose, that at one of those points the function $u = \partial_z \vartheta_{\varepsilon} + \omega \partial_x \vartheta_{\varepsilon}$, for $\omega > 0$, have a maximum, say at a point (ℓ, z_0) , $z_0 \in [z_+, z_-]$. Then

$$0 = \partial_z u = \partial_{zz}^2 \vartheta_{\varepsilon} + \omega \partial_{xz}^2 \vartheta_{\varepsilon} = \nu \partial_z [\beta_{\varepsilon}(\vartheta_{\varepsilon})] - \partial_{xx}^2 \vartheta_{\varepsilon} + \omega \partial_{xz}^2 \vartheta_{\varepsilon} \text{ at } (\ell, z_0)$$

On the other hand, at that point (ℓ, z_0) ,

$$0 < \partial_x u = \partial_{xz}^2 \vartheta_\varepsilon + \omega \partial_{xx}^2 \vartheta_\varepsilon = (\omega^2 + 1) \partial_{xz}^2 \vartheta_\varepsilon + \nu \omega \partial_z [\beta_\varepsilon(\vartheta_\varepsilon)].$$

Since $\partial_z [\beta_\varepsilon(\vartheta_\varepsilon)] \leq 0$ and using the assumption (2.4), at (ℓ, z_0) , we have

$$0 < \partial_{xz}^2 \vartheta_\varepsilon = \partial_z f_\varepsilon + \partial_\theta f_\varepsilon \partial_z \vartheta_\varepsilon \leq -\zeta - C_0 \partial_z \vartheta_\varepsilon,$$

where C_0 is the Lipschitz constant of f in F_0 (i.e. $\partial_\theta f_\varepsilon \geq -C_0$). Because $|\partial_x \vartheta_\varepsilon| \leq C_*$ on $\partial\mathcal{O} \cap \{x = \ell\}$, we conclude also

$$\partial_z \vartheta_\varepsilon + \omega \partial_x \vartheta_\varepsilon \leq 0 \text{ on } \partial\mathcal{O} \cap \{x = \ell\} \text{ for } \omega \leq \zeta C_*/C_0.$$

Consequently, we can choose $\omega_0 = \inf(\omega_\pm, \zeta C_*/C_0)$ and conclude $\partial_z \vartheta_\varepsilon + \omega_0 \partial_x \vartheta_\varepsilon \leq 0$ in $\{|\vartheta_\varepsilon| < \delta\}$. Letting $\varepsilon \rightarrow 0$ and since $\delta < \delta_0$ is arbitrary, we conclude (2.6). $\square$

Remark 1. The assumption (2.5) is only a mild regularity condition, while (2.4) is a restriction on the flux, near the solidification temperature only if it is a heating flux. For instance, in the linear case

$$(2.8) \quad f(\theta, z) = \gamma[f_0(z) - \theta], \quad \gamma > 0,$$

the condition (2.5) is empty if $f_0(z) \leq -\delta_0 < 0$. More generally (2.4) is, of course, not necessary if the set $F_0 = \emptyset$ for a particular solution ϑ , which can be controlled, for instance, by estimating the position of the free boundary.

Remark 2. From the proof of the Proposition 3, the constant ω_0 appearing in (2.6), depends essentially on the L^∞ -bound of $\partial_x \vartheta$ on $S_+ \cup S_- \cup (\partial\mathcal{O} \cap \{x = \ell\})$ and the behaviour of $\partial_z \vartheta$ on $\bar{S}_+ \cup \bar{S}_-$, besides the assumptions (2.4), (2.5) and (1.10)–(1.12) on f .

Following the argument of [Z1,2], from (2.2) and (2.6) and recalling $\vartheta \in C^{1,\alpha}(\Omega \setminus \{\vartheta = 0\})$ it is easy to see that each level line

$$(2.9) \quad \Phi_\delta = \{(x, z) \in \Omega : \vartheta(x, z) = \delta\}, \quad \text{for any } \delta \in]-\delta_0, \delta_0[, \delta \neq 0,$$

is given by a differentiable curve. It follows that in the new system of coordinates $\hat{x} = x/\omega_0$ and $\hat{z} = z - 2x/\omega_0$, Φ_δ admits the representation

$$(2.10) \quad \hat{x} = \varphi_\delta(\hat{z}), \quad \text{with } |\varphi'_\delta| \leq 1.$$

By the monotonicity properties of ϑ , it is immediate that there are unique Lipschitz curves Φ_+ and Φ_- , such that

$$\Phi_\delta \rightarrow \Phi_+ \text{ as } \delta \downarrow 0 \text{ and } \Phi_\delta \rightarrow \Phi_- \text{ as } \delta \uparrow 0,$$

and with representations similar to (2.10) verifying $\varphi_+ \leq \varphi_-$. We cannot have $\varphi_+ < \varphi_-$ anywhere, since otherwise, it would be some open disc D centered, say, at a point of Φ_+ , where $\Delta \vartheta \leq 0$ and $\vartheta \geq 0$ on ∂D ; but then, by the strong minimum principle, it would be $\vartheta > 0$ in D , which is impossible. Therefore we conclude $\varphi_+ = \varphi_- = \varphi$ and we have the following free boundary representation:

$$(2.11) \quad \Phi \equiv \{\vartheta = 0\} = \{(\hat{x}, \hat{z}) \in \Omega_L : \hat{x} = \varphi(\hat{z})\}, \text{ with } |\varphi'| \leq 1 \text{ a.e.}$$

We can now state the main result of this section.

THEOREM 1. *Under the preceding assumptions, namely (2.1), (2.4) and (2.5) the free boundary corresponding to the Stefan problem (1.8) (1.9) is a Lipschitz curve verifying (2.11) in suitable coordinates and ϑ is actually the a.e.-classical solution.*

Proof. It remains to show that ϑ is a.e.-classical. Since the free boundary regularity implies that each open subset $\Omega_+ = \{\vartheta > 0\}$ and $\Omega_- = \{\vartheta < 0\}$ have Lipschitz boundary, we can apply integration by parts formula to (2.7). Notice that since $b \in C^{0,1}$, we have $b(\vartheta) \in H^1(\Omega_L) \cap C^{0,\alpha}(\overline{\Omega}_L)$ and $\Delta\vartheta \in L^2(\Omega_\pm)$. Then

$$(2.12) \quad \int_{\Omega_\pm} \{\nabla\vartheta \cdot \nabla v + v\Delta\vartheta\} = \int_{\partial\Omega_\pm} v\partial_{n_\pm}\vartheta, \quad \forall v \in H^1(\Omega)$$

where the right hand side integral is interpreted as the usual duality pairing between $H^{1/2}(\partial\Omega_\pm)$ and $H^{-1/2}(\partial\Omega_\pm)$ and $\partial_{n_\pm}\vartheta$ represents the outer normal derivative in this later space.

Since equations (1.2) and the conditions (1.4) follow from (1.9) in a standard way, we need only to verify (1.3) a.e. on the Lipschitz curve Φ , where each n^+ and n^- is defined a.e.

Remarking that, if χ_{Ω_+} denotes the characteristic function of the set Ω_+ , we have

$$(2.13) \quad \eta = b(\vartheta) + \lambda\chi_{\Omega_+} \text{ a.e. in } \Omega_L.$$

From (1.9) and (2.12), using (1.2) and (1.4), we find

$$-\nu\lambda \int_{\Omega_+} \partial_z v = \nu\lambda \int_{\Phi} v n_z = \int_{\Phi} \nu[\nabla\vartheta \cdot \vec{n}]_+^+, \quad \forall v \in V_0,$$

where $[\nabla\vartheta \cdot \vec{n}]_+^+ = \partial_{n_+}\vartheta + \partial_{n_-}\vartheta \in H^{-1/2}(\Phi)$. Since v is arbitrary, we conclude (1.3) first in the $H^{-1/2}(\Phi)$ sense, and since $n_z \in L^\infty(\Phi)$, also almost everywhere in Φ , afterwards. $\square$

Remark 3. Multiplying (1.2) by $\partial_z\vartheta$ and by $\partial_x\vartheta$ and integrating by parts in $\Omega_{+\delta} = \{\vartheta > \delta\}$ and in $\Omega_{-\delta} = \{\vartheta < -\delta\}$, as in [Z2], we can obtain the estimates $\int_{\Phi_\delta} |\nabla\vartheta|^2 \leq C$ and $\int_{\Phi_{-\delta}} |\nabla\vartheta|^2 \leq C$, respectively. By letting $\delta \rightarrow 0$, it can be shown that each one $\partial_{n_+}\vartheta$ and $\partial_{n_-}\vartheta$ is in $L^2(\Phi)$. $\square$

Further regularity of the free boundary can be obtained if the nonlinear function $b = b(\theta)$ is smoother than merely Lipschitz continuous for $\theta \neq 0$.

THEOREM 2. *In addition to the assumptions of Theorem 1, suppose*

$$(2.14) \quad b(\theta) = b_+\theta \text{ if } \theta > 0 \text{ and } b(\theta) = b_-\theta \text{ if } \theta < 0,$$

for some constants $b_+, b_- > 0$. Then in Ω_L the free boundary Φ is a $C^{1,\alpha}$ -curve and ϑ is the classical solution. Moreover Φ is analytic in every point where $\partial_z \vartheta^+ < 0$.

Proof. For the $C^{1,\alpha}$ smoothness it is enough to apply the theory of Caffarelli [C1]. We remark that our free boundary condition (1.3) is of the form $\partial_{n^+} \vartheta = G(\partial_n \vartheta, n^+)$ in the sense and under the assumptions of Section 7 of [C1]. However in [C1] the results are shown only for the Laplace equation. Nevertheless, a simple adaptation of the monotonicity formula Lemma 1 of [C2] shows that the theory is still valid for the equation (1.2), written in the divergence form

$$\nabla \cdot (e^{-\nu b_{\pm} z} \nabla \vartheta) = 0 \quad \text{in } \Omega_{\pm}.$$

Then we also have $\vartheta \in C^{1,\alpha}(\Omega_{\pm} \cup \Phi)$ and the same argument of the Theorem 1 shows that (1.3) is satisfied everywhere.

Finally the regularity theory of Kinderlehrer, Nirenberg and Spruck applies: it is a simple variant of Theorem 3.2 of [KNS], where the nondegeneracy condition associated with the free boundary condition (1.3) is now $n_z^+ \neq 0$, which is implied by the condition $\partial_z \vartheta^+ < 0$ at the free boundary. $\square$

3. Stability Estimates With Respect to the Strip Length. In order to study the variation of the temperature $\vartheta = \vartheta_L$ and of its free boundary $\Phi = \Phi_L$ with respect to the length parameter L , we recall the following result of [RY].

PROPOSITION 4. *Let (ϑ, η) and $(\hat{\vartheta}, \hat{\eta})$ denote the solutions of (1.8) (1.9), under the assumptions of the Proposition 1, corresponding to different Dirichlet data τ, μ and $\hat{\tau}, \hat{\mu}$, respectively. Then, for any $p, 1 \leq p < \infty$, there exists a constant $K_p > 0$, independent of $L \geq L_* > 0$, such that,*

$$(3.1) \quad b_* \int_{\Omega_L} |\vartheta - \hat{\vartheta}|^p \leq \int_{\Omega_L} |\eta - \hat{\eta}|^p \leq K_p \left(\int_{S_{-L}} |\tau - \hat{\tau}| + \int_{S_L} |\mu - \hat{\mu}| \right)$$

where $S_{\pm L} =]0, \ell[\times \{\pm L\}$.

Proof. As in [RY], integrating by parts, we obtain for any function $\psi \in V$, such that $\Delta \psi \in L^2(\Omega_L)$, the identity

$$(3.2) \quad - \int_{\Omega_L} (\eta - \hat{\eta})(\nu \partial_z \psi + \alpha \Delta \psi) + \int_{\Gamma_L} (\vartheta - \hat{\vartheta})(\partial_x \psi + \gamma \psi) = \int_{S_{-L}} (\tau - \hat{\tau}) \partial_z \psi - \int_{S_L} (\mu - \hat{\mu}) \partial_z \psi$$

where the nonnegative functions $\alpha = \alpha(x, z)$ and $\gamma = \gamma(z)$ are defined, respectively in Ω_L and Γ_L , by $\alpha = (\vartheta - \hat{\vartheta})/(\eta - \hat{\eta})$ if $\vartheta \neq \hat{\vartheta}$, $\alpha = 1/b'(\vartheta)$ if $\vartheta = \hat{\vartheta} \neq 0$, $\alpha = 0$ if $\vartheta = \hat{\vartheta} = 0$; $\gamma = [f(\vartheta) - f(\hat{\vartheta})]/(\hat{\vartheta} - \vartheta)$ if $\vartheta \neq \hat{\vartheta}$, $\gamma = 0$ if $\vartheta = \hat{\vartheta}$.

Choosing appropriate regularizing functions $\alpha_n \rightarrow \alpha$ and $\gamma_n \rightarrow \gamma$ (see [RY], for details) we take for ψ the unique solution of the elliptic problem for some $\xi \in \mathcal{D}(\Omega_L)$

$$\begin{cases} \nu \partial_z \psi_n + \alpha_n \Delta \psi_n = \xi & \text{in } \Omega_L \\ \psi_n = 0 \text{ at } z = \pm L, \partial_n \psi_n = 0 \text{ at } x = 0 \text{ and } \partial_x \psi_n + \gamma_n \psi_n = 0 \text{ at } x = \ell. \end{cases}$$

By local elliptic estimates and using local barriers at $z = \pm L$ it can be shown there is a constant $C_\xi = C(\|\xi\|_{L^\infty(\Omega_L)}) > 0$, such that $|\partial_z \psi_n| \leq C_\xi$ on $S_{\pm L}$, independently of n and of L , since in a small neighbourhood of $S_{\pm L}$ we can take α_n uniformly positive and smooth. Using these estimates in (3.2), letting $n \rightarrow \infty$, we obtain

$$(3.3) \quad \int_{\Omega_L} (\eta - \hat{\eta}) \xi \leq C_\xi \ell \left(\int_{S_{-L}} |\tau - \hat{\tau}| + \int_{S_L} |\mu - \hat{\mu}| \right).$$

Since $\eta, \hat{\eta} \in L^\infty(\Omega_L)$ independently of L ; we can choose in (3.3) $\xi_n \in \mathcal{D}(\Omega_L)$ such that $\xi_n \rightarrow |\eta - \hat{\eta}|^{p-1} \text{sign}(\eta - \hat{\eta})$ in $L^\infty(\Omega_L)$ -weakly*, and (3.1) follows easily by remarking that the first inequality follows from the assumption $b'(\theta) \geq b_* > 0$. $\square$

Remark 4. The estimate (3.1) does not require the Dirichlet data on S_{-L} and on S_L to be constants but only strictly positive at $z = -L$ and strictly negative at $z = +L$ (see [RY]), provided they are smooth, say Lipschitz continuous.

The result can be used in the same way as in Theorem 3 of [R2], in order to show the stability of the free boundary with respect to L . In fact, if we assume the solutions ϑ_L and $\vartheta_{\hat{L}}$ a.e.-classical for two values of L , say $\hat{L} > L$, we can guarantee that $\tau > \vartheta_{\hat{L}} \geq \tau - c(\hat{L} - L) > 0$ at $z = -L$ and $\mu < \vartheta_{\hat{L}} \leq \mu + c(\hat{L} - L) < 0$ at $z = L$, provided $\hat{L}$ is sufficient close to L (since $\vartheta_{\hat{L}}$ is locally $C^{0,1}$ near $S_{\pm}$). We can also suppose that, in a suitable common coordinate system, the free boundaries Φ_L and $\Phi_{\hat{L}}$ have a Lipschitz-continuous representation in terms of φ_L and $\varphi_{\hat{L}}$ (recall (2.11)). Hence remarking that, by (2.13), we have

$$(3.4) \quad \lambda \int_I |\varphi_L - \varphi_{\hat{L}}| \leq \int_{\Omega_L} |\eta_L - \eta_{\hat{L}}| \leq K_1 \left(\int_{S_{-L}} |\tau - \vartheta_{\hat{L}}| + \int_{S_L} |\mu - \vartheta_{\hat{L}}| \right)$$

and using an interpolating argument, as in [R2], we can state the following consequence of the estimate (3.1).

THEOREM 3. *for each fixed L , under the assumptions of Theorem 2 there exists L^* , $L < L^* \leq \infty$, such that, for every $\hat{L} \in]L, L^*[$, the associated free boundaries verify the estimate, for every $\alpha \in [0, 1[$*

$$(3.5) \quad \|\varphi_L - \varphi_{\hat{L}}\|_{C^{0,\alpha}} \leq C_\alpha |L - \hat{L}|^{(1-\alpha)/2}$$

In particular, the mapping $L \mapsto \varphi_L$ is continuous, with respect to the $C^{0,\alpha}$ -topology, for every $\alpha < 1$.

In order to consider the infinite strip case by letting $L \rightarrow +\infty$, we need, in particular, to obtain an “a priori” frame for the free boundary position. Assume we can write the lateral flux in the form

$$(3.6) \quad f(z, \theta) = \gamma(z)[\Xi(z) - \theta] + \rho(z, \theta),$$

under the additional conditions

$$(3.7) \quad \gamma(t) \geq \gamma_* > 0 \quad \text{for } |z| \geq L_* > 0;$$

$$(3.8) \quad \Xi \in C^{0,1}(\mathbf{R}), \Xi(z) = \tau \text{ for } z \leq -L_*, \Xi(z) = \mu \text{ for } z \geq L_*;$$

$$(3.9) \quad \exists \rho_* \in L^2(\mathbf{R}) : \sup_{\mu < \theta < \tau} |\rho(z, \theta)| \leq \rho_*(z) \quad \text{a.e. } z \in \mathbf{R}.$$

PROPOSITION 5. *Let (2.1) and (3.6)–(3.9) hold. Then there are constants $C, C_* > 0$, independent of $L > L_*$, such that*

$$(3.10) \quad \int_{\Omega_L} |\nabla \vartheta_L|^2 + \gamma_* \int_{\Gamma_L} (\Xi - \vartheta_L)^2 \leq C$$

$$(3.11) \quad \max_{0 \leq x \leq \ell} |\vartheta_L(x, z) - \Xi(z)| \leq \frac{C_*}{\sqrt{|z| - L_*}} \text{ for } |z| > L_*.$$

Proof. Taking $v = \vartheta_L - \Xi \in V_0$ in (1.9), we obtain

$$\int_{\Omega_L} |\nabla \vartheta_L|^2 + \int_{\Omega} \gamma(\Xi - \vartheta_z)^2 = \int_{\Omega_L} \partial_z \vartheta_L \Xi' + \nu \int_{\Omega_L} \eta_L \partial_z (\vartheta_L - \Xi) + \int_{\Gamma_L} \rho(\vartheta_L)(\vartheta_L - \Xi).$$

Noting that $\text{supp } \Xi'$ is contained in $\Omega_* = \overline{\Omega}_L \cap \{|z| \leq L_*\}$, recalling the boundedness of ϑ_L and η_L , we can estimate each integral in the left hand side as follows

$$\begin{aligned} \left| \int_{\Omega_L} \partial_z \vartheta_L \Xi' \right| &\leq \frac{1}{2} \int_{\Omega_L} |\partial_z \vartheta_L|^2 + \frac{1}{2} \int_{\Omega_*} |\Xi'|^2 \\ \left| \int_{\Omega_L} \eta_L \partial_z (\vartheta_L - \Xi) \right| &= \left| \int_{S_{-L}} (B(\tau) + \lambda) - \int_{S_L} B(\mu) - \int_{\Omega_*} \eta_L \Xi' \right| \\ &\leq \ell[B(\tau) + \lambda - B(\mu)] + \|\eta_L\|_{L^\infty} \|\Xi'\|_{L^1} \left(B(s) = \int_0^s b(t) dt \right) \\ \left| \int_{\Gamma_L} \rho(\vartheta_L)(\vartheta_L - \Xi) \right| &\leq \int_{\Gamma_L} \rho_* |\vartheta_L - \Xi| \leq \frac{1}{2\gamma_*} \int_{\mathbf{R}} \rho_*^2 + \frac{\gamma_*}{2} \int_{\Gamma_L} (\vartheta_L - \Xi)^2 \end{aligned}$$

Then using (3.7), the estimate (3.10) follows immediately.

For each $x \in]0, \ell[$, we have the estimate

$$(3.12) \quad \int_{-L}^L |\Xi(z) - \vartheta_L(x, z)|^2 dz \leq 2\ell \int_{-L}^L \int_0^\ell |\partial_x \vartheta_L|^2 dx dz + 2 \int_{-L}^L |\Xi(z) - \vartheta_L(\ell, z)|^2 dz \\ \leq 2\ell c + 2c/\gamma_* \equiv c_*^2 \text{ (by (3.10))}.$$

By Proposition 2, ϑ_L is non-increasing in z , so that, for instance, for $z < -L_*$, we have

$$(L_* - z)(\tau - \vartheta_L(x, z))^2 \leq \int_z^{-L_*} |\Xi(z') - \vartheta_L(x, z')|^2 dz' \leq C_*^2, \quad \forall x \in]0, \ell[.$$

A similar argument for $z > L_*$, yields

$$(\vartheta_L(x, z) - \mu)^2 \leq C_*^2/(z - L_*),$$

and the conclusion (3.11) follows immediately. $\square$

We note that, since $\tau \geq \vartheta \geq \mu$ in Ω_L , the estimate (3.11) is only interesting for large $|z|$. It can be used to estimate the region where the free boundary is located, since (3.11) implies that

$$(3.13) \quad \vartheta_L < 0 \quad \text{for } z > L_* + c_*^2/\mu^2 \equiv M^*, \text{ and}$$

$$(3.14) \quad \vartheta_L > 0 \quad \text{for } z < -L_* - c_*^2/\tau^2 \equiv M_*.$$

Therefore we have proved the following interesting result.

PROPOSITION 6. *Under the conditions of Proposition 5, we have*

$$(3.15) \quad \Phi_L = \{\vartheta_L = 0\} \subset [0, \ell] \times]M_*, M^*[, \text{ for all } L > M \equiv \max(M^*, -M_*),$$

where M^* and M_* are given by (3.13) and (3.14) respectively.

4. The Case of an Infinite Strip. In this section we consider the infinite strip Ω_∞ and we want to extend the previous results to the corresponding Stefan problem (1.2)–(1.4) and (1.6). Keeping the preceding notations, in particular, with a fixed Ξ as in (3.8), we say that (ϑ, η) is a *weak solution* in Ω_∞ , if

$$(4.1) \quad \vartheta - \Xi \in H^1(\Omega_\infty) \quad \text{and} \quad \int_0^\ell |\vartheta(x, z) - \Xi(z)| dx \rightarrow 0 \text{ as } |z| \rightarrow \infty$$

$$(4.2) \quad \eta \in L_{\text{loc}}^2(\Omega_\infty) \quad \text{and} \quad \eta \in \beta(\vartheta) \text{ a.e. in } \Omega_\infty$$

$$(4.3) \quad \int_{\Omega_\infty} \{\nabla \vartheta \cdot \nabla v - \nu \eta \partial_z v\} dx dz = \int_{\Gamma_\infty} f(\vartheta) v dz, \quad \forall v \in H^1(\Omega_\infty), \text{ with bounded support .}$$

Analogously to the definition in bounded strips, we say that ϑ is an *a.e.-classical solution* in Ω_∞ , if its free boundary is given by a Lipschitz-continuous bounded curve, ϑ verifies (1.2), (1.3) and (1.4) in the almost everywhere sense and (1.6) in the following sense

$$(4.4) \quad \sup_{0 < x < \ell} |\vartheta(x, z) - \Xi(z)| \rightarrow 0 \text{ as } |z| \rightarrow \infty.$$

THEOREM 4. *Let (1.7), (1.10), (1.11), (1.12), (2.1) and (3.6)–(3.9) hold. Then there exists a unique weak solution ϑ to (4.1)–(4.3). Moreover $\vartheta \in C^{0,\alpha}(\overline{\Omega}_\infty)$, for some $0 < \alpha < 1$, it satisfies (1.13) and*

$$(4.5) \quad \max_{0 \leq x \leq \rho} |\vartheta(x, z) - \Xi(z)| = 0(1/\sqrt{z}) \text{ as } |z| \rightarrow \infty.$$

Proof. First we prove uniqueness. Assume (ϑ, η) and $(\hat{\vartheta}, \hat{\eta})$ are two weak solutions in the infinite strip. For an arbitrary $\varepsilon > 0$, by (4.1) there exists $L_\varepsilon < \infty$, such that

$$\int_0^\ell |\vartheta(x, z) - \hat{\vartheta}(x, z)| dx < \varepsilon, \quad \text{for } |z| \geq L_\varepsilon.$$

Then considering ϑ and $\hat{\vartheta}$ as weak solutions of the problem in the finite strip Ω_{L_ε} , with Dirichlet data given by their respective traces at $z = \pm L_\varepsilon$, from (3.1), we obtain for any $L \leq L_\varepsilon$:

$$b_* \int_{\Omega_L} |\vartheta - \hat{\vartheta}| \leq \int_{\Omega_L} |\eta - \hat{\eta}| \leq 2K_1 \varepsilon;$$

which, by the arbitrariness of ε , yields $\vartheta = \hat{\vartheta}$ and $\eta = \hat{\eta}$.

To prove the existence part we consider a sequence of approximating solutions ϑ_L of the Stefan problem (1.8) (1.9) in the finite strip Ω_L and we take $L \rightarrow +\infty$. In fact, defining $\tilde{\theta}_L$ and $\tilde{\eta}_L$ as the natural extensions to Ω_∞ of each ϑ_L and η_L , respectively,

$$\tilde{\vartheta}_L = \vartheta_L \text{ and } \tilde{\eta}_L = \eta_L \text{ for } |z| \leq L, \tilde{\vartheta}_L = \Xi \text{ and } \tilde{\eta}_L = \beta(\Xi) \text{ for } |z| \geq L,$$

it is clear that (3.8), (3.10) and (3.12) imply the “a priori” estimate

$$(4.6) \quad \|\tilde{\vartheta}_L - \Xi\|_{H^1(\Omega_\infty)} \leq C, C \text{ independent of } L.$$

Hence, there exists a pair $(\theta_\infty, \eta_\infty)$, such that $\vartheta_\infty - \Xi \in H^1(\Omega_\infty)$, $\eta_\infty \in L^\infty(\Omega_\infty)$, and a subsequence $L \rightarrow \infty$, satisfying $\tilde{\eta}_L \rightharpoonup \eta_\infty$ in $L^\infty(\Omega_\infty)$ -weak* and $\tilde{\vartheta}_L \rightarrow \vartheta_\infty$ in $H^1(\Omega_M)$ -weak and in $C^0(\overline{\Omega}_M)$ for each $M < \infty$. Notice that, in each $\overline{\Omega}_M$, the $\tilde{\vartheta}_L$ also belong to a bounded set of $C^{0,\alpha}(\overline{\Omega}_M)$.

From (3.11), we have that ϑ_∞ satisfies (4.5) and, consequently also (4.1). Since, by monotonicity of β , it is easily seen that (4.2) holds for η_∞ , if $(\vartheta_\infty, \eta_\infty)$ satisfies (4.3), by uniqueness, we have that $\vartheta_\infty = \theta$ and $\eta_\infty = \eta$ are in fact the unique solution to the problem in the infinite strip.

To verify (4.3), for each $v \in H^1(\Omega_\infty)$, with $\text{supp } v \subset \overline{\Omega}_M$ for some $M < \infty$, we may take the limit as $L \rightarrow \infty$ in

$$\int_{\Omega_\infty} \{\nabla \tilde{\theta}_L \cdot \nabla v - \nu \tilde{\eta}_L \partial_z v\} dx dz = \int_{\Gamma} f(\tilde{\vartheta}_L) v dz,$$

which holds for every $\tilde{\vartheta}_L$ such that $L > M$. $\square$

As a consequence of Proposition 4, taking in (3.1) $\vartheta = \vartheta_L$, for L sufficiently large, and $\hat{\vartheta} = \vartheta_\infty$ considered as a solution of the problem in the finite strip Ω_L , taking Dirichlet values $\vartheta_\infty = \hat{\tau}$ at $z = -L$ and $\vartheta_\infty = \hat{\mu}$ at $z = L$, we immediately obtain the following result.

PROPOSITION 7. *Under the assumptions of Theorem 3, for all $1 \leq p < \infty$:*

$$(4.7) \quad \|\vartheta_L - \vartheta_\infty\|_{L^p(\Omega_L)} = o(L^{-1/2p}) \text{ and } \|\eta_L - \eta_\infty\|_{L^p(\Omega_L)} = o(L^{-1/2p}) \text{ as } L \rightarrow \infty.$$

Considering our approximation procedure, we can use the Proposition 6 to conclude that the free boundary Φ_∞ for the infinite strip case is also such that

$$(4.8) \quad \Phi_\infty = \{\theta_\infty = 0\} \subset [0, \ell] \times]M_*, M^*[.$$

Similarly, we can use this type of information to impose the additional assumptions (2.4) and (2.5) localized only for those z , such that, $M_* - 1 \leq z \leq M^* + 1$, for instance. We have the following result.

THEOREM 5. *Let the assumptions of Theorem 1 and 3 hold. Then the free boundary corresponding to the infinite strip case is also given by a Lipschitz curve of the form $\Phi_\infty : \hat{x} = \varphi_\infty(\hat{z})$, in some coordinate system $(\hat{x}, \hat{z})$, and ϑ_∞ is the a.e.-classical solution. Moreover, as $L \rightarrow \infty$, we have*

$$(4.9) \quad \|\varphi_L - \varphi_\infty\|_{L^1} = o(L^{-1/2}) \text{ and } \|\varphi_L - \varphi_\infty\|_{C^{0,\alpha}} = o(L^{(\alpha-1)/4}), 0 \leq \alpha < 1.$$

Proof. Owing to (3.11) we can guarantee that the subsets

$$\Lambda_{0L} \equiv \{(x, t) \in \overline{\Omega}_\infty : |\vartheta_L(x, z)| < \delta_0\} \subset \overline{\Omega}_M,$$

for some fixed M and for every L sufficiently large. Hence we can find suitable neighbourhoods N_+ and N_- of the level sets $\{\vartheta_\infty = \delta_0\}$ and $\{\vartheta_\infty = -\delta_0\}$, where $\vartheta_L \rightarrow \vartheta_\infty$ in $C^{1,\alpha}$,

by local elliptic estimates. Then repeating the argument of the Proposition 3, we can show the existence of a constant $\omega_0 > 0$, independent of L , for all L sufficiently large, such that

$$\partial_z \vartheta_L + \omega_0 \partial_x \vartheta_L \leq 0 \text{ in } \Lambda_{0\infty} = \{|\vartheta_\infty| < \delta_0\}.$$

This inequality yields a uniform bound on the Lipschitz curves Φ_L and imply that $\Phi_L \rightarrow \Phi_\infty$ as $L \rightarrow \infty$, where Φ_∞ is also a Lipschitz curve. It follows that ϑ_∞ is a a.e.-classical solution, exactly as in the Theorem 1.

Finally, the conclusion (4.9) is proved as in the Theorem 2 (see also [R2]). $\square$

COROLLARY. *Under the assumption (2.1) the same conclusions of Theorem 2 also hold in the case of an infinite strip.*

Acknowledgements. This work was initiated at the CMAF/INIC (University of Lisbon) during a visit of the second author in the framework of the research project n.87046/JNICT. The first author also thanks the IMA (University of Minnesota) for providing the good environment which allowed the completion of this paper. Special acknowledgements are due to L. Wang and J. Spruck for helpful discussions on the applicability of the theory of [C1,2] and [KNS].

REFERENCES

- [AD] H.W. ALT AND C.J. VAN DUIJN, *A stationary flow in fresh and salt groundwater in a coastal aquifer*, Nonlinear Analysis Th. Math. Appl. 14 (1990), pp. 625–656.
- [C1] L. CAFFARELLI, *A Harnack Inequality Approach to the Regularity of Free Boundaries, Part I: Lipschitz Free Boundaries are $C^{1,\alpha}$* , Rev. Mat. Iberoamericana 3 (1987), pp. 139–162.
- [C2] L. CAFFARELLI, *A Harnack Inequality Approach to the Regularity of Free Boundaries, Part III: Existence Theory, Compactness, and Dependence on X* , Ann. S Norm. Sup. Disc (IV) 15 (1988), pp. 583–602.
- [D] I.I. DANILYUK, *On the Stefan problem*, Russian Math. Surveys 40:5 (1985), pp. 157–223.
- [F] A. FRIEDMAN, *Variational Principles and Free-Boundary Problems*, J. Wiley, New York (1982).
- [KNS] D. KINDERLEHRER, L. NIRENBERG AND J. SPRUCK, *Regularity in Elliptic Free Boundary Problems I*, J. Anal. Math. 34 (1978), pp. 86–119.
- [M] A.M. MEIRMANOV, *Stefan Problems*, (in Russian) Nauka, Novosibirsk (1986); (English translation by Walter de Gruyter, Berlin (1991)).
- [R1] J.F. RODRIGUES, *The Stefan problem revisited*, Birkhäuser Verlag, Basel (1989), pp. 129–190.
- [R2] J.F. RODRIGUES, *On a Steady-state two-phase Stefan problem with extraction*, Birkhäuser Verlag, Basel (1990), pp. 229–240.
- [RY] J.F. RODRIGUES AND F. YI, *On a two-phase continuous casting Stefan problem with nonlinear flux*, European J. Appl. Math. 1 (1990), pp. 259–279.
- [S1] B. ZALTZMAN, *Multidimensional two-phase quasistationary Stefan problem in an unbounded region*, (in Russian) Chapter 1 of Ph.D. Thesis, University of Novosibirsk, URSS (1989).
- [S2] B. ZALTZMAN, Appendix to the English Translation of [M].

#	Author/s	Title
721	Ian M. Anderson, Niky Kamran and Peter J. Olver,	Internal, external and generalized symmetries
722	C. Foias and J.C. Saut,	Asymptotic integration of Navier–Stokes equations with potential forces. I
723	Ling Ma,	The convergence of semidiscrete methods for a system of reaction-diffusion equations
724	Adelina Georgescu,	Models of asymptotic approximation
725	A. Makagon and H. Salehi,	On bounded and harmonizable solutions on infinite order arma systems
726	San-Yih Lin and Yan-Shin Chin,	An upwind finite-volume scheme with a triangular mesh for conservation laws
727	J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart,	On the dynamics of fine structure
728	KangPing Chen and Daniel D. Joseph,	Lubrication theory and long waves
729	J.L. Ericksen,	Local bifurcation theory for thermoelastic Bravais lattices
730	Mario Taboada and Yuncheng You,	Some stability results for perturbed semilinear parabolic equations
731	A.J. Lawrance,	Local and deletion influence
732	Bogdan Vernescu,	Convergence results for the homogenization of flow in fractured porous media
733	Xinfu Chen and Avner Friedman,	Mathematical modeling of semiconductor lasers
734	Yongzhi Xu,	Scattering of acoustic wave by obstacle in stratified medium
735	Songmu Zheng,	Global existence for a thermodynamically consistent model of phase field type
736	Heinrich Freistühler and E. Bruce Pitman,	A numerical study of a rotationally degenerate hyperbolic system part I: the Riemann problem
737	Epifanio G. Virga,	New variational problems in the statics of liquid crystals
738	Yoshikazu Giga and Shun'ichi Goto,	Geometric evolution of phase-boundaries
739	Ling Ma,	Large time study of finite element methods for 2D Navier–Stokes equations
740	Mitchell Luskin and Ling Ma,	Analysis of the finite element approximation of microstructure in micromagnetics
741	M. Chipot,	Numerical analysis of oscillations in nonconvex problems
742	J. Carrillo and M. Chipot,	The dam problem with leaky boundary conditions
743	Eduard Harabetian and Robert Pego,	Efficient hybrid shock capturing schemes
744	B.L.J. Braaksma,	Multisummability and Stokes multipliers of linear meromorphic differential equations
745	Tae Il Jeon and Tze-Chien Sun,	A central limit theorem for non-linear vector functionals of vector Gaussian processes
746	Chris Grant,	Solutions to evolution equations with near-equilibrium initial values
747	Mario Taboada and Yuncheng You,	Invariant manifolds for retarded semilinear wave equations
748	Peter Rejto and Mario Taboada,	Unique solvability of nonlinear Volterra equations in weighted spaces
749	Hi Jun Choe,	Holder regularity for the gradient of solutions of certain singular parabolic equations
750	Jack D. Dockery,	Existence of standing pulse solutions for an excitable activator-inhibitory system
751	Jack D. Dockery and Roger Lui,	Existence of travelling wave solutions for a bistable evolutionary ecology model
752	Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo,	Singular perturbation problems with a compact support semilinear term
753	Emad A. Fatemi,	Numerical schemes for constrained minimization problems
754	Y. Kuang and H.L. Smith,	Slowly oscillating periodic solutions of autonomous state-dependent delay equations
755	Emad A. Fatemi,	A new splitting method for scalar conservation laws with stiff source terms
756	Hi Jun Choe,	A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities
757	Haitao Fan,	A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids
758	T.A. Osborn and F.H. Molzahn,	The Wigner–Weyl transform on tori and connected graph propagator representations
759	Avner Friedman and Bei Hu,	A free boundary problem arising in superconductor modeling
760	Avner Friedman and Wenxiong Liu,	An augmented drift-diffusion model in semiconductor device
761	Avner Friedman and Miguel A. Herrero,	Extinction and positivity for a system of semilinear parabolic variational inequalities
762	David Dobson and Avner Friedman,	The time-harmonic Maxwell equations in a doubly periodic structure
763	Hi Jun Choe,	Interior behaviour of minimizers for certain functionals with nonstandard growth
764	Vincenzo M. Tortorelli and Epifanio G. Virga,	Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation
765	Nikan B. Firoozye and Robert V. Kohn,	Geometric parameters and the relaxation of multiwell energies
766	Haitao Fan and Marshall Slemrod,	The Riemann problem for systems of conservation laws of mixed type
767	Joseph D. Fehribach,	Analysis and application of a continuation method for a self-similar coupled Stefan system
768	C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi,	Dissipativity of numerical schemes
769	D.D. Joseph, T.Y.J. Liao and J.-C. Saut,	Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity

- 770 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
- 771 **B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu**, Convergence of finite volume methods
- 772 **N.G. Lloyd and J.M. Pearson**, Computing centre conditions for certain cubic systems
- 773 **João Palhoto Matos**, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
- 774 **L.A. Peletier & W.C. Troy**, Self-similar solutions for infiltration of dopant into semiconductors
- 775 **H. Scott Dumas and James A. Ellison**, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
- 776 **Stathis Filippas and Robert V. Kohn**, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
- 777 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
- 778 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximal smoothness of solutions to certain Euler-Lagrange equations from nonlinear elasticity
- 779 **Jack Carr and Robert Pego**, Self-similarity in a coarsening model in one dimension
- 780 **J.M. Greenberg**, The shock generation problem for a discrete gas with short range repulsive forces
- 781 **George R. Sell and Mario Taboada**, Local dissipativity and attractors for the Kuramoto-Sivashinsky equation in thin 2D domains
- 782 **T. Subba Rao**, Analysis of nonlinear time series (and chaos) by bispectral methods
- 783 **Nicholas Baumann, Daniel D. Joseph, Paul Mohr and Yuriko Renardy**, Vortex rings of one fluid in another free fall
- 784 **Oscar Bruno, Avner Friedman and Fernando Reitich**, Asymptotic behavior for a coalescence problem
- 785 **Johannes C.C. Nitsche**, Periodic surfaces which are extremal for energy functionals containing curvature functions
- 786 **F. Abergel and J.L. Bona**, A mathematical theory for viscous, free-surface flows over a perturbed plane
- 787 **Gunduz Caginalp and Xinfu Chen**, Phase field equations in the singular limit of sharp interface problems
- 788 **Robert P. Gilbert and Yongzhi Xu**, An inverse problem for harmonic acoustics in stratified oceans
- 789 **Roger Fosdick and Eric Volkmann**, Normality and convexity of the yield surface in nonlinear plasticity
- 790 **H.S. Brown, I.G. Kevrekidis and M.S. Jolly**, A minimal model for spatio-temporal patterns in thin film flow
- 791 **Chao-Nien Chen**, On the uniqueness of solutions of some second order differential equations
- 792 **Xinfu Chen and Avner Friedman**, The thermistor problem for conductivity which vanishes at large temperature
- 793 **Xinfu Chen and Avner Friedman**, The thermistor problem with one-zero conductivity
- 794 **E.G. Kalnins and W. Miller, Jr.**, Separation of variables for the Dirac equation in Kerr Newman space time
- 795 **E. Knobloch, M.R.E. Proctor and N.O. Weiss**, Finite-dimensional description of doubly diffusive convection
- 796 **V.V. Pukhnachov**, Mathematical model of natural convection under low gravity
- 797 **M.C. Knaap**, Existence and non-existence for quasi-linear elliptic equations with the p-laplacian involving critical Sobolev exponents
- 798 **Stathis Filippas and Wenxiong Liu**, On the blowup of multidimensional semilinear heat equations
- 799 **A.M. Meirmanov**, The Stefan problem with surface tension in the three dimensional case with spherical symmetry: non-existence of the classical solution
- 800 **Bo Guan and Joel Spruck**, Interior gradient estimates for solutions of prescribed curvature equations of of parabolic type
- 801 **Hi Jun Choe**, Regularity for solutions of nonlinear variational inequalities with gradient constraints
- 802 **Peter Shi and Yongzhi Xu**, Quasistatic linear thermoelasticity on the unit disk
- 803 **Satyanad Kichenassamy and Peter J. Olver**, Existence and non-existence of solitary wave solutions to higher order model evolution equations
- 804 **Dening Li**, Regularity of solutions for a two-phase degenerate Stefan Problem
- 805 **Marek Fila, Bernhard Kawohl and Howard A. Levine**, Quenching for quasilinear equations
- 806 **Yoshikazu Giga, Shun'ichi Goto and Hitoshi Ishii**, Global existence of weak solutions for interface equations coupled with diffusion equations
- 807 **Mark J. Friedman and Eusebius J. Doedel**, Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study
- 808 **Mark J. Friedman**, Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds
- 809 **Peter W. Bates and Songmu Zheng**, Inertial manifolds and inertial sets for the phase-field equations
- 810 **J. López Gómez, V. Márquez and N. Wolanski**, Global behavior of positive solutions to a semilinear equation with a nonlinear flux condition
- 811 **Xinfu Chen and Fahuai Yi**, Regularity of the free boundary of a continuous casting problem
- 812 **Eden, A., Foias, C., Nicolaenko, B. and Temam, R.**, Inertial sets for dissipative evolution equations Part I: Construction and applications
- 813 **Jose-Francisco Rodrigues and Boris Zaltzman**, On classical solutions of the two-phase steady-state Stefan problem in strips
- 814 **Viorel Barbu and Srdjan Stojanovic**, Controlling the free boundary of elliptic variational inequalities on a variable domain