**REGULARITY OF THE FREE BOUNDARY
OF A CONTINUOUS CASTING PROBLEM**

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REGULARITY OF THE FREE BOUNDARY OF A CONTINUOUS CASTING PROBLEM

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Abstract. We prove the existence and uniqueness of weak solutions of a steady-state, two-phase continuous casting problem under the Dirichlet boundary conditions. In particular, we establish the C^α regularity of the free boundary and, under some stronger conditions, the Lipschitz continuity of the free boundary.

Key words. Casting problem, Stefan problem, free boundary, convection.

1. Introduction. The continuous casting problem is a simplified model describing the solidification of a material being cast continuously in a cylindrical domain $\Omega \times (-\infty, \infty)$, where Ω is a Lipschitz bounded domain in $\mathbb{R}^n$ ($n \geq 1$). Denote by $u(x, z, t)$ the temperature and by $\vec{v}(x, z, t)$ the velocity of the material at position $(x, z) \in \Omega \times (-\infty, \infty)$ and time t . Then the law of the conservation of energy implies that

$$\frac{\partial}{\partial t} \beta(u) + \operatorname{div}(\beta(u) \vec{v}) - \Delta u = 0 \quad (1.1)$$

where $\beta(u)$ is the enthalpy defined by

$$\beta(u) := \begin{cases} u + l & \text{if } u > 0 \\ [0, l] & \text{if } u = 0 \\ u & \text{if } u < 0 \end{cases} \quad (1.2)$$

and $l > 0$ is the latent heat.

In this paper, we are interested in the steady-state solutions of (1.1) with a prescribed velocity $\vec{v}(x, z, t) = -\vec{e}_z$ (the unit vector in z -direction) in the cylindrical domain $Q := \Omega \times (0, 1)$. More precisely, we study the boundary value problem:

$$-\Delta u(x, z) - \frac{\partial}{\partial z} \beta(u(x, z)) = 0 \quad \text{in } Q := \Omega \times (0, 1), \quad (1.3)$$

$$u(x, z) = g(x, z) \quad \text{on } \partial Q \quad (1.4)$$

where $g \in H^1(Q)$ is a given function and ∂Q stands for the boundary of Q .

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The continuous casting problem (with a prescribed velocity $\vec{v}$) has been studied by several authors; see, for instance, Brière [1], Chipot & Rodrigues [3], Rodrigues [7, 8], Rulla [14] and the references therein. Recently, Yi [13] and Rodrigues & Yi [12] considered the time dependent problem (1.1) with $\vec{v} = \vec{e}_z$, subject to the Dirichlet boundary conditions and to the mixed boundary conditions, respectively. They established the existence and uniqueness of weak solutions, as well as the stabilities of the solutions. Their proof of the existence and uniqueness can be, with minor changes, carried out here.

In this paper, we are particularly interested in the regularity of the free boundary defined by

$$\Gamma_u := \{(x, z) \in \overline{Q} \mid u(x, z) = 0\}. \quad (1.5)$$

Towards this direction, Rodrigues [10] and Rodrigues & Zaltzman [11] have recently obtained the Lipschitz regularity of the free boundary in the two dimensional case (i.e., $\Omega \in \mathbf{R}^1$). Here, we shall establish the C^α regularity of the free boundary under the assumption that

$$g \in H^1(Q) \cap C^\alpha(\overline{Q}), \quad (1.6)$$

$$g(x, 1) = m^+ > 0, \quad g(x, 0) = -m^- < 0, \quad x \in \Omega, \quad (1.7)$$

$$\inf_{x \in \Omega, 0 \leq z_1 < z_2 \leq 1} \frac{g(x, z_1) - g(x, z_2)}{z_1 - z_2} > 0. \quad (1.8)$$

We shall also establish the Lipschitz continuity of the free boundary under some stronger conditions. By doing this, we can apply the results of Caffarelli [2] to conclude that the free boundary is $C^{1+\alpha}$ and the weak solution is $C^{1+\alpha}$ up to the free boundary, therefore the weak solution is a classical solution.

All our results hold for any dimension.

The key method used here is the construction of sub- and supersolutions.

In §2 we shall establish the existence and uniqueness of weak solutions of (1.3), (1.4), as well as comparison lemmas which will play an important role in establishing the regularity of the free boundary. We prove the C^α regularity of the free boundary in §3 and the Lipschitz regularity in §4.

2. Existence and uniqueness of weak solutions. We begin with defining the weak solution of (1.3), (1.4).

DEFINITION 2.1. A pair (u, η) is called a weak solution of (1.3), (1.4) if $u \in H^1(Q)$, $\eta \in \beta(u)$, $u = g$ on ∂Q , and

$$\iint_Q (\nabla u \nabla \zeta + \eta \zeta_z) = 0 \quad \forall \zeta \in H_0^1(Q). \quad (2.1)$$

The free boundary of the solution is defined as

$$\Gamma_u := \{(x, z) \in \overline{Q} \mid u(x, z) = 0\}. \quad (2.2)$$

Observe that if Γ_u is smooth and is given by the zero level set of a $C^1(\overline{Q})$ function ϕ , $|\nabla\phi| \neq 0$ on Γ_u , then one can easily deduce from (2.1) the *jump relation*

$$(\nabla u^- - \nabla u^+) \cdot \nabla \phi = l\phi_z \quad \text{on } \Gamma_u \quad (2.3)$$

where “ u^- ” and “ u^+ ” represent the restriction of u in the domain $Q_u^- := \{u < 0\}$ and $Q_u^+ := \{u > 0\}$, respectively. In particular, if $\phi = z - \varphi(x)$, then (2.3) can be written as

$$\frac{\partial u^-}{\partial n} - \frac{\partial u^+}{\partial n} = \frac{l}{\sqrt{1 + |\nabla_x \varphi|^2}} \quad \text{on } \Gamma_u \quad (2.4)$$

where n is the unit vector normal to Γ_u , pointing to the region Q_u^+ , and ∇_x is the gradient with respect to $x \in \Omega \subset \mathbb{R}^n$. Noting that the tangential derivatives of u along Γ_u vanish, one can further write the relation (2.4) as

$$\frac{\partial u^-}{\partial z} - \frac{\partial u^+}{\partial z} = \frac{l}{1 + |\nabla_x \varphi|^2} \quad \text{on } \Gamma_u. \quad (2.5)$$

We call u a classical solution to (1.3) if

$$\begin{aligned} u &\in C^2(Q \setminus \Gamma_u) \cap C^1(Q_u^+ \cup \Gamma_u) \cap C^1(Q_u^- \cup \Gamma_u), \\ -\Delta u - u_z &= 0 \quad \text{in } Q \setminus \Gamma_u, \end{aligned}$$

and u satisfies (2.3) for some function $\phi \in C^1(Q)$ satisfying

$$\phi = 0, \quad |\nabla\phi| \neq 0 \quad \text{on } \Gamma_u.$$

In the sequel, we shall assume that Ω is a Lipschitz bounded domain in $\mathbb{R}^n$, $n \geq 1$.

THEOREM 1. (Existence and Regularity of Weak Solutions)

(1) If $g \in H^1(Q)$, then there exists at least one weak solution to (1.3), (1.4).

(2) If (u, η) is a weak solution of (1.3), (1.4), then $u \in C^\alpha(Q)$ for any $\alpha \in (0, 1)$. In addition, if $g \in C^\alpha(S)$ for some hypersurface $S \subset \partial Q$, then $u \in C^\alpha(Q \cup S^0)$ where S^0 is the interior of S in ∂Q .

Proof. The existence proof is based on a regularization method which is quite standard; see, for instance, Friedman [4, pp 684–692]. We shall only sketch the proof here since most of the details can be deduced from Yi [13] or Rodrigues & Yi [12].

For any $\varepsilon > 0$, consider the nonlinear elliptic problem:

$$-\Delta u^\varepsilon - u_z^\varepsilon = \frac{l}{2} \left(1 + \tanh \frac{u^\varepsilon}{\varepsilon}\right)_z \quad \text{in } Q, \quad (2.6)$$

$$u^\varepsilon = g \quad \text{on } Q \quad (2.7)$$

where $\tanh s$ is the hyperbolic tangent function $(e^s - e^{-s})/(e^s + e^{-s})$ and it can be replaced by any monotone increasing function approximating the sign function. By the elliptic theory [5], the problem (2.6), (2.7) admits a unique solution $u^\varepsilon \in H^1(Q) \cap C^\infty(Q)$.

Multiplying (2.6) by $u^\varepsilon - g$ and integrating over Q , one can get, after routine calculation (i.e., integration by parts, Hölder inequality, Sobolev embedding theorem), the a priori estimate

$$\iint_Q |\nabla u^\varepsilon|^2 \leq C \left(\iint_Q (|\nabla g|^2 + g^2) + \int_\Omega (g^2(x, 1) + g^2(x, 0)) + l^2 \right) \quad (2.8)$$

where the constant C depends only on n and Ω . Consequently, one can use the standard limit taken process (see Yi [13]) to obtain a weak solution of (1.3), (1.4), thereby establishing the first assertion of the theorem.

Noticing that the integral identity (2.1) implies that

$$-\Delta u - u_z = (f)_z \quad \text{in } H^{-1}(Q) \quad (2.9)$$

where $f := \eta - u$ only takes values in $[0, l]$, one can apply the linear elliptic theory [5, p 203] to conclude the second assertion of the theorem. $\square$

To establish the uniqueness of the weak solution, we shall first prove the following:

LEMMA 2.1. (First Comparison) *Assume that $g \in H^1(Q)$, and let (u, η) be a weak solution of (1.3), (1.4). Let $\bar{u} \in H^1(Q)$, $\bar{\eta} \in \beta(\bar{u})$ be a pair satisfying $\bar{u} \geq g$ on ∂Q and*

$$\iint_Q \nabla \bar{u} \nabla \zeta + \bar{\eta} \zeta_z \geq 0 \quad \forall \zeta \in H_0^1(Q), \zeta \geq 0. \quad (2.10)$$

Further, assume that there exists a constant $\delta \in (0, 1)$ such that

$$|u| + |\bar{u}| \geq \delta \quad \text{in } S_\delta := \Omega \times (1 - \delta, 1). \quad (2.11)$$

Then, $(\bar{u}, \bar{\eta})$ is a supersolution to (1.3), (1.4), i.e.,

$$u \leq \bar{u}, \quad \eta \leq \bar{\eta} \quad \text{a.e. in } Q. \quad (2.12)$$

REMARK 2.1. The assumption (2.11), which is called “*sufficient condition for stability*” in [12], will play a crucial role in the proof. We don’t know if (2.11) can be relaxed by the “natural” condition

$$\eta \leq \bar{\eta} \quad \text{on } \partial Q. \quad (2.13)$$

Proof of Lemma 2.1. The following proof is based on the classical method developed by Kamenomostskaja (Kamin) [6].

Subtracting (2.1) from (2.10), one gets

$$\iint_Q \nabla(\bar{u} - u) \nabla \zeta + (\bar{\eta} - \eta) \zeta_z \geq 0 \quad \forall \zeta \in H_0^1(Q), \zeta \geq 0. \quad (2.14)$$

If $\Delta \zeta \in L^2(Q)$, then integrating by parts (see Rodrigues [9, p 76] for a justification) and using the fact that $\frac{\partial \zeta}{\partial n}|_{\partial Q} \leq 0$ (n the outward unit normal), one obtains

$$\iint_Q (\bar{\eta} - \eta) \left(-\alpha \Delta \zeta + \zeta_z \right) \geq 0 \quad \forall \zeta \in H_0^1(Q), \Delta \zeta \in L^2(Q), \zeta \geq 0 \quad (2.15)$$

where

$$\alpha = \begin{cases} \frac{\bar{u} - u}{\bar{\eta} - \eta} & \text{if } \bar{\eta} \neq \eta \\ 1 & \text{if } \bar{\eta} = \eta. \end{cases} \quad (2.16)$$

It is easy to see that $0 < \alpha \leq 1$.

In order to choose ζ , we shall first make some preparations.

Let $\alpha_n \in C^\infty(Q)$, $n \in \mathbb{N}^+$, be a sequence of smooth functions satisfying

$$\frac{1}{n} \leq \alpha_n \leq 2 \quad \forall n \in \mathbb{N}^+, \quad (2.17)$$

$$\left\| (\bar{\eta} - \eta) \frac{\alpha_n - \alpha}{\sqrt{\alpha_n}} \right\|_{L^2(Q)} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.18)$$

(Since $\bar{u}, u \in H^1(Q)$, one has $\bar{\eta} - \eta \in L^p$ for some $p > 2$ by the Sobolev embedding theorem, and therefore such a sequence α_n can be constructed.)

Note that (2.11), (2.16) imply $\alpha \geq \frac{\delta}{l+\delta}$ in S_δ , so that we can assume

$$\alpha_n \geq \frac{\delta}{l+\delta} \quad \text{in } S_\delta. \quad (2.19)$$

For any $f \in C_0^\infty(Q)$, $f \geq 0$, we define ζ^n as the unique solution of the linear elliptic problem:

$$-\alpha_n \Delta \zeta^n + \zeta_z^n = f \quad \text{in } Q, \quad (2.20)$$

$$\zeta^n = 0 \quad \text{on } \partial Q. \quad (2.21)$$

Firstly, one can directly verify that 0 is a subsolution to (2.20), (2.21), and $z\|f\|_{C^0(Q)}$ is a supersolution, so that one has

$$0 \leq \zeta^n(x, z) \leq z\|f\|_{C^0(Q)} \quad \forall (x, z) \in Q. \quad (2.22)$$

Secondly, using the condition (2.19) to compare ζ^n with $\|f\|_{C^0(Q)}[1 - \tilde{\delta}^{-2}(z - 1 + \tilde{\delta})^2]$ in the set $S_{\tilde{\delta}}$ where $\tilde{\delta} := \frac{\delta}{t+1+\delta}$, one can conclude that

$$\zeta^n \leq \|f\|_{C^0(Q)}[1 - \tilde{\delta}^{-2}(z - 1 + \tilde{\delta})^2] \quad \text{in } S_{\tilde{\delta}}.$$

Consequently,

$$\left| \frac{\partial \zeta^n}{\partial z} \right| \leq \frac{2}{\tilde{\delta}} \|f\|_{C^0(Q)} \quad \text{on } \Omega \times \{1\}. \quad (2.23)$$

Finally, multiplying both sides of (2.20) by $\Delta \zeta^n$ and integrating over Q , we get

$$\begin{aligned} \iint_Q \alpha_n |\Delta \zeta^n|^2 &= \iint_Q (\zeta_z^n \Delta \zeta^n - f \Delta \zeta^n) \\ &= \int_{\partial Q} \frac{\partial \zeta^n}{\partial n} \zeta_z^n - \iint_Q \nabla \zeta_z^n \nabla \zeta^n - \iint_Q \zeta^n \Delta f \\ &= \frac{1}{2} \int_\Omega \zeta_z^n(x, 1)^2 - \frac{1}{2} \int_\Omega \zeta_z^n(x, 0)^2 - \iint_Q \zeta^n \Delta f \\ &\leq C(\delta) \left(\|f\|_{C^0(Q)}^2 + \|\Delta f\|_{L^1(Q)}^2 \right) \end{aligned} \quad (2.24)$$

by the estimates (2.22) and (2.23).

Now we can take $\zeta = \zeta^n$ in (2.15), obtaining

$$\iint (\eta - \bar{\eta}) f \leq \iint_Q (\bar{\eta} - \eta)(\alpha_n - \alpha) \Delta \zeta^n \leq \|\sqrt{\alpha_n} \Delta \zeta^n\|_{L^2(Q)} \left\| (\bar{\eta} - \eta) \frac{\alpha_n - \alpha}{\sqrt{\alpha_n}} \right\|_{L^2(Q)} \longrightarrow 0$$

as $n \rightarrow \infty$, by (2.24) and (2.18). By the arbitrariness of f , we conclude that $\eta \leq \bar{\eta}$ and therefore $u \leq \bar{u}$, thereby proving the lemma. $\square$

THEOREM 2. (Uniqueness) *Assume that $g \in H^1(Q)$ and that for some $\delta > 0$, g satisfies*

$$g \geq \delta \quad \left(\text{ or } g \leq -\delta \right) \quad \text{ in } \partial S_\delta \cap \partial Q. \quad (2.25)$$

Then there exists a unique weak solution to (1.3), (1.4).

Proof. We need only to prove the uniqueness since the existence has been established by Theorem 1. In view of Lemma 2.1, it suffices to show that for some $\tilde{\delta} \in (0, \delta)$, any weak solution to (1.3), (1.4) satisfies the stability condition

$$u \geq \tilde{\delta} \quad \left(\text{ or } u \leq -\tilde{\delta} \right) \quad \text{ in } S_{\tilde{\delta}}. \quad (2.26)$$

We shall only consider the case where $g \geq \delta$ in S_δ since the other case can be similarly treated.

Let $\tilde{u}$ be a weak solution to (1.3) with the boundary condition

$$\tilde{u} = \min\{g, \delta\} \quad \text{ on } \partial Q.$$

Then by the second assertion of Theorem 1, $\tilde{u} \in C^\alpha(\overline{S}_{\delta/2})$, and therefore for some $\tilde{\delta} \in (0, \delta/2)$,

$$\tilde{u} \geq \tilde{\delta} > 0 \quad \text{ in } S_{\tilde{\delta}}.$$

Now we can apply Lemma 2.1 to u and $\tilde{u}$ to deduce that $\tilde{u} \leq u$ in Q , and therefore (2.26) holds. This completes the proof of Theorem 2. $\square$

We conclude this section with a weaker version of the comparison Lemma 2.1.

LEMMA 2.2. (Second Comparison) *Let (u, η) be a weak solution of (1.3) and (1.4). Assume that $w \in C^0(\overline{Q})$, $\varphi \in C^1(\overline{\Omega})$ satisfy*

$$\Gamma_w := \{(x, z) \in \overline{Q} \mid w(x, z) = 0\} = \{(x, z) \in \overline{Q} \mid z = \varphi(x)\}, \quad (2.27)$$

$$Q_w^\pm := \{(x, z) \in \overline{Q} \mid w(x, z) \gtrless 0\} = \{(x, z) \in \overline{Q} \mid z \gtrless \varphi(x)\}, \quad (2.28)$$

$$w \in C^2(Q \setminus \Gamma_w) \cap C^1(Q_w^+ \cup \Gamma_w) \cap C^1(Q_w^- \cup \Gamma_w), \quad (2.29)$$

$$-\Delta w - w_z \geq 0 \quad \text{ in } Q \setminus \Gamma_w, \quad (2.30)$$

$$w \geq g \quad \text{ on } \partial Q, \quad (2.31)$$

$$\frac{\partial w^-}{\partial z} - \frac{\partial w^+}{\partial z} \geq \frac{l}{1 + |\nabla_x \varphi|^2} \quad \text{ on } \Gamma_w. \quad (2.32)$$

Finally, assume that u, w satisfy, for some $\delta \in (0, 1)$, the inequality

$$|u| + |w| \geq \delta \quad \text{ in } S_\delta. \quad (2.33)$$

Then one has the inequality

$$u \leq w \quad \text{in } Q. \quad (2.34)$$

If all the inequality signs in (2.30)–(2.32) are reversed, then

$$w \leq u \quad \text{in } Q. \quad (2.35)$$

Proof. From the differential inequality (2.30), one gets, for any $\zeta \in H_0^1(Q)$ satisfying $\zeta \geq 0$, the inequality

$$\begin{aligned} 0 &\leq \iint_{Q \setminus \Gamma_w} (-\Delta w - w_z) \zeta \\ &= - \int_{\Gamma_w} \zeta \left[\frac{\partial w}{\partial n} \right]_+^- + \iint_Q \nabla w \nabla \zeta + w \zeta_z \\ &= - \int_{\Omega} \zeta \left(\sqrt{1 + |\nabla_x \varphi|^2} \left[\frac{\partial w}{\partial n} \right]_+^- - l \right) \Big|_{z=\varphi(x)} dx + \iint_Q (\nabla w \nabla \zeta + \beta(w) \zeta_z) \\ &\leq \iint_Q (\nabla w \nabla \zeta + \beta(w) \zeta_z) \end{aligned}$$

by (2.32). Inequality (2.34) thus follows from the first comparison lemma. Similarly, we can show (2.35) in case when all the inequality signs in (2.30)–(2.32) are reversed. $\square$

3. C^α regularity of the free boundary. In this section, we shall study the regularity of the free boundary under the conditions:

$$\|g\|_{H^1(Q)} + \|g\|_{C^\alpha(\overline{Q})} \leq C_0, \quad (3.1)$$

$$g(x, 1) = m^+, \quad g(x, 0) = -m^-, \quad x \in \Omega, \quad (3.2)$$

$$\inf_{x \in \Omega, 0 \leq z_1 < z_2 \leq 1} \frac{g(x, z_1) - g(x, z_2)}{z_1 - z_2} > L_0. \quad (3.3)$$

where $\alpha \in (0, 1)$ and C_0, m^+, m^-, L_0 are all given positive constants.

Note that by the second assertion of Theorem 1, the assumption (3.1) implies that

$$\|u\|_{C^\alpha(\overline{Q})} =: M < \infty. \quad (3.5)$$

In the sequel, we shall always denote by (u, η) the unique solution of (1.3), (1.4) established in §2.

LEMMA 3.1. *For any $(x_1, z_1), (x_2, z_2) \in Q, (x_1, z_1) \neq (x_2, z_2)$, one has the inequality*

$$u(x_1, z_1) \leq u(x_2, z_2) \quad (3.6)$$

provided that

$$z_2 - z_1 \geq \frac{M}{L_0} |x_1 - x_2|^\alpha \quad (3.7)$$

where $M = M(\Omega, C_0)$ is the constant defined in (3.5).

Proof. The idea of the proof is based on a shifting domain technique; that is, by comparing the functions $u(x, z)$ with $u(x + a, z + b)$ ($a \in \mathbb{R}^n, b \in (0, 1)$) in the domain $Q_{a,b} := \{(x, z) \in Q \mid (x + a, z + b) \in Q\}$, we want to show that

$$u(x, z) \leq u(x + a, z + b) \quad \forall (x, z) \in Q_{a,b} \quad (3.8)$$

as long as

$$b \geq \frac{M}{L_0} |a|^\alpha. \quad (3.9)$$

Using the first comparison lemma, we need only to verify (3.8) on the boundary of $Q_{a,b}$.

One can easily deduce, via the first comparison lemma and the assumption (3.2), (3.3), that $-m^-$ and m^+ are the minimum and the maximum of u respectively, so that (3.8) holds on the top and bottom of $Q_{a,b}$. It remains to verify (3.8) on the lateral boundary of $Q_{a,b}$.

When (x, z) belongs to the lateral boundary of $Q_{a,b}$, there are only two possibilities: (i) $x \in \partial\Omega$ and (ii) $x + a \in \partial\Omega$. In the first case, one has $u(x, z) = g(x, z)$ and $u(x, z + b) = g(x, z + b)$, so that

$$\begin{aligned} u(x + a, z + b) - u(x, z) &= u(x + a, z + b) - u(x, z + b) + g(x, z + b) - g(x, z) \\ &\geq -M|a|^\alpha + L_0 b \geq 0 \end{aligned}$$

by (3.9) and (3.5). The case (ii) can be similarly treated. Therefore, inequality (3.8) holds for $(x, z) \in \partial Q_{a,b}$, and by the first comparison lemma, (3.8) also holds for all $(x, z) \in Q_{a,b}$. Taking $a = x_2 - x_1$ and $b = z_2 - z_1$, the assertion of the lemma follows. $\square$

Taking $a = 0$ in (3.8), one finds that u is monotone increasing in z , so that there exists two functions φ^- and φ^+ , $0 < \varphi^- \leq \varphi^+ < 1$, such that

$$u(x, z) > 0 \quad \text{if } x \in \overline{\Omega}, \varphi^+(x) < z \leq 1, \quad (3.10)$$

$$u(x, z) = 0 \quad \text{if } x \in \overline{\Omega}, \varphi^-(x) \leq z \leq \varphi^+(x), \quad (3.11)$$

$$u(x, z) < 0 \quad \text{if } x \in \overline{\Omega}, 0 \leq z < \varphi^-(x). \quad (3.12)$$

LEMMA 3.2. *The two functions φ^- and φ^+ are Hölder continuous; more precisely, one has the bounds*

$$\|\varphi^+\|_{C^\alpha(\bar{\Omega})} \leq 1 + \frac{M}{L_0}, \quad (3.13)$$

$$\|\varphi^-\|_{C^\alpha(\bar{\Omega})} \leq 1 + \frac{M}{L_0}. \quad (3.14)$$

Proof. Note that (3.6) implies that for any $\varepsilon > 0$

$$u(x_2, z_2) \geq u(x_1, \varphi^+(x_1) + \varepsilon) > 0$$

if

$$z_2 \geq \varphi^+(x_1) + \varepsilon + \frac{M}{L_0}|x_1 - x_2|^\alpha.$$

It follows that

$$\varphi^+(x_2) \leq \varphi^+(x_1) + \varepsilon + \frac{M}{L_0}|x_1 - x_2|^\alpha,$$

and, by letting $\varepsilon \rightarrow 0$, that

$$\varphi^+(x_2) \leq \varphi^+(x_1) + \frac{M}{L_0}|x_1 - x_2|^\alpha.$$

Exchanging the roles of x_1 and x_2 , we then get

$$|\varphi^+(x_2) - \varphi^+(x_1)| \leq \frac{M}{L_0}|x_1 - x_2|^\alpha.$$

This proves (3.13). Similarly, one can prove (3.14). $\square$

THEOREM 3. (C^α Regularity of the Free Boundary) *Assume that g satisfies (3.1)–(3.3), and let (u, η) be the unique solution of (1.3), (1.4). Then there exists a function $\varphi \in C^\alpha(\bar{\Omega})$, $0 < \varphi < 1$, such that*

$$u(x, z) > 0 \quad \text{if } x \in \bar{\Omega}, \varphi(x) < z \leq 1, \quad (3.15)$$

$$u(x, z) = 0 \quad \text{if } x \in \bar{\Omega}, z = \varphi(x), \quad (3.16)$$

$$u(x, z) < 0 \quad \text{if } x \in \bar{\Omega}, 0 \leq z < \varphi(x). \quad (3.17)$$

Proof. It suffices to show that $\varphi^- \equiv \varphi^+$. Assume that this is not true; then there exists a point $x_0 \in \Omega$ such that

$$\varphi^-(x_0) < \varphi^+(x_0).$$

Set $z_0 := \frac{1}{2}(\varphi^+(x_0) + \varphi^-(x_0))$. Since both φ^+ and φ^- are continuous, there exists a ball $B_r(x_0) \in \Omega$ (where $B_r(x_0)$ represents a ball of radius r centered at x_0) such that

$$\varphi^-(x) < z_0 < \varphi^+(x) \quad \forall x \in B_r(x_0). \quad (3.18)$$

Let w be the solution of

$$\begin{aligned} -\Delta w &= 0 & \text{in } \tilde{Q} &:= B_r(x_0) \times (z_0, 1), \\ w &= u & \text{on } \partial\tilde{Q}. \end{aligned}$$

Recall that u is monotone in z , so that w is also monotone in z , i.e.,

$$w_z \geq 0 \quad \text{in } \tilde{Q}.$$

Since (3.10), (3.11), and (3.18) imply that

$$w > 0 \quad \text{in } \tilde{Q}, \quad (3.19)$$

one can calculate

$$\iint_{\tilde{Q}} \nabla w \nabla \zeta + \beta(w) \zeta_z = - \iint_{\tilde{Q}} w_z \zeta \leq 0 \quad \forall \zeta \in H_0^1(\tilde{Q}), \zeta \geq 0.$$

By the first comparison lemma, one concludes that

$$w \leq u \quad \text{in } \tilde{Q} = B_r(x_0) \times (z_0, 1).$$

Consequently, by (3.19), $u > 0$ in $\tilde{Q}$, contradicting to the second inequality in (3.18). This contradiction shows that $\varphi^- = \varphi^+$, thereby establishing Theorem 3. $\square$

REMARK 3.1. Using the same method as in the proof of Theorem 3, one can also show that the function φ is continuous if (3.3) is replaced by the weaker condition:

$$\inf_{x \in \partial\Omega} (g(x, z_2) - g(x, z_1)) \geq \omega(z_2 - z_1) \quad \text{if } 0 \leq z_1 < z_2 \leq 1$$

where $\omega(\cdot)$ is any monotone increasing function satisfying $\omega(t) > 0$ if $t > 0$.

4. Lipschitz regularity of the free boundary. From Theorem 3, we know that there exists a function $\varphi \in C^\alpha(\overline{\Omega})$ such that the free boundary is given by $z = \varphi(x)$. We shall now proceed to establish the Lipschitz continuity of φ under some stronger conditions.

Before we state the condition, we shall first deduce a necessary condition for the boundary data g . Assume that $\varphi \in C^1(\overline{\Omega})$, then, from (2.5), we know that g has to satisfy the equation

$$\frac{\partial g}{\partial z}(x, \varphi(x) - 0) - \frac{\partial g}{\partial z}(x, \varphi(x) + 0) = \frac{l}{1 + |\nabla \varphi|^2} = \frac{l}{1 + |\nabla_\tau \varphi|^2 + |\partial \varphi / \partial n|^2} \quad \forall x \in \partial \Omega$$

where ∇_τ stands for the derivatives tangential to $\partial \Omega$, and n represents the outward unit normal to $\partial \Omega$.

Let $\varphi^0(x)$, $x \in \partial \Omega$, be the function implicitly defined by

$$g(x, \varphi^0(x)) = 0 \quad \forall x \in \partial \Omega. \quad (4.1)$$

Clearly, the restriction of φ on $\partial \Omega$ is φ^0 , so that $\nabla_\tau \varphi = \nabla_\tau \varphi^0$. Therefore a necessary condition for φ to be in $C^1(\overline{\Omega})$ is that

$$0 < \frac{\partial g}{\partial z}(x, \varphi^0(x) - 0) - \frac{\partial g}{\partial z}(x, \varphi^0(x) + 0) \leq \frac{l}{1 + |\nabla_\tau \varphi^0|^2} \quad \forall x \in \partial \Omega. \quad (4.2)$$

Recall that $\partial \Omega$ is called satisfying the uniform exterior ball condition if there exists a positive constant R_0 such that at every point $y \in \partial \Omega$, there exists a ball $B_{R_0}(y_0) \in \mathbb{R}^n \setminus \Omega$ with $|y - y_0| = R_0$.

To prove the Lipschitz continuity of the free boundary, we shall assume that

$$\partial \Omega \in C^{1+\alpha} \quad \text{and} \quad \partial \Omega \text{ satisfies the uniform exterior ball condition,} \quad (4.3)$$

$$\frac{\partial g}{\partial z}(x, z) \geq L_0 > 0 \quad \forall x \in \partial \Omega, 0 \leq z \leq 1, \quad (4.4)$$

$$\|\varphi^0\|_{C^{1+\alpha}(\partial \Omega)} + \|g\|_{C^{2+\alpha}(Q_g^+)} + \|g\|_{C^{2+\alpha}(Q_g^-)} \leq C_0, \quad (4.5)$$

$$\frac{\partial g}{\partial z}(x, \varphi^0(x) - 0) - \frac{\partial g}{\partial z}(x, \varphi^0(x) + 0) > \alpha_0 > 0 \quad \forall x \in \partial \Omega \quad (4.6)$$

and

$$\frac{\partial g}{\partial z}(x, \varphi^0(x) - 0) - \frac{\partial g}{\partial z}(x, \varphi^0(x) + 0) \leq \frac{l}{1 + 2K^2} \quad \forall x \in \partial \Omega \quad (4.7)$$

for some $K = K(C_0, L_0, \Omega)$ given below in (4.25); here Q_g^+ and Q_g^- denote the regions $\{g > 0\}$ and $\{g < 0\}$, respectively.

LEMMA 4.1. Assume that $\partial\Omega$ satisfies (4.3) and g satisfies (3.2), (4.4) and (4.5). Let (u, η) be the unique solution of (1.3), (1.4). Then one has the following:

- (1) if g satisfies (4.6) for some $\alpha_0 > 0$ then there exists a positive constant $C = C(C_0, L_0, \alpha_0, \Omega)$ such that

$$u(x, z) \leq g(y, z) + C|x - y| \quad \forall (x, z) \in Q, y \in \partial\Omega; \quad (4.8)$$

- (2) there exists a constant $K = K(C_0, L_0, \Omega)$ such that if g satisfies (4.7) for such given K , then

$$u(x, z) \geq g(y, z) - C|x - y| \quad \forall (x, z) \in Q, y \in \partial\Omega \quad (4.9)$$

for some constant C depending on K, C_0, L_0 , and Ω .

REMARK 4.1. Note that the condition (4.7) is stronger than the necessary condition (4.2) we derived. We do not know what the necessary and sufficient condition is.

Proof of Lemma 4.1. We shall prove this lemma via the construction of sub- and supersolutions. To do this, we first introduce several functions.

Extend φ^0 into Ω such that

$$\Delta_x \varphi^0 = 0 \quad \text{in } \Omega. \quad (4.10)$$

Since $\partial\Omega \in C^{1+\alpha}$, one has $\varphi^0 \in C^{1+\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$.

Introduce functions $A^+, A^- \in C^{1+\alpha}(\partial Q)$ satisfying

$$A^+ = \frac{g(x, z)}{z - \varphi^0(x)} \quad \forall x \in \partial\Omega, \varphi^0(x) < z \leq 1, \quad (4.11)$$

$$A^- = \frac{g(x, z)}{z - \varphi^0(x)} \quad \forall x \in \partial\Omega, 0 \leq z < \varphi^0(x). \quad (4.12)$$

Since g is $C^{2+\alpha}$ in $\overline{Q}_g^\pm$, such functions A^+ and A^- can be constructed. In addition, we can assume, by the assumption (4.4), that

$$A^\pm(x, z) \geq L_0 \quad \forall (x, z) \in \partial Q. \quad (4.13)$$

Further, we can assume that $A^\pm$ satisfy

$$A^-(x, z) - A^+(x, z) \geq \alpha_0/2 \quad \forall (x, z) \in \partial Q \quad (4.14a)$$

under the assumption (4.6), and/or satisfy

$$A^- - A^+ \leq \frac{l}{1 + K^2} \quad \forall (x, z) \in \partial Q \quad (4.14b)$$

under the assumption (4.7). We extend A^+ , A^- into Q such that they satisfy the equation

$$-\Delta A^\pm - A_z^\pm = 0 \quad \text{in } Q. \quad (4.15)$$

Clearly, $A^\pm \in C^\infty(Q) \cap C^{1+\alpha}(\overline{Q})$, and, by (4.13),

$$A^\pm \geq L_0 \quad \text{in } \overline{Q}. \quad (4.16)$$

Furthermore, if (4.6) holds then, by (4.14a),

$$A^- - A^+ \geq \alpha_0/2 \quad \text{in } \overline{Q}, \quad (4.17a)$$

and if (4.7) holds then, by (4.14b),

$$A^- - A^+ \leq \frac{l}{1 + K^2} \quad \text{in } \overline{Q}. \quad (4.17b)$$

For any fixed $y \in \partial\Omega$, let $\psi(\cdot, y)$ be a smooth function in $\overline{\Omega}$ satisfying

$$\psi(y, y) = 0, \quad (4.18)$$

$$\psi(x, y) \geq 0 \quad \forall x \in \Omega, \quad (4.19)$$

$$|\nabla_x \psi(x, y)| \geq 1 \quad \forall x \in \Omega, \quad (4.20)$$

$$-\Delta_x \psi \geq 1 + \tilde{C}_0 |\nabla_x \psi| \quad \forall x \in \Omega \quad (4.21)$$

where

$$\tilde{C}_0 := 2 \max \left\{ \left\| \frac{\nabla A^+}{A^+} \right\|_{C^0(Q)}, \left\| \frac{\nabla A^-}{A^-} \right\|_{C^0(Q)} \right\}. \quad (4.22)$$

For example, one can take

$$\psi(x, y) = M(1 - e^{\tilde{C}_0(R_0 - |x - y_0|)})$$

for some M large enough, where $B_{R_0}(y_0)$ is an exterior ball of Ω such that $R_0 = |y_0 - y|$.

Now we want to show that the function

$$w^\lambda(x, z) := \begin{cases} A^+(x, z)(z - \varphi^0(x) + \lambda\psi(x, y)) & \text{if } (x, z) \in Q, z \geq \varphi^0(x) - \lambda\psi(x, y) \\ A^-(x, z)(z - \varphi^0(x) + \lambda\psi(x, y)) & \text{if } (x, z) \in Q, z \leq \varphi^0(x) - \lambda\psi(x, y) \end{cases}$$

is a supersolution to (1.3), (1.4) if λ is large enough, and is a subsolution if $-\lambda$ and K in (4.7) are large enough.

One can calculate, for $z \geq \varphi^0(x) - \lambda\psi(x, y)$,

$$\begin{aligned} -\Delta w^\lambda - w_z^\lambda &= -(\Delta A^+ + A_z^+)(z - \varphi^0 + \lambda\psi) - 2\nabla A^+ \nabla(z - \varphi^0 + \lambda\psi) \\ &\quad - A^+(1 - \Delta_x \varphi^0 + \lambda \Delta_x \psi) \\ &= A^+ \left\{ \lambda \left[-\Delta_x \psi - \frac{2\nabla A^+}{A^+} \nabla \psi \right] - \left[\frac{2\nabla A^+}{A^+} \nabla(z - \varphi^0) + 1 \right] \right\} \end{aligned} \quad (4.23)$$

by (4.10) and (4.15). Similar calculation also holds for the case when $z \leq \varphi^0(x) - \lambda\psi(x, y)$.

In view of (4.21) and (4.22), one finds that if

$$\lambda \geq 1 + 2 \max \left\{ \left\| \frac{\nabla A^+}{A^+} \nabla(z - \varphi^0) \right\|_{C^0(Q)}, \left\| \frac{\nabla A^-}{A^-} \nabla(z - \varphi^0) \right\|_{C^0(Q)} \right\}$$

then

$$-\Delta w^\lambda - w_z^\lambda \geq 0 \quad \text{in } Q \setminus \Gamma_{w^\lambda}$$

where $\Gamma_w := \{w = 0\} = \{(x, z) \mid x \in \overline{\Omega}, z = \varphi^0(x) - \lambda\psi(x, y)\}$. Since $|\nabla_x \psi| \geq 1$ (by (4.20)), one can deduce that

$$\left[\frac{\partial w}{\partial z} \right]_+^- = A^- - A^+ \geq \alpha_0/2 \geq \frac{l}{1 + |\nabla_x \varphi^0 + \lambda \nabla_x \psi|^2} \quad \text{on } \Gamma_{w^\lambda}$$

if λ is large enough. Finally, from (4.11), (4.12), and (4.19), one can easily verify that

$$w^\lambda \geq g = u \quad \text{on } \partial Q,$$

and hence, by applying the second comparison lemma, we get

$$u(x, z) \leq w^\lambda(x, z) \quad \text{in } Q$$

if λ is large enough. Noting that (4.18) implies that $w^\lambda(y, z) = g(y, z)$ for all $z \in [0, 1]$, one can use the Lipschitz continuity of the function w^λ to deduce (4.8), thereby establishing the first assertion of the lemma.

To prove the second assertion of the lemma, we take

$$\lambda = \lambda_- := -2 \max \left\{ \left\| \frac{\nabla A^+}{A^+} \nabla(z - \varphi^0) \right\|_{C^0(Q)}, \left\| \frac{\nabla A^-}{A^-} \nabla(z - \varphi^0) \right\|_{C^0(Q)} \right\}. \quad (4.24)$$

Then, the last inequality in (4.23) yields

$$-\Delta w^\lambda - w_z^\lambda \leq 0 \quad \text{in } Q \setminus \Gamma_{w^\lambda}.$$

Now, if we take

$$K = \sup_{x \in \Omega, y \in \partial\Omega} |\nabla_x \varphi^0(x) + \lambda_- \nabla_x \psi(x, y)|, \quad (4.25)$$

then by (4.17b),

$$\left[\frac{\partial w}{\partial z} \right]_+^- = A^- - A^+ \leq \frac{l}{1 + K^2} \leq \frac{l}{1 + |\nabla_x \varphi^0 + \lambda_- \nabla_x \psi|^2} \quad \text{on } \Gamma_{w^\lambda}$$

and hence we can proceed as before to conclude that w^λ is a subsolution to (1.3), (1.4). The inequality (4.9) thus follows as before. This completes the proof of the lemma. $\square$

REMARK 4.2. One may notice that the harmonic function with boundary value g is a subsolution to (1.3), (1.4); however we cannot use it since its zero level surface is not uniformly Lipschitz continuous.

LEMMA 4.2. *Assume that (3.2), (4.3)–(4.7) hold. Then there exists a positive constant C such that*

$$u(x, z) \leq u(y, z + C|x - y|) \quad \text{if } (x, z) \in Q \text{ and } (y, z + C|x - y|) \in Q. \quad (4.26)$$

The proof is exactly the same as that of Lemma 3.1 except that we use lemma 4.1 at the place where we use the Hölder continuity of u .

Note that lemma 4.2 indicates that u is monotone along any direction (e_x, C) where e_x is any vector in $\mathbb{R}^n$ with $|e_x| \leq 1$. Therefore we have the following:

THEOREM 4. *Assume that (3.2), (4.3)–(4.7) hold. Let (u, η) be the unique solution of (1.3), (1.4), and let φ be the function given in Theorem 3. Then φ is Lipschitz continuous.*

The proof is similar to that of Theorem 3 and it is omitted.

Now we can use the results of Caffarelli [2] to conclude the following:

THEOREM 5. *Assume that the conditions of Theorem 4 hold. Then the free boundary of the weak solution to (1.3), (1.4) is a $C^{1+\alpha}$ z -graph and the solution is $C^{1+\alpha}$ in each side of the free boundary, up to the free boundary. Consequently, the weak solution is a classical solution.*

Here a z -graph means that the free boundary Γ_u can be written as $z = \varphi(x)$, $x \in \overline{\Omega}$.

REMARK 4.3. All our results extend to the case when $\beta(u)$ defined in (1.2) is replaced by $\alpha(u) + lH(u)$ where $\alpha(u)$ is any Lipschitz function with $\alpha' \geq c_0 > 0$ and $H(u)$ is the Heaviside function taking value 0 when $u < 0$ and value 1 when $u > 0$.

REFERENCES

- [1] BRIÈRE, T., *Application des méthodes variationnelles à la cristallisation d'un métal par passage dans une gaine de refroidissement*, Ann. Fac. Sc. Toulouse, 2 (1980), pp. 219–247.
- [2] CAFFARELLI, L. A., *A Harnack inequality approach to the regularity of free boundaries. Part I: Lipschitz boundaries are $C^{1,\alpha}$* , Revista Matematica Iberoamericana, 3 (1987), pp. 139–162.
- [3] CHIPOT, M & RODRIGUES, J.F., *On the steady-state continuous casting Stefan problem with non-linear cooling*, Quart. Appl. Math., 40 (1983), pp. 476–491.
- [4] FRIEDMAN, A., *Variational Principle And Free Boundary Problems*, John Wiley & Sons Inc., New York, 1982.
- [5] GILBARG, D & TRUDINGER, N. S., *Elliptic Partial Differential Equations Of Second Order*, 2nd Ed., Springer-Verlag, New York, 1983.
- [6] KAMENOMOSTSKAJA, S. L., *On Stefan's problem*, Mat.Sb. 53(95) (1965), pp. 485–514.
- [7] RODRIGUES, J. F., *An evolutionary continuous casting problem of Stefan type*, Quart. Appl. Math., 44 (1986), pp. 109–131.
- [8] RODRIGUES, J. F., *Aspects of the variational approach to the continuous casting problem*, In Research Notes in Math (Pitman), 120 (1985), pp. 72–83.
- [9] RODRIGUES, J. F., *Obstacle Problems In Mathematical Physics*, North-Holland-Amsterdam, New York, 1987.
- [10] RODRIGUES, J. F., *On the free boundary of a two-phase Stefan problem with convection*, CMAF Preprint, University of Lisbon.
- [11] RODRIGUES, J. F. & ZALTZMAN, B., *The two-phase steady-state Stefan problem in strips*, IMA preprint, University of Minnesota.
- [12] RODRIGUES, J. F. & YI, F., *On a two-phase continuous casting Stefan problem with nonlinear flux*, European Journal of Applied Mathematics, 1 (1990), pp. 259–278.
- [13] YI, F., *An evolutionary continuous casting problem of two-phases and its periodic behavior*, J. Partial Differential Equations, 2 (1989), pp. 7–22.
- [14] RULLA, J., *Weak solutions to Stefan problems with prescribed convection*, SIAM J. Math. Anal., 18 (1987), pp. 1784–1800.

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