**NUMERICAL ANALYSIS AND ACCURATE COMPUTATION
OF HETEROCLINIC ORBITS IN THE CASE
OF CENTER MANIFOLDS**

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Numerical analysis and accurate computation of heteroclinic orbits in the case of center manifolds.

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Abstract: In earlier papers (see [8] and the references there) Doedel and the author have developed a numerical method for the computation of branches of heteroclinic orbits for a system of autonomous ordinary differential equations in $\mathbb{R}^n$ in the case that the solution approaches the fixed points exponentially. The idea of the method is to reduce a boundary value problem on the real line to a boundary value problem on a finite interval by using linear approximation of the unstable and stable manifolds. Using the fact that the linearized operator of the problem is Fredholm in Banach spaces with exponential weights, the authors employed the general theory of approximation of nonlinear problems to show that the errors in the approximate solution decay exponentially with the length of the approximating interval. In this paper we extend the analysis in [8] to the case of center manifolds which requires the refinement of the analysis in [8]. The algorithm is applied to a model problem: the DC Josephson Junction. Computations are done using the software package AUTO.

Key words. heteroclinic orbits, center manifolds, computation and continuation, Banach spaces, error analysis.

AMS(MOS) subject classification. 34B15, 34C30, 34C35, 65J15, 65L10, 68-04.

1. Introduction. Global analysis of *homoclinic and heteroclinic* orbits, i.e., orbits of an infinite period connecting two fixed points of an associated system of autonomous ordinary differential equations, is important in analysis of dynamical systems. Applications include, in particular, the problem of finding traveling wave front solutions of constant speed to nonlinear parabolic partial differential equations, etc...

In earlier papers [5, 6, 8] Doedel and the author have developed an accurate, robust, and systematic method for computing branches of heteroclinic orbits. Specifically, we have considered the problem of finding a branch of solutions of the system of autonomous ordinary differential equations

$$(1.1) \quad \begin{aligned} a) & \quad x'(t) - f(x(t), \lambda) = 0, \quad x(\cdot), f(\cdot, \cdot) \in \mathbb{R}^n, \quad \lambda \in \mathbb{R}^{n_\lambda}, \\ b) & \quad \lim_{t \rightarrow -\infty} x(t) = x_-, \quad \lim_{t \rightarrow \infty} x(t) = x_+. \end{aligned}$$

The method utilizes linear approximation of unstable (for $t < T_-$, $-T_- > 0$, large) and stable (for $t > T_+$, $T_+ > 0$, large) manifolds, under the assumption that solutions of (1.1) decay exponentially to their limits at $\pm\infty$. Since every translation of a solution of Eq. (1.1) is also a solution, to remove this “phase” indeterminacy we need to add a constraint.

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Equation

$$(1.2) \quad \int_{-\infty}^{\infty} (f(x(t), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(x(t), \lambda) dt = 0$$

seems to be, computationally, the most appropriate way to do this. It is obtained by requiring that the current solution $x(t)$ be as “close” as possible to the previously computed one $q(t)$ (see [5] for the discussion). Our principle result, Theorem 2 in [8] can be summarized as follows:

Let (q, λ^0) be a solution of (1.1), (1.2). Assume that $n_\lambda = 2 - (n_u^- + n_s^+ - n) \geq 0$, where n_u^- and n_s^+ are dimensions of the unstable and stable manifolds of x_- and x_+ , respectively. Under appropriate assumptions on f and appropriate transversality conditions, in a neighborhood of (q, λ^0) there exists a unique solution branch $(x(s), \lambda(s))$, $(x(0), \lambda(0)) = (q, \lambda^0)$, of (1.1), (1.2) and for sufficiently large $-T_-, T_+$ a unique branch $(x(s), \lambda_T(s))$ of approximate solutions. Here s is the pseudo-arclength continuation parameter (employed by AUTO).

Moreover, we have an error estimate

$$(1.3) \quad \|\lambda(s) - \lambda_T(s)\|_{\mathbb{R}^{n_\lambda}} + \|x(s) - x_T(s)\|_{W_\infty^1(\mathbb{R})} \leq C(e^{2T-\mu_0} + e^{2T+\mu_1}),$$

for some $\mu_0 > 0 > \mu_1$.

Similar results were obtained in Beyn [1, 2]. Higher order approximation of the unstable and stable manifolds in combination with the shooting method was developed in Hassard [12] for the computation of heteroclinic orbits. See also Hagstrom and Keller [10] and references there for approximation of PDEs and ODEs on infinite domains by the problems with appropriate boundary conditions on finite domains.

In this paper we extend the numerical method and its analysis to the case of center manifolds. Specifically, we consider the situation when during the continuation process one of the positive eigenvalues of $f_x(x_-, \lambda)$ (corresponding to the unstable manifold of x_-) approaches zero, and thus the exponential rate of decay is lost, i.e. a center manifold appears. The results trivially extend to the situation when an eigenvalue is zero during the continuation process.

The algorithm is applied to a model problem: the DC Josephson Junction, see Friedman and Doedel [9] for computational details. This problem has been chosen due to the fact that it exhibits typical nontrivial behavior (and is also of practical interest) and at the same time it has been studied theoretically, see Chow and Lin [3], Schecter [13] and references there.

In § 2 we first state our assumptions and describe the exact problem. It is convenient to look for the solution in a Banach space of functions which are $W_\infty^1(\mathbb{R})$ and decay exponentially to zero at $\pm\infty$. In [8], in order to be able to formulate our problem as an operator equation in such a Banach space, we “subtracted” the fixed points x_- , x_+ from the solution $x(t)$ so that to force the solution to decay (exponentially) to zero at $\pm\infty$. In this paper, to achieve the same goal, we “subtract” the fixed points x_- , x_+ together with the *center manifold*. The latter idea was employed in [3] in their analysis of the

saddle node bifurcation and was suggested to the author by Lin. Extending the analysis in [8], we then show that the linearized operator of our problem is Fredholm. We next derive the solution algorithm by first constructing an appropriate approximation to the exponentially decaying part of the solution (with the linear approximation of the unstable and stable manifolds) and then constructing a (higher order) approximation of the center manifold. The advantage of our somewhat lengthy derivation of the solution algorithm is that it provides us with insights valuable for the error analysis in § 3.

In § 3 we present the error analysis. To estimate the error in the solution we split it into three terms: the first term is the error in the exponentially decaying part of the solution of exact problem due to the approximation of the unstable and stable manifolds of the exponentially decaying part of the solution only, the second term is the additional error in the exponentially decaying part of the solution due to the approximation of the center manifold, and the third term is the error in the center manifold itself, due to its approximation. The derivation of the estimate for the first error term follows the error analysis in [8]. The estimate for the third error term is standard. The major difficulty lies in the estimation of the second error term. This difficulty is due to the fact that this term decays slower than exponentially with $|T_-|$, as $T_- \rightarrow -\infty$, namely, as a positive power of $\frac{1}{|T_-|}$, and hence a Banach space of exponentially decaying functions is not appropriate in this case. On the other hand, an attempt to use a larger space, such as W_∞^1 , seems to lead to insurmountable difficulties. We resolve this difficulty by constructing an appropriate subspace of W_∞^1 of exponentially decaying functions.

In § 4 our method is applied to a model problem: the DC Josephson Junction. We briefly describe the solution algorithm and the numerical results, see [9] for the details. Computations are done using the software package AUTO [7] developed by Doedel.

2. Formulation of the exact problem and its analysis. Derivation of the solution algorithm. Let X, Y be Banach spaces. For a linear operator $A \in L(X, Y)$ we denote $\|A\|_{L(X, Y)}$ the norm of A , $\mathcal{N}(A) = \text{Null space of } A$, $\mathcal{R}(A) = \text{Range of } A$. We denote by C, c generic positive constants.

We consider the problem of finding a branch of solutions $(x(t, s), \lambda(s))$, for some parametrization s , given a solution $(q(t), \lambda^0) \equiv (x(t, 0), \lambda(0))$, of the system

$$(2.1) \quad \begin{aligned} a) & \quad x'(t) - f(x(t), \lambda) = 0, \quad -\infty < t < \infty, \quad x(\cdot) \in \mathbb{R}^n, \\ b) & \quad \lim_{t \rightarrow -\infty} x(t) = x_-, \quad \lim_{t \rightarrow \infty} x(t) = x_+, \end{aligned}$$

$$(2.2) \quad \begin{aligned} a) & \quad f(x_-, \lambda) = 0, \\ b) & \quad f(x_+, \lambda) = 0, \end{aligned}$$

$$(2.3) \quad \int_{-\infty}^{\infty} (x'(t) - q'(t)) \cdot q''(t) dt = 0,$$

where $f : \mathbb{R}^n \times \Lambda \rightarrow \mathbb{R}^n$, $\lambda = (\lambda_1, \lambda_2) \in \Lambda$, and Λ is a neighborhood of $\lambda = \lambda^0$ in $\mathbb{R}^2$, and (2.3) is a “phase condition”. We obtain this phase condition by requiring the current

heteroclinic orbit $x(t, \lambda)$ to “look like” the previously computed orbit $q(t)$ as much as possible (see [5], [8]), which is convenient numerically. Here for $x, y \in \mathbb{R}^n$ $x \cdot y$ denotes the inner product in $\mathbb{R}^n$, and we denote by $|\cdot|$ the l^2 norm in $\mathbb{R}^n$; we shall keep the same notation for the inner product and the norm for finite dimensional Euclidian spaces of dimension other than n .

Consider the following hypotheses:

- (i) $f \in C^k$, $k \geq 4$; $|f_x|$, $|f_\lambda|$, $|f_{xx}|$, $|f_{xx\lambda}|$ are bounded uniformly in x, λ ;
 (2.2) can be solved to find $x_- = x_-(\lambda)$, $x_+ = x_+(\lambda)$, smooth functions of λ .

- (ii) $\sigma(f_x(x_-(s), \lambda(s))) = \left\{ \mu_{-n_s}^-, \dots, \mu_{-1}^-, \mu_0^-, \mu_1^-, \dots, \mu_{n_u}^- \right\}$ with
 $\operatorname{Re} \mu_{-n_s}^- \leq \dots \leq \operatorname{Re} \mu_{-1}^- < 0 < \mu_0^- < \operatorname{Re} \mu_1^- \leq \dots \leq \operatorname{Re} \mu_{n_u}^-$,
 $\sigma(f_x(x_+(\lambda), \lambda)) = \left\{ \mu_{-n_s}^+, \dots, \mu_{-1}^+, \mu_1^+, \dots, \mu_{n_u}^+ \right\}$ with
 $\operatorname{Re} \mu_{-n_s}^+ \leq \dots \leq \operatorname{Re} \mu_{-1}^+ < 0 < \operatorname{Re} \mu_1^+ \leq \dots \leq \operatorname{Re} \mu_{n_u}^+$,
 $\lambda \in \tilde{\Lambda}$, for some set $\tilde{\Lambda} \in \Lambda$, $n_u^- + n_s^+ + 1 = n$.

- For each $s \in [0, s^*]$ (2.1) has a heteroclinic orbit $\Gamma(s) : x(t, s)$;
 in the process of continuation as $s \rightarrow s^*$, $\lambda \rightarrow \lambda(s^*) \equiv \lambda^*$, $x(t, s) \rightarrow$
 (iii) $x(t, s^*) \equiv x^*(t)$, a heteroclinic orbit $\Gamma(s^*)$, such that $x^*(t) \in W_{loc}^c(x_-^*)$
 as $t \rightarrow -\infty$, and $x^*(t) \in W_{loc}^{ss}(x_+^*)$ as $t \rightarrow \infty$.

- for $\lambda \in \operatorname{Cl}(\tilde{\Lambda})$, $\mu_0^-(\lambda)$ is a simple eigenvalue, and
 (iv) $\operatorname{Re} \mu_{-1}^+ < \gamma_2 < 0 \leq \mu_0^- < \gamma_1 < \operatorname{Re} \mu_1^-$, for some fixed $\gamma_1, \gamma_2 \in \mathbb{R}$;
 in the process of continuation as $\lambda \rightarrow \lambda^*$, $\mu_0^- \rightarrow 0^+$.

- (v) $W^{cu}(x_-)$ and $W^{cs}(x_+)$ intersect transversely along $\Gamma(s)$, $0 \leq s \leq s^*$.

Remark 1. The assumption (i) on f can be relaxed (see Remark 2 in [8]).•

The hypothesis (i) implies that f satisfies the Lipschitz condition

$$(2.4a) \quad |f(x_1, \lambda) - f(x_2, \lambda)| \leq \gamma |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

From (2.2) and boundedness of $|f_{xx}|$ we also have

$$(2.4b) \quad \begin{cases} |f(x_- + x, \lambda) - f_x(x_-, \lambda)x| \leq C|x|^2, & \forall x \in \mathbb{R}^n, \\ |f(x_+ + x, \lambda) - f_x(x_+, \lambda)x| \leq C|x|^2, & \forall x \in \mathbb{R}^n. \end{cases}$$

For $\gamma = (\gamma_1, \gamma_2)^T$ as in hypothesis (iv) we introduce a weight function

$$(2.5) \quad w_\gamma(t) = \begin{cases} e^{-\gamma_1 t}, & t \leq 0, \\ e^{-\gamma_2 t}, & t > 0, \end{cases}$$

and denote by $B_{\gamma,0} \equiv B_{\gamma,0}(\mathbb{R}^n)$ the set of L_∞ functions $x : \mathbb{R} \rightarrow \mathbb{R}^n$ such that $|w_\gamma(t)x(t)|$ is finite. The set of such functions form a Banach space $B_{\gamma,0} \equiv B_{\gamma,0}(\mathbb{R})$ with the norm $\|x\|_{\gamma,0} = \|w_\gamma x\|_{L_\infty}$. Let $B_{\gamma,j} \equiv B_{\gamma,j}(\mathbb{R})$ be the Banach space of all the L_∞ functions $x(t)$ such that $D^i x(\cdot) \in B_{\gamma,0}$, $i = 0, 1, \dots, j$, with the norm

$$(2.6) \quad \|u\|_{\gamma,j} = \sum_{i=0}^j \|D^i x\|_{\gamma,0}.$$

From the conditions on the eigenvalues of $f_x(x_-, \lambda)$, there exists a real matrix $P \equiv P(\lambda)$ such that its first column w_0^- is the eigenvector for μ_0^- , the next n_u^- columns w_i^- , $i = 1, \dots, n_u^-$, span the union of the generalized eigenspaces for μ_1^- through $\mu_{n_u^-}^-$ and the last n_s^- columns span the union of the generalized eigenspaces for $\mu_{-n_s^-}^-$ through μ_{-1}^- . Upon changing variables according to

$$(2.7) \quad x = x_-(\lambda) + P(\lambda) \begin{bmatrix} y \\ u \\ v \end{bmatrix}, \quad y \in \mathbb{R}, \quad u \in \mathbb{R}^{n_u^-}, \quad v \in \mathbb{R}^{n_s^-},$$

(2.1a) becomes

$$(2.8) \quad \begin{aligned} y' &= \mu_0^- y + h(y, u, v, \lambda), \\ u' &= A_1 u + g_1(y, u, v, \lambda), \\ v' &= A_2 v + g_2(y, u, v, \lambda), \end{aligned}$$

where

$$(2.9) \quad \begin{bmatrix} \mu_0^- y + h(y, u, v, \lambda) \\ A_1 u + g_1(y, u, v, \lambda) \\ A_2 v + g_2(y, u, v, \lambda) \end{bmatrix} = P^{-1}(\lambda) f \left(x_-(\lambda) + P(\lambda) \begin{bmatrix} y \\ u \\ v \end{bmatrix}, \lambda \right),$$

h , g_1 and g_2 are real valued, A_1 is an $n_u^- \times n_u^-$ matrix and A_2 is an $n_s^- \times n_s^-$ matrix, $\operatorname{Re} \sigma(A_1) > 0$, $\operatorname{Re} \sigma(A_2) < 0$, $h(0, 0, 0, \lambda) = 0$, $D_{(y,u,v)} h(0, 0, 0, \lambda) = 0$, for $g \equiv (g_1, g_2)$, $g(0, 0, 0, \lambda) = 0$, $D_{(y,u,v)} g(0, 0, 0, \lambda) = 0$.

By assumption (ii) $\sigma(A_1) = \{\mu_1^-, \dots, \mu_{n_u^-}^-\}$ and $\sigma(A_2) = \{\mu_{-n_s^-}^-, \dots, \mu_{-1}^-\}$. We look for a “slowly growing” solution of (2.1a) in the form

$$(2.10) \quad \begin{aligned} x_{sl} &= x_-(\lambda) + P(\lambda) \begin{bmatrix} y \\ U(y, \lambda) \\ V(y, \lambda) \end{bmatrix}, \\ y &= Y(t, \bar{\lambda}) \in \mathbb{R}, \quad \bar{\lambda} = (\lambda, \lambda_3), \quad u = U(y, \lambda) \in \mathbb{R}^{n_u^-}, \quad v = V(y, \lambda) \in \mathbb{R}^{n_s^-}, \end{aligned}$$

where for $A = \text{diag}(A_1, A_2)$, and $p \equiv (U, V)$ solves

$$(2.11) \quad D_y p(y, \lambda) [\mu_0^- y + h(y, p(y, \lambda), \lambda)] = Ap(y, \lambda) + g(y, p(y, \lambda), \lambda),$$

and $y = Y(t, \bar{\lambda})$ solves

$$(2.12) \quad \begin{aligned} y' &= \mu_0^- y + h(y, p(y), \lambda), \quad -\infty < t < T_- + 1, \\ y(T_-) &= \lambda_3. \end{aligned}$$

Note that by (2.8)-(2.10) x_{sl} satisfies

$$(2.13) \quad x'_{sl} = P \begin{bmatrix} \mu_0^- & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} P^{-1}(x_{sl} - x_-(\lambda)) + P \begin{bmatrix} h(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_1(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_2(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \end{bmatrix}.$$

Let $\chi(t)$ be a smooth real valued function satisfying

$$(2.14) \quad \chi(t) = \begin{cases} 1, & t \leq T_-, \\ 0, & t > T_- + 1. \end{cases}$$

Then a solution $x(t)$ of (2.1) can be written as

$$(2.15) \quad x(t) = z(t) + \chi(t)x_{sl}(t, \bar{\lambda}).$$

This implies $x(T_-) = z(T_-) + x_{sl}(T_-, \bar{\lambda})$, or, using (2.10) and (2.12),

$$(2.16) \quad x(T_-) = z(T_-) + x_-(\lambda) + P(\lambda) \begin{bmatrix} \lambda_3 \\ U(\lambda_3, \lambda) \\ V(\lambda_3, \lambda) \end{bmatrix}.$$

Substituting (2.15) into (2.1), the equation for $z(t)$ becomes

$$(2.17) \quad \begin{aligned} z'(t) - \tilde{f}(t, z(t), \bar{\lambda}) &= 0, \quad -\infty < t < \infty, \\ \tilde{f}(t, z(t), \bar{\lambda}) &\equiv f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) - \frac{\partial}{\partial t}(\chi(t)x_{sl}(t, \bar{\lambda})). \end{aligned}$$

Note that upon changing variables according to

$$(2.18) \quad x \equiv z + \chi x_{sl} = \chi x_-(\lambda) + P \begin{bmatrix} y + \chi Y \\ u + \chi U \\ v + \chi V \end{bmatrix}, \quad y \in \mathbb{R}, \quad u \in \mathbb{R}^{n_u}, \quad v \in \mathbb{R}^{n_s},$$

(2.1a) becomes

$$(2.19) \quad \begin{aligned} (z + \chi x_{sl})' &= P \begin{bmatrix} \mu_0^- & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} P^{-1}(z + \chi x_{sl} - \chi x_-(\lambda)) \\ &+ P \begin{bmatrix} h(P^{-1}(z + \chi x_{sl} - \chi x_-(\lambda)), \lambda) \\ g_1(P^{-1}(z + \chi x_{sl} - \chi x_-(\lambda)), \lambda) \\ g_2(P^{-1}(z + \chi x_{sl} - \chi x_-(\lambda)), \lambda) \end{bmatrix}. \end{aligned}$$

In particular, in view of (2.13) and (2.14), for $t < T_-$ (2.19), and hence (2.17), reduces to

$$(2.20) \quad \begin{aligned} z' = & P \begin{bmatrix} \mu_0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} P^{-1} z \\ & + P \left(\begin{bmatrix} h(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \\ g_1(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \\ g_2(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \end{bmatrix} - \begin{bmatrix} h(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_1(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_2(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \end{bmatrix} \right), \quad t < T_-. \end{aligned}$$

which is convenient for the derivation of the solution algorithm and the error analysis below. We consider (2.17) as an operator equation

$$(2.21) \quad F(\cdot, z(\cdot), \bar{\lambda}) = 0,$$

where

$$F : B_{\gamma,1} \times \bar{\Lambda} \rightarrow B_{\gamma,0}, \quad \bar{\Lambda} \equiv \Lambda \times \mathbb{R}.$$

Let $(z_0, \bar{\lambda}^0) \in B_{\gamma,1} \times \bar{\Lambda}$ be a solution of (2.21) with

$$(2.22) \quad z_0(t) = q(t) - \chi(t)x_{sl}(t, \bar{\lambda}^0),$$

and define the linearized operator $F_z^0 : B_{\gamma,1} \rightarrow B_{\gamma,0}$ by

$$(2.23) \quad \begin{aligned} (F_z^0 z)(t) &\equiv (D_z F(\cdot, z_0(\cdot), \bar{\lambda}^0) z)(t) \\ &= z'(t) - f_x(q(t), \bar{\lambda}^0) z(t). \end{aligned}$$

The following lemma was proved in [3] (see also [11], [8]).

Lemma 1. *The operator $F_z^0 : B_{\gamma,1} \rightarrow B_{\gamma,0}$ is a Fredholm operator of index -1 . Moreover,*

$$(2.24) \quad \begin{aligned} a) \quad & \dim \mathcal{N}(F_z^0) = 0, \\ b) \quad & \text{codim} \mathcal{R}(F_z^0) = 1. \end{aligned}$$

Proof. We only note that (2.24a) follows from the assumption (v), and (2.24b) follows from the first part of Lemma 1 and (2.24a).•

We next rewrite the phase condition (2.3) (which essentially fixes the parameter λ_3 in (2.21)) in an equivalent form as

$$(2.25) \quad \begin{aligned} l(z, \bar{\lambda}) &\equiv \int_{-\infty}^{\infty} (f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) - f(q(t), \lambda^0)) \\ &\cdot \frac{\partial}{\partial t} f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) dt = 0, \end{aligned}$$

where the functional $l : B_{\gamma,0} \times \bar{\Lambda} \rightarrow \mathbb{R}$ is clearly well defined. Combining this with (2.21) we obtain

$$(2.26) \quad \Phi(\cdot, z, \bar{\lambda}) \equiv (F(\cdot, z, \bar{\lambda}), l(z, \bar{\lambda})) = 0,$$

where $\Phi : B_{\gamma,1} \times \bar{\Lambda} \rightarrow B_{\gamma,0} \times \mathbb{R}$. Φ is obviously differentiable with

$$(2.27) \quad \Phi'(\cdot, z_0, \bar{\lambda}^0) = \begin{bmatrix} F_z^0 & F_{\lambda_1}^0 & F_{\lambda_2}^0 & F_{\lambda_3}^0 \\ l_z^0 & l_{\lambda_1}^0 & l_{\lambda_2}^0 & l_{\lambda_3}^0 \end{bmatrix},$$

where

$$(2.28) \quad \begin{aligned} F_z^0 v &= v' - \tilde{f}_x(\cdot, z_0(\cdot), \bar{\lambda}^0)v, \\ F_{\lambda_i}^0 &\equiv D_{\lambda_i} F(\cdot, z_0(\cdot), \bar{\lambda}^0) = -\frac{\partial}{\partial \lambda_i} \tilde{f}(\cdot, z_0(\cdot), \bar{\lambda}^0), \quad i = 1, 2, 3; \end{aligned}$$

$$(2.29) \quad \begin{aligned} l_z^0 v &\equiv (D_z l(z_0, \lambda^0))v = \int_{-\infty}^{\infty} f_x(q(t), \lambda^0)v(t) \cdot q''(t)dt, \\ l_{\lambda_i}^0 &\equiv D_{\lambda_i} l(z_0, \lambda^0) = \int_{-\infty}^{\infty} \frac{\partial}{\partial \lambda_i} f(z_0(t) + \chi(t)x_{sl}(t, \bar{\lambda}^0), \lambda^0) \cdot q''(t)dt, \quad i = 1, 2, 3. \end{aligned}$$

Lemma 2. (bordering Lemma, see e.g. [1, Lemma 2.4]). Let X, Y be Banach spaces and consider the operator

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : X \times \mathbb{R}^p \rightarrow Y \times \mathbb{R}^q$$

with bounded linear operators $A : X \rightarrow Y, B : C : X \rightarrow \mathbb{R}^q, \mathbb{R}^p \rightarrow Y, D : \mathbb{R}^p \rightarrow \mathbb{R}^q$. If A is Fredholm of index N then S is Fredholm of index

$$(2.30) \quad \text{Ind}(S) = N + p - q.$$

Lemma 3. The linearized operator $\Phi'(\cdot, z_0, \bar{\lambda}^0) : B_{\gamma,1} \times \bar{\Lambda} \rightarrow B_{\gamma,0} \times \mathbb{R}$ is a Fredholm operator of index 1.

Moreover,

$$(2.31) \quad \text{codim } \mathcal{R}(\Phi'(\cdot, z_0, \bar{\lambda}^0)) = 0.$$

Proof. The first part of Lemma 3 follows from (2.30) with $N = -1, p = 3, q = 1$.

Next assume that $(v, \nu_1, \nu_2, \nu_3) \in \mathcal{N}(\Phi'(\cdot, z_0, \bar{\lambda}^0))$. Then by (2.27), (2.28) and (2.29) we have

$$(2.32) \quad \begin{aligned} &F_z^0 v + F_{\lambda_1}^0 \nu_1 + F_{\lambda_2}^0 \nu_2 + F_{\lambda_3}^0 \nu_3 \\ &= v' - \tilde{f}_x(\cdot, z_0(\cdot), \bar{\lambda}^0)v - \sum_{i=1}^3 \nu_i \frac{\partial}{\partial \lambda_i} \tilde{f}(\cdot, z_0(\cdot), \bar{\lambda}^0) = 0, \end{aligned}$$

where

$$\tilde{f}(t, z_0(t), \bar{\lambda}^0) \equiv f(z_0(t) + \chi(t)x_{sl}(t, \bar{\lambda}^0), \lambda^0) - \frac{\partial}{\partial t}(\chi(t)x_{sl}^0(t, \lambda^0));$$

$$(2.33) \quad \begin{aligned} & l_z^0 v + l_{\lambda_1}^0 \nu_1 + l_{\lambda_2}^0 \nu_2 + l_{\lambda_3}^0 \nu_3 \\ &= \int_{-\infty}^{\infty} \left[f_x(q(t), \lambda^0) v(t) + \sum_{i=1}^3 \nu_i \frac{\partial}{\partial \lambda_i} f(z_0(t) + \chi(t)x_{sl}(t, \bar{\lambda}^0), \lambda^0) \right] \cdot q''(t) dt = 0. \end{aligned}$$

From (2.32) we see that $F_z^0 v \in B_{\gamma,0}$, while $F_{\lambda_1}^0 \nu_1 + F_{\lambda_2}^0 \nu_2 + F_{\lambda_3}^0 \nu_3 \notin B_{\gamma,0}$. Hence by (2.32) $F_z^0 v = 0$. Since also by Lemma 1 $\dim \mathcal{N}(F_z^0) = 0$, we conclude that $v = 0$. Substituting $v = 0$ into (2.32), (2.33), reduces them to

$$(2.34) \quad \begin{aligned} & \sum_{i=1}^3 \nu_i \frac{\partial}{\partial \lambda_i} \left[f(z_0(t) + \chi(t)x_{sl}(t, \bar{\lambda}^0), \lambda^0) - \frac{\partial}{\partial t}(\chi(t)x_{sl}(t, \lambda^0)) \right] = 0, \\ & \int_{-\infty}^{\infty} \left[\sum_{i=1}^3 \nu_i \frac{\partial}{\partial \lambda_i} f(z_0(t) + \chi(t)x_{sl}(t, \bar{\lambda}^0), \lambda^0) \right] \cdot q''(t) dt = 0. \end{aligned}$$

It is easy to see that (2.34) implies $\dim \mathcal{N}(\Phi'(\cdot, z_0, \bar{\lambda}^0)) = 1$. Combining this result with the first part of Lemma 3 completes the proof of the second part of Lemma 3. •

We approximate the problem (2.26) by

$$(2.35) \quad \Phi_T(\cdot, z, \bar{\lambda}) \equiv (F_T(\cdot, z, \bar{\lambda}), l_T(z, \bar{\lambda})) = 0,$$

where $\Phi_T : B_{\gamma,1} \times \bar{\Lambda} \rightarrow B_{\gamma,0} \times \mathbb{R}$; and the operators $F_T : B_{\gamma,1} \times \bar{\Lambda} \rightarrow B_{\gamma,0}$ and $l_T : B_{\gamma,0} \times \bar{\Lambda} \rightarrow \mathbb{R}$ are defined, respectively, by

$$(2.36) \quad F_T(t, z, \bar{\lambda}) = \begin{cases} z' - P(\lambda) \begin{bmatrix} \mu_0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & A_2 \end{bmatrix} P^{-1}(\lambda) z, & t < T_-, \\ z' - f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) + \frac{\partial}{\partial t}(\chi(t)x_{sl}(t, \bar{\lambda})), & t \in [T_-, T_+], \\ z' - f_x(x_+, \lambda) z, & t > T_+, \end{cases}$$

$$(2.37) \quad \begin{aligned} & l_T(z, \bar{\lambda}) \equiv \int_{T_-}^{T_+} (f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) - f(q(t), \lambda^0)) \\ & \cdot \frac{\partial}{\partial t} f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) dt = 0. \end{aligned}$$

Define also

$$(2.38) \quad \begin{aligned} x_{sl}^m &= x_-(\lambda) + P(\lambda) \begin{bmatrix} y \\ U^m(y, \lambda) \\ V^m(y, \lambda) \end{bmatrix}, \\ y &= Y_m(t, \bar{\lambda}) \in \mathbb{R}, \quad \bar{\lambda} = (\lambda, \lambda_3), \quad u = U^m(y, \lambda) \in \mathbb{R}^{n_u}, \quad v = V^m(y, \lambda) \in \mathbb{R}^{n_s}, \end{aligned}$$

to be an approximation to a “slowly growing” solution (2.10) of (2.1) such that for $p_m \equiv (U^m, V^m)$, $p_m(y, \lambda)$ is the Taylor polynomial of order m about $y = 0$ for $p(y, \lambda)$ which is obtained by solving

$$(2.39) \quad \begin{aligned} D_y p_m(y, \lambda) [\mu_0 y + h(y, p_m(y, \lambda), \lambda)] \\ = A p_m(y, \lambda) + g(y, p_m(y, \lambda), \lambda), \end{aligned}$$

and $y = Y_m(t, \bar{\lambda})$ solves

$$(2.40) \quad \begin{aligned} y' &= \mu_0 y + h(y, p_m(y, \lambda), \lambda), \quad -\infty < t < T_-, \\ y(T_-) &= \lambda_3. \end{aligned}$$

Then, (see e.g. Hassard [12])

$$(2.41) \quad \begin{aligned} |p(y, \lambda) - p_m(y, \lambda)| &= O(|y|^{m+1}), \\ \sup_{t \leq T_-} |y(t, \bar{\lambda}) - Y_m(t, \bar{\lambda})| &= O(|\lambda_3|^{m+1}), \end{aligned}$$

and hence

$$(2.42) \quad |p(y, \lambda) - p_m(y_m, \lambda)| = O(|y|^{m+1}),$$

where y solves (2.12). Therefore

$$(2.43) \quad \sup_{t \leq T_-} |x_{sl}(t, \bar{\lambda}) - x_{sl}^m(t, \bar{\lambda})| = O(|\lambda_3|^{m+1}).$$

Next note that since the solution to (2.35) $z \in B_{\gamma,1}$, we can rewrite F_T for $t \notin [T_-, T_+]$ as

$$F_T(t, z, \bar{\lambda}) = \begin{cases} z' - P(\lambda) \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1}(\lambda) z, & t < T_-, \\ z' - f_x(x_+, \lambda) z, & t > T_+. \end{cases}$$

Hence we can rewrite the approximate problem (2.35) in the form more convenient for the derivation of the solution algorithm: find $(z, \bar{\lambda}) = (z_T, \bar{\lambda}_T)$ from

(2.44)

$$\begin{aligned}
a) \quad & z'(t) - f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) + \frac{\partial}{\partial t}(\chi(t)x_{sl}(t, \bar{\lambda})) = 0, \quad T_- < t < T_+, \\
& \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \bar{\Lambda}, \\
b) \quad & z(t) = \sum_{i=1}^{n_u^-} c_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(t, \lambda) \\ 0 \end{bmatrix}, \quad t \leq T_-, \\
c) \quad & z(t) = \sum_{i=-n_s^+}^{-1} c_{1i} v_i(t, \lambda), \quad t \geq T_+, \\
d) \quad & \int_{T_-}^{T_+} (f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(z(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) dt = 0,
\end{aligned}$$

where $u_i(t, \lambda) \in \mathbb{R}^{n_u^-}$, $i = 1, \dots, n_u^-$, and $v_i(t, \lambda) \in \mathbb{R}^n$, $i = -n_s^+, \dots, -1$, are assumed to be computed analytically. Here u_i form a fundamental set of solutions for the linear system (with constant coefficients)

$$(2.45a) \quad u'(t) = A_1 u(t)$$

and v_i are the solutions of the linear system (with constant coefficients)

$$(2.45b) \quad v'(t) = f_x(x_+, \lambda) v(t),$$

which correspond to the eigenvalues μ_i^+ , $i = -n_s^+, \dots, -1$. Introduce also an auxiliary problem: find $(z, \bar{\lambda}) = (z_{T,m}, \bar{\lambda}_{T,m})$ from

(2.46)

$$\begin{aligned}
a) \quad & z'(t) - f(z(t) + \chi(t)x_{sl}^m(t, \bar{\lambda}), \lambda) + \frac{\partial}{\partial t}(\chi(t)x_{sl}^m(t, \bar{\lambda})) = 0, \\
& \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \bar{\Lambda}, \quad T_- < t < T_+, \\
b) \quad & z(t) = \sum_{i=1}^{n_u^-} c_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(t, \lambda) \\ 0 \end{bmatrix}, \quad t \leq T_-, \\
c) \quad & z(t) = \sum_{i=-n_s^+}^{-1} c_{1i} v_i(t, \lambda), \quad t \geq T_+, \\
d) \quad & \int_{T_-}^{T_+} (f(z(t) + \chi(t)x_{sl}^m(t, \bar{\lambda}), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(z(t) + \chi(t)x_{sl}^m(t, \bar{\lambda}), \lambda) dt = 0.
\end{aligned}$$

or, in the operator form,

$$(2.47) \quad \Phi_{T,m}(\cdot, z, \bar{\lambda}) \equiv (F_{T,m}(\cdot, z, \bar{\lambda}), l_{T,m}(z, \bar{\lambda})) = 0,$$

where the operator $F_{T,m}$ is obtained from the operator F_T by replacing in (2.36) x_{sl} by its approximation x_{sl}^m . and $l_{T,m}(z, \bar{\lambda})$ is given by

Actually, we are interested in an approximation, call it $x_{T,m}$, of the solution x of (2.1)–(2.3). We define this approximation by

$$(2.48) \quad x_{T,m} \equiv z_{T,m} + \chi x_{sl}^m.$$

Once x_{sl}^m is found, we can compute $x_{T,m}$ directly. Substituting $z_{T,m}$ from (2.48) into (2.46) we obtain the problem for $(x, \bar{\lambda}) = (x_{T,m}, \bar{\lambda}_{T,m})$:

$$\begin{aligned}
(2.49) \quad & a) \quad x'(t) - f(x(t), \lambda) = 0, \quad \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \bar{\Lambda}, \quad T_- < t < T_+, \\
& b) \quad x(t) = \sum_{i=1}^{n_u^-} c_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(t, \lambda) \\ 0 \end{bmatrix} + x_{sl}^m(t, \bar{\lambda}), \quad t \leq T_-, \\
& c) \quad x(t) = x_+(\lambda) + \sum_{i=-n_s^+}^{-1} c_{1i} v_i(t, \lambda), \quad t \geq T_+, \\
& d) \quad \int_{T_-}^{T_+} (f(x(t), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(x(t), \lambda) dt = 0.
\end{aligned}$$

Restricting t to the interval $[T_-, T_+]$ in the (2.48) we obtain an equivalent problem

$$\begin{aligned}
(2.50) \quad & a) \quad x'(t) - f(x(t), \lambda) = 0, \quad \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \bar{\Lambda}, \quad T_- < t < T_+, \\
& b) \quad x(T_-) = \sum_{i=1}^{n_u^-} c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix} + x_{sl}^m(T_-, \bar{\lambda}), \\
& c) \quad x(T_+) = x_+(\lambda) + \sum_{i=-n_s^+}^{-1} c'_{1i} v_i(T_+, \lambda), \\
& d) \quad \int_{T_-}^{T_+} (f(x(t), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(x(t), \lambda) dt = 0,
\end{aligned}$$

where we replaced, for convenience, c_{0i} , c_{1i} by c'_{0i} , c'_{1i} , respectively. From (2.50) we obtain the solution algorithm as follows. We set $T = |T_-| + T_+$ and then scale the independent variable t so that it varies from 0 to 1. The actual period T therefore appears explicitly in (2.52a) below, whereas T_- and T_+ do not appear explicitly in (2.52). Next, setting (2.50a)

$$\begin{aligned}
(2.50a) \quad & \epsilon_- = \sqrt{\sum_{i=1}^{n_u^-} \left| c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix} \right|^2}, \quad c_{0i} = \left| c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix} \right| / \epsilon_-, \quad i = 1, \dots, n_u^-, \\
(2.50b) \quad & \epsilon_+ = \sqrt{\sum_{i=-n_s^+}^{-1} |c'_{1i} v_i(T_+, \lambda)|^2}, \quad c_{1i} = |c'_{1i} v_i(T_+, \lambda)| / \epsilon_+, \quad i = -n_s^+, \dots, -1,
\end{aligned}$$

$$(2.51) \quad w_i^- = \frac{c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix}}{\left| c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix} \right|}, \quad i = 1, \dots, n_u^-, \quad w_i^+ = \frac{c'_{1i} v_i(T_+, \lambda)}{|c'_{1i} v_i(T_+, \lambda)|}, \quad i = -n_s^+, \dots, -1,$$

we rewrite the terms $c'_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(T_-, \lambda) \\ 0 \end{bmatrix}$, $c'_{1i} v_i(T_+, \lambda)$ in (2.50b) and (2.50c), respectively, as $\epsilon_- c_{0i} w_i^-$ and $\epsilon_+ c_{1i} w_i^+$ and add the normalization condition (2.52d), below. Finally,

replacing x_{sl}^m , in (2.50b) by its expression in (2.38) we arrive at the equations for our solution algorithm:

$$\begin{aligned}
(2.52) \quad & a) \quad x'(t) - Tf(x(t), \lambda) = 0, \quad \bar{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \in \bar{\Lambda}, \quad 0 < t < 1, \\
& b) \quad x(0) = x_-(\lambda) + \epsilon_- \sum_{i=1}^{n_u^-} c_{0i} w_i^-(\lambda) + P(\lambda) \begin{bmatrix} \lambda_3 \\ U^m(\lambda_3, \lambda) \\ V^m(\lambda_3, \lambda) \end{bmatrix}, \\
& c) \quad x(1) = x_+(\lambda) + \epsilon_+ \sum_{i=-n_s^+}^{-1} c_{1i} w_i^+(\lambda), \\
& d) \quad \sum_{i=1}^{n_u^-} |c_{0i}|^2 = 1, \quad \sum_{i=-n_s^+}^{-1} |c_{1i}|^2 = 1, \\
& e) \quad \int_0^1 (f(x(t), \lambda) - f(q(t), \lambda^0)) \cdot \frac{\partial}{\partial t} f(x(t), \lambda) dt = 0.
\end{aligned}$$

Equations (2.52) represent n coupled differential equations subject to $n_c = 2n + 3$ constraints, of which e) is an integral condition. In addition to the vector function variable $x(t) \in \mathbb{R}^n$ we have scalar variables

$$\begin{aligned}
(2.53) \quad & \bar{\lambda} \in \mathbb{R}^3, \quad \epsilon_-, \epsilon_+ \in \mathbb{R}, \\
& c_{0i} \in \mathbb{R}, \quad i = 1, \dots, n_u^-, \\
& c_{1i} \in \mathbb{R}, \quad i = -n_s^+, \dots, -1.
\end{aligned}$$

Here the normalized vectors $w_i^-(\lambda)$ and $w_i^+(\lambda)$ span the union of the generalized eigenspaces for μ_1^- through $\mu_{n_u^-}^-$ and for $\mu_{-n_s^+}^+$ through μ_{-1}^+ , respectively (see also (2.45)). In practice, they can be computed from the corresponding eigenvalue problems at each step of the continuation of the continuation process. By hypothesis (ii) $n_u^- + n_s^+ = n - 1$, Hence the total number of scalar variables equals $n_v \equiv n + 4$. Formally we need $n_v = n_c - n$ for a single heteroclinic connection. Here we shall be interested in computing an entire *branch* (one dimensional continuum) of orbits, in which case $n_v = n_c - n + 1$. This requirement is obviously satisfied here.

3. Error analysis. The error analysis is an extension of the analysis in [8]. To estimate the error $\|x - x_{T,m}\|_{0,1} + |\bar{\lambda} - \bar{\lambda}_{T,m}|$, we split it into three terms:

$$\begin{aligned}
(3.1) \quad & \|x - x_{T,m}\|_{0,1} + |\bar{\lambda} - \bar{\lambda}_{T,m}| \leq \|(z + \chi x_{sl}) - (z_{T,m} + \chi x_{sl}^m)\|_{0,1} + |\bar{\lambda} - \bar{\lambda}_{T,m}| \\
& \leq \|(z + \chi x_{sl}) - (z_T + \chi x_{sl})\|_{0,1} + \|(z_T + \chi x_{sl}) - (z_{T,m} + \chi x_{sl}^m)\|_{0,1} + |\bar{\lambda} - \bar{\lambda}_{T,m}| \\
& \leq (\|z - z_T\|_{0,1} + |\bar{\lambda} - \bar{\lambda}_T|) + (\|z_T - z_{T,m}\|_{0,1} + |\bar{\lambda}_T - \bar{\lambda}_{T,m}|) + \|\chi(x_{sl} - x_{sl}^m)\|_{0,1}
\end{aligned}$$

and then estimate each term in the right hand side of (3.1) separately.

We first analyze the exact problem (2.26) together with the approximate problem (2.35) and derive an estimate for $\|z - z_T\|_{\gamma,1} + |\bar{\lambda} - \bar{\lambda}_T|$. As in [8], to establish existence and uniqueness for the exact problem and error estimates for the approximate problem we

shall use a version of the Implicit Function Theorem in Descoux and Rappaz [4]:

Theorem 1. *Let X and Y be two real Banach spaces, $\Phi, \Phi_n : X \rightarrow Y$ be a C^p mapping and a family of mappings, respectively, with $p \geq 1$, and $x_0 \in X$ be such that $\Phi(x_0) = 0$. Assume that x_0 is regular point in the following sense:*

$$(3.2) \quad \begin{aligned} a) & \Phi'(x_0) : X \rightarrow Y \text{ is a Fredholm operator of index 1,} \\ b) & \mathcal{R}(\Phi'(x_0)) = Y, \end{aligned}$$

Assume that $\mathcal{N}(\Phi'(x_0))$ is spanned by some vector $\varphi_0 \in X$, $\varphi_0 \neq 0$, and let $\bar{l} : X \rightarrow \mathbb{R}$ be a linear functional that satisfies

$$(3.3) \quad \bar{l}(\varphi_0) \neq 0.$$

Assume also that for any $0 \leq k \leq p-1$ and for any fixed $x, \xi_1, \dots, \xi_k \in X$, we have

$$(3.4) \quad \lim_{n \rightarrow \infty} \left\| \left(\Phi^{(k)}(x) - \Phi_n^{(k)}(x) \right) (\xi_1, \dots, \xi_k) \right\| = 0.$$

and that

$$(3.5) \quad \lim_{n \rightarrow \infty} \left\| \Phi'(x_0) - \Phi'_n(x_0) \right\| = 0.$$

Then there exist positive constants n_0, s_0, α, K and two unique C^1 maps $x(s), x_n(s) \in X$, $|s| < s_0$, satisfying for $|s| < s_0$, respectively, the conditions:

$$(3.6) \quad (\Phi(x(s)), \bar{l}(x(s) - x_0) - s) = 0, \quad \|x(s) - x_0\| < \alpha,$$

$$(3.7) \quad (\Phi_n(x_n(s)), \bar{l}(x_n(s) - x_0) - s) = 0, \quad \|x_n(s) - x_0\| < \alpha, \quad n > n_0.$$

Moreover $x(0) = x_0$ and for $|s| < s_0$, $n > n_0$, $0 \leq k \leq p-1$ we have an error estimate

$$(3.8) \quad \left\| x^{(k)}(s) - x_n^{(k)}(s) \right\| \leq C \sum_{l=0}^k \left\| \frac{d^l}{ds^l} \Phi_n(x(s)) \right\|,$$

where C is a positive constant independent of n .

Theorem 1 requires a consistency condition (a pointwise convergence) and a stability condition. These conditions are given by the following lemma.

Lemma 4. *Let $\bar{z} = (z, \lambda, \lambda_3)$, $(z, \lambda) \in B_{\gamma,1} \times \mathbb{R}^2$, where $\bar{z}_0 = (z_0, \lambda^0, \lambda_3^0)$ solves (2.21); and assume that f is C^p , $p \geq 1$,*

$$(3.9) \quad \sup_{t < T_-} |x_{sl}(t)| \leq C |\lambda_3| = O\left(\frac{1}{|T_-|^\alpha}\right), \quad \alpha > 0, \text{ as } T_- \rightarrow -\infty,$$

$$\text{and } \|P(\lambda)\|_\infty, \|P^{-1}(\lambda)\|_\infty \leq C.$$

Then for $-T_-, T_+ \rightarrow \infty$ we have: for any $0 \leq k \leq p-1$ and for any fixed $\bar{z}, \xi_1, \dots, \xi_k \in B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right]$, for some $C_1 > 0$, fixed,

$$(3.10) \quad \left\| \left(\Phi^{(k)}(\bar{z}) - \Phi_T^{(k)}(\bar{z})(\xi_1, \dots, \xi_k) \right) \right\|_{\gamma,0} \leq C \|z\|_{\gamma,1} \left(|\lambda_3| + e^{T_+ \gamma_2} \right) \rightarrow 0,$$

$$(3.11) \quad \left\| (\Phi'(\bar{z}_0) - \Phi_T'(\bar{z}_0))z \right\|_{\gamma,0} \leq C \|z\|_{\gamma,1} \left(|\lambda_3| + e^{T_+ \gamma_2} \right) \rightarrow 0,$$

where $\gamma_1 > 0$, $\gamma_2 < 0$ are as in hypothesis (iv), and C does not depend on λ , T_- and T_+ .

Proof. We verify (3.10) for $k = 0$. For $k > 0$ the proof is similar. From the definitions (2.21) and (2.36) of F and F_T , respectively, and the expression (2.20) for F when $t \leq T_-$ we have

$$(3.12) \quad \begin{aligned} & \|F(\bar{z}) - F_T(\bar{z})\|_{\gamma,0} \\ & \leq \|F(\bar{z}) - F_T(\bar{z})\|_{B_{\gamma,0}(-\infty, T_-]} + \|F(\bar{z}) - F_T(\bar{z})\|_{B_{\gamma,0}[T_+, \infty)}, \end{aligned}$$

$$(3.13) \quad \begin{aligned} & \|F(\bar{z}) - F_T(\bar{z})\|_{B_{\gamma,0}(-\infty, T_-]} \leq \\ & \left\| P \begin{bmatrix} h(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \\ g_1(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \\ g_2(P^{-1}(x_{sl} - x_-(\lambda) + z), \lambda) \end{bmatrix} - \begin{bmatrix} h(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_1(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \\ g_2(P^{-1}(x_{sl} - x_-(\lambda)), \lambda) \end{bmatrix} \right\|_{B_{\gamma,0}(-\infty, T_-]} \\ & \leq C \|(|x_{sl}| |z|)\|_{B_{\gamma,0}(-\infty, T_-]} \leq C \sup_{t < T_-} |x_{sl}(t)| |w_\gamma(t) z(t)| \\ & \leq C \|z\|_{\gamma,0} \sup_{t < T_-} |x_{sl}(t)| \leq C \|z\|_{\gamma,0} |\lambda_3|. \end{aligned}$$

The estimate

$$(3.14) \quad \|F(\bar{z}) - F_T(\bar{z})\|_{B_{\gamma,0}[T_+, \infty)} \leq C \|z\|_{\gamma,1} e^{T_+ \gamma_2}$$

is obtained in the same way as the estimate (3.9) in [8]. Combining the above, we obtain (3.10). The estimate (3.11) is obtained similarly. •

Combining Lemmas 3–4 with Theorem 1, we obtain Theorem 2 and Corollary 1 in the same way as Theorem 2 and Corollary 1 in [8].

Theorem 2. Assume that the hypotheses (i)–(v) hold. Let $\bar{z} = (z, \lambda, \lambda_3)$, $(z, \lambda) \in B_{\gamma,1} \times \mathbb{R}^2$, where $\bar{z}_0 = (z_0, \lambda^0, \lambda_3^0)$ solves (2.21); and assume that f is C^p , $p \geq 1$,

$$(3.15) \quad \sup_{t < T_-} |x_{sl}(t)| \leq C |\lambda_3| = O\left(\frac{1}{|T_-|^\alpha}\right), \quad \alpha > 0, \text{ as } T_- \rightarrow -\infty,$$

and $\|P(\lambda)\|_\infty, \|P^{-1}(\lambda)\|_\infty \leq C$.

Then the operator $\Phi : B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right] \rightarrow B_{\gamma,0} \times \mathbb{R}$ defined by (2.26) satisfies:

$$(3.16) \quad \begin{aligned} & a) \quad \mathcal{R}(\Phi'(\cdot, \bar{z}_0)) = B_{\gamma,0} \times \mathbb{R}, \\ & b) \quad \dim \mathcal{N}(\Phi'(\cdot, \bar{z}_0)) = 1. \end{aligned}$$

Let $(p, \nu) \in B_{\gamma,1} \times \mathbb{R}^3$ be a vector that spans $\mathcal{N}(\Phi'(\cdot, \bar{z}_0))$. Define a linear functional $\bar{l} : B_{\mu,1} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ by

$$(3.17) \quad \bar{l}(z, \lambda) = \int_{T_-}^{T_+} z(t) \cdot p(t) dt + \lambda \cdot \nu, \quad T_- < 0 < T_+.$$

Note that

$$\bar{l}(p, \nu) \neq 0.$$

Define also the operators $\bar{\Phi}, \bar{\Phi}_T : B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right] \times \mathbb{R} \rightarrow B_{\gamma,0} \times \mathbb{R}^2$ by

$$(3.18) \quad \begin{aligned} \bar{\Phi}(\bar{z}, s) &\equiv (\Phi(\bar{z}(s)), \bar{l}(\bar{z}(s) - \bar{z}_0) - s), \\ \bar{\Phi}_T(\bar{z}, s) &\equiv (\Phi_T(\bar{z}(s)), \bar{l}(\bar{z}(s) - \bar{z}_0) - s). \end{aligned}$$

Then the linearized operator $D_{\bar{z}}\bar{\Phi}(\bar{z}_0, s_0) : B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right] \rightarrow B_{\gamma,0} \times \mathbb{R}^2$ is 1-1.

There exist positive constants $-T_-^0, T_+^0, s_0, \alpha, C$ and the two unique C^1 maps $\bar{z}(s) \equiv (z(s), \bar{\lambda}(s)), \bar{z}_T(s) \equiv (z_T(s), \bar{\lambda}_T(s)) \in B_{\gamma,1} \times \mathbb{R}^3, |s| < s_0$, satisfying for $|s| < s_0$, respectively, the conditions:

$$(3.19) \quad \bar{\Phi}(\bar{z}, s) = 0, \quad \|\bar{z}(s) - \bar{z}_0\|_{B_{\gamma,1} \times \mathbb{R}^3} < \alpha;$$

the linearized operator $D_{\bar{z}}\bar{\Phi}_T(\bar{z}_0, s_0) : B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right] \rightarrow B_{\gamma,0} \times \mathbb{R}^2$ is 1-1, and for any $\bar{z} \equiv (z, \bar{\lambda}) \in B_{\gamma,1} \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right]$ there holds an a priori estimate

$$(3.20) \quad \begin{aligned} &\|D_{\bar{z}}\bar{\Phi}_T(\bar{z}_0, s_0)\bar{z}\|_{B_{\epsilon,0} \times \mathbb{R}^2} \geq C \|\bar{z}\|_{B_{\epsilon,1} \times \mathbb{R}^3}, \\ &\forall T_- < T_-^0, \forall T_+ > T_+^0, \quad \forall \epsilon = (\epsilon_1, \epsilon_2) \text{ with } \gamma_2 < \epsilon_2 < 0 \leq \mu_0^- < \epsilon_1 < \gamma_1, \end{aligned}$$

where C does not depend on $\lambda, T_-, T_+, \epsilon$;

$$(3.21) \quad \bar{\Phi}(\bar{z}_T, s) = 0, \quad \|\bar{z}_T(s) - \bar{z}_0\|_{B_{\gamma,1} \times \mathbb{R}^3} < \alpha, \quad T_- < T_-^0, T_+ > T_+^0.$$

Moreover, $\bar{z}(0) = \bar{z}_0$, and for $|s| < s_0, T_- < T_-^0, T_+ > T_+^0, 0 \leq k \leq p-1$ we have an error estimate

$$(3.22) \quad \begin{aligned} &\left| \bar{\lambda}^{(k)}(s) - \bar{\lambda}_T^{(k)}(s) \right| + \left\| z^{(k)}(s) - z_T^{(k)}(s) \right\|_{\gamma,1} \\ &\leq C \sum_{l=0}^k \left\| z^{(l)}(s) \right\|_{\gamma,1} \left(|\lambda_3| + e^{T_+ \gamma_2} \right), \end{aligned}$$

where $\gamma_2 < 0$ is as in hypothesis (iv).

Proof. We only remark on the proof of (3.20). For $\epsilon = (\gamma_1, \gamma_2)$ it follows from [8, Lemma 4]. For other ϵ it follows from the examination of the proof of [8, Lemma 4] and from the analysis in [4, pp. 325–326]. •

Corollary 1. *Let the assumptions of Theorem 2 hold.*

(i) *For $|s| < s_0$, $T_- < T_-^0$, $T_+ > T_+^0$, $0 \leq k \leq p-1$ we have*

$$(3.23) \quad \begin{aligned} & \left| \bar{\lambda}^{(k)}(s) - \bar{\lambda}_T^{(k)}(s) \right| + \left\| z^{(k)}(s) - z_T^{(k)}(s) \right\|_{0,1} \\ & \leq C \sum_{l=0}^k \left\| z^{(l)}(s) \right\|_{\gamma,1} \left(|\lambda_3| e^{T-\gamma_1} + e^{2T+\gamma_2} \right), \end{aligned}$$

where $\gamma_1 > 0$, $\gamma_2 < 0$ are as in hypothesis (iv).•

To analyze the approximate problem (2.47) and to estimate the error term $\|z_T - z_{T,m}\|_{0,1} + |\bar{\lambda}_T - \bar{\lambda}_{T,m}|$ in ((3.1)) we shall use subspaces $B_{0,i}^T$ of $B_{0,i}$ of functions that decay exponentially at $\pm\infty$. These subspaces are defined below. We first formulate an extension of Theorem 1 which easily follows from the analysis in [4].

Theorem 3. *Let X_n and Y_n be two sequences of real Banach spaces; for each $n = 1, 2, \dots$ and $m = 1, 2, \dots, M > 0$, for some M , integer, let $\Phi_n, \Phi_{n,m} : X_n \rightarrow Y_n$ be a C^p mapping and a family of mappings, respectively, with $p \geq 1$, and let $x_n^0 \in X_n$ be such that $\Phi_n(x_n^0) = 0$. Assume that x_n^0 , $n = 1, 2, \dots$, is a regular point in the following sense:*

$$(3.24) \quad \begin{aligned} & a) \Phi_n'(x_n^0) : X_n \rightarrow Y_n \text{ is a Fredholm operator of index } 1, \\ & b) \mathcal{R}(\Phi_n'(x_n^0)) = Y_n. \end{aligned}$$

Assume that for each n $\mathcal{N}(\Phi_n'(x_n^0))$ is spanned by some vector $\varphi_n^0 \in X_n$, $\varphi_n^0 \neq 0$, and let $\bar{l}_n : X_n \rightarrow \mathbb{R}$ be a linear functional that satisfies

$$(3.25) \quad \bar{l}_n(\varphi_n^0) \neq 0.$$

Assume also that for any $0 \leq k \leq p-1$, $0 \leq m \leq M$ and for any $x_n, \xi_n^1, \dots, \xi_n^k \in X_n$ with $\|x_n\|, \|\xi_n^1\|, \dots, \|\xi_n^k\| \leq C$, we have

$$(3.26) \quad \lim_{n \rightarrow \infty} \left\| \left(\Phi_n^{(k)}(x) - \Phi_{n,m}^{(k)}(x) \right) \left(\xi_n^1, \dots, \xi_n^k \right) \right\| = 0,$$

$$(3.27a) \quad \lim_{n \rightarrow \infty} \left\| \Phi_n'(x_n^0) - \Phi_{n,m}'(x_n^0) \right\| = 0,$$

and for some $C > 0$, independent of n ,

$$(3.27b) \quad \left\| \Phi_n'(x_n^0) \xi \right\| \geq C \|\xi\|, \quad \forall \xi \in \{x \in X_n : \langle x, \varphi_n^0 \rangle = 0\}.$$

Then there exist positive constants n_0, s_0, α, K and two unique C^1 maps $x_n(s), x_{n,m}(s) \in X_n$, $|s| < s_0$, satisfying for $|s| < s_0$, respectively, the conditions:

$$(3.28) \quad (\Phi_n(x_n(s)), \bar{l}_n(x_n(s) - x_n^0) - s) = 0, \quad \|x_n(s) - x_n^0\| < \alpha,$$

$$(3.29) \quad (\Phi_{n,m}(x_{n,m}(s)), \bar{l}_n(x_{n,m}(s) - x_n^0) - s) = 0, \|x_{n,m}(s) - x_n^m\| < \alpha, n > n_0.$$

Moreover $x_n(0) = x_n^0$, and for $|s| < s_0$, $n > n_0$, $0 \leq k \leq p-1$ we have an error estimate

$$(3.30) \quad \|x_n^{(k)}(s) - x_{n,m}^{(k)}(s)\| \leq C \sum_{l=0}^k \left\| \frac{d^l}{ds^l} \Phi_{n,m}(x_n(s)) \right\|,$$

where C is a positive constant independent of n .

We next define subspaces $B_{0,i}^T$ of $B_{0,i}$ as follows. For each pair (T_-, T_+) let $B_{0,i}^T$ be the closure in the $B_{0,i}$ norm of the set of functions

$$\left\{ z \in B_{0,i} : z(t) = \begin{cases} \sum_{i=1}^{n_u^-} c_{0i} P(\lambda) \begin{bmatrix} 0 \\ u_i(t, \lambda) \\ 0 \end{bmatrix}, & c_{0i} \in \mathbb{R}, \lambda \in \tilde{\Lambda}, \text{ for } t \leq T_-, \\ \sum_{i=-n_s^+}^{-1} c_{1i} v_i(t, \lambda), & c_{1i} \in \mathbb{R}, \lambda \in \tilde{\Lambda}, \text{ for } t \geq T_+, \end{cases} \right\}$$

where u_i and v_i are the same as in (2.44b) and (2.44c). Now define the operator $\bar{\Phi}_{T,m} : B_{0,1}^T \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right] \rightarrow B_{0,0}^T \times \mathbb{R}^2$ by

$$(3.31) \quad \bar{\Phi}_{T,m}(\bar{z}, s) \equiv (\Phi_{T,m}(\bar{z}(s)), \bar{l}(\bar{z}(s) - \bar{z}_0) - s),$$

where $\Phi_{T,m}$ and $\bar{l}$ are defined, respectively, by (2.47) and (3.17).

Theorem 3 requires a consistency condition (a pointwise convergence) and a stability condition. These conditions are given by the following lemma.

Lemma 5. Assume that the conditions of Theorem 2 hold. Let $\bar{z}_T = (z_T, \lambda_T, \lambda_{3,T}) \in B_{0,1}^T \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right]$, where $\bar{z}_T^0 = (z_T^0, \lambda_T^0, \lambda_{3,T}^0)$ solves the approximate problem (2.35) (or (3.21)).

Then for $-T_-, T_+ \rightarrow \infty$ we have: for any $0 \leq k \leq p-1$ and for any fixed $\bar{z}_T$, $\xi_1, \dots, \xi_k \in B_{0,1}^T \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right]$, for some $C_1 > 0$, fixed,

$$(3.32) \quad \left\| \left(\bar{\Phi}_T^{(k)}(\bar{z}_T, s) - \bar{\Phi}_{T,m}^{(k)}(\bar{z}_T, s)(\xi_1, \dots, \xi_k) \right) \right\|_{B_{0,0}^T \times \mathbb{R}^2} = O(|\lambda_3|^{m+1}) = O\left(\frac{1}{|T_-|^{\alpha(m+1)}} \right),$$

(3.33)

$$\left\| D_{\bar{z}} \bar{\Phi}_T(\bar{z}_T^0, s) - D_{\bar{z}} \bar{\Phi}_{T,m}(\bar{z}_T^0, s) \right\|_{\mathcal{L}(B_{0,1}^T \times \mathbb{R}^3, B_{0,0}^T \times \mathbb{R}^2)} = O(|\lambda_3|^{m+1}) = O\left(\frac{1}{|T_-|^{\alpha(m+1)}} \right).$$

Proof. We verify (3.32) for $k = 0$. For $k > 0$ the proof is similar. From the definitions (3.18) and (3.31) of $\bar{\Phi}_T$ and $\bar{\Phi}_{T,m}$, respectively, we have

$$(3.34) \quad \left\| \bar{\Phi}_T(\bar{z}_T, s) - \bar{\Phi}_{T,m}(\bar{z}_T, s) \right\|_{B_{0,0}^T \times \mathbb{R}^2} \leq \left\| f(z_T(t) + \chi(t)x_{sl}(t, \bar{\lambda}), \lambda) \right\|$$

$$\begin{aligned}
& \left\| -f(z_T(t) + \chi(t)x_{sl}^m(t, \bar{\lambda}), \lambda) - \frac{\partial}{\partial t}\chi(t)(x_{sl}(t, \bar{\lambda}) - x_{sl}^m(t, \bar{\lambda})) \right\|_{L_\infty[T_-, T_+ + 1]} \\
& \leq C \left(\|x_{sl}(t, \bar{\lambda}) - x_{sl}^m(t, \bar{\lambda})\|_{L_\infty[T_-, T_+ + 1]} + \left\| \frac{\partial}{\partial t}(x_{sl}(t, \bar{\lambda}) - x_{sl}^m(t, \bar{\lambda})) \right\|_{L_\infty[T_-, T_+ + 1]} \right) \\
& \leq C \|x_{sl}(t, \bar{\lambda}) - x_{sl}^m(t, \bar{\lambda})\|_{L_\infty[T_-, T_+ + 1]}.
\end{aligned}$$

where in the second inequality we used the fact, see (2.4a), that f is Lipschitz and the definition (2.14) of χ . Combining this with (2.43) and (3.9) we arrive at (3.32). The stability condition (3.33) is obtained in a similar way (see also [4, pp. 325–326]) using (2.4b) and a stability estimate

$$(3.35) \quad \|D_{\bar{z}}\bar{\Phi}_T(\bar{z}_T^0, s_0)\bar{z}_T\|_{B_{0,0}^T \times \mathbb{R}^2} \geq C \|\bar{z}_T\|_{B_{0,1}^T \times \mathbb{R}^3}, \quad \forall T_- < T_-^0, \forall T_+ > T_+^0,$$

where C does not depend on λ , T_- , T_+ . The latter estimate is obtained from (3.20) by passing to the limit as $\epsilon = (\epsilon_1, \epsilon_2) \rightarrow (0^+, 0^-)$. •

Theorem 4. *Assume that the conditions of Theorem 2 hold. Then for, perhaps, larger positive constants $-T_-^0$, T_+^0 , C and, perhaps, smaller positive constants, s_0 , α , then in Theorem 2, there exist a unique C^1 map $\bar{z}_{T,m}(s) \equiv (z_{T,m}(s), \bar{\lambda}_{T,m}(s)) \in B_{0,1}^T \times \mathbb{R}^2 \times \left[\frac{-C_1}{|T_-|^\alpha}, \frac{C_1}{|T_-|^\alpha} \right]$, $|s| < s_0$, satisfying for $|s| < s_0$ the conditions*

$$(3.36) \quad \begin{aligned} \bar{\Phi}_{T,m}(\bar{z}_{T,m}, s) &= 0, \quad \|\bar{z}_{T,m}(s) - \bar{z}_T^0\|_{B_{0,1}^T \times \mathbb{R}^3} < \alpha, \\ T_- &< T_-^0, \quad T_+ > T_+^0. \end{aligned}$$

Moreover, for $|s| < s_0$, $T_- < T_-^0$, $T_+ > T_+^0$, $0 \leq k \leq p-1$ we have error estimates

$$(3.37) \quad \begin{aligned} & \left| \bar{\lambda}_T^{(k)}(s) - \bar{\lambda}_{T,m}^{(k)}(s) \right| + \left\| z_T^{(k)}(s) - z_{T,m}^{(k)}(s) \right\|_{B_{0,1}^T} \\ & \leq C \sum_{l=0}^k \left\| z^{(l)}(s) \right\|_{B_{0,1}^T} |\lambda_3|^{m+1} = C \frac{1}{|T_-|^\alpha(m+1)}, \end{aligned}$$

$$(3.38) \quad \|x - x_T^m\|_{B_{0,1}^T} \leq C \left(|\lambda_3| e^{T_- \gamma_1} + e^{2T_+ \gamma_2} + |\lambda_3|^{m+1} \right).$$

Proof. Combining Theorem 3 with Lemma 5 we obtain (3.36) and (3.37). Combining the estimates (3.1), (3.23) and (3.37) we arrive at our final error estimate (3.38). •

4. Example. The DC Josephson Junction equations, in dimensionless form, are [13]:

$$(4.1) \quad x' = f(x, \rho, \beta),$$

with $x = (x_1, x_2)^T$, $\lambda = (\rho, \beta)$,

$$\begin{aligned} f(x, \rho, \beta) &= (f_1(x_1, x_2, \rho, \beta), f_2(x_1, x_2, \rho, \beta))^T \\ &= \left(x_2, \frac{1}{\beta}(-x_2 - \sin x_1 + \rho) \right)^T. \end{aligned}$$

Assume that $\rho \leq 1$, $\rho^* = 1$. If $\rho < 1$ then (4.1) has a heteroclinic orbit connecting two saddles: $x_- = (\pi - \arcsin \rho, 0)^T$ and $x_+ = (3\pi - \arcsin \rho, 0)^T$. We note that

$$(4.2) \quad f_x(x_-, \rho, \beta) = f_x(x_+, \rho, \beta) = \begin{bmatrix} 0 & 1 \\ \frac{\sqrt{1-\rho^2}}{\beta} & -\frac{1}{\beta} \end{bmatrix},$$

which has eigenvalues

$$(3.3) \quad \mu_0^- \equiv \mu_0 = \frac{-1 + \Delta}{2\beta} \approx \sqrt{2(1-\rho)}, \text{ as } \rho \rightarrow 1, \quad \mu_{-1}^+ \equiv \mu_{-1} = \frac{-1 - \Delta}{2\beta},$$

where $\Delta \equiv \sqrt{1 + 4\beta\sqrt{1-\rho^2}}$. Corresponding right eigenvectors and the matrix P are

$$(4.4) \quad w_{-1} = \frac{1}{\sqrt{1 + \mu_{-1}^2}} \begin{bmatrix} 1 \\ \mu_{-1} \end{bmatrix}, \quad w_0 = \frac{1}{\sqrt{1 + \mu_0^2}} \begin{bmatrix} 1 \\ \mu_0 \end{bmatrix},$$

$$P = [w_0 \quad w_{-1}] = \begin{bmatrix} \frac{1}{\sqrt{1+\mu_0^2}} & \frac{1}{\sqrt{1+\mu_{-1}^2}} \\ \frac{\mu_0}{\sqrt{1+\mu_0^2}} & \frac{\mu_{-1}}{\sqrt{1+\mu_{-1}^2}} \end{bmatrix}.$$

Upon changing the variables according to

$$(4.5) \quad x \equiv \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + P \begin{bmatrix} y \\ v \end{bmatrix}, \quad y, v \in \mathbb{R},$$

we rewrite (4.1) in the form (2.8) as

$$(4.6) \quad \begin{aligned} y' &= \mu_0 y + h(y, v, \rho, \beta), \\ v' &= \mu_{-1} v + g(y, v, \rho, \beta), \end{aligned}$$

where

$$(4.7) \quad \begin{aligned} & \begin{bmatrix} \mu_0 y + h(y, v, \rho, \beta) \\ \mu_{-1} v + g(y, v, \rho, \beta) \end{bmatrix} = P^{-1} f(x_1, x_2, \rho, \beta) \\ &= \begin{bmatrix} \mu_0 y + \frac{\rho(y+v)^2}{2\Delta(1+\mu_0^2)} - \frac{\sqrt{1-\rho^2}(y+v)^3}{6\Delta(1+\mu_0^2)^{3/2}} + O((y+v)^4) \\ \mu_{-1} v - \frac{\rho(y+v)^2}{2\Delta(1+\mu_0^2)} + \frac{\sqrt{1-\rho^2}(y+v)^3}{6\Delta(1+\mu_0^2)^{3/2}} + O((y+v)^4) \end{bmatrix}. \end{aligned}$$

Hence the “slowly growing” solution (see (2.10) and its approximation (see (2.38), correspondingly, have the form:

$$(4.9) \quad x_{sl} = \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + P \begin{bmatrix} y \\ V(y, \rho, \beta) \end{bmatrix}, \quad y = Y(t, \rho, \beta, \lambda_3), \quad V(y, \rho, \beta) \in \mathbb{R},$$

$$(4.10) \quad x_{sl}^m = \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + P \begin{bmatrix} y \\ V^m(y, \rho, \beta) \end{bmatrix}, \quad y = Y_m(t, \rho, \beta, \lambda_3), \quad V^m(y, \rho, \beta) \in \mathbb{R},$$

where

$$(4.11) \quad V^m(y, \rho, \beta) = A(\rho, \beta)y^2 + B(\rho, \beta)y^3 + \dots + O(y^m)$$

is the Taylor polynomial of order m about $y = 0$ for $V(y, \rho, \beta)$ which is obtained, according to (2.39), by solving

$$(4.12) \quad V_y^m[\mu_0 y + h(y, V^m, \rho, \beta)] = \mu_{-1}v + g(y, V^m, \rho, \beta).$$

Substituting (4.7) and (4.11) into (4.12), we have

$$(4.13) \quad \begin{aligned} & (2Ay + 3By^2 + O(y^3)) \left[\mu_0 y + \frac{\rho}{2\Delta} y^2 - \frac{\sqrt{1-\rho^2}}{6\Delta} (y + Ay^2)^3 + O(y^3) \right] \\ &= \mu_{-1} (Ay^2 + By^3 + O(y^4)) - \frac{\rho}{2\Delta} (y + Ay^2)^2 + \frac{\sqrt{1-\rho^2}}{6\Delta} y^3 + O(y^4) \end{aligned}$$

Equating the coefficients for the second and third powers of y , we obtain

$$(4.15) \quad \begin{aligned} A &= \frac{\rho}{2\Delta(\mu_{-1} - 2\mu_0)} = \frac{\rho\beta}{\Delta(1 - 3\Delta)} < 0, \\ B &= \frac{\sqrt{1-\rho^2} - 6A\rho}{3\Delta(3\mu_0 - \mu_{-1})} = \frac{\sqrt{1-\rho^2} - 6A\rho}{3\Delta(2\Delta - 1)}\beta > 0. \end{aligned}$$

Substituting A and B into (4.11) and then (4.11) into (4.10) gives (see (2.38))

$$(4.16a) \quad x_{sl}^2 = \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + P \begin{bmatrix} Y_2 \\ V^2(Y_2, \lambda) \end{bmatrix}, \quad V^2(Y_2, \lambda) = Y_2 + AY_2^2,$$

$$(4.16b) \quad x_{sl}^3 = \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + P \begin{bmatrix} Y_3 \\ V^3(Y_3, \lambda) \end{bmatrix}, \quad V^3(Y_3, \lambda) = Y_2 + AY_3^2 + BY_3^3.$$

Hence the boundary conditions (2.52b) and (2.52c), respectively, take the form:

$$(4.17) \quad \begin{aligned} b) \quad x(0) &= x_-(\lambda) + P(\lambda) \begin{bmatrix} \lambda_3 \\ V^m(\lambda_3, \lambda) \end{bmatrix} \equiv \begin{bmatrix} \pi - \arcsin \rho \\ 0 \end{bmatrix} + \lambda_3 w_0 + (A\lambda_3^2 + B\lambda_3^3 + \dots)w_{-1}, \\ c) \quad x(1) &= x_+(\lambda) - P(\lambda) \begin{bmatrix} 0 \\ \epsilon_+ \end{bmatrix} \equiv \begin{bmatrix} 3\pi - \arcsin \rho \\ 0 \end{bmatrix} - \epsilon_+ w_{-1}, \end{aligned}$$

Known theoretical results. (Schechter [13] and Chow *et al.* [3]). We identify x_1 and $x_1 + 2\pi$, so that (4.1) defines a vector field on a cylinder. For an equilibrium x_- of (4.1) we denote by μ_{-1} and μ_0 the eigenvalues of $f_x(x_-, \rho, \beta)$. We shall be interested in the situation when during the continuation procedure $\mu_{-1} < 0$ and $\mu_0 \geq 0$. For $(\rho, \beta) = (1, \beta^*)$ for some (unknown) β^* (4.1) has an equilibrium $(\pi/2, 0)$ of saddle-node type, with a separatrix loop Γ , where $\mu_{-1} = \mu_{-1}^* < 0$ and $\mu_0 = \mu_0^* \equiv 0$. Denote $\nu_1 = \rho - 1$, $\nu_2 = \beta - \beta^*$. The bifurcation diagram [13] is depicted on Figure 1.

Figure 1.

The phase portrait in a neighborhood of Γ (Figure 1) is as follows:

- 1) $\nu_1 = 0, \nu_2 = 0$; a saddle-node and a separatrix loop.
- 2) $\nu_1 > 0$; no equilibria, a unique stable closed orbit near Γ .
- 3) $\nu_1 = 0, \nu_2 < 0$; a saddle-node.
- 4) $\nu_1 < 0, (\nu_1, \nu_2)$ below C ; a saddle and a node.
- 5) $\nu_1 < 0, (\nu_1, \nu_2)$ on C ; a saddle and a node; the saddle has a separatrix loop.
- 6) $\nu_1 < 0, (\nu_1, \nu_2)$ above C ; a saddle and a node; there is a unique stable closed orbit near Γ .

- 7) $\nu_1 = 0, \nu_2 > 0$; a saddle-node and a unique stable closed orbit near Γ .

Solution algorithm and the numerical results (see [9] for the details).

Step 1. (Doedel and Kernevez [7, pp. 50–54]). A homotopy from a problem with known periodic solution (on a cylinder) to a periodic solution of period T , for some T , of (4.1) in the region 2) ($\beta > \beta^*, \rho > 1$).

Step 2 [7, pp. 50–54]. Computation of a branch $(x(s), T(s), \rho(s))$ of the periodic solutions in the direction of decreasing ρ and increasing T , while $\beta > \beta^*$ is kept frozen; passing through the regions 7) and 6) and approaching a heteroclinic orbit (a homoclinic orbit on a cylinder) connecting the saddles x_- and x_+ in the region 5).

Step 3. Switching from the periodic approximation of the heteroclinic orbit to a tangent one.

Step 4. Switching from the tangent approximation to a higher order approximation for the “slow” manifold.

Step 5. Computation of a branch $(x(s), \rho(s), \beta(s))$ of solutions in the region 5) in the direction of increasing ρ and decreasing β , while T is frozen at a large value, and thus approaching the point $(\rho, \beta) = (1, \beta^*)$. See Figure 2, left. The figure agrees with the

theoretical result [13]:

$$(4.18) \quad \rho = 1 - l^2(\beta - \beta^*)^2 + o((\beta - \beta^*)^2) \text{ as } \rho \rightarrow 1.$$

To obtain a better approximation of β^* , in view of (4.18), we make the following change of variables:

$$(4.19) \quad \rho_1 = \sqrt{1 - \rho^2},$$

which implies $\mu_0 \approx \rho_1$, as $\rho_1 \rightarrow 0$. The results of computations are shown on Figure 3, left.

Step 6. Investigating the dependence of the accuracy of approximation of β on the period T for fixed ρ .

The dependence of the accuracy of approximation of β on the period T for fixed ρ close to 1 is shown on Figure 4, right, and for fixed ρ_1 close to 0 on Figure 4, left.

We have noticed that the tangent approximation is considerably more accurate than the periodic approximation, specifically, one requires $T=33.5, 135, 10600$ in the case of the periodic approximation to obtain the same accuracy in β as for $T=12.5, 25, 50$, correspondingly, in the case of the tangent approximation, for $\beta (\approx \beta^*)$ for $\rho = 1 - .51 \times 10^{-6}$ ($\mu_0 = 10^{-3}$). At the same time, we have not noticed significant improvement in accuracy when the tangent approximation of the slow (center) manifold is replaced by the quadratic or cubic approximation. Perhaps, this is due to the fact of some kind of superconvergence in the case of the tangent approximation.

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References

- [1] W.-J. Beyn, The numerical computation of connecting orbits in dynamical systems, to appear in IMA J. Numer. Anal.
- [2] W.-J. Beyn, Global bifurcations and their numerical computation, in: D. Rose, Ed. Continuation and bifurcations: *Numerical Integration Techniques and Applications*, (NATO ASI Series, Leuven, 1989).
- [3] S.N. Chow and X.B. Lin, Bifurcation of a homoclinic orbit with a saddle-node equilibrium (preprint).
- [4] J. Descloux and J. Rappaz, Approximation of solution branches of nonlinear equations, R.A.I.R.O. **16** (1982) 319–349.
- [5] E.J. Doedel and M.J. Friedman, Numerical computation of heteroclinic orbits, J. Comput. and Appl. Math. **26** (1989) 159–170.
- [6] E.J. Doedel and M.J. Friedman, Numerical computation and continuation of invariant manifolds connecting fixed points with application to computation of combustion fronts, *Proc. 7th Int. Conf. on Finite Element Methods in flow problems*, April 1989.

- [7] E.J. Doedel and J.P. Kernevez, AUTO: Software for continuation and bifurcation problems in ordinary differential equations, Applied Mathematics Report, California Institute of technology, 1986, 226 pages.
- [8] M.J. Friedman and E.J. Doedel, Numerical computation and continuation of invariant manifolds connecting fixed points, to appear in SIAM J. Numer. Anal.
- [9] M.J. Friedman and E.J. Doedel, Computational methods for global analysis of homoclinic and heteroclinic orbits: a case study, submitted to J. of Dynamics and Dif. Equations..
- [10] T.M. Hagstrom and H.B. Keller, Asymptotic boundary conditions and numerical methods for nonlinear elliptic problems on unbounded domains, Math. Comput. **48** (1987) 449–470.
- [11] J.K. Hale and X.-B. Lin, Heteroclinic orbits for retarded functional differential equations, J. Differential equations, **65** (1986) 175–202.
- [12] B.D. Hassard, Computation of invariant manifolds, in: P.J. Holmes, Ed., *New Approaches to Nonlinear Problems in Dynamics* (SIAM, Philadelphia, PA, 1980) 27–42.
- [13] S. Schecter, The saddle-node separatrix-loop bifurcation, SIAM J. Math. Anal., **18** (1987) 1142–1157.

Figure 2. Left (Step 5): a branch of heteroclinic orbits with $T = 20$, fixed, starting at the point $(\rho, \beta) = (0.9718606, 0.9916909)$ ($\mu_0 \approx 0.24$) and terminating at the point $(\rho, \beta) = (0.99999949, 0.7028165)$ ($\mu_0 \approx 0.001$). Right (Step 6): (T, ρ) plane representation of a branch of heteroclinic orbits parametrized by the period T for fixed $\rho = 0.9718606$ ($\mu_0 \approx 0.24$).

Figure 3. Left (Step 5): a branch of heteroclinic orbits with $T = 20$, fixed, starting at the point $(\rho_1, \beta) = (0.167, 0.9004192)$ ($\mu_0 \approx 0.167$), and terminating at the point $(\rho_1, \beta) = (10^{-6}, 0.6890448)$. Right (Step 6): a branch of heteroclinic orbits with $\rho_1 = 10^{-6}$ ($\mu_0 \approx 10^{-6}$), fixed, starting at the point $(T, \beta) = (10, 0.5951997)$ and terminating at the point $(T, \beta) = (121.3, 0.7025648)$.

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