QUENCHING FOR QUASILINEAR EQUATIONS

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1. Introduction and Motivation

Suppose that $\beta > 0$. Consider the initial boundary value problems (1.1) (1.2a) (1.3) and (1.1) (1.2b) (1.3).

$$u_t - (\phi(u_x))_x = -u^{-\beta} \quad \text{for } x \in (0, a), t > 0 \quad (1.1)$$

$$u_x(t, 0) = 0, \quad u(t, a) = 1 \quad \text{for } t > 0, \quad (1.2a)$$

$$u_x(t, 0) = 0, \quad u_x(t, a) = 0 \quad \text{for } t > 0, \quad (1.2b)$$

$$u(0, x) = u_0(x) > 0 \quad \text{for } x \in (0, a), \quad (1.3)$$

Model cases are $\phi(s) = s$, where (1.1) becomes $u_t - u_{xx} = -u^{-\beta}$, and $\phi(s) = \arctan s$, where (1.1) becomes

$$u_t - \frac{u_{xx}}{1 + u_x^2} = -u^{-\beta}. \quad (1.4)$$

Equation (1.4) plays an important role in the description of mean-curvature flow and has motivated us to study the general quasilinear equation (1.1). Boundary condition (1.2b) occurs for instance, if one studies (1.1) for $x \in \mathbb{R}$, $t > 0$, with periodic initial data. Solutions of (1.1) (1.2a) (1.3) can be continued as even functions and satisfy the Dirichlet conditions $u(t, \pm a) = 1$.

It is well-known that solutions of (1.1) do not need to exist globally in time as classical solutions. In fact they can reach zero and then u_t tends to $-\infty$ in finite time. This phenomenon is called quenching, and the first time at which this occurs is called the quenching time T .

Let us now motivate (1.4). Imagine the surface between two physical phases, say fluid and solid. According to Brakke [B] such a surface moves (in the simplest physical model) with a normal velocity v proportional to its mean curvature. If the initial surface is convex, then G. Huisken has shown in [H] that it stays convex for positive times and shrinks to a point in a strong topology. In general, however, a single surface can split into two separate ones. To see this imagine a dumbbell domain, i.e. an initial surface which consists essentially of two large disjoint spheres and of a thin pipe or handle connecting them. We expect the spheres to shrink slower than the pipe under this flow; and it has in fact been shown by M. Grayson in [G] that the dumbbell will experience a “necking”, “pinching”, or “quenching” phenomenon. These abstract considerations are confirmed by numerical experiments, e.g. by G. Dziuk [D] or by J.A. Sethian [S], and are illustrated in Figure 1 below.

Figure 1: Mean curvature flow of a dumbbell

For the reader's convenience let us derive the differential equation which describes the flow. We learned this from M.Grayson, see also [G]. Let x be the axial coordinate, t denote time and $u(t, x)$ denote the radial distance of the surface to its

axis of rotation. Then the normal velocity v of the surface is $u_t (1 + u_x^2)^{-1/2}$. Notice that u_t is the velocity in radial direction, see Figure 2. The principle curvatures of the surface are $-u_{xx} (1 + u_x^2)^{-3/2}$ in axial direction and $(u \sqrt{1 + u_x^2})^{-1}$ in the other direction. Therefore u satisfies (1.4) with $\beta = 1$:

$$u_t - \frac{u_{xx}}{1 + u_x^2} = -\frac{1}{u} \quad \text{where } u > 0. \quad (1.5)$$

Figure 2: Derivation of (1.5)

The paper is organized as follows. In Section 2 we derive sufficient conditions for quenching in finite time. In Section 3 we study the asymptotics of u near the quenching point. In general $u(t, 0)$ behaves like a fractional power of $T - t$. Our results, as applied to the particular equation (1.5), suggest that $u(T, x)$ behaves like cx near $x = 0$. In fact, on one hand $u(T, x) \geq C_\gamma x^{2/(\gamma+1)}$ for any $\gamma < 1$, but on the other hand $\int_0^a u^{-\lambda}(T, x) dx = \infty$ for any $\lambda > 1$. Section 4 contains results on single point quenching under Neumann boundary conditions. These results are not obvious, because diffusion counteracts the development of a single quenching point and tends to flatten out the solution of (1.1). In fact, for constant initial data the solution quenches in every point. Moreover the results of section 4 show that by periodic continuation one can construct solutions with exactly

$k \in \mathbb{N}$ quenching points. Theorem 3.4 and Lemma 5.2 contain results on single point quenching, which considerably extend previous results ([AK],[DK, Thm. 5]). Incidentally, X.Y.Chen has kindly informed us, that he has obtained results on the finite number of quenching points for equation (1.5) under very general initial data. Finally, in Section 5 we investigate the existence and stability of stationary solutions in the spirit of [L2].

2. A Sufficient Criterion for Finite Time Quenching

Solutions of equation (1.1) under Neumann boundary conditions will quench in finite time, provided the initial data are finite. This follows from a simple comparison argument, using constant initial data. Under Dirichlet boundary conditions, however, it is conceivable that solutions converge asymptotically to positive equilibria. This is the case, for instance, if the spatial domain is "small", or if the initial data are "large" enough, see e.g. [L2,AK]. Conversely, experience suggests that we can expect quenching in finite time for "large" domains or "small" initial data. This is in fact reflected in the following result, which is obtained by a simple variant of Kaplan's method, cf [L1].

Theorem 2.1. *Let $u(t, x)$ be a solution of*

$$u_t - (\phi(u_x))_x = -u^{-\beta} \quad \text{for } x \in (0, a), t > 0 \quad (2.1)$$

$$u_x(t, 0) = 0, \quad u(t, a) = 1 \quad \text{for } t > 0, \quad (1.2a)$$

$$1 \geq u(0, x) = u_0(x) > 0 \quad \text{for } x \in (0, a), \quad (2.2)$$

and let T be its maximal time of existence as a classical solution. Let $v(x) = \frac{2a}{\pi} \cos(\frac{\pi}{2a}x)$ and suppose that

$$\phi(0) = 0 \quad \text{and} \quad \phi(s) \leq \kappa_1 s + \kappa_2. \quad (2.3)$$

If (i) $\int_0^a u_0(x)v(x) dx$ is small enough, or if (ii) a is large enough ($a > 3\kappa_2/2$), then T is finite.

Proof: Set $I_1(t) = \int_0^a u(t, x)v(x) dx$. Then, using (2.1) and Jensen's inequality, we obtain

$$\begin{aligned}
I_1'(t) &= \int_0^a u_t(t, x)v(x) dx = \int_0^a \{(\phi(u_x))_x - u^{-\beta}\}v dx \\
&\leq - \int_0^a \phi(u_x)v_x dx - (I_1(t))^{-\beta} \\
&\leq -\kappa_1 \frac{\pi^2}{4a^2} I_1(t) + \kappa_1 + \kappa_2 \frac{2a}{\pi} - (I_1(t))^{-\beta}.
\end{aligned} \tag{2.4}$$

So for fixed a and sufficiently small u_0 the right hand side of (2.4) will be negative initially. By a continuity argument $I_1'(t) < 0$ for all $t \in [0, T)$. A simple quadrature shows that $T < \infty$ and that I_1 tends to zero as t increases to T . This proves (i). To prove (ii) we multiply (2.1) by $w(x) = a^2 - x^2$, set $I_2(t) = \int_0^a u(t, x)w(x) dx$ and estimate

$$\begin{aligned}
I_2'(t) &= \int_0^a u_t(t, x)w(x) dx = \int_0^a \{(\phi(u_x))_x - u^{-\beta}\}w dx \\
&\leq - \int_0^a \phi(u_x)w_x dx - \int_0^a w(x) dx \\
&\leq -2\kappa_1 \int_0^a u dx + \kappa_2 a^2 - (a^2 x - x^3/3)|_0^a \\
&\leq a^2(\kappa_2 - 2a/3) < 0.
\end{aligned}$$

Thus I_2 must decrease to zero in finite time. ■

3. Space and Time Asymptotics

Throughout this section we consider the problem (1.1) (1.2a) (1.3). We assume that the solution quenches at time $T < \infty$. We derive results on the asymptotic profile of u near the quenching point $(T, 0)$, which correspond to analogous results for the special case $\phi(s) = s$ in [FK].

One or both of following hypotheses will be used in the sequel.

H-1. $\phi \in C^3((0, \infty))$ and $\phi' > 0$ in $(0, \infty)$.

H-2. $\phi \in C^1([0, \infty))$ and $\phi' > 0$ in $[0, \infty)$.

H-1 and H-2 are ellipticity assumptions. H-2 is violated for $\phi(s) = |s|^{p-2}s$ with $p \neq 2$.

For later use we state the next Lemma also for the Neumann problem.

Lemma 3.1. *Let u be a solution of (1.1) (1.2a) (1.3) or (1.1) (1.2b) (1.3) and suppose that H-1 holds.*

- (i) *If $u'_0 \geq 0$, then $u_x(t, x) \geq 0$ for $0 \leq t < T$, $0 \leq x \leq a$.*
- (ii) *If $\phi'(u'_0)u''_0 - u_0^{-\beta} \leq 0$, then $u_t(t, x) \leq 0$ for $0 \leq t < T$, $0 \leq x \leq a$.*
- (iii) *If $(\phi'(u'_0)u''_0 - u_0^{-\beta})' \geq 0$ in addition to the assumptions of (i), (ii), then $u_{xt}(t, x) \geq 0$ for $0 \leq t < T$, $0 \leq x \leq a$.*

Proof: These facts follow from the maximum principle. In the case $\phi(s) = \arctan s$ and $\beta = 1$ the proofs can be found in [DK]. For the reader's convenience we give a proof of (iii). The function $v(t, x) = u_{xt}(t, x)$ satisfies the equation

$$\begin{aligned} v_t - \phi'(u_x)v_{xx} - \phi''(u_x)u_{xx}v_x - (\phi'''(u_x)u_{xx}^2 + \beta u^{-\beta-1})v \\ = -\beta(\beta+1)u^{-\beta-2}u_xu_t > 0, \end{aligned}$$

and, using (i) (ii) and the boundary conditions one can see that $v \geq 0$ on the parabolic boundary. Hence $u_{xt} \geq 0$ in $(0, T) \times (0, a)$. ■

Theorem 3.2. *Let u_0 satisfy the assumptions of Lemma 3.1 (i)(ii) (iii). Assume further H-1 and $\phi'' \leq 0$ in $(0, \infty)$. Then there are constants $c_1, c_2 > 0$ such that*

$$c_1 \leq u(t, 0)(T - t)^{-1/(1+\beta)} \leq c_2, \quad \text{for } 0 < t < T.$$

Proof: We shall first prove the estimate from below. Similarly as in [DL] (see also [FM]) we want to show that for any sufficiently small $\eta > 0$ there is a $\delta > 0$

such that $J(t, x) := u_t + \delta u^{-\beta} \leq 0$ in $(\eta, T) \times (0, a - \eta)$. We assume that u quenches, which means that $\phi'(u'_0)u''_0 - u_0^{-\beta} \neq 0$. From this it follows that $u_t < 0$ for $t \in (0, T)$, $x \in [0, a]$. Therefore, for any positive and small η there is a $\delta > 0$ such that $J(\eta, x) \leq 0$. From Lemma 3.1 (iii) it follows that u quenches at a single point, at $x = 0$ only. Hence $J(t, a - \eta) \leq 0$ if δ is small enough. It is easy to see that $J_x(t, 0) = 0$. Now we derive a parabolic differential inequality for J in $(\eta, T) \times (0, a - \eta)$.

$$\begin{aligned} J_t - \phi'(u_x)J_{xx} &= u_{tt} - \phi'(u_x)u_{xxt} - \beta\delta u^{-\beta-1}(u_t - \phi'(u_x)u_{xx}) \\ &\quad - \phi'(u_x)\beta(\beta+1)\delta u^{-\beta-2}u_x^2 \\ &= \left(\frac{\phi''(u_x)}{\phi'(u_x)}u_{xt} + \beta u^{-\beta-1} \right) J + \frac{\phi''(u_x)}{\phi'(u_x)}(1-\delta)u_{xt}u^{-\beta} \\ &\quad - \phi'(u_x)\beta(\beta+1)\delta u^{-\beta-2}u_x^2. \end{aligned}$$

The second and third term on the right hand side of the last equality are nonpositive. Therefore the maximum principle implies that $J \leq 0$ in $(\eta, T) \times (0, a - \eta)$. Integrating this inequality we obtain the desired lower bound. Since $u(t, \cdot)$ is minimal at $x = 0$ (Lemma 3.1(i)), the upper bound follows easily upon integrating $u_t(t, 0) \geq -u^{-\beta}(t, 0)$. This completes the proof of Theorem 3.2. ■

Remark 3.3. In the following Theorem 3.4 we shall assume that

$$\phi(u'_0) \geq \varepsilon x u_0^{-\gamma} \tag{3.1}$$

holds for some $\varepsilon > 0$ and $0 < \gamma < \beta$. This assumption is satisfied for instance if $u_0(x) = 1 - \eta + \eta x^2$ with positive and small η . In fact, since $u'_0 = 2\eta x$, we have, assuming $\phi(s) \geq \alpha s$ for small s , that

$$\phi(u'_0) \geq 2\alpha\eta x \geq \frac{\varepsilon}{(1-\eta)^\gamma} x \geq \varepsilon x u_0^{-\gamma},$$

provided $\varepsilon < 2\alpha\eta(1-\eta)^\gamma$.

Theorem 3.4. Assume H-1. Assume $0 < \gamma < \beta$, $\varepsilon \leq \min\{(\beta - \gamma)/2\gamma, 1\}$, $u'_0 \geq 0$ and that (3.1) holds.

(i) Suppose that (3.2) holds for $s > 0$:

$$\left. \begin{aligned} \phi(s) &\leq s, & \text{and} \\ \frac{1}{\phi'(s)} &\leq \delta s^2 + 1 & \delta \in [0, \gamma + 1], \text{ and} \\ \beta &\leq 1 & \text{if } \delta > 0. \end{aligned} \right\} \quad (3.2)$$

Then

$$u(t, x) \geq C(\gamma, \varepsilon) x^{2/(1+\gamma)}. \quad (3.3)$$

(ii) Suppose that (3.4) holds for $s > 0$:

$$\phi(s) \leq \min \{ \kappa s^q, s\phi'(s) \} \quad \kappa > 0, q > 1. \quad (3.4)$$

Then

$$u(t, x) \geq C(\gamma, \varepsilon, \kappa, q) x^{(q+1)/(q+\gamma)}. \quad (3.5)$$

Before we proceed with the proof let us remark that (3.2) holds for the model problem $\phi(s) = \arctan s$, $\beta = 1$ with $\delta = 1$, and (3.4) holds for $\phi(s) = |s|^{p-2}s$ with $p \geq 2$ and $q = p - 1$.

To prove Theorem 3.4 we set $J(t, x) = \phi(u_x) - \varepsilon x u^{-\gamma}$. By assumption (3.1) we have $J(0, x) \geq 0$ in $[0, a]$, and due to boundary condition (1.2a) $J(t, 0) \geq 0$. Moreover $J_x(t, a) \geq 0$. To prove (3.3) it suffices to show that $J(t, x) \geq 0$ in $(0, T) \times [0, a]$. Again the maximum principle will be of use, once we derive the right differential inequality for J . A straightforward calculation gives

$$\begin{aligned} \frac{1}{\phi'(u_x)} J_t - J_{xx} &= (\beta u^{-\beta-1} - 2\varepsilon \gamma u^{-\gamma-1}) u_x \\ &\quad - \varepsilon \gamma x \frac{1}{\phi'(u_x)} u^{-\gamma-\beta-1} \\ &\quad + \varepsilon \gamma x (\gamma + 1) u^{-\gamma-2} u_x^2. \end{aligned}$$

The first term on the right hand side can be estimated below by $\gamma u^{-\beta-1} u_x$. To estimate the second term under assumption (3.2) one proceeds as follows

$$\begin{aligned} \frac{1}{\phi'(u_x)} J_t - J_{xx} &\geq \gamma u^{-\beta-1} u_x - \delta \varepsilon \gamma x u^{-\gamma-\beta-1} u_x^2 \\ &\quad - \varepsilon \gamma x u^{-\gamma-\beta-1} + \varepsilon \gamma x (\gamma + 1) u^{-\gamma-2} u_x^2 \\ &= \gamma u^{-\beta-1} u_x + \varepsilon \gamma x u^{-\gamma-2} u_x^2 ((\gamma + 1) - \delta u^{1-\beta}) \\ &\quad + \gamma u^{-1-\beta} J - \gamma u^{-\beta-1} \phi(u_x) \\ &\geq \gamma u^{-1-\beta} J. \end{aligned}$$

Notice that the constraint $\beta \leq 1$ is used if $\delta > 0$. This proves statement (3.3).

Under assumption (3.4), however

$$\begin{aligned}
\frac{1}{\phi'(u_x)} J_t - J_{xx} &\geq \gamma u^{-\beta-1} u_x + \frac{\gamma}{\phi'(u_x)} u^{-1-\beta} J \\
&\quad - \frac{\gamma}{\phi'(u_x)} u^{-1-\beta} \phi(u_x) + \varepsilon \gamma x (\gamma + 1) u^{-\gamma-2} u_x^2 \\
&= bJ + \gamma [u^{-1-\beta} (u_x - \frac{\phi(u_x)}{\phi'(u_x)}) + \varepsilon x (\gamma + 1) u^{-\gamma-2} u_x^2] \\
&\geq bJ,
\end{aligned}$$

which shows again that $J(t, x) \geq 0$. Hence $\kappa u_x^q \geq \varepsilon x u^{-\gamma}$. This implies (3.5) and completes the proof of Theorem 3.4. $\blacksquare$

Remark 3.5. Let the assumptions of Theorem 3.4 and Lemma 3.1(ii) be satisfied. Relation (3.3) implies that $u_{xx}(T, 0)$ cannot exist. To see this suppose that $u(T, \cdot) \in C^2[0, a]$. Fix $\gamma < \min\{1, \beta\}$ and denote the right hand side of (3.3) by $v(x)$. Notice that $v_x(0) = 0 = u_x(T, 0)$ and that $u(T, \cdot) - v(\cdot)$ attains a minimum at $x = 0$. Hence for sufficiently small $x > 0$ we have $u_x(T, x) \geq v_x(x)$. Taking difference quotients and observing that $v_{xx}(x) \rightarrow \infty$ as x decreases to zero we conclude that $u_{xx}(T, 0)$ cannot exist, a contradiction.

Using Theorem 3.2 and the fact that $u_t \leq 0$ we can even estimate the blow up rate of $u_{xx}(t, 0)$ as follows:

$$\phi'(0) u_{xx}(t, 0) \leq u^{-\beta}(t, 0) \leq c_1^{-\beta} (T - t)^{-\beta/(\beta+1)} \quad \text{for } 0 < t < T.$$

Theorem 3.4 is complemented by the following two Theorems 3.6 and 3.8.

Theorem 3.6. Assume H-2. Let $\beta < 1$ and set

$$P(t, x) = \int_0^{u_x} r \phi'(r) dr - \frac{1}{1-\beta} u^{1-\beta}.$$

Suppose that $P(0, x) \leq 0$. Then $P(t, x) \leq 0$ in $(0, T) \times [0, a]$. In particular

$$u^{(\beta+1)/2}(t, x) \leq C_\beta x + u^{(\beta+1)/2}(t, 0). \quad (3.6)$$

Proof: $P < 0$ at the points where $u_x = 0$. A calculation shows that

$$P_t - \phi'(u_x)P_{xx} = -u_t(u_t - 2u^{-\beta}) \leq -\frac{2u^{-\beta}}{u_x}P_x,$$

so that P can only attain a positive maximum at $x = 0$ or $x = a$. But by definition $P(t, 0) \leq 0$; and $P_x(t, a) = u_t u_x = 0$ because of (1.2a). To show (3.6) let us first introduce the notation

$$\Phi(s^2) := \int_0^s r \phi'(r) dr \quad (3.7)$$

and observe that $(\Phi^{-1}(\sigma))' = 2/\phi'(\sqrt{\Phi^{-1}(\sigma)}) > 0$, so that Φ is invertible and Φ^{-1} is monotone increasing. Since $P \leq 0$, or equivalently,

$$\Phi(u_x^2) \leq \frac{1}{1-\beta} u^{1-\beta}(t, x),$$

we have, using the mean value theorem and $\Phi^{-1}(0) = 0$,

$$\begin{aligned} u_x^2 &\leq \Phi^{-1}\left(\frac{1}{1-\beta} u^{1-\beta}(t, x)\right) \\ &\leq \frac{2}{\phi'(\theta)} \left(\frac{1}{1-\beta} u^{1-\beta}(t, x)\right) \\ &\leq C(\beta) u^{1-\beta}. \end{aligned} \quad (3.8)$$

Notice that θ lies between 0 and $\sqrt{\Phi^{-1}(1/(1-\beta))}$ and depends on β . After taking the square root, an integration of (3.8) leads to (3.6). $\blacksquare$

Notice that $P(0, x) \leq 0$ if u_0 satisfies the assumptions of Lemma 3.1 (i) (ii). An alternative proof of Theorem 3.6 under these slightly stronger assumptions is contained in the following lemma, which will be used in the proof of Theorem 3.8. Notice also that for $\beta \geq 1$ the function P is nonnegative.

Lemma 3.7. *Assume H-1. Let u_0 satisfy the assumptions of Lemma 3.1 (i) (ii) and let Φ be defined as in (3.7). Then*

$$\Phi(u_x^2) \leq \begin{cases} \frac{1}{1-\beta}(u^{1-\beta}(t, x) - u^{1-\beta}(t, 0)) & \text{if } \beta \neq 1, \\ \ln u(t, x) - \ln u(t, 0) & \text{if } \beta = 1. \end{cases} \quad (3.9)$$

Proof: Let $\beta \neq 1$, then

$$\Phi(u_x^2) = \int_0^x (\phi(u_x))_x u_x \, dx = \int_0^x (u_t + u^{-\beta}) u_x \, dx \leq \frac{1}{1-\beta} \int_0^x (u^{1-\beta})_x \, dx,$$

where we have used the facts that $u_t \leq 0$ and $u_x \geq 0$. The case $\beta = 1$ follows in the same way. ■

Theorem 3.8. *Assume H-1 and H-2. Let $\beta = 1$ and let u_0 be as in the preceding lemma. Then for $\lambda > 1$*

$$\int_0^a u^{-\lambda}(t, x) \, dx \rightarrow \infty \quad \text{as } t \rightarrow T.$$

Proof: We follow a line of reasoning similar to that of Theorem 3.4 in [FK], see also [FM]. From (3.9) we have

$$\Phi(u_x^2) \leq \underline{u}^{-1}(u - \underline{u}), \quad \text{where } \underline{u} := u(t, 0).$$

Let t_0 be such that $u(t_0, 0) < 1/2$. Then for $t \geq t_0$ there exists a point $x_1(t)$ for which $u(t, x_1(t)) = 2\underline{u}$. Since $0 \leq \underline{u}^{-1}(u - \underline{u}) \leq 1$ for $x \leq x_1$ and $0 < t < T$ the following holds:

$$u_x^2 \leq \Phi^{-1}\left(\frac{u - \underline{u}}{\underline{u}}\right) - \Phi^{-1}(0) \leq \left(\frac{u - \underline{u}}{\underline{u}}\right) \max_{0 \leq s \leq 1} (\Phi^{-1})'(s). \quad (3.10)$$

We have used the fact that Φ^{-1} is increasing, $\Phi(0) = 0$ and $(\Phi^{-1})'$ is bounded. From (3.10) it follows that

$$\frac{u_x}{\sqrt{u - \underline{u}}} \leq \frac{c_3}{\sqrt{\underline{u}}}.$$

Integrating the last inequality from 0 to $x_1(t)$ we get that $x_1 \geq c_4 \underline{u}$. Now

$$\int_0^a u^{-\lambda}(t, x) \, dx \geq \int_0^{x_1} u^{-\lambda}(t, x) \, dx \geq \int_0^{x_1} (2\underline{u})^{-\lambda} \, dx \geq (2\underline{u})^{-\lambda} x_1 \geq c_5 \underline{u}^{1-\lambda} \rightarrow \infty$$

if $\lambda > 1$. ■

4. Single Point Quenching under Neumann Boundary Conditions

In this section we consider the problem (1.1) (1.2b) (1.3). It is obvious that any solution quenches in finite time and that there are solutions which quench everywhere (solutions with $u_0 \equiv c > 0$). The aim of this section is to show the existence of initial data for which single point quenching occurs. This was done in [DK] for $\phi(s) = \arctan s$ and $\beta = 1$. For functions ϕ which describe stronger diffusion (e.g. $\phi(s) = s$) this question seems to be more difficult.

Theorem 4.1. *Assume H-1. Let u_0 satisfy the assumptions of Lemma 3.1 (i)–(iii), $u'_0 \not\equiv 0$. Then u quenches only at $x = 0$.*

Proof: Under our assumptions on u_0 an inspection of the proof of Lemma 3.1 (iii) reveals that $u_{xt} > 0$ in $(0, T) \times (0, a)$. This means that u quenches only at $x = 0$ (cf. [AK, DK]). ■

In order to show that the previous theorem is not vacuous we have to prove

Theorem 4.2. *Assume H-2. Then there exist initial functions u_0 which satisfy the assumptions of Lemma 3.1 (i)–(iii), $u'_0 \not\equiv 0$.*

Proof: Consider the problem

$$\phi'(v')v'' - v^{-\beta} = -\delta \quad \text{for } x > 0, \quad (4.1)$$

$$v(0) = \mu, \quad (4.2)$$

$$v'(0) = 0. \quad (4.3)$$

We will be done with the proof if we show the existence of $\delta, \mu > 0$ such that the corresponding solution $v(x; \delta, \mu)$ satisfies the conditions: $v'(x; \delta, \mu) > 0$ for $x \in (0, a)$, $v'(a; \delta, \mu) = 0$. First we observe that (4.1) admits a first integral. Namely multiplying (4.1) by v' and integrating we obtain that

$$\Phi(v'^2) = \int_0^{v'} r \phi'(r) dr = F(v) - F(\mu), \quad F'(v) = v^{-\beta} - \delta.$$

Equation (4.1) can be rewritten as the following system

$$\begin{aligned} v' &= w, \\ w' &= \frac{1}{\phi'(w)}(v^{-\beta} - \delta). \end{aligned}$$

The unique critical point in the halfplane $\{v > 0\}$ is $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} \delta^{-1/\beta} \\ 0 \end{pmatrix}$. The matrix of the linearization at this critical point is

$$\begin{pmatrix} 0 & 1 \\ -d & 0 \end{pmatrix}, \quad d := \frac{\beta}{\phi'(0)} \delta^{(\beta+1)/\beta}$$

hence $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix}$ is a center. If $\mu \in (0, \delta^{-1/\beta})$, then, since we have a first integral, the solution $v(x; \delta, \mu)$ is periodic and according to the Liapunov center theorem the minimal period tends to $2\pi d^{-1/2}$ as $\mu \rightarrow \delta^{-1/\beta}$. Now set

$$S(\delta, \mu) = \sup\{y > 0 \mid v'(x; \delta, \mu) > 0 \text{ for } x \in (0, y)\}.$$

S is defined on $M := \{(\delta, \mu) \mid \delta > 0; \mu \in (0, \delta^{-1/\beta})\}$ and the standard theorem on continuous dependence on parameters and initial values yields that S is continuous on M . Since $S(\delta, \mu) \rightarrow \pi d^{-1/2}$ as $\mu \rightarrow \delta^{-1/\beta}$ and $\pi d^{-1/2} < a$ ($\pi d^{-1/2} > a$) if δ is large (resp. small) enough, it is obvious that we can choose δ, μ such that $S(\delta, \mu) = a$. ■

5. Equilibria and their stability

Throughout this paragraph we restrict our attention to the case $\beta = 1$. Furthermore we assume

H-3. $\phi'(r) > 0$ for $r > 0$, $\phi \in C([0, \infty)) \cap C^1((0, \infty))$,

H-4. $\lim_{r \rightarrow \infty} \Psi(r) = \lim_{r \rightarrow \infty} \int_0^r \eta \phi'(\eta) d\eta = +\infty$,

H-5. $\int_0^1 \frac{d\eta}{\Psi^{-1}(\eta)} < \infty$.

The existence of Ψ^{-1} on $[0, \infty)$ is guaranteed by (H-3), (H-4). These assumptions are satisfied for the model cases $\phi(s) = |s|^{p-2}s$ with $p \geq 2$ and for $\phi(s) = \arctan s$. Consider the boundary value problem

$$(\phi(v_x))_x = \frac{1}{v} \quad \text{in } (0, a), \quad (5.1)$$

$$v_x(0) = 0, \quad (5.2)$$

$$v(a) = 1, \quad (5.3)$$

where $v > 0$ in $[0, a]$. Notice that the pure Neumann problem, with (5.3) replaced by $v_x(t, a) = 0$, has no solution.

Theorem 5.1. *Suppose that $\beta = 1$ and that ϕ satisfies H-3, H-4, H-5. Then there exists $a_0 > 0$ such that for $a < a_0$ there are exactly two solutions of (5.1) (5.2) (5.3), for $a > a_0$ there is none, and for $a = a_0$ there is exactly one solution. Moreover, if $a < a_0$, the two solutions $v^1(x, a)$ and $v^2(x, a)$ are ordered ($v^1 > v^2$), and v^1 is a stable and v^2 an unstable equilibrium of (1.1) (1.2a) (1.3). For $a = a_0$ the functions $v^1(x, a)$ and $v^2(x, a)$ coalesce and form an equilibrium which is unstable from below and stable from above.*

Proof: Since v is convex we have $v(0) < v(a)$ and $v'(x) > 0$ on $(0, a)$. For ease of notation let $v_0 = v(0)$. Then v solves (5.1) (5.2) iff

$$\Psi(v_x) = \ln\left(\frac{v(x)}{v_0}\right)$$

or, equivalently,

$$\int_{v_0}^{v(x)} \frac{d\eta}{\Psi^{-1}(\ln(\eta/v_0))} = x.$$

Therefore the number of solutions to (5.1) (5.2) (5.3) is the same as the number of solutions v_0 of

$$\int_{v_0}^1 \frac{d\eta}{\Psi^{-1}(\ln(\eta/v_0))} = a. \quad (5.4)$$

If we substitute $y = \ln(\eta/v_0)$, relation (5.4) becomes

$$v_0 \int_0^{\ln(1/v_0)} \frac{e^y}{\Psi^{-1}(y)} dy = a.$$

Notice that here we have made use of H-5. Setting $z = 1/v_0$ we now must count the number of solutions $z \in (1, \infty)$ of

$$H(z) := \int_0^{\ln z} \frac{e^y}{\Psi^{-1}(y)} dy = az. \quad (5.5)$$

Straightforward calculations show that $H(z)$ is strictly increasing and strictly concave in z , since

$$H'(z) = \frac{1}{\Psi^{-1}(\ln z)} > 0, \quad (5.6)$$

and, recalling that $\Psi' > 0$ implies $(\Psi^{-1})' > 0$,

$$H''(z) = \frac{-[\Psi^{-1}(\ln z)]'}{[\Psi^{-1}(\ln z)]^2} = -\frac{1}{z} \cdot \frac{[\Psi^{-1}]'}{[\Psi^{-1}]^2}(\ln z) < 0.$$

To determine the asymptotic slope of H as $z \rightarrow \infty$ we use (5.6) and observe that

$\lim_{z \rightarrow \infty} \Psi^{-1}(\ln z) = +\infty$ due to H-4. Therefore $H'(z) \rightarrow 0$ as $z \rightarrow \infty$ and there is

a unique a_0 such that the graph of H is tangent to the line $y = a_0 z$, see Figure

3.

Figure 3: Graphs of $H(z)$ and az

This proves the first statement of Theorem 5.1. For $a < a_0$ let us denote the two solutions of $H(z) = az$ by $z_1(a)$ and $z_2(a)$, $z_1(a) < z_2(a)$. It is obvious from

Figure 3 that $z_1(a)$ is increasing in a while $z_2(a)$ is decreasing in a . Moreover, $\lim_{a \rightarrow 0} z_1(a) = 1$ and $\lim_{a \rightarrow 0} z_2(a) = +\infty$. Correspondingly there are two stationary solutions $v^1(x, a)$ and $v^2(x, a)$ such that $v^i(0, a) = 1/z_i(a)$ and $v^1(0, a) > v^2(0, a)$. The solution branches are illustrated in Figure 4. They are continuous and monotone as indicated.

Figure 4: Bifurcation diagram for solutions of (5.1) (5.2) (5.3)

Let us now show that $v^1(0, a) > v^2(0, a)$ implies $v^1(x, a) > v^2(x, a)$ for every $x \in (0, a)$. Otherwise $v^1(x, a) - v^2(x, a)$ has at least two zeroes in $(0, a]$. Let $y \in (0, a]$ be the second zero. Then $v_x^1(y, a) \geq v_x^2(y, a)$. Therefore

$$\Psi(v_x^1(y, a)) - \ln v^1(y, a) \geq \Psi(v_x^2(y, a)) - \ln v^2(y, a).$$

Since the left and right hand sides of this inequality are constant in y , we obtain

$$-\ln v^1(0, a) \geq -\ln v^2(0, a)$$

in contrast to $v^1(0, a) > v^2(0, a)$.

Finally we have to study the stability of v^1 and v^2 . We shall show in detail that v^2 is unstable from below, i.e. solutions of (1.1) (1.2a) (1.3) which start in any arbitrarily small one sided neighbourhood of v^2 do not necessarily return to v^2 . The remaining cases can be proved by the same kind of arguments. Suppose

that the initial data u_0 from (1.3) satisfy $u_0(x) \leq v^2(x, a)$ and $u_0(0) < v^2(0, a)$. Then after a short time $\eta > 0$ we have

$$u(\eta, x) < v^2(x, a) \quad \text{for } x \in [0, a)$$

for the solution $u(t, x)$ of (1.1) (1.2a) (1.3). Since v^2 depends continuously and monotonically on a , there exists a $\delta > 0$ such that

$$u(\eta, x) < v^2(x, a - \delta) \quad \text{for } x \in [0, a - \delta).$$

We use the differential equation (5.1) to extend $v^2(x, a - \delta)$ onto the interval $[0, a]$. Now $v^2(x, a - \delta)$ and $v^2(x, a)$ are both solutions of (5.1) on $(0, a)$. Hence

$$\hat{v}(x) := \min\{ v^2(x, a), v^2(x, a - \delta) \} \quad (5.7)$$

is a supersolution of (5.1) (5.3) and $\hat{v}(0) < v^2(0, a)$, see Figure 5.

Figure 5: Construction of $\hat{v}(x)$

Therefore $u(t, x) < \hat{v}(x)$ for all $t \in (\eta, T)$ and $v^2(x, a)$ is unstable. This completes the proof of Theorem 5.1. ■

What happens to a solution u of (1.1) (1.2a) (1.3) if the initial data $u_0(x)$ are taken from the unstable regime $0 < u_0(x) \leq v^2(x, a)$; $u_0(x) \neq v^2(x, a)$? Does it quench in finite time? The answer to this question is in general positive but requires several arguments.

Lemma 5.2. *Suppose that ϕ satisfies the assumptions of Theorem 5.1 and that ϕ is convex and of class C^2 or that $s \geq \phi(s)$ for $s > 0$. Let u be a solution of (1.1) (1.2a) (1.3) and suppose that $u'_0(x) > 0$ for $x \in (0, a]$ and $u_x(t, a) \geq \delta > 0$ for every $t \in (0, T)$. If u quenches at time $T \in (0, \infty]$, then u quenches only at the origin.*

For the proof we follow ideas from [DL] and [FM]. It amounts to showing that for every $b \in (0, a)$ and for some small but positive $\varepsilon(b, \delta)$ the inequality

$$J(t, x) := u_x(t, x) - \varepsilon(x - b) \geq 0 \quad (5.8)$$

holds in $Q_b := (0, T) \times (b, a)$. In fact, an integration of (5.8) yields

$$u(t, x) - u(t, b) \geq \varepsilon \frac{(x - b)^2}{2} \quad \text{for } x \in (b, a),$$

and so u cannot quench in (b, a) . To control the sign of J in Q_b one derives differential inequalities for expressions such as J . In order to cover the model problems $\phi(s) = |s|^{p-2}s$ and $\phi(s) = \arctan s$ we distinguish two cases.

Case a) $\phi'' \geq 0$. It is easily checked that

$$J_t - \phi'(u_x)J_{xx} = \frac{1}{u^2}u_x + \phi''(u_x)u_{xx}^2 \geq 0 \quad \text{in } Q_b.$$

Therefore (5.8) holds provided $J \geq 0$ on the parabolic boundary of Q_b .

Case b) $s \geq \phi(s)$. Then $\tilde{J}(t, x) := \phi(u_x) - \varepsilon(x - b)$ satisfies

$$\tilde{J}_t - \phi'(u_x)\tilde{J}_{xx} = \phi'(u_x)\frac{u_x}{u^2} \geq 0 \quad \text{in } Q_b,$$

and (5.8) holds provided $\tilde{J} \geq 0$ on the parabolic boundary of Q_b . Notice that $\tilde{J}(t, b) \geq 0$.

It remains to check the sign of J and $\tilde{J}$ for $x = a$ or $t = 0$. By assumption $u_x(t, a) \geq \delta > 0$. Hence $J(t, a) \geq 0$ and $\tilde{J}(t, a) \geq 0$ for a suitably small ε and for every $t \in (0, T)$. Since $u'_0 > 0$ on $[b, a]$, one has also $J(0, x) \geq 0$ and $\tilde{J}(0, x) \geq 0$ on $[b, a]$ for a suitably small ε . This completes the proof of Lemma 5.2. ■

Similarly to [L2, Thm. 3.1.A] we can now state

Theorem 5.3. *Suppose that H-3, H-4, H-5 hold.*

- i) *If $a \leq a_0$ and $0 < u_0(x) \leq v^2(x, a)$, $u_0(0) < v^2(0, a)$, then u quenches in finite time.*
- ii) *If $a \leq a_0$ and $v^2(x, a) \leq u_0(x) \leq 1$, $u_0(x) \not\equiv v^2(x, a)$, then u converges asymptotically to $v^1(x, a)$ as $t \rightarrow \infty$.*
- iii) *If $a > a_0$ and $0 < u_0(x) \leq 1$, then u quenches in finite time.*

Again we give a detailed proof only for the most interesting case i). The proofs for ii) and iii) are easier and will be left as an exercise for the reader. By the same comparison argument as in the proof of Theorem 5.1 it suffices to show finite time quenching for the solution starting from $\hat{v}(x)$, see (5.7). If u does not quench in finite time, then a) either it converges to a positive equilibrium, or b) it quenches in infinite time. Case a) is ruled out, since there is no positive equilibrium below $v^2(x, a)$. In case b) we first observe that u decreases monotonically (it starts from a supersolution), and that u satisfies the assumptions of Lemma 5.2. Therefore it converges pointwise, as $t \rightarrow \infty$, to a singular stationary solution $w(x)$, i.e. a function w that vanishes in 0 and satisfies $w(a) = 1$ and differential equation (5.1) in $(0, a)$, see Lemma 5.2. Consequently $P(x) = \Psi(w_x(x)) - \ln w(x)$ is constant in $(0, a)$. However, as x tends to zero, $P(x)$ becomes unbounded, a contradiction which rules out b). Therefore u must quench in finite time and the proof is complete. ■

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