**THE STEFAN PROBLEM WITH SURFACE TENSION IN
THE THREE DIMENSIONAL CASE WITH SPHERICAL
SYMMETRY: NON-EXISTENCE
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THE STEFAN PROBLEM WITH SURFACE TENSION IN THE THREE DIMENSIONAL CASE WITH SPHERICAL SYMMETRY: NON-EXISTENCE OF THE CLASSICAL SOLUTION

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Abstract. The Stefan problem with surface tension in three dimensional case with spherical symmetry is considered. We first establish the existence and uniqueness of the classical solution of the Stefan problem with surface tension and kinetic undercooling effects

$$\theta = \theta_* - \gamma k + \beta V_\nu$$

for all time and then pass to the limit as $\beta \rightarrow 0$. The limiting solution be the global-time weak solution and the local-time classical solution of the Stefan problem with surface tension. These solution cannot be the global-time classical solution. If $S(t)$ be the radius of the solid ball, then there exist time $\bar{t}_* > 0$ such that

$$S(\bar{t}_* + 0) < S(\bar{t}_* - 0).$$

Key words. Supercooling, Stefan problem, Gibbs-Thomson law.

AMS(MOS) subject classifications. 35R35, 35K20, 35K85.

Introduction. This work arose as an answer to a question of J.R. Ockendon and S. Howison.

The question being, how to calculate the time of disappearance of a solid ball in a supercooling liquid?

When we say “supercooling liquid”, we mean that the process is modeled by the Stefan problem with surface tension.

Before giving a formulation of this problem let me recall the Stefan problem.

We have a continuous medium occupying a domain Ω , θ is the temperature and the domains $\Omega^+(t)$, $\Omega^-(t)$ correspond to the liquid and the solid parts of the domain Ω at the moment t .

In each domains $\Omega^\pm(t)$ the temperature θ satisfies the heat equation

$$(0.1) \quad C \frac{\partial \theta}{\partial t} = \kappa \Delta \theta, \vec{x} \in \Omega^\pm(t), \quad t > 0,$$

Let $\Gamma(t)$ be the free boundary between the liquid and the solid. We have

$$(0.2) \quad [\theta] = 0, \quad \left[\kappa \frac{\partial \theta}{\partial \nu} \right] = L \cdot V_\nu, \quad \vec{x} \in \Gamma(t), \quad t > 0,$$

and

$$(0.3) \quad \theta = \theta_*, \quad \vec{x} \in \Gamma(t), \quad t > 0.$$

Here c is a heat capacity, κ is a heat conductivity, $[f]$ is a jump of the function f on the free boundary $\Gamma(t)$, L is the latent heat (all constants are positive), $\vec{\nu}$ is a normal vector to the free boundary, V_ν is the normal speed of the free boundary and θ_* is the melting temperature.

The problem (0.1)-(0.3) is completed with the addition of initial conditions and a boundary condition on the known boundary $\partial\Omega$.

If one introduces a phase function φ , such that

$$(0.4) \quad \varphi = 1 \text{ in } \Omega^+(t), \quad \varphi = 0 \text{ in } \Omega^-(t),$$

then equation (0.1) together with conditions (0.2) may be rewritten in the form

$$(0.5) \quad \frac{\partial}{\partial t}(c\theta + L\varphi) = \kappa\Delta\theta, \quad \vec{x} \in \Omega, \quad t > 0.$$

Equation (0.5) together with corresponding initial and boundary conditions can be considered as a problem of finding only the temperature θ . Then the condition (0.3) on the free boundary $\Gamma(t)$ is reduced to a problem of determining on the phase function φ .

If one introduces Heaviside's graph $H(\xi)$, then the condition (0.3) may be rewritten in the form

$$(0.6) \quad \varphi \in H(\theta - \theta_*), \quad \vec{x} \in \Omega, t > 0$$

The problem (0.5), (0.6) together with initial and boundary conditions on the boundary $\partial\Omega$ is a weak formulation of the Stefan problem.

From the physical point of view the Stefan problem is the simplest equilibrium phase-transition model, but this is not a simple problem from the mathematical point of view.

Perhaps more natural and more interesting for applications are the nonequilibrium phase-transitions models. The Stefan problem with surface tension can be the simplest model of such problems.

There are two independent thermodynamic parameters: the temperature θ and the phase function φ . Temperature θ as in the Stefan problem satisfies equation (0.5), and phase function φ , instead of equation (0.6) in the Stefan problem, satisfies the Gibbs-Thomson law on the free boundary

$$(0.7) \quad \theta = \theta_* \mp \gamma k, \quad \vec{x} \in \Gamma(t), \quad t > 0,$$

where positive constant γ is the surface tension, k is the mean curvature of the interface. If a line segment $x_0\vec{z}_0$ of the segment $x_0\vec{y}_0$ connecting the interface point $\vec{x}_0$ to the center of curvature $\vec{y}_0$ lies in the solid, then in (0.7) we take the negative sign.

More general form of the Gibbs-Thomson law on the free boundary is the condition

$$(0.8) \quad \theta = \theta_* - k\gamma + \beta V_\nu, \quad \vec{x} \in \Gamma(t), \quad t > 0,$$

where a small constant β is a positive, if a normal $\vec{\nu}$ be the outside to the solid.

A. Visintin [1] proved the existence of at least one weak solution of problem (0.5), (0.8). S. Luckhaus [2] proved the existence of a weak solution of the problem (0.5), (0.7) and showed the nonuniqueness of this solution. X. Chen and F. Reitich [3] proved the local-time existence of the classical solution of the problem (0.5), (0.8) and E. Radkevitch [4] proved the local-time existence of the classical solution of problems (0.5), (0.7) and (0.5), (0.8). One-phase problem with spherical symmetry is considered by J. Chadam, S. Howison and P. Ortoleva [5].

Now, let us consider a spherical solid ball in $\mathbb{R}^3$ surrounded by a supercooling liquid. Both the liquid and the solid are contained in the spherical domain Ω with the same center as the solid ball.

We suppose, that the initial temperature depends only on radius $x = |\vec{x}|$ and temperature on the boundary $\partial\Omega$ depends only on time t . Then the problem (0.5), (0.7) (or (0.5), (0.8)) has a solution $\{\theta, \varphi\}$ depending only on the radius x and the time t .

Firstly we make a change of variables to dimensionless form

$$t \rightarrow t/t_0, \quad x \rightarrow x/x_0, \quad \theta \rightarrow \frac{\theta - \theta_*}{\theta_0}$$

and choose

$$(0.9) \quad \theta_0 = \frac{2\gamma}{x_0}, \quad t_0 = \frac{cx_0^2}{\kappa}, \quad \lambda = \frac{Lx_0}{2\gamma}$$

Then (keeping for simplicity the same notations $x, t, \theta, \varphi, \Omega, \Omega^\pm(t)$) we obtain

$$(0.10) \quad \frac{\partial}{\partial t}(\theta + \lambda\varphi) = \frac{1}{x} \frac{\partial^2}{\partial x^2}(x\theta), \quad x \in \Omega, \quad t > 0,$$

$$(0.11) \quad \theta(x, 0) = \tilde{\theta}_0(x), \quad \varphi(x, 0) = \varphi_0(x), \quad x \in \Omega,$$

$$(0.12) \quad \lim_{x \rightarrow 0} |\theta| < \infty, \quad \theta(\ell, t) = \theta_1(t), \quad t > 0,$$

$$(0.13) \quad \theta = -\frac{1}{S(t)}, \quad x = S(t), \quad t > 0,$$

and instead of conditions (0.5), (0.8) we have (0.10), (0.11), (0.12) and

$$(0.14) \quad \theta = -\frac{1}{S(t)} + \beta \frac{dS}{dt}, \quad x = S(t), t > 0.$$

Here $\Omega = \{x|0 < x < \ell\}$, $\ell = \text{const} > 1$, $S(t)$ is the radius of the solid ball, $\Omega^-(t) = \{x|0 < x < S(t)\}$, $\Omega^+(t) = \{x|S(t) < x < \ell\}$.

The main results of the present work are:

1. Existence and uniqueness of the classical solution of the problem (0.10) - (0.12), (0.14) for any $\beta > 0$ and all time $t > 0$.
2. Local-time existence and uniqueness of the classical solution of the problem (0.10)-(0.13).
3. Existence of at least one weak solution of the problem (0.10) - (0.13) with monotone free boundary $x = S(t)$ for all time $t > 0$.
4. Nonexistence of the classical solution of the problem (0.10)-(0.13) for all time $t > 0$.
5. Nonuniqueness of the solution of the problem (0.10)-(0.13) when $\varphi_0 \equiv 0$.

Let us introduce a new unknown function

$$(0.15) \quad u = x \cdot \theta(x, t) + 1.$$

Then the problems (0.10)-(0.13) or (0.10)-(0.12), (0.14) have the form

$$(0.16) \quad \frac{\partial}{\partial t}(u + \lambda x \varphi) = \frac{\partial^2 u}{\partial x^2}, \quad x \in \Omega, \quad t > 0,$$

$$(0.17) \quad u(x, 0) = u_0(x), \quad \varphi(x, 0) = \varphi_0(x), \quad x \in \Omega,$$

$$(0.18) \quad u(0, t) = 1, \quad u(\ell, t) = u^1(t), \quad t > 0,$$

$$(0.19) \quad u = 0, \quad x = S(t), \quad t > 0,$$

or, in the second case, we replace (0.19) with

$$(0.20) \quad u = -\beta S \frac{dS}{dt}, \quad x = S(t), t > 0$$

The problem (0.16)-(0.19) is exactly the Stefan problem, but with a negative and nonconstant latent heat $\tilde{L} = -\lambda x$.

Actually, if the solution of the problem (0.16)-(0.19) is smooth enough, then the equation (0.16) is equivalent to the heat equation in each domain $\Omega^\pm(t)$

$$(0.21) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \Omega^\pm(t), \quad t > 0,$$

with the condition

$$(0.22) \quad [u] = 0, \quad x = S(t), t > 0,$$

and the Stefan condition with negative latent heat

$$(0.23) \quad \lambda S \frac{dS}{dt} = \left[\frac{\partial u}{\partial x} \right]_{S(t)+0}^{S(t)-0}, \quad x = S(t), t > 0,$$

We choose new variables such that $S(0) = 1$.

If we suppose that $u^1(t) \equiv 0$, $u_0(x) \geq 0$ for $x \in (0, 1)$, $u_0(x) \equiv 0$ for $x \in (1, \ell)$, $\varphi_0(x) \equiv 0$ for $x \in (0, 1)$, $\varphi_0(x) \equiv 1$ for $x \in (1, \ell)$ then we obtain a one-phase problem

$$(0.24) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < S(t), \quad t > 0$$

$$(0.25) \quad u = 0, \quad \lambda S \cdot \frac{dS}{dt} = \frac{\partial u}{\partial x}, \quad x = S(t), t > 0,$$

$$(0.26) \quad u(0, t) = 1, \quad u(x, 0) = u_0(x), \quad S(0) = 1.$$

Our conclusion concerning the nonexistence of the classical solution of the Stefan problem with surface tension for all time is based upon a very simple identity; multiplying equation (0.25) by x and integrating by parts, one obtains after several simple transformations

$$(0.27) \quad \int_0^{S(t)} x u(x, t) dx - \frac{\lambda}{3} S^3(t) = \int_0^1 x u_0(x) dx - \frac{\lambda}{3} + t.$$

First of all note that if

$$(0.28) \quad \lambda \leq 3 \int_0^1 x u_0(x) dx,$$

then the radius of the solid ball $S(t)$ is strictly greater than zero as long as the classical solution exists.

Actually, the classical solution of the problem (0.24)-(0.26) satisfies the maximum principle

$$(0.29) \quad 0 < u(x, t) < \max\{\max u_0(x), 1\}, \quad x \in \Omega^-(t), \quad t > 0$$

$$(0.30) \quad \frac{\partial u}{\partial x}(S(t) - 0, t) < 0, \quad t > 0.$$

Applying inequality (0.29) to the identity (0.27) one has

$$(0.31) \quad \begin{aligned} \frac{1}{2} \max |u| \cdot S^2(t) &= \max |u| \cdot \int_0^{S(t)} x dx \geq \int_0^{S(t)} x u(x, t) dx \geq \\ &\int_0^{S(t)} x u(x, t) dx - \frac{\lambda}{3} S^3(t) = \int_0^1 x u_0(x) dx - \lambda 3 + t \geq t. \end{aligned}$$

So, the radius of the solid ball is strictly greater than zero for all time.

On the other hand, this radius is bounded for all time:

$$(0.32) \quad S(t) \leq 1, \quad t > 0.$$

This follows from (0.30) and Stefan condition (0.25). If one supposes now that the classical solution of the problem (0.24)-(0.26) exist for all time, then one obtains a contradiction to (0.27): the left hand side of the identity (0.27) is bounded, while the right hand side increases with time t .

Hence, there exists a time t_* such that

$$\lim_{t \rightarrow t_* - 0} \left| \frac{dS}{dt}(t) \right| = \infty.$$

It is difficult to see what happens with the solution when $t > t_*$. After several discussions with J. Ockendon he prompted me use an approximating law (0.20) and then pass to the limit as $\beta \rightarrow 0$.

Let us follow this way. (All notations of functional spaces and norms in these spaces from [6]).

§1. Approximating problem (0.16)-(0.18), (0.20) ($\beta > 0$). So, one has to determine functions $u(x, t)$ and $S(t)$ such that

$$(1.1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \Omega^\pm(t), \quad t > 0$$

$$(1.2) \quad [u] = 0, \beta \left[\frac{\partial u}{\partial x} \right]_{S(t)+0}^{S(t)-0} + \lambda u = 0, \quad x = S(t), t > 0$$

$$(1.3) \quad u(0, t) = 1, \quad u(\ell, t) = u^1(t), \quad t > 0,$$

$$(1.4) \quad u(x, 0) = u_0(x), x \in \Omega, \quad S(0) = 1,$$

$$(1.5) \quad u = -\beta S \cdot \frac{dS}{dt}, \quad x = S(t), t > 0.$$

We used condition (1.5) for changing the Stefan condition to the form (1.2).

First of all let us consider the linear problem (1.1)-(1.4) with given boundary $x = \bar{S}(t)$ such that

$$(1.6) \quad \delta \leq \bar{S}(t) \leq \ell - \delta$$

with sufficiently small positive constant δ .

Denote as usual

$$\Omega^-(t) = \{x \mid 0 < x < \bar{S}(t)\}, \Omega^+(t) = \{x \mid \bar{S} < x < \ell\}, \Omega_T^\pm = \bigcup_{0 < t < T} \Omega^\pm(t).$$

Assume, that the functions $u_0(x)$ and $u^1(t)$ satisfy the conditions of concordance at the points $x = 0, x = 1, x = \ell$:

$$(1.7) \quad \begin{cases} u_0(0) = 1, & u_0''(0) = 0; & u_0(1-0) = u_0(1+0), \\ \beta[u_0']_{1+0}^{1-0} + \lambda u_0(1) = 0; & \lambda[u_0'']_{1-0}^{1+0} = ([u_0']_{1+0}^{1-0})^2, \\ \lambda u_0''(1-0) + u_0'(1-0) \cdot [u_0']_{1+0}^{1-0} = 0; & u_0(\ell) = u^1(0), u_0''(\ell) = \dot{u}^1(0). \end{cases}$$

LEMMA 1. Let $u_0 \in H^{2+\alpha}(\overline{\Omega^\pm(0)})$, $u^1, \bar{S} \in H^{\frac{2+\alpha}{2}}[0, \infty)$, $0 < \alpha < 1$, $\bar{S}(0) = 1$, $\beta \bar{S}(0) = -u_0(1)$ and $\bar{S}, u_0, u^1$ satisfy conditions (1.6), (1.7).

Then there exists a unique solution

$$w \in H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega_T^\pm})$$

of the problem (1.1)-(1.4) with given boundary $\bar{S}(t)$ for all $T > 0$.

The proof of this lemma is based upon the methods in [6] and the solvability of the problem

$$(1.8) \quad \begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, & x \neq 0, t > 0; v|_{t=0} = 0; \\ [v] = 0, & \left[\frac{\partial v}{\partial x} \right]_{+0}^{-0} + a(t)v = b(t), x = 0, t > 0 \end{cases}$$

in $R_T = \{(x, t) \mid |x| \leq \infty, 0 < t < T\}$ with $a, b \in H^{\frac{1+\alpha}{2}}[0, T]$, $a(t) \geq 0$.

The problem (1.8) is solved with the help of the problem

$$(1.9) \quad \begin{cases} \frac{\partial v^+}{\partial t} = \frac{\partial^2 v^+}{\partial x^2}, & (x, t) \in R_T^+ = \{x > 0, 0 < t < T\}, \\ v|_{t=0} = 0, & -\frac{\partial v}{\partial x}(0, t) + \frac{a(t)}{2}v(0, t) = \frac{b(t)}{2}, \quad t > 0 \end{cases}$$

if we suppose

$$v(x, t) = \begin{cases} v^+(x, t), & x > 0, t > 0, \\ v^+(-x, t), & x < 0, t > 0. \end{cases}$$

And finally, the problem (1.9) has a unique solution $v \in H^{2+\alpha, \frac{2+\alpha}{2}}(\bar{R}_T^+)$, if $b(0) = 0$. This follows from [6].

THEOREM 1. Let us assume, that

$$u_0 \in H^{2+\alpha}(\overline{\Omega^\pm(0)}), \quad u^1 \in H^{\frac{2+\alpha}{2}}[0, \infty)$$

and u_0, u^1 satisfy conditions (1.7).

Then there exists a unique solution

$$u \in H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega_T^\pm}), \quad S \in H^{2+\alpha/2}[0, T]$$

of the problem (1.1)–(1.5) with sufficiently small $T > 0$.

Proof. Let

$$\max\{|u_0|_{\Omega^+(0)}^{(2+\alpha)}, |u_0|_{\Omega^-(0)}^{(2+\alpha)}, |u^1|_{[0, \infty)}^{(\frac{2+\alpha}{2})}\} \leq C_0.$$

One can use a fixed point theorem to prove this result. Consider the set

$$\mathcal{M} = \{\bar{u} \in H^{\frac{1+\alpha}{2}}[0, T] \mid |\bar{u}|_{[0, T]}^{(\frac{1+\alpha}{2})} \leq 2C_0\}$$

and let

$$(1.10) \quad \bar{S}^2(t) = 1 - \frac{2}{\beta} \int_0^t \bar{u}(\tau) d\tau,$$

with $\bar{u} \in \mathcal{M}$. Choose time $T > 0$ such that $\bar{S}(t)$ satisfies condition (1.6) on the interval $[0, T]$.

One can easily show that

$$(1.11) \quad T \leq T_1 = \frac{\beta}{2C_0} \max\{1 - \delta, \ell - 1 - \delta\}$$

In accordance with lemma 1 the problem (1.1)–(1.4) with $\bar{S}(t)$ given by the formula (1.10) has a unique solution $w(x, t)$ satisfying

$$(1.12) \quad |w|_{\Omega_T^+}^{(2+\alpha)} \leq C_1(C_0, \delta).$$

The function

$$(1.13) \quad \bar{w}(t) = w(\bar{S}(t), t) = F(\bar{u})$$

defines a compact operator F mapping the set $\mathcal{M}$ into the space $H^{\frac{2+\alpha}{2}}[0, T]$. Hence, if one can choose a time T such that F transform $\mathcal{M}$ into $\mathcal{M}$ then, in accordance with Schauder's fixed point theorem one has proven the existence of at least one classical solution of the problem (1.1) - (1.5).

Since

$$\bar{w}(t) = \bar{w}(0) + \int_0^t \bar{w}'(\tau) d\tau,$$

and

$$\frac{|\overline{w}(t+h) - \overline{w}(t)|}{|h|} \leq |\overline{w}'(t_1)|, \quad t_1 \in [t, t+h],$$

then

$$\begin{aligned} |w|_{[0,T]^{\frac{(1+\alpha)}{2}}} &\leq |\overline{w}(0)| + T|\overline{w}|_{[0,T]}^{(1)} + 2T^{\frac{1-\alpha}{2}} |\overline{w}|_{[0,T]}^{(1)} \leq \\ &\leq C_0 + C_1(T + 2T^{\frac{1-\alpha}{2}}) \leq 2C_0 \end{aligned}$$

if

$$(1.14) \quad T \leq T_2 = \left(\frac{C_0}{C_1(T^{\frac{1+\alpha}{2}} + 2)} \right)^{\frac{2}{1-\alpha}}$$

Choosing

$$T \leq \min\{T_1, T_2\}$$

we see that F transforms $\mathcal{M}$ into $\mathcal{M}$.

Before proving the uniqueness of the solution of (1.1)-(1.5) let us transform this problem to a problem on a fixed domain.

The transformations

$$y = \frac{x}{S(t)}, \quad \tau = t \text{ in } \Omega_T^-$$

map the domain Ω_T^- into the domain

$$\Pi_T^- = \{(y, \tau) \mid 0 < y < 1, \quad 0 < \tau < T\}$$

and

$$y = 1 + (\ell - 1) \frac{x - S(t)}{\ell - S(t)}, \quad \tau = t \text{ in } \Omega_T^+(t)$$

map the domain Ω_T^+ into the domain

$$\Pi_T^+ = \{(y, \tau) \mid 1 < y < \ell, \quad 0 < \tau < T\}.$$

The new unknown functions

$$v(y, \tau) \equiv u(x, t) \text{ and } S(\tau)$$

satisfy the problem similar to (1.1) - (1.5) but in the known domains $\Pi_T^\pm$.

If there exist two different solutions of the problem (1.1) - (1.5) then their difference

$$\overline{v} = v^1 - v^2, \quad \overline{S} = S^1 - S^2$$

satisfies the following problem

$$(1.15) \quad \frac{\partial \bar{v}}{\partial \tau} - \bar{a}^2 \frac{\partial^2 \bar{v}}{\partial y^2} + \bar{b} \frac{\partial \bar{v}}{\partial y} = A \cdot \bar{S} + B \cdot \frac{d\bar{S}}{d\tau}, \quad (y, \tau) \in \Pi_T^\pm,$$

$$(1.16) \quad [v] = 0, [\bar{a} \frac{\partial \bar{v}}{\partial y}] + \frac{\lambda}{\beta} \bar{v} = 0, \quad y = 1, \tau > 0,$$

$$(1.17) \quad \bar{v} = C \cdot \bar{S} + D \cdot \frac{d\bar{S}}{d\tau}, \quad y = 1, \tau > 0,$$

$$(1.18) \quad \bar{v}(0, \tau) = \bar{v}(\ell, \tau) = 0, \tau > 0, \quad \bar{v}(y, 0) = 0, y \in \Omega^\pm(0),$$

with smooth functions $\bar{a}, \bar{b}, A, B, C, D, \bar{a} \geq a_0 = \text{const} > 0$.

Applying results similar to those of lemma 1, one obtains from (1.15), (1.16), (1.18) the following inequality

$$(1.19) \quad |\bar{v}|_{\Pi_T^\pm}^{(2+\alpha)} \leq C_2(C_0) |\bar{S}|_{[0, T]}^{(\frac{2+\alpha}{2})}$$

On the other hand, considering (1.17) as an ordinary differential equation with respect to $\bar{S}$, one can conclude that

$$(1.20) \quad |\bar{S}|^{(\frac{2+\alpha}{2})}[0, T] \leq C_3(C_0) |\bar{v}|_{\Pi_T}^{(\alpha)}.$$

Applying the simple inequalities

$$|\bar{v}|_{\Pi_T}^{(\alpha)} = \left(|\bar{v}|_{\Pi_T}^{(0)} + \langle \bar{v} \rangle_{\Pi_T}^{(\alpha)} \right) \leq T \cdot |\bar{v}|_{\Pi_T}^{(2+\alpha)} + T^{\frac{2-\alpha}{2}} |\bar{v}|_{\Pi_T}^{(2+\alpha)}$$

to (1.20) and (1.19) one obtains that

$$|\bar{S}|_{[0, T]}^{(\frac{2+\alpha}{2})} \leq C_2 C_3 (T + T^{\frac{2-\alpha}{2}}) |\bar{S}|_{[0, T]}^{(\frac{2+\alpha}{2})}$$

If

$$c_2 c_3 (T + T^{\frac{2-\alpha}{2}}) < 1$$

then there only exists the one possibility $\bar{S}(t) \equiv 0$.

Step by step we prove the existence and the uniqueness of the solution (1.1)-(1.5) up to time T_* where

$$\lim_{T \rightarrow T_*} |U|_{Q_T^\pm}^{(2+\alpha)} = \infty, \text{ or } \lim_{t \rightarrow T_*} S(t) = 0, \ell.$$

One can exclude the case $S(T_* - 0) = 0, \ell$. Because in this situation after time T_* solution of the problem (1.1) - (1.5) is the solution of the heat equation with the boundary condition (1.3) and with the initial data $u(x, T_* - 0)$. Hence,

$$(1.21) \quad \lim_{T \rightarrow T_*} |u|_{\Omega_T^\pm}^{(2+\alpha)} = \infty, \quad \delta < S(t) < \ell - \delta, t \in [0, T_*].$$

THEOREM 2. Under assumption theorem 1 $T_* = \infty$.

Proof. First of all we prove the maximum principle:

$$(1.22) \quad \min\{\min u^1, \min u_0\} \leq u(x, t) \leq \max\{\max u^1, \max u_0\}.$$

If we suppose that $u(x, t)$ has its absolute positive maximum at the point $(S(t_0), t_0)$ on the free boundary, then

$$u(S(t_0), t_0) > 0, \\ \frac{\partial u}{\partial x}(S(t_0) - 0, t_0) > 0, \quad \frac{\partial u}{\partial x}(S(t_0) + 0, t_0) < 0.$$

But these inequalities contradict the condition (1.2). The same applied to the case of the absolute negative minimum.

The main estimate results from multiplying equation (1.1) by u_t and integrating by parts in each domain $\Omega^\pm(t)$ with the help of conditions (1.2) and (1.5):

$$(1.23) \quad \int_{\Omega} u_t^2(x, t) dx + \frac{1}{2} \frac{d}{dt} \int_{\Omega} u_x^2(x, t) dx + \frac{1}{4} \beta \lambda \frac{d}{dt} \left[\frac{dS^2}{dt} \right]^2 = \\ \frac{1}{2} \frac{dS}{dt} \cdot [u_x^2(S + 0, t) - u_x^2(S - 0, t)].$$

Applying the heat equation and the condition (1.2) one has

$$(1.24) \quad \int_{\Omega} (u_t^2 + u_{xx}^2) dx + \frac{d}{dt} \int_{\Omega} u_x^2 dx + \frac{1}{2} \beta \lambda \frac{d}{dt} \left[\frac{dS^2}{dt} \right] = \\ = - \frac{\lambda}{\beta^2 S(t)} \cdot [u_x(S + 0, t) + u_x(S - 0, t)] \cdot u^2(S, t).$$

This identity, together with the inequality

$$u_x^2(S \pm 0, t) \leq C_4(C_0, \delta) + \varepsilon \int_{\Omega^\pm(t)} u_{xx}^2 dx + \frac{C_5}{\varepsilon} \int_{\Omega} u_x^2 dx$$

gives us our main estimate

$$(1.25) \quad \iint_{\Omega_T^\pm} (u_t^2 + u_{xx}^2) dx dt + \max_{t \in (0, T)} \int_{\Omega} u_x^2 dx \leq C_6(C_0, \delta).$$

The Sobolev space $W_2^{2,1}(\Omega_T^\pm)$ is contained in the space $H^{\alpha_0, \frac{\alpha_0}{2}}(\overline{\Omega_T^\pm})$ with some $\alpha_0 > 0$. Hence, comparing this with condition (1.5) one has that

$$S \in H^{\frac{2+\alpha_0}{2}}[0, T].$$

From lemma 1 it follows that

$$u \in H^{2+\alpha_0, \frac{2+\alpha_0}{2}}(\overline{\Omega_T^\pm})$$

and reapplying all this once more if $\alpha_0 < \alpha$, we obtain the final result.

§2. The exact problem (0.16)-(0.19) ($\beta = 0$). The classical formulation of the problem (0.16)-(0.19) has the form:

$$(2.1) \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \Omega^\pm(t), t > 0,$$

$$(2.2) \quad [u] = 0, \quad u = 0 \quad x = S(t), t > 0,$$

$$(2.3) \quad u(0, t) = 1, \quad u(\ell, t) = u^1(t), t > 0$$

$$(2.4) \quad u(x, 0) = u_0(x), \quad x \in \Omega, \quad S(0) = 1,$$

$$(2.5) \quad [u_x]_{S+0}^{S-0} = \lambda S \cdot \frac{dS}{dt}, \quad x = S(t), t > 0.$$

In a very similar manner to theorem 1, one can prove the small-time classical solvability of the problem (2.1)-(2.5).

THEOREM 3. *Let $u_0 \in H^{2+\alpha}(\overline{\Omega^\pm(0)})$, $u^1 \in H^{\frac{2+\alpha}{2}}[0, \infty)$ and u_0, u^1 satisfy conditions (1.7) with $\beta = 0$.*

Then there exists a unique solution

$$u \in H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega_T^\pm}), \quad S \in H^{\frac{2+\alpha}{2}}[0, T]$$

of the problem (2.1) -(2.5), where $T \leq T_3$ and T_3 sufficiently small.

The proof of this theorem is also based upon the fixed point theorem for the set

$$\mathcal{M} = \{\bar{S} \in H^{\frac{2+\alpha}{2}}[0, T] \mid |\bar{S}|_{[0, T]}^{(\frac{2+\alpha}{2})} \leq 2C_0\}$$

where $\bar{S}(t)$ satisfies the condition (1.6). Let $\bar{S}(t)$ be the boundary between the domains $\Omega^-(t)$ and $\Omega^+(t)$ and $\bar{u}(x, t)$ be the solution of the linear problem (2.1)-(2.4). Then the operator

$$F(\bar{S}) \equiv \left(1 + \frac{2}{\lambda} \int_0^t \left[\frac{\partial \bar{u}}{\partial x}(\bar{S}(\tau) - 0, \tau) - \frac{\partial \bar{u}}{\partial x}(\bar{S}(\tau) + 0, \tau) \right] d\tau \right)^{\frac{1}{2}}$$

maps the set $\mathcal{M}$ into the space $H^{\frac{3+\alpha}{2}}[0, \tau]$, and hence is a compact operator from $\mathcal{M}$ into $H^{\frac{2+\alpha}{2}}[0, T]$. Choosing further the time T sufficiently small, one can show that F transforms $\mathcal{M}$ into $\mathcal{M}$ and, hence, has at least one fixed point.

Uniqueness is proved very similarly to theorem 1.

The time of existence of classical solution can be controlled with the help of the distance between the free boundary and the boundary $\{x = 0\}$.

Let us explain this by considering the one-phase problem

LEMMA 2. Let u_0 be the same as in theorem 3, $u^1 \equiv 0$, $u_0(x) \equiv 0$ where $1 < x < \ell$ and

$$(2.6) \quad \lambda > \frac{9}{4} \int_0^1 |u'_0(x)| dx$$

Then the classical solution of the problem (2.1) - (2.5) exists up to the time T_0 , where

$$(2.7) \quad S(t) \geq \frac{9}{4\lambda} \int_0^1 |u'_0(x)| dx = \delta_0, \quad 0 < t < T_0$$

Proof. One can easily show that

$$(2.8) \quad \int_0^{S(t)} |u_x(x, t)| dx \leq \int_0^1 |u'_0(x)| dx.$$

This estimate is obtained by multiplying equation (2.1) for $q = u_x$ by the function

$$\psi_\varepsilon(q) = q(q^2 + \varepsilon^2)^{-1/2},$$

and integrating by parts and passing to the limit when $\varepsilon \rightarrow 0$.

For the next step, we consider the identity (1.23) with $\beta = 0$ and substitute the derivative $u_x(S - 0, t)$ from Stefan condition (2.5) instead of $\frac{dS}{dt}$ and the derivative u_{xx} from heat equation instead of the derivative u_t :

$$(2.9) \quad \int_0^{S(t)} u_{xx}^2 dx + \frac{1}{2} \frac{d}{dt} \int_0^{S(t)} u_x^2 dx = \frac{1}{2\lambda S(t)} \cdot |u_x^3(S - 0, t)|.$$

Since

$$\begin{aligned} |u_x(S - 0, t)|^{3/2} &= |u_x(0, t)|^{3/2} + \frac{3}{2} \int_0^{S(t)} u_{xx} |u_x|^{1/2} dx \leq \\ &\leq |u_x(0, t)|^{3/2} + \frac{3}{2} \left(\int_0^S u_{xx}^2 dx \right)^{1/2} \left(\int_0^S |u_x| dx \right)^{1/2} \leq \\ &\leq |u_x(0, t)|^{3/2} + \frac{3}{2} \left(\int_0^1 |u'_0(x)| dx \right)^{\frac{1}{2}} \cdot \left(\int_0^S |u_{xx}^2| dx \right)^{\frac{1}{2}}, \end{aligned}$$

then

$$\begin{aligned} \int_0^{S(t)} u_{xx}^2 dx + \frac{1}{2} \frac{d}{dt} \int_0^{S(t)} u_x^2 dx &\leq \frac{1}{\lambda S(t)} |u_x(0, t)|^3 + \\ &+ \frac{9}{4\lambda S(t)} \cdot \int_0^1 |u'_0(x)| dx \cdot \int_0^{S(t)} u_{xx}^2 dx \end{aligned}$$

Hence

$$(2.10) \quad \iint_{\Omega_t^-} (u_{xx}^2 + u_t^2) dx dt + \max_{t \in (0, T)} \int_{\Omega^-(t)} u_x^2 dx \leq C_7(C_0, \delta_0)$$

Using the inequality

$$|u_x(S - 0, t)|^4 \leq 2|u_x(0, t)|^4 + \int_0^{S(t)} u_{xx}^2 dx + C_s \left(\int_0^{S(t)} u_x^2 dx \right)^3$$

and the Stefan condition (2.5) one can obtain that

$$(2.11) \quad \int_0^T \left| \frac{dS}{dt}(t) \right|^4 dt \leq C_9(C_0, S_0).$$

It is known from [6], that the linear problem (2.1)-(2.4) with known boundary $S \in W_4^1[0, T]$ has a unique solution

$$u \in W_4^{2,1}(\Omega_T^-).$$

The Sobolev space $W_4^{2,1}(\Omega_T^-)$ is contained in the space $H^{1+\alpha_1, \frac{1+\alpha_1}{2}}(\overline{\Omega_T^-})$. Hence from the Stefan condition one has

$$S \in H^{\frac{2+\alpha_1}{2}}[0, T], \quad 0 < \alpha_1 < \frac{1}{4}.$$

Returning to the linear problem (2.1)-(2.4) with known boundary $x = S(t)$ in the space $H^{\frac{2+\alpha_1}{2}}[0, T]$ one can see as in [6] that

$$u \in H^{2+\alpha_2, \frac{2+\alpha_2}{2}}(\overline{\Omega_T^-}), \quad \alpha_2 = \min(\alpha, \alpha_1).$$

If $\alpha_1 < \alpha$ then reapplying the above, one obtains the final result.

LEMMA 3. *Let $u^1 \in H^{\frac{1+\alpha}{2}}[0, \infty)$, $u_0, u^\beta \in H^{2+\alpha}(\overline{\Omega^\pm(0)})$, u^1, u_0^β satisfy the conditions (1.7) with $\beta > 0$, u^1, u_0 satisfy the conditions (1.7) with $\beta = 0$ and*

$$\lim_{\beta \rightarrow 0} |u_0^\beta - u_0|_{\Omega^\pm(0)}^{(2+\alpha)} = 0$$

Then the solution $\{u^\beta(x, t), S^\beta(t)\}$ of the problem (1.1) - (1.5) with the data $\{u^1, u_0^\beta\}$ converges to the problem (2.1) - (2.5) with data $\{u^1, u_0\}$ as $\beta \rightarrow 0$ in $H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega_T^\pm})$, where $T \leq T_4$ and T_4 is sufficiently small.

Proof. Let us consider the identity (1.23) and substitute the derivatives $u_x^\beta(S^\beta \pm 0, t)$ from Stefan condition (2.5) instead of $\frac{dS}{dt}$. (note, that $u^\beta(x, t)$ satisfies the condition (2.5)). Further estimating the right hand side of this identity with the help of the inequality

$$\begin{aligned} |u_x^\beta(S(t) \pm 0, t)|^3 &\leq C_{10} + 3 \int_{\Omega^\pm(t)} |u_{xx}^\beta|^2 dx^{1/2} \cdot \left(\int_{\Omega^\pm(t)} |u_x^\beta|^4 dx \right)^{1/2} \leq \\ &\leq C_{10} + \varepsilon \int_{\Omega^\pm(t)} |u_{xx}^\beta|^2 dx + C_{11}(C_0, \varepsilon) \left(\int_{\Omega^\pm(t)} |u_{xx}^\beta|^2 dx \right)^3 \end{aligned}$$

and using the heat equation one obtains that

$$\begin{aligned} & \int_{\Omega} \{|u_{xx}^{\beta}|^2 + |u_t^{\beta}|^2\} dx + \frac{d}{dt} \left\{ \int_{\Omega} |u_x^{\beta}|^2 dx + \lambda \beta \left| \frac{d|S^{\beta}|^2}{dt} \right|^2 \right\} \leq \\ & \leq C_{12}(C_0, \delta) + C_{13}(C_0, \delta) \left(\int_{\Omega} |u_x^{\beta}|^2 dx \right)^3 \end{aligned}$$

and

$$(2.12) \quad \iint_{\Omega_T} \{|u_{xx}^{\beta}|^2 + |u_t^{\beta}|^2\} dx dt + \max_{0 < t < T} \int_{\Omega} |u_x^{\beta}(x, t)| dx \leq C_{14}(C_0, \delta)$$

for $T \leq T_4(C_0, \delta)$.

Similarly to lemma 2 one can show that

$$(2.13) \quad S^{\beta} \in W_4^1[0, T]$$

The estimates (2.12), (2.13), allow one to choose subsequence $\{u^{\beta k}(x, t), S^{\beta k}(t)\}$ which converges weakly in $W_2^{2,1}(\Omega_T^{\pm}) \otimes W_4^1[0, T]$ to the solution

$$\bar{u} \in W_2^{2,1}(\Omega_T^{\pm}), \quad \bar{S} \in W_4^1[0, T]$$

of the problem (2.1) - (2.5) as $\beta \rightarrow 0$. Almost identically to lemma 2, one can show that

$$\bar{u} \in H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega_T^{\pm}}), \quad \bar{S} \in H^{\frac{2+\alpha}{2}}[0, T]$$

and, hence, it coincides with the unique classical solution $\{u(x, t), S(t)\}$ corresponding to the data $\{u_0, u^1\}$.

Let us prove now that problem (0.16) - (0.19) has at least one global-time weak solution with monotone free boundary. This corresponds to the case of a melting solid ball.

THEOREM 4. *Under the assumptions of theorem 3 let $u_0(x) \geq 0$, $u_0(x) \equiv 0$ when $x \in \Omega^-(0)$ and $u^1(t) \equiv 0$.*

Then there exists at least one weak solution $\{u(x, t), \varphi(x, t)\}$ of the problem (0.16) - (0.19) such that $\varphi \equiv 0$ in the domain

$$\Omega_{\infty}^{-} = \{(x, t) \mid 0 < x < \ell, \quad 0 < t < g(x)\}$$

and $\varphi \equiv 1$ in the domain

$$\Omega_{\infty}^{+} = \{(x, t) \mid 0 < x < \ell, g(x) < t < \infty\}$$

where $g(x)$ is a continuous monotone function on the interval (X_*, ℓ) , $g(x) \equiv 0$ when $x \in (1, \ell)$,

$$\lim_{x \rightarrow X_* + 0} g(x) = \infty \text{ and } g(x) = \infty \text{ when } x \in [0, X_*) \text{ if} \\ X_* > 0 (g(x) \text{ may be bounded for all } x \in [0, \ell))$$

Proof. Let the $u_0^\beta(x)$ be nonnegative sufficiently smooth functions satisfying conditions (1.7) with $\beta > 0$ and converging to $u_0(x)$ in $H^{2+\alpha}(\overline{\Omega^\pm(0)})$. In accordance with theorem 2 there exists a unique classical solution $\{u^\beta(x, t), \bar{S}^\beta(t)\}$, where $\bar{S}^\beta = S^\beta(t)$ when $0 < S^\beta(t) < \ell$ and $\bar{S}^\beta \equiv 0, \ell, t > T_*^\beta$, when $S^\beta(T_*^\beta) = 0, \ell$.

Let $g^\beta(x)$ be the inverses of the functions $S^\beta(t)$ on the interval $(0, T_*^\beta)$:

$$x \equiv S^\beta(g^\beta(x)).$$

If $T_*^\beta = \infty$, then suppose $g^\beta(x) = \infty$ when $x \in (0, X_*^\beta)$, $X_*^\beta = \lim_{t \rightarrow \infty} S^\beta(t)$.

Suppose $\varphi^\beta(x, t) = 0$ in the domain $\Omega_{\beta, \infty}^- = \{(x, t) | 0 < x < \ell, 0 < t < g^\beta(x)\}$ and $\varphi^\beta(x, t) = 1$ in the domain $\Omega_{\beta, \infty}^+ = \{(x, t) | 0 < x < \ell, g^\beta(x) < t < \infty\}$. For all $\beta > 0$ the functions u^β, φ^β satisfy the identity

$$(2.14) \quad \int_{\Omega} (u^\beta + \lambda x \varphi^\beta) \eta|_{t=T} dx + \iint_{\Omega_T} \{-(u^\beta + \lambda x \varphi^\beta) \eta_t + u_x^\beta \cdot \eta_x\} dx dt = \\ = \int_{\Omega} (u_0^\beta + \lambda x \varphi_0^\beta) \eta|_{t=0} dx + \int_0^T (\eta u_x^\beta)_{x=\ell} dt$$

where η is an arbitrary smooth function such that $\eta(0, t) = 0$.

The maximum principle gives us

$$(2.15) \quad 0 \leq u^\beta(x, t) \leq \max\{1, \max |U_0^\beta|\} = A_0$$

$$(2.16) \quad -\frac{1}{\beta S^\beta(t)} \cdot u^\beta(S^\beta(t), t) = \frac{dS^\beta}{dt}(t) \leq 0,$$

$$(2.17) \quad 0 \leq S^\beta(t) \leq 1.$$

From (2.17), lemma 3 and local estimates for linear equation from [6] it follows that in the domain $G = \{(x, t) | 1 < x < \ell\}$

$$(2.18) \quad |u^\beta|_G^{(2+\alpha)} \leq A_1(C_0)$$

If in (2.14) $\eta = xu^\beta(x, t)$ then one can easily obtain that

$$(2.19) \quad \iint_{\Omega_T} x |u_x^\beta|^2 dx dt \leq A_2(C_0).$$

The estimates (2.15)–(2.19) allow one to choose subsequences $\{u^{\beta_k}(x, t)\}$, $\{g^{\beta_k}(x)\}$, $\{\varphi^{\beta_k}(x, t)\}$ such that when $\beta \rightarrow 0$ $g^{\beta_k}(x)$ converges to a monotone function $g(x)$ almost everywhere; $\varphi^{\beta_k}(x, t)$ converges to a function $\varphi(x, t)$ almost everywhere; and $u^{\beta_k}(x, t)$ converges to a function $u(x, t)$ weakly in $\tilde{W}_2^{1,0}(\Omega_T)$, where

$$\tilde{W}_2^{1,0}(\Omega_T) = \{w \mid \iint_{\Omega_T} (w^2 + xw_x^2) dx dt < \infty\}.$$

Moreover $u^{\beta_k}(x, t)$ converges to $u(x, t)$ in

$$H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{\Omega}_{T_4}^\pm) \cap H^{2+\alpha, \frac{2+\alpha}{2}}(\overline{G})$$

and $S^{\beta_k}(t)$ converges to a monotone function $S(t)$ almost everywhere. The functions $u(x, t)$ and $\varphi(x, t)$ satisfy the identity (2.14) and $u(x, t)$ satisfies the condition (2.2) in the following sense

$$(2.20) \quad u(x_0, t) = \int_{S(t)}^{x_0} u_x(x, t) dx$$

for almost every $t \in [0, T_*]$, $T_* = \lim_{\beta_k \rightarrow 0} T_*^{\beta_k}$.

The proof of the equality (2.20) is based upon the identity

$$(2.21) \quad \begin{aligned} \iint_{\Omega_T} \eta(t) u_x^\beta \varphi^\beta dx dt &= \beta \int_0^T \eta S^\beta \frac{dS^\beta}{dt} dt = \\ &= -\frac{\beta}{2} \int_0^T (S^\beta)^2 \cdot \frac{d\eta}{dt} dt \end{aligned}$$

where $\eta(t)$ is an arbitrary smooth function satisfying conditions $\eta(0) = \eta(T) = 0$. If one passes to the limit as $\beta \rightarrow 0$ then

$$\int_0^T \eta(t) \left\{ \int_{S(t)}^\ell u_x(x, t) dx \right\} dt = 0,$$

and, hence

$$\int_{S(t)}^\ell u_x(x, t) dx = 0,$$

for almost every $t \in [0, T_*]$. The last equality is equivalent to (2.20) if $x_0 > S(t)$. The case $x_0 < S(t)$ is handled similarly.

Now, let us show that $g(x)$ is a continuous function on the interval $(X_*, 1)$.

If $g(x)$ is discontinuous at some point $\bar{x} \in (X_*, 1)$, then there exists a positive constants $h, \bar{t}$ such that the domain $Q^- = \{(x, t) \mid \bar{x} - h < x < \bar{x}, \bar{t} < t < \bar{t} + h\}$ is contained in Ω_∞^-

and the domain $Q^+ = \{(x, t) \mid \bar{x} < x < \bar{x} + h, \bar{t} < t < \bar{t} + h\}$ is contained in Ω_∞^+ . From (2.14) it follows that

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} \text{ in } \overline{Q^-} \cup \overline{Q^+}, \\ u(\bar{x} - h, t) &> 0, \quad u(\bar{x} + h, t) > 0, \quad t \in [\bar{t}, \bar{t} + h], \\ u(x, \bar{t}) &\geq 0, \quad x \in [\bar{x} - h, \bar{x} + h]. \end{aligned}$$

and from (2.20) it follows that

$$u(\bar{x}, t) = 0, \quad t \in [\bar{t}, \bar{t} + h].$$

The last equality contradicts the maximum principle.

Hence $g(x)$ is a continuous function and the monotone continuous functions $g^\beta(x)$ uniformly converge to $g(x)$ on the interval $[x_*, 1]$.

It is known from lemma 3 that the weak solution $\{u, \varphi\}$ is actually a classical solution on the small interval $[0, T_4]$.

Now we prove that this solution cannot be a classical solution for large time.

More precisely we prove that $S(t)$ is discontinuous at some point t_* .

Let t_* be such that $g(x)$ is strictly monotone on the interval $(x_*, 1)$, where $g(x_*) = t_*$ (or $S(t)$ is strictly monotone on the interval $(0, t_*)$). We prove that $x_* > X_*$ if $g(x)$ is unbounded and $x_* > 0$ if $g(x)$ is bounded.

The last statement is equivalent to the discontinuity of the function $S(t)$.

First of all we prove that in the domain $\Omega_T^+ = \{(x, t) \mid S(t) < x < \ell, 0 < t < T\}$, $T < t_*$ we have $u(x, t) \equiv 0$.

If $\bar{u} = \varphi u$, then $\bar{u} \in \mathring{W}_2^{1,0}(\Omega_T^+)$ and

$$(2.22) \quad \int_{S(T)} \bar{u} \eta|_{t=T} dx + \iint_{\Omega_T^+} (-\bar{u} \eta_t + \bar{u}_x \eta_x) dx dt = 0$$

with $\eta \in \mathring{W}_2^{1,1}(\overline{\Omega_T^+})$ an arbitrary function.

In the new variables

$$\tau = t, \quad y = x - S(t)$$

the functions

$$v(y, \tau) \equiv u(x, t), \quad \zeta(y, \tau) \equiv \eta(x, t)$$

satisfy the identity

$$\begin{aligned} (2.23) \quad & \int_0^T \left(\int_0^{\ell-S(\tau)} (-v \zeta_\tau + v_y \zeta_y) dy \right) d\tau + \int_0^{\ell-S(\tau)} v \zeta|_{\tau=T} dy = \\ & = - \int_0^T \left(\int_0^{\ell-S(\tau)} v \zeta_y dy \right) dS(\tau). \end{aligned}$$

The integral on the right hand side of the inequality (2.23) makes sense since $\dot{S}$ is a measure.

Let

$$v_h = \frac{1}{h} \int_{\tau-h}^{\tau} v(y, t) dt, \quad v_{\bar{h}} = \frac{1}{h} \int_{\tau}^{\tau+h} v(y, t) dt,$$

and

$$\zeta = (v_{\bar{h}})_h.$$

Since

$$\int_0^T v \omega_h d\tau = \int_0^T v_{\bar{h}} \cdot \omega d\tau,$$

then after some transformations (2.23) one obtains that

$$\begin{aligned} & \frac{1}{2} \int_0^{\ell-S(T)} |v_{\bar{h}}(y, T)|^2 dy + \int_0^T \left(\int_0^{\ell-S(\tau)} |v_{\bar{h}y}|^2 dy \right) d\tau = \\ & = \int_0^T (v_y v_{\bar{h}h})|_{y=\ell-S(\tau)} d\tau - \frac{1}{2} \int_0^T (v_{\bar{h}})^2|_{y=\ell-S(\tau)} dS(\tau). \end{aligned}$$

Passing to the limit as $h \rightarrow 0$ and taking into account that $v(y, \tau)$ sufficiently smooth near $y = \ell - S(\tau)$ and that $v = 0$ when $y = \ell - S(\tau)$ one has that

$$v(y, \tau) \equiv 0 \text{ when } 0 < y < \ell - S(\tau), \quad 0 < \tau \leq T \leq t_*.$$

$$\text{Hence } u(x, t) \equiv 0 \text{ in } \Omega_{T_*}^+.$$

If in (2.14) $\eta = x$ then this identity is equivalent to the identity (0.27). Hence the next statement is valid.

THEOREM 5. *Under the assumptions of theorem 4, let λ satisfy the condition (0.28). Then the weak solution of the problem (0.16) - (0.19) cannot be a classical solution: there exists a point $\bar{t}_*$ such that*

$$S(\bar{t}_* + 0) < S(\bar{t}_* - 0).$$

Now our last result.

Lemma 4. Let $\ell = 1, \varphi_0 = 0, u^1 \equiv 0$. Then the problem (0.16) - (0.19) has at least two different solutions.

Proof. The first solution is the limit of the solutions of the problem (0.16) - (0.19) with $\ell > 1, u_0(x) \equiv 0, x \in (1, \ell); \varphi_0 \equiv 0, x \in (0, 1), \varphi_0 \equiv 1, x \in (1, \ell)$ as $\ell \rightarrow 1$.

The second solution is the solution $\overset{\circ}{u}(x, t)$ of the heat equation in the domain Ω_T with corresponding initial and boundary data and $\overset{\circ}{\varphi}(x, t) \equiv 0$ in Ω_T , that is the medium is in the solid phase at every point and for all time.

Remark. About the question of J. Ockendon and S. Howison: let t_0 be the time of disappearance of the solid ball and the solution of the problem (0.24) - (0.26) is classical for $t < t_0$. Then we get from (0.27)

$$t_0 = \frac{\lambda}{3} - \int_0^1 x \cdot u_0(x) dx.$$

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