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FOR THE KURAMOTO-SIVASHINSKY EQUATION
IN THIN 2D DOMAINS**

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Local Dissipativity and Attractors for the Kuramoto-Sivashinsky equation in thin 2D domains

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Abstract

We study the global existence, regularity and boundedness of solutions of the two-dimensional periodic Kuramoto-Sivashinsky equation with a thin periodicity rectangle $\Omega_\varepsilon = [0, 2\pi] \times [0, 2\pi\varepsilon]$. The main result is that for a large set of initial conditions, the solution exists and is uniformly bounded. This implies the existence of a local compact attractor with a basin of attraction which expands to the whole space as $\varepsilon \rightarrow 0$.

Key Words and Phrases: Nonlinear parabolic equations, global existence, Kuramoto-Sivashinsky equation, compact attractors.

1. Introduction

Our goal in this paper is to obtain results on the global asymptotic behavior of the Kuramoto-Sivashinsky (K-S) equation

(1.1)

$$u_t + \nu \Delta^2 u + \Delta u + \frac{1}{2} |\nabla u|^2 = 0,$$

in spatial dimension two, where $u = u(y, t) = u(y_1, y_2, t)$ satisfies the periodic boundary condition

$$(1.2) \quad u(y_1 + 2\pi, y_2, t) = u(y_1, y_2 + 2\pi\varepsilon, t) = u(y_1, y_2, t) \quad \text{for all } y \text{ in } \mathbb{R}^2 \text{ and } t \geq 0$$

and the periodic initial condition

$$(1.3) \quad u(y, 0) = u_0(y).$$

Here $0 < \varepsilon \leq 1$ is assumed to be sufficiently small, so that the basic periodicity cell $\Omega_\varepsilon = [0, 2\pi] \times [0, 2\pi\varepsilon]$ is a thin domain. We shall also mention other types of boundary conditions for which the same methods are applicable. The dissipativity of the general two (and higher) dimensional problem has been open for some time, the essential difficulty being the lack of a proof of the existence of an absorbing set. In fact, if such a set exists, then there exists a global attractor, and one can prove very precise regularity

results, (excellent references for these, as well as many refined results on the geometry of the attractor, are the paper by Nicolaenko, Scheurer and Témam [NST2], Ilyashenko [I], and the books by Témam [Te] and Constantin, Foias, Nicolaenko and Témam [CFNT]).

The approach we shall adopt is based on the intuitive idea that (1.1)-(1.3) should be close to one-dimensional, and is close to the methods introduced by Hale and Raugel [HR1,2] and Raugel and Sell [RS]. There are many nontrivial results available for the periodic one-dimensional problem. The odd-periodic case was studied by [NST1,2]. The same authors treat [NST1] a Neumann type of problem without symmetry, and the rigid Dirichlet problem is studied by [Ta], again without any symmetry assumptions. The general periodic one-dimensional problem has only recently been proved to be dissipative [I]. Our main result is the following:

Theorem 1.1

Consider the Kuramoto-Sivashinsky equation with periodic boundary conditions given by the thin periodicity cell $[0, 2\pi] \times [0, 2\pi\epsilon]$ and a periodic initial condition. Then there exists ϵ_0 , $0 < \epsilon_0 < 1$ and a constant B_0 , and on this range there are real-valued functions $R(\epsilon)$ and $K(\epsilon)$ such that $R(\epsilon) > 0$, $K(\epsilon) \geq 1$ and $R(\epsilon) \rightarrow \infty$ as $\epsilon \rightarrow 0^+$ and such that for all $U_0 \in D(A^{1/4})$ such that $\|U_0\|_{1/4} \leq R(\epsilon)$ one has $U(t) \in D(A^{1/4})$ for all $t \geq 0$ and

$$\|A^{1/4}U(t)\|_{L^2(\Omega_\epsilon)} \leq K(\epsilon)R(\epsilon)$$

for all $t \geq 0$, and

$$\limsup_{t \rightarrow \infty} \|A^{1/4}U(t)\|_{L^2(\Omega_\epsilon)} \leq B_0.$$

We start by changing (1.1) into a system by means of the hodograph transformation

$$D_{y_i} u = U_i, \quad i = 1, 2 \quad \text{and } U = (U_1, U_2).$$

Equality of the mixed partials requires the condition $\text{curl } U = 0$. Notice also that the average of U over the periodicity cell Ω is zero. After this transformation we obtain

(1.4)

$$U_t + \nu \Delta^2 U + \Delta U + (U \cdot \nabla) U = 0.$$

This is the form that seems most appropriate for our purpose. Note that the nonlinearity is now formally the same as that in the Navier-Stokes equations, but is subject to the condition $\text{curl } U = 0$.

1.1 Functional Setting

Let $Q = [0, 2\pi]$ and $\Omega_\epsilon = Q \times [0, 2\pi\epsilon]$. A point in Ω_ϵ will be denoted by $y = (y_1, y_2)$. For an $\mathbb{R}^2$ -valued function $U \in L^p(\Omega_\epsilon)$ we shall introduce the *renormalized norm*

$$\|U\|_{L^p(\Omega_\epsilon)} = \epsilon^{-\frac{1}{p}} \|U\|_{L^p(\Omega_\epsilon)},$$

where $\|\cdot\|$ is the standard L^p norm. This norm is well-suited to the study of problems on thin rectangles, since for small ϵ it is close to $\|U(\cdot, 0)\|_{L^p(0, 2\pi)}$ and, for functions of y_1 alone, it coincides with the L^p norm in $[0, 2\pi]$.

For $p = 2$, $\|\cdot\|$ has a corresponding renormalized inner product given by

$$\langle U, V \rangle_{L^2(\Omega_\epsilon)} = \epsilon^{-1} \langle U, V \rangle_{L^2(\Omega_\epsilon)},$$

where $(\cdot, \cdot)_{L^2(\Omega)}$ denotes the standard inner product in $L^2(\Omega)$.

Given $U \in L^2(\Omega_\epsilon)$ we define the *projection operator* M as follows

$$V = MU, \text{ where } V = V(y_1) = \frac{1}{2\pi\epsilon} \int_0^{2\pi\epsilon} U dy_2.$$

This averaging operation maps $L^2(\Omega_\epsilon)$ onto the closed subspace formed by functions of y_1 alone and it is an orthogonal projection. The complementary projection $I - M$ defines $W = (I - M)U$. Notice that $MW = 0$, so that W has zero average with respect to y_2 . Also, M and $I - M$ commute with the Laplacian on its domain, and they preserve periodicity. As a consequence of this orthogonal decomposition one has

$$\|U\|_{L^2(\Omega_\epsilon)}^2 = \|V\|_{L^2(\Omega_\epsilon)}^2 + \|W\|_{L^2(\Omega_\epsilon)}^2.$$

1.2 The Rescaled Equation

Let us define $x = (x_1, x_2)$ by the linear mapping $x_1 = y_1$ and $x_2 = \epsilon^{-1} y_2$ which transforms Ω_ϵ into the square $Q_2 = [0, 2\pi]^2$. Let us also define the rescaled operators

$$\nabla_\epsilon = (D_{x_1}, \epsilon^{-1} D_{x_2}), \Delta_\epsilon = D_{x_1}^2 + \epsilon^{-2} D_{x_2}^2.$$

Note that these differential operators become singular when ϵ is small. Finally, let us define a new function $u = u(x)$ by $u(x) = U(y)$, where x and y are linearly related as

above. In the sequel, we write $(.,.)$ for the standard inner product in $L^2(Q_2)$. One checks that, for any $p > 1$, the following equality holds

$$\|U\|_{L^p(\Omega_\epsilon)} = \|u\|_{L^p(Q_2)},$$

and also that the Sobolev norms satisfy the inequalities

$$\|u\|_{H^1(Q_2)} \leq \|U\|_{H^1(\Omega_\epsilon)} \leq \epsilon^{-1} \|u\|_{H^1(Q_2)},$$

$$\|u\|_{H^2(Q_2)} \leq \|U\|_{H^2(\Omega_\epsilon)} \leq \epsilon^{-2} \|u\|_{H^2(Q_2)}.$$

and similar bounds hold for the higher order Sobolev norms.

Just as before one can define a projection operator M for functions defined on Q_2 by

$$v = Mu, \text{ where } v = v(x_1) = \frac{1}{2\pi} \int_0^{2\pi} u(x) dx_2 \text{ and } w = (I - M)u.$$

After rescaling, Eq. (1.4) becomes

(1.5)

$$u_t + v \Delta_\epsilon^2 u + \Delta_\epsilon u + (u \cdot \nabla_\epsilon) u = 0$$

$$u = u(x, t) \quad 2\pi \text{ periodic in } x_1, x_2,$$

$$u(x, 0) = u_0(x), \text{ also } 2\pi \text{ periodic.}$$

Without loss of generality, we shall set $v = 1$ (as we shall see, none of our estimates depends crucially on the size of v .) This is essential since, as is well-known, the turbulent behavior of the solutions occurs precisely when v becomes small.

Let us now apply the projections M and $I - M$ to (1.5). This gives

(1.6)

$$v_t + \Delta_\epsilon^2 v + \Delta_\epsilon v + M[(u \cdot \nabla_\epsilon) u] = 0$$

$$w_t + \Delta_\epsilon^2 w + \Delta_\epsilon w + (I - M)[(u \cdot \nabla_\epsilon) u] = 0.$$

We wish to associate with (1.6) a simpler problem which, for small ϵ , will turn out to govern to a large extent the dynamics of the system. We thus define the *reduced problem*

to be the one obtained from (1.6) by setting $(v,w) = (\bar{v},0)$ in (1.6) and taking as initial condition the projection of u_0 onto the v -space. This gives

(1.7)

$$\bar{v}_t + \Delta_\varepsilon^2 \bar{v} + \Delta_\varepsilon \bar{v} + (\bar{v} \cdot \nabla_\varepsilon) \bar{v} = 0$$

$$\bar{v}(x,0) = v_0 = Mu_0.$$

Let us write v_i , $i = 1,2$ for the components of v . Then these satisfy

$$\bar{v}_{1,t} + \Delta_\varepsilon^2 \bar{v}_1 + \Delta_\varepsilon \bar{v}_1 + \bar{v}_1 D_{x_1} \bar{v}_1 = 0$$

$$\bar{v}_{2,t} + \Delta_\varepsilon^2 \bar{v}_2 + \Delta_\varepsilon \bar{v}_2 + \bar{v}_1 D_{x_1} \bar{v}_2 = 0.$$

Notice that the first component satisfies the one-dimensional K-S equation (the dependence of the equations on ε is illusory), while the second one satisfies a *linear* equation.

We now recall some results from Ilyashenko [I], and Nicolaenko, Scheurer and Témam [NST2] concerning the asymptotic behavior of the one dimensional K-S equation. According to these results, if the initial condition $u_0 \in H^k(\Omega)$, and $0 \leq s \leq k$, there exists an absorbing set in H^s . Moreover, these same properties of regularity and boundedness of the higher Sobolev norms hold, *even for dimensions 2 and 3*, as soon as a solution is uniformly bounded in time in $L^2(\Omega)$.

The above dissipativity results for the 1D problem, together with general theorems on attractors of asymptotically smooth nonlinear semigroups (see Hale [H]) imply the existence of a global attractor. This attractor lies in $L^2(\Omega)$ and for smooth initial conditions, it will lie in the corresponding Sobolev space. It has also been proved that this attractor has finite Hausdorff and fractal dimensions, which have been estimated in terms of the period [I], [NST2].

1.3 The evolutionary equation

In what follows, we will always assume that the initial condition has zero average over the periodicity cell; it is then easy to prove that the solutions have the same property for all times for which they exist. Let us define the unbounded linear operator A_ε by $A_\varepsilon u = \Delta_\varepsilon^2 u$ on the closed subspace X of $L^2(\Omega_\varepsilon)$ consisting of functions with zero average and whose $(\varepsilon$ -rescaled) curl is zero. Then $D(A)$ will be a subspace of X determined by the boundary conditions plus some other assumptions such as symmetry. In particular, we will concentrate on the periodic case, since the technicalities become

more tractable. The operator defined above is self-adjoint, has a dense domain and, when restricted to the subspace of periodic functions with zero average, is positive definite. It is also well-known that its resolvent is compact. These properties imply that the fractional powers of A_ε are well-defined. In particular, one has

$$A_\varepsilon^2 = -\Delta_\varepsilon, \text{ and } \|A_\varepsilon^{\frac{1}{4}} u\|^2 = \|\nabla_\varepsilon u\|^2.$$

We will also introduce rescaled versions of the classical bilinear and trilinear forms associated with the nonlinear term in our equation (see [Te], or [L]). We thus consider

$$B_\varepsilon(u, v) = (u, \nabla_\varepsilon) v, \text{ and } b_\varepsilon(u, v, w) = (B_\varepsilon(u, v), w)$$

Explicitly, we have

$$b_\varepsilon(u, v, w) = \sum_{i, k \leq 2} \int_{Q_2} \varepsilon^{-\{k\}} u_k D_k v_i w_i dx, \text{ where } \{1\} = 0 \text{ and } \{2\} = 1.$$

If we define

$$b_\varepsilon(U, V, W) = \sum_{i, k \leq 2} \int_{Q_\varepsilon} \varepsilon^{-1} U_k D_k V_i W_i dy, \text{ then one checks that } b_\varepsilon(u, v, w) = b_\varepsilon(U, V, W).$$

It is easy to see that the subspace $H_{\text{per}}^1(\Omega_\varepsilon)$ of $H^1(\Omega_\varepsilon)$ formed by periodic functions of zero average is continuously embedded in $L^4(\Omega_\varepsilon)$ and (see Agmon [A]) the following estimate holds

(1.8)

$$\|U\|_{L^4(\Omega_\varepsilon)} \leq C \|U\|_{L^2(\Omega_\varepsilon)}^{\frac{1}{2}} \|DU\|_{L^2(\Omega_\varepsilon)}^{\frac{1}{2}} \text{ for } U \in H_{\text{per}}^1(\Omega_\varepsilon), \text{ where } C \text{ is independent of } \varepsilon.$$

We now establish a bound for the trilinear forms b_ε introduced above. We recall the inequality

(1.9)

$$\left| \int_{\Omega} U_k (D_k V_i) W_i dx \right| \leq \|U_k\|_{L^4(\Omega)} \|D_k V_i\|_{L^2(\Omega)} \|W_i\|_{L^4(\Omega)}, \text{ valid for periodic functions,}$$

where Ω is the periodicity rectangle in $\mathbb{R}^2$.

By combining this with (1.8) one obtains the following ($\|\cdot\|$ = norm in $L^2(\Omega_\varepsilon)$.)

(1.10)

$$|b_\varepsilon(U, V, W)| \leq C |U|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} U|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} V|^{\frac{1}{2}} |W|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} W|^{\frac{1}{2}}$$

where C is a constant independent of ε . By our previous discussion on norms, the same inequality is true if we replace Ω_ε by Q_2 and (U, V, W) by (u, v, w) . This gives the inequality

(1.11)

$$|b_\varepsilon(u, v, w)| \leq C |u|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} u|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} v|^{\frac{1}{2}} |w|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} w|^{\frac{1}{2}}$$

(from now on, we will work in $L^2(Q_2)$ and the norm will be denoted simply by $|\cdot|$.)

In particular, when $u = v$ this gives

(1.12)

$$|b_\varepsilon(u, u, w)| \leq C |u|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} u|^{\frac{3}{2}} |w|^{\frac{1}{2}} |A_\varepsilon^{\frac{1}{4}} w|^{\frac{1}{2}}.$$

With the above definitions, equation (1.5) can be written in evolutionary form as follows

$$u' + A_\varepsilon u - A_\varepsilon^{\frac{1}{2}} u + B_\varepsilon(u, u) = 0$$

and the projections v and w satisfy the system

(1.13)

$$\begin{aligned} v' + A_\varepsilon v - A_\varepsilon^{\frac{1}{2}} v + M B_\varepsilon(u, u) &= 0. \\ w' + A_\varepsilon w - A_\varepsilon^{\frac{1}{2}} w + (I - M) B_\varepsilon(u, u) &= 0. \end{aligned}$$

Here we have used the fact that M commutes with A_ε on its domain. Notice also that since v is independent of x_2 , the linear part of the v -equation is independent of ε .

2. Regularity results for a thin domain

We now turn to the problem of global existence and regularity of solutions of the K-S equation on a thin domain (local existence and uniqueness are a consequence of classical theorems for sectorial evolutionary equations [P]). Our strategy is to prove existence and regularity for solutions with initial conditions in a large set, over intervals

whose length is independent of the initial condition chosen in this set. This enables us to patch up local solutions and thus construct a global solution. In order to do this we shall need some results on the 1D equation, as well as on the reduced 2D equations.

2.1 The dynamics of the reduced 2D problem

We recall the reduced 2D Kuramoto-Sivashinsky equations

(2.1)

$$\bar{v}_{1,t} + \Delta_\varepsilon^2 \bar{v}_1 + \Delta_\varepsilon \bar{v}_1 + \bar{v}_1 D_{x_1} \bar{v}_1 = 0$$

$$\bar{v}_{2,t} + \Delta_\varepsilon^2 \bar{v}_2 + \Delta_\varepsilon \bar{v}_2 + \bar{v}_1 D_{x_1} \bar{v}_2 = 0 .$$

We observe that (2.1) is independent of ε and $\bar{v}_1, \bar{v}_2$ are independent of x_2 . Notice that $\bar{v}_1$ satisfies the 1D K-S equation. In order to study the dynamics of the system, we first solve the equation for $\bar{v}_1$. We first note that the absorbing property holds for $\bar{v}_1$ in $L^2[0, 2\pi]$. We also see that, since the ε -curl of v is zero, $\bar{v}_2$ is also independent of x_1 , hence this function depends only on time. It then follows from the equation satisfied by $\bar{v}_2$ that it must be constant. However, its average is zero, hence $\bar{v}_2$ is identically zero. Therefore, the reduced 2D K-S equation has a global attractor, namely $A \times \{0\}$.

2.2 Growth Estimates for the reduced equation

In order to proceed with the theory, we need some results regarding the solutions of the reduced 2D equations (or, equivalently, the 1D problem). Let us thus consider the one-dimensional equation

(2.2)

$$u_t + \Delta^2 u + \Delta u + uu_x = 0.$$

We know ([I], [NST1,2]) that (2.2) has a global attractor A in X , and that there exist constants ρ_0 and ρ_1 such that

$$\limsup_{t \rightarrow \infty} |S(t)u_0|^2 \leq \rho_0^2$$

$$\limsup_{t \rightarrow \infty} |A^{\frac{1}{4}} S(t)u_0|^2 \leq \rho_1^2 ,$$

where $S(t)$ is the solution semigroup of (2.2). Moreover, there exist absorbing balls of radii $2\rho_0$ and $2\rho_1$ in X and $X^{1/4} = D(A^{1/4})$. The next results are of a technical nature, and make more precise the above absorbing properties.

Lemma 2.1

Consider Eq. (2.2) in 1D with a periodic boundary condition and initial condition. Then there exist absolute positive constants γ and L , and a function D , real analytic in $|u_0|$, such that the solution semigroup satisfies the estimate

$$|A^{1/4}S(t)u_0|^2 \leq e^{-2\gamma t} D + L \text{ for } t \geq 0.$$

This result is an easy consequence of the dissipativity estimates for (2.2) [NST1,2].

Lemma 2.2

Given k such that $0 < k < 1$, there exist absolute positive constants b_i , $i = 1, 2$, such that for all $u_0 \in X^{1/4}$ one has

$$|A^{1/4}S(t)u_0|^2 \leq L + k |A^{1/4}u_0|^2 \text{ for } t \geq T_0, \text{ where } L \text{ is a constant and } T_0 = b_1 \exp(b_2 |A^{1/4}u_0|^4).$$

Lemma 2.3

Assume that $B_1 > L$, with L as above. Then there exists a $K_0 \geq 1$ such that for all η such that $0 \leq \eta \leq 1$ and for all $u_0 \in D(A^{1/4})$ satisfying $L < |A^{1/4}u_0|^2 \leq B_1^2$ the following holds for $t \geq 0$:

$$|A^{1/4}S(t)u_0|^2 \leq K_0 |A^{1/4}u_0|^2 \eta^{*-2} \text{ where } \eta^{*-2} = \exp(a_1 \exp(a_2 B_1^4 \eta^{-4}))$$

and a_1, a_2 are constants.

The proofs of these results are omitted, since they are very similar to those in [RS].

2.3 The growth condition "G"

We will often consider functions $\eta = \eta(\epsilon)$ for which η^{-1} blows up at a certain "sufficiently slow" rate as $\epsilon \rightarrow 0^+$. This motivates the following definition :

Let the a_i , $i = 1$, given by Lemma (2.3) and let $\eta = \eta(\epsilon)$ be a monotone increasing function on the interval $0 \leq \epsilon < \infty$. Define also $\eta^* = \eta(\epsilon)$ as above, B_1 being a constant to be introduced later. We shall impose the following conditions:

$$(G_1) \quad \eta \rightarrow 0 \text{ as } \varepsilon \rightarrow 0^+,$$

$$(G_2) \quad \eta \leq 1 \text{ for } 0 < \varepsilon \leq 1.$$

$$(G_3) \quad \varepsilon \eta^{-2} \leq 1, \quad \varepsilon \eta^{*-2} \leq 1 \text{ for } 0 \leq \varepsilon \leq 1.$$

Finally, the following hold for $\varepsilon \rightarrow 0^+$:

$$(G_4) \quad \varepsilon \eta^{-4} \rightarrow 0,$$

$$(G_5) \quad \varepsilon^3 \log(\varepsilon \eta^{-4}) \rightarrow 0,$$

$$(G_6) \quad \varepsilon^3 \eta^{-2} \log(\varepsilon^3 \eta^{-1}) \rightarrow 0,$$

$$(G_7) \quad \varepsilon \exp(A \eta^{*-2} \eta^{-2} e^{B \eta^{-4}}) \rightarrow 0$$

for any positive constants A and B. A function that satisfies these growth conditions is, for example,

$$\eta(\varepsilon) = \{A + B \log \log \log \log(C \varepsilon^{-1})\}^{-\frac{1}{4}} \text{ for appropriate constants } A, B, C.$$

We are now ready to state our first result.

2.4 Estimates for the blow-up time

Let us define, for a given initial condition u_0 , the "blow-up" time in $D(A_\varepsilon^{1/4})$

$$T^* = T^*(u_0) = \sup \{ t : \sup_{0 \leq s \leq t} |A_\varepsilon^{\frac{1}{4}} u(s)| < \infty \}.$$

We wish to obtain an estimate for T^* in terms of u_0 . In order to do this, let us start with

$$u' + A_\varepsilon u - A_\varepsilon^{\frac{1}{2}} u + B_\varepsilon(u, u) = 0$$

By taking the inner product with $A_\varepsilon^{1/2} u$, and using the self-adjointness of A_ε , we arrive at

$$\frac{1}{2} \frac{d}{dt} |A_\varepsilon^{\frac{1}{4}} u|^2 + |A_\varepsilon^{\frac{3}{4}} u|^2 - |A_\varepsilon^{\frac{1}{2}} u|^2 \leq |b_\varepsilon(u, u, A_\varepsilon^{\frac{1}{2}} u)|.$$

By (1.12) this implies

$$\frac{1}{2} \frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} u|^2 + |A_{\varepsilon}^{\frac{3}{4}} u|^2 - |A_{\varepsilon}^{\frac{1}{2}} u|^2 \leq |u|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}} u|^{\frac{3}{2}} |A_{\varepsilon}^{\frac{1}{2}} u|^{\frac{3}{2}} |A_{\varepsilon}^{\frac{3}{4}} u|^{\frac{1}{2}}.$$

From now on, we shall encounter various constants, which are independent of ε , and which will generically be denoted by C (with an occasional added index to avoid confusion.) By Poincaré's inequality, we see that the right-hand side of this inequality is bounded by

$$C |A_{\varepsilon}^{\frac{1}{4}} u|^2 |A_{\varepsilon}^{\frac{1}{2}} u|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}} u|^{\frac{1}{2}}.$$

By the weighted Young inequality with exponents $4/3$ and 4 this is

$$\leq \frac{1}{2} |A_{\varepsilon}^{\frac{3}{4}} u|^2 + C |A_{\varepsilon}^{\frac{1}{4}} u|^{\frac{8}{3}} |A_{\varepsilon}^{\frac{1}{2}} u|^{\frac{2}{3}}.$$

By using Young once again, this time with exponents $3/2$ and 3 , we can break up the second summand to obtain

$$\leq \frac{1}{2} |A_{\varepsilon}^{\frac{3}{4}} u|^2 + C |A_{\varepsilon}^{\frac{1}{2}} u|^2 + \frac{1}{2} |A_{\varepsilon}^{\frac{1}{4}} u|^4.$$

If we insert this estimate into the previous differential inequality, we arrive at

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} u|^2 + |A_{\varepsilon}^{\frac{3}{4}} u|^2 - 2(1 + C) |A_{\varepsilon}^{\frac{1}{2}} u|^2 \leq |A_{\varepsilon}^{\frac{1}{4}} u|^4.$$

We now recall [P] the interpolation inequality for fractional powers of an operator, which implies

$$|A_{\varepsilon}^{\frac{1}{2}} u| \leq |A_{\varepsilon}^{\frac{1}{4}} u|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}} u|^{\frac{1}{2}}.$$

This gives, via the weighted Young inequality

$$|A_{\varepsilon}^{\frac{1}{2}} u|^2 \leq [2(1 + C)]^{-1} |A_{\varepsilon}^{\frac{3}{4}} u|^2 + C_1 |A_{\varepsilon}^{\frac{1}{4}} u|^2.$$

The combination of this with (1.14) gives

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} u|^2 - C_2 |A_{\varepsilon}^{\frac{1}{4}} u|^2 \leq |A_{\varepsilon}^{\frac{1}{4}} u|^4.$$

By considering the differential inequality $\phi' \leq \phi^2 + C_2 \phi$ it is immediately seen that

$$T^* \geq \frac{K}{\frac{1}{|A_\varepsilon^{1/4} u_0|^2}} .$$

where K is a constant, independent of the initial conditions. As an easy consequence of the differential inequality we have :

Lemma 2.4

Let R_0 and $N > 1$ be given . Then, for any $u_0 \in D(A^{1/4})$ such that $|A^{1/4} u_0|^2 \leq R_0$ the following holds:

$$|A_\varepsilon^{1/4} S(t)u_0|^2 \leq NR_0 \quad \text{for } 0 \leq t \leq \frac{N-1}{N} \cdot \frac{K}{R_0} \quad \text{where } K \text{ is a constant, independent of } u_0.$$

For simplicity, we shall henceforth assume that $K = 1$.

Lemma 2.5

Let $B_0 > 0$, $C_0 > 0$ be given, and consider a function η satisfying the growth hypothesis (G) indicated above. Then there exist ε_0 , $0 < \varepsilon_0 \leq 1$, $B_1 \geq B_0$, $C_1 > C_0$ and $T_1 = T_1(\varepsilon) > 0$ such that the following hold for all ε such that $0 < \varepsilon < \varepsilon_0$:

$$\begin{aligned} \text{If } |A_\varepsilon^{1/4} v_0|^2 \leq B_0^2 \eta^{-2} \quad \text{and} \quad |A_\varepsilon^{1/4} w_0|^2 \leq C_0^2 \varepsilon^{-1} \eta^{-1} , \text{ then} \\ |A_\varepsilon^{1/4} v(T_1)|^2 \leq B_1^2 \eta^{-2} \quad \text{and} \quad |A_\varepsilon^{1/4} w(T_1)|^2 \leq C_1^2 \varepsilon^2 \eta^{-2} . \end{aligned}$$

The quantities ε_0 , B_1 , C_1 , depend on B_0 , C_0 , *but not on* ε , and $T_1(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0^+$.

Proof

We shall let D_i , $i = 0, 1, \dots$, denote quantities which depend on the parameters of the problem but are uniformly bounded for ε in $(0, 1]$. Let us define T^N by

$$T^N = \frac{N-1}{N} \cdot \frac{1}{R_0} , \quad \text{and notice that } T^N < T^*(u_0), \quad \text{where}$$

$$R_0 = B_0^2 \eta^{-2} + C_0^2 \varepsilon^{-1} \eta^{-1} \quad \text{and} \quad \text{also that } R_0^2 \leq 2B_0^4 \eta^{-4} + 2C_0^4 \varepsilon^{-2} \eta^{-2} . \quad \text{We also have}$$

$$R_0 \leq \varepsilon^{-1} \eta^{-2} D_0^2 , \quad \text{where } D_0^2 = B_0^2 + C_0^2 .$$

Notice also that if $T^* < \infty$, then $T^N \rightarrow T^*$ as $N \rightarrow \infty$.

We now turn our attention to the equation satisfied by w on the interval $[0, T^N)$, where $N \geq 2$ will be fixed. On this interval we have the inequality (2.3)

$$|\frac{1}{\epsilon} \frac{d}{dt} u(t)|^2 \leq N R_0.$$

By taking the inner product with $A_\epsilon^{1/2} w$ and proceeding as above, we arrive at

$$\frac{1}{2} \frac{d}{dt} |A_\epsilon^{\frac{1}{4}} w|^2 + |A_\epsilon^{\frac{3}{4}} w|^2 - |A_\epsilon^{\frac{1}{2}} w|^2 \leq |b_\epsilon(u, u, A_\epsilon^{\frac{1}{2}} w)|.$$

By using (1.12) this implies

$$\frac{1}{2} \frac{d}{dt} |A_\epsilon^{\frac{1}{4}} w|^2 + |A_\epsilon^{\frac{3}{4}} w|^2 - |A_\epsilon^{\frac{1}{2}} w|^2 \leq |u|^2 |A_\epsilon^{\frac{1}{4}} u|^2 |A_\epsilon^{\frac{1}{2}} w|^2 |A_\epsilon^{\frac{3}{4}} w|^2.$$

and by Poincaré's inequality this is

$$\leq C |A_\epsilon^{\frac{1}{4}} u|^2 |A_\epsilon^{\frac{1}{2}} w|^2 |A_\epsilon^{\frac{3}{4}} w|^2.$$

By using $2ab \leq a^2 + b^2$ and the bound (2.3) on $|A_\epsilon^{1/4} u|^2$ we obtain

$$\begin{aligned} &\leq \frac{1}{4} C^2 N^2 R_0^2 + |A_\epsilon^{\frac{1}{2}} w|^2 |A_\epsilon^{\frac{3}{4}} w|^2 \\ &\leq \frac{1}{4} C^2 N^2 R_0^2 + \frac{1}{2} (|A_\epsilon^{\frac{1}{2}} w|^2 + |A_\epsilon^{\frac{3}{4}} w|^2). \end{aligned}$$

Hence we arrive at the inequality

$$\frac{d}{dt} |A_\epsilon^{\frac{1}{4}} w|^2 + |A_\epsilon^{\frac{3}{4}} w|^2 - 3 |A_\epsilon^{\frac{1}{2}} w|^2 \leq \frac{1}{2} C^2 N^2 R_0^2.$$

We now notice that, since the average of w with respect to x_2 is zero, we have

$$\epsilon^{-2} |A_\epsilon^{\frac{1}{2}} w|^2 \leq |A_\epsilon^{\frac{3}{4}} w|^2 \quad \text{and} \quad \epsilon^{-2} |A_\epsilon^{\frac{1}{4}} w|^2 \leq |A_\epsilon^{\frac{1}{2}} w|^2.$$

By taking ϵ sufficiently small, we can ensure that

$$|A_\epsilon^{\frac{3}{4}} w|^2 - 3 |A_\epsilon^{\frac{1}{2}} w|^2 \geq \epsilon^{-2} |A_\epsilon^{\frac{1}{2}} w|^2 \geq \epsilon^{-4} |A_\epsilon^{\frac{1}{4}} w|^2.$$

We thus have the inequality

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} w|^2 + \varepsilon^{-4} |A_{\varepsilon}^{\frac{1}{4}} w|^2 \leq \frac{1}{2} C^2 N^2 R_0^2.$$

By integration, this yields

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 \leq e^{-\varepsilon^4 t} |A_{\varepsilon}^{\frac{1}{4}} w_0|^2 + \frac{\varepsilon^4}{2} C^2 N^2 R_0^2.$$

By using the previous bound on R_0 and our assumption on $|A_{\varepsilon}^{1/4} w_0|$ we see that

$$\begin{aligned} |A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 &\leq e^{-\varepsilon^4 t} C_0^2 \varepsilon^{-1} \eta^{-1} + \varepsilon^4 C^2 N^2 (B_0^4 \eta^{-4} + C_0^4 \varepsilon^{-2} \eta^{-2}) = \\ &= e^{-\varepsilon^4 t} C_0^2 \varepsilon^{-1} \eta^{-1} + C^2 N^2 \varepsilon^2 \eta^{-2} (B_0^4 \varepsilon^2 \eta^{-2} + C_0^4) \end{aligned}$$

By Hypothesis (G), we see that

$$D_1 = C^2 N^2 (B_0^4 \varepsilon^2 \eta^{-2} + C_0^4) \text{ is uniformly bounded for } 0 < \varepsilon \leq 1.$$

We thus have

$$\begin{aligned} |A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 &\leq e^{-\varepsilon^4 t} C_0^2 \varepsilon^{-1} \eta^{-1} + \varepsilon^2 \eta^{-2} D_1. \text{ Now solve for } t = T_1 \text{ by setting} \\ \varepsilon^{-1} \eta^{-1} C_0^2 e^{-\varepsilon^4 t} &= \varepsilon^2 \eta^{-2} D_1 \text{ to obtain } T_1 = -\varepsilon^4 \log \left(\frac{\varepsilon^3 \eta^{-1} D_1}{C_0^2} \right). \end{aligned}$$

We want this T_1 to be $\leq T^N$, and this will motivate our choice of ε_0 . Since we have

$$R_0 \leq \varepsilon^{-1} \eta^{-2} D_0^2, \text{ it follows that } R_0 T_1 \leq -D_0^2 \varepsilon^3 \eta^{-2} \log \left(\frac{\varepsilon^3 \eta^{-1} D_1}{C_0^2} \right).$$

By our choice of η , we see that the right-hand side tends to zero as $\varepsilon \rightarrow 0^+$ so $R_0 T_1 \rightarrow 0$ and also $T_1 \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Hence $T_1(\varepsilon_0) \leq (N-1)/NR_0$ is true for some $\varepsilon_0 > 0$ and so $T_1 \leq T^N$. It then follows that

$$|A_{\varepsilon}^{\frac{1}{4}} w(T_1)|^2 \leq C_1^2 \varepsilon^2 \eta^{-2}, \text{ where } C_1^2 = 2 D_1.$$

We now look at the v -equation. By taking the inner product of this equation with $A_{\varepsilon}^{1/2} v$ and proceeding as before, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |A_\epsilon^{\frac{1}{4}} v|^2 + |A_\epsilon^{\frac{3}{4}} v|^2 - |A_\epsilon^{\frac{1}{2}} v|^2 &\leq |b_\epsilon(u, u, A_\epsilon^{\frac{1}{2}} v)| \leq \\ &\leq |u|^{\frac{1}{2}} |A_\epsilon^{\frac{1}{4}} u|^{\frac{3}{2}} |A_\epsilon^{\frac{1}{2}} v|^{\frac{1}{2}} |A_\epsilon^{\frac{3}{4}} v|^{\frac{1}{2}} \leq C |A_\epsilon^{\frac{1}{4}} u|^2 |A_\epsilon^{\frac{1}{2}} v|^2 |A_\epsilon^{\frac{3}{4}} v|^2. \end{aligned}$$

(The last step is a consequence of Poincaré's inequality).

By using the weighted Young inequality with exponents 4 and 4/3 we obtain the bound

$$\frac{1}{2} |A_\epsilon^{\frac{1}{4}} v|^2 + C |A_\epsilon^{\frac{1}{4}} u|^{\frac{8}{3}} |A_\epsilon^{\frac{1}{2}} v|^{\frac{2}{3}}$$

By using Young's inequality again with exponents 3/2 and 3, this is bounded by

$$\frac{1}{2} |A_\epsilon^{\frac{1}{4}} v|^2 + C |A_\epsilon^{\frac{1}{2}} v|^2 + \frac{1}{2} |A_\epsilon^{\frac{1}{4}} u|^4.$$

By putting all these estimates together, we arrive at

$$\frac{d}{dt} |A_\epsilon^{\frac{1}{4}} v|^2 + |A_\epsilon^{\frac{3}{4}} v|^2 - (2 + C) |A_\epsilon^{\frac{1}{2}} v|^2 \leq |A_\epsilon^{\frac{1}{4}} u|^4.$$

The interpolation inequality (applied in the same manner as before) gives

$$\frac{d}{dt} |A_\epsilon^{\frac{1}{4}} v|^2 - \alpha |A_\epsilon^{\frac{1}{4}} v|^2 \leq |A_\epsilon^{\frac{1}{4}} u|^4, \text{ where } \alpha \text{ is a constant independent of } \epsilon.$$

Now notice that

$$|A_\epsilon^{\frac{1}{4}} u|^4 = |A_\epsilon^{\frac{1}{4}} (v + w)|^4 \leq [(|A_\epsilon^{\frac{1}{4}} v|^2 + |A_\epsilon^{\frac{1}{4}} w|^2)]^2 \leq 2 (|A_\epsilon^{\frac{1}{4}} v|^4 + |A_\epsilon^{\frac{1}{4}} w|^4).$$

Since A_ϵ commutes with M , so do its fractional powers, therefore

$$\begin{aligned} |A_\epsilon^{\frac{1}{4}} v|^2 &\leq |A_\epsilon^{\frac{1}{4}} u|^2, \text{ so that the inequality becomes} \\ \frac{d}{dt} |A_\epsilon^{\frac{1}{4}} v|^2 - \alpha |A_\epsilon^{\frac{1}{4}} v|^2 &\leq 2 |A_\epsilon^{\frac{1}{4}} v|^2 |A_\epsilon^{\frac{1}{4}} u|^2 + 2 |A_\epsilon^{\frac{1}{4}} w|^4. \end{aligned}$$

Since $|A_\epsilon^{1/4} u|^2 \leq NR_0$ on the interval of t under consideration, we have

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} v|^2 - \alpha |A_{\varepsilon}^{\frac{1}{4}} v|^2 \leq 2NR_0 |A_{\varepsilon}^{\frac{1}{4}} v|^2 + 2|A_{\varepsilon}^{\frac{1}{4}} w|^4.$$

By defining $\beta = \alpha + 2NR_0$, this becomes

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} v|^2 - \beta |A_{\varepsilon}^{\frac{1}{4}} v|^2 \leq 2 |A_{\varepsilon}^{\frac{1}{4}} w|^4.$$

Recall now that

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 \leq e^{-\varepsilon^4 t} C_0^2 \varepsilon^{-1} \eta^{-1} + \varepsilon^2 \eta^{-2} D_1, \text{ which implies}$$

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^4 \leq 2C_0^4 e^{-2\varepsilon^4 t} \varepsilon^{-2} \eta^{-2} + 2\varepsilon^4 \eta^{-4} D_1^2.$$

We can write

$$\begin{aligned} \beta &= \alpha + 2N R_0 = \alpha + 2N \varepsilon^{-1} \eta^{-2} D_0^2 = \varepsilon^{-1} \eta^{-2} (\alpha \varepsilon \eta^2 + 2ND_0^2) = \\ &= \varepsilon^{-1} \eta^{-2} D_2, \text{ where } D_2 \text{ is uniformly bounded for } 0 < \varepsilon \leq 1. \text{ Thus we arrive at} \\ \frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} v|^2 - \varepsilon^{-1} \eta^{-2} D_2 |A_{\varepsilon}^{\frac{1}{4}} v|^2 &\leq 2 C_0^4 e^{-2\varepsilon^4 t} \varepsilon^{-2} \eta^{-2} + 2\varepsilon^4 \eta^{-4} D_1^2. \end{aligned}$$

We can rewrite this as

$$\frac{d}{dt} \left(e^{-\varepsilon^{-1} \eta^{-2} D_2 t} |A_{\varepsilon}^{\frac{1}{4}} v|^2 \right) \leq e^{-\varepsilon^{-1} \eta^{-2} D_2 t} (2C_0^4 e^{-2\varepsilon^4 t} \varepsilon^{-2} \eta^{-2} + 2\varepsilon^4 \eta^{-4} D_1^2).$$

By integrating between 0 and T_1 and using the definition of R_0 , we obtain

$$\begin{aligned} e^{-\varepsilon^{-1} \eta^{-2} D_2 T_1} |A_{\varepsilon}^{\frac{1}{4}} v|^2 - |A_{\varepsilon}^{\frac{1}{4}} v_0|^2 &\leq \varepsilon^5 \eta^{-2} D_1^2 D_2^{-1} + C_0^4 \varepsilon^2 \eta^{-2} \leq \\ &\leq D_3 \eta^{-2}, \text{ where } D_3 = \varepsilon^5 D_1^2 D_2^{-1} + C_0^4 \varepsilon^2. \end{aligned}$$

Hence

$$|A_{\varepsilon}^{\frac{1}{4}} v(T_1)|^2 \leq \exp(\varepsilon^{-1} \eta^{-2} D_2 T_1) \cdot \{ |A_{\varepsilon}^{\frac{1}{4}} v_0|^2 + D_3 \eta^{-2} \}.$$

By our choice of T_1 , we see that

$\varepsilon^{-1}\eta^{-2}D_2T_1 \rightarrow 0$ as $\varepsilon \rightarrow 0$, hence there exists ε_0 such that

$$\exp(\varepsilon^{-1}\eta^{-2}D_2T_1) \leq 2 \text{ for } 0 < \varepsilon < \varepsilon_0.$$

Hence for $0 < \varepsilon < \varepsilon_0$ we have

$$|A_\varepsilon^{\frac{1}{4}}v(T_1)|^2 \leq 2|A_\varepsilon^{\frac{1}{4}}v_0|^2 + 2D_3\eta^{-2} \leq B_1^2\eta^{-2}, \text{ where } B_1^2 = \sup_{0 < \varepsilon < \varepsilon_0} (2B_0^2 + 2D_3).$$

This concludes the proof.

As a direct consequence of Lemma 2.5 we have the following result:

Lemma 2.6

Let $B_0 > 0$ and $C_0 > 0$ be given, and let η satisfy hypothesis (G). Then there exist $\varepsilon_0, 0 < \varepsilon_0 < 1, B_1 > B_0, C_1 > C_0$ and $T_1 > 0$ such that the following holds for all ε satisfying $0 < \varepsilon < \varepsilon_0$:

$$\text{If } |A_\varepsilon^{\frac{1}{4}}v_0|^2 \leq B_0^2\eta^{-2} \text{ and } |A_\varepsilon^{\frac{1}{4}}w_0|^2 \leq C_0^2\varepsilon^{-1}\eta^{-1} \text{ then}$$

$$|A_\varepsilon^{\frac{1}{4}}v(T_1)|^2 \leq B_1^2\eta^{-2} \text{ and } |A_\varepsilon^{\frac{1}{4}}w(T_1)|^2 \leq C_1^2\varepsilon.$$

Proof.

By an application of Lemma 2.5 we obtain

$$|A_\varepsilon^{\frac{1}{4}}v(T_1)|^2 \leq B_1^2\eta^{-2} \text{ and } |A_\varepsilon^{\frac{1}{4}}w(T_1)|^2 \leq C_1^2\varepsilon^2\eta^{-2}.$$

By condition (G), we know that $\varepsilon\eta^{-2} \leq 1$, so we have

$$|A_\varepsilon^{\frac{1}{4}}w(T_1)|^2 \leq C_1^2\varepsilon, \text{ which proves the assertion.}$$

We are now ready to state a lemma which plays a key role in the proof of our main result.

Lemma 2.8

Fix B_1 and C_1 and let $u_0 = (v_0, w_0)$ be chosen such that

$$|A_\varepsilon^{\frac{1}{4}}v_0|^2 \leq B_1^2\eta^{-2} \text{ and } |A_\varepsilon^{\frac{1}{4}}w_0|^2 \leq C_1^2\varepsilon.$$

Let $K_0 \geq 1$ be given by Lemma 2.2 with, say, $k = 1/8$ and set $N = 4K_0$, and

let us define $T_0 = T_0(\varepsilon) = b_1 \exp(b_2 B_1^4 \eta^{-4})$ so that one has

$$|A_{\varepsilon}^{\frac{1}{4}} v(t)|^2 \leq L + \frac{1}{8} |A_{\varepsilon}^{\frac{1}{4}} v_0|^2, t \geq T_0.$$

Next, define τ_N by

$$\tau_N = \sup \{ \tau > 0 : |A_{\varepsilon}^{\frac{1}{4}} u(t)|^2 \leq N D_4^2 \eta^{*-2} \eta^{-2} \text{ for } 0 \leq t \leq \tau \}$$

where $D_4^2 = B_1^2 + C_1^2$.

Then there exists ε_0 , $0 < \varepsilon_0 < 1$, such that for all ε satisfying $0 < \varepsilon < \varepsilon_0$ the following hold

$$\begin{aligned} T_0 &\leq \tau_N \\ |A_{\varepsilon}^{\frac{1}{4}} v(T_0)|^2 &\leq \frac{3}{4} B_1^2 \eta^{-2} \\ |A_{\varepsilon}^{\frac{1}{4}} w(T_0)|^2 &\leq C_1^2 \varepsilon. \end{aligned}$$

Proof.

From our previous estimates we have

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}} w|^2 + \varepsilon^{-4} |A_{\varepsilon}^{\frac{1}{4}} w|^2 \leq \frac{1}{2} C^2 N^2 D_4^2 \eta^{-4} \eta^{*-4} \text{ on } 0 < t < \tau_N.$$

By Gronwall's inequality, we obtain

$$\begin{aligned} |A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 &\leq e^{-\varepsilon^{-4}t} |A_{\varepsilon}^{\frac{1}{4}} w_0|^2 + \varepsilon^4 \left[\frac{1}{2} C^2 N^2 D_4^2 \eta^{*-4} \eta^{-4} \right] \\ &\leq \varepsilon C_1^2 e^{-\varepsilon^{-4}t} + \varepsilon^4 \eta^{*-4} \eta^{-4} \left(\frac{1}{2} C^2 N^2 D_4^2 \right). \end{aligned}$$

By using condition (G) one has

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 \leq \varepsilon C_1^2 e^{-\varepsilon^{-4}t} + \varepsilon D_5^2 \text{ where } D_5^2 = \varepsilon^3 \eta^{*-4} \eta^{-4} C^2 N^2 D_4^2.$$

(Notice that D_5 is uniformly bounded.) It follows that

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 \leq D_6^2 \varepsilon \text{ for } 0 \leq t \leq \tau_N, \text{ where } D_6^2 = C_1^2 + D_5^2.$$

Now solve the equation

$C_1^2 e^{-\epsilon^{-4}t} = D_5^2$ for $t = T_2$ to obtain

$$T_2 = -\epsilon^4 \log \frac{D_5^2}{C_1^2}.$$

It follows that

$$R_0 T_2 \leq -D_4^2 \eta^{-2} \epsilon^4 \log \frac{D_5^2}{C_1^2} \rightarrow 0 \text{ and } T_2 \rightarrow 0 \text{ as } \epsilon \rightarrow 0^+.$$

Notice that then $T_2(\epsilon_0) \leq \min(T_0, \tau_N)$ for some $\epsilon_0 > 0$.

Therefore we get

$$|A_\epsilon^{\frac{1}{4}} w(t)|^2 \leq 2D_5^2 \epsilon, \text{ for } T_2 \leq t \leq \tau_N. \text{ This implies that}$$

$$|A_\epsilon^{\frac{1}{4}} w(t)|^2 \leq C_1^2 \epsilon \text{ provided } \epsilon_0 \text{ is chosen so that } \sup_{0 < \epsilon < \epsilon_0} 2D_5^2 \leq C_1^2.$$

We now turn to the estimates for v . As pointed out above, we wish to compare v with the solution $\bar{v}$ of the reduced 2D equations satisfying the same initial condition, that is $\bar{v}(0) = v(0)$.

By subtraction, we obtain

(2.4)

$$\begin{aligned} (v - \bar{v})' + A_\epsilon (v - \bar{v}) - A_\epsilon^{\frac{1}{2}}(v - \bar{v}) + B_\epsilon(v, v) - B_\epsilon(\bar{v}, \bar{v}) = \\ = -M[B_\epsilon(w, v) + B_\epsilon(v, w) + B_\epsilon(w, w)] \end{aligned}$$

Let

$$|b_\epsilon(v, v, A_\epsilon^{\frac{1}{2}}(v - \bar{v})) - b_\epsilon(\bar{v}, \bar{v}, A_\epsilon^{\frac{1}{2}}(v - \bar{v}))| = (\text{def}) R.$$

We will need the following estimate for R :

$$\begin{aligned}
& |b_{\varepsilon}(v, v, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})) - b_{\varepsilon}(\bar{v}, \bar{v}, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v}))| = \\
& = |b_{\varepsilon}(v - \bar{v}, v, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})) + b_{\varepsilon}(\bar{v}, v - \bar{v}, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v}))| \quad \text{and, by inequality (1.11), this is}
\end{aligned}$$

$$\begin{aligned}
& \leq C |v - \bar{v}|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}v|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^{\frac{1}{2}} + \\
& + C |\bar{v}|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}\bar{v}|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^{\frac{1}{2}}.
\end{aligned}$$

By using the Poincaré inequality, we see that R is bounded by

$$C (|A_{\varepsilon}^{\frac{1}{4}}v| + |A_{\varepsilon}^{\frac{1}{4}}\bar{v}|) |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|.$$

Before taking the inner product of (2.4) with $A_{\varepsilon}^{1/2}(v - \bar{v})$, let us note that inequality (1.11) provides the following bounds for the trilinear terms that involve w :

$$S_1 = |b_{\varepsilon}(w, v, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v}))| \leq C |w|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}w|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}v|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^{\frac{1}{2}}$$

By Poincaré's inequality, we have

$$|w| \leq \varepsilon |A_{\varepsilon}^{\frac{1}{4}}w| \leq |A_{\varepsilon}^{\frac{1}{4}}w| \quad \text{and} \quad |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})| \leq |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|, \quad \text{hence}$$

$$S_1 \leq C |A_{\varepsilon}^{\frac{1}{4}}w| |A_{\varepsilon}^{\frac{1}{4}}v| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|.$$

Similarly,

$$\begin{aligned}
S_2 & = |b_{\varepsilon}(v, w, A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v}))| \leq C |v|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}v|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{4}}w|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^{\frac{1}{2}} |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^{\frac{1}{2}} \\
& \leq C |A_{\varepsilon}^{\frac{1}{4}}v| |A_{\varepsilon}^{\frac{1}{4}}w| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|.
\end{aligned}$$

By taking the inner product of (2.4) with $A_{\varepsilon}^{1/2}(v - \bar{v})$ we obtain

$$\frac{1}{2} \frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 + |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^2 - |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^2 \leq R + S_1 + S_2.$$

We now recall that, by Lemma 2.3, we have

$$(2.5) \quad |A_{\varepsilon}^{1/4} v(t)|^2 \leq K_0 |A_{\varepsilon}^{1/4} v_0|^2 \eta^{*-2} \leq K_0 B_1^2 \eta^{*-2} \eta^{-2} \text{ for } t \geq 0.$$

(Here we have used the fact that $\bar{v}_0 = v_0$.)

We also have

(2.6)

$$|A_{\varepsilon}^{\frac{1}{4}} v(t)|^2 \leq |A_{\varepsilon}^{\frac{1}{4}} u(t)|^2 \leq N \eta^{*-2} R_0 = N D_4^2 \eta^{*-2} \eta^{-2} \text{ for } 0 \leq t \leq \tau_N.$$

We shall use these to obtain more precise bounds on R, S_1, S_2 . First, we have

$$\begin{aligned} R &\leq C (K_0^{\frac{1}{2}} B_1 + N^{\frac{1}{2}} D_4) \eta^{*-1} \eta^{-1} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})| \text{ which, by Young's inequality, is} \\ &\leq \frac{1}{6} |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^2 + C_1 (K_0 B_1^2 + N D_4^2) \eta^{*-2} \eta^{-2} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2. \end{aligned}$$

where C_1 is an absolute constant. In a similar fashion, we obtain

$$\begin{aligned} S_1 &\leq C N^{\frac{1}{2}} D_4 \eta^{*-1} \eta^{-1} |A_{\varepsilon}^{\frac{1}{4}} w| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})| \leq \\ &\leq \frac{1}{6} |A_{\varepsilon}^{\frac{3}{4}} w|^2 + C_2 N D_4^2 \eta^{*-2} \eta^{-2} |A_{\varepsilon}^{\frac{1}{4}} w|^2. \end{aligned}$$

and the same bound holds for S_2 .

By putting all these estimates together, we arrive at

$$\begin{aligned} \frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 + |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^2 - 2 |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^2 - D_7 \eta^{*-2} \eta^{-2} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 \leq \\ \leq 4C_2 N D_4^2 \eta^{*-2} \eta^{-2} |A_{\varepsilon}^{\frac{1}{4}} w|^2, \text{ where } D_7 = 2C_1 (K_0 B_1^2 + N D_4^2). \end{aligned}$$

Let us now recall the previous estimate

$$|A_{\varepsilon}^{\frac{1}{4}} w(t)|^2 \leq D_6^2 \varepsilon \text{ for } 0 \leq t \leq \tau_N.$$

By the interpolation inequality, one has

$$2 |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^2 \leq 2 |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})| |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})| \leq |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 + |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^2,$$

$$\text{hence } |A_{\varepsilon}^{\frac{3}{4}}(v - \bar{v})|^2 - 2 |A_{\varepsilon}^{\frac{1}{2}}(v - \bar{v})|^2 \geq - |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2.$$

Hence we have the inequality

$$\frac{d}{dt} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 - D_8 \eta^{*-2} \eta^{-2} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|^2 \leq D_9 \varepsilon \eta^{*-2} \eta^{-2}, \text{ where}$$

$$D_8 = \eta^{*2} \eta^2 + D_7 \quad \text{and} \quad D_9 = 4 C_2 N D_4^2 D_6^2.$$

By integration of this differential inequality we obtain

$$|A_{\varepsilon}^{\frac{1}{4}}(v(t) - \bar{v}(t))|^2 \leq \varepsilon D_8^{-1} D_9 e^{D_8 \eta^{*-2} \eta^{-2} t} \text{ for } 0 \leq t < \tau_N.$$

We now claim that $T_0 \leq \tau_N$, with T_0 defined as in the statement of the Lemma. Let us assume that this is false, and define

$$\Delta(t) = \varepsilon D_8^{-1} D_9 e^{D_8 \eta^{*-2} \eta^{-2} t}.$$

Then, by the inequalities just proved, one has

$$\begin{aligned} |A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v})|_{t=\tau_N}^2 &\leq \Delta(T_0) = \varepsilon D_8^{-1} D_9 \exp(D_8 \eta^{*-2} \eta^{-2} T_0) \\ &= \varepsilon D_8^{-1} D_9 \exp\{D_8 \eta^{*-2} \eta^{-2} b_4 e^{b_5 B_1^4 \eta^{-4}}\}. \end{aligned}$$

By condition (G₇), this tends to 0 as $\varepsilon \rightarrow 0^+$. Hence there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ one has :

$$\Delta \leq \min\left(\frac{1}{6} D_4^2, \frac{1}{4} (B_1^2 \eta^{-2} - 4L)\right) \text{ for } 0 < \varepsilon < \varepsilon_0.$$

Next, notice that

$$A_{\varepsilon}^{\frac{1}{4}} u = A_{\varepsilon}^{\frac{1}{4}} v + A_{\varepsilon}^{\frac{1}{4}} w = A_{\varepsilon}^{\frac{1}{4}}(v - \bar{v}) + A_{\varepsilon}^{\frac{1}{4}} w + A_{\varepsilon}^{\frac{1}{4}} \bar{v}.$$

Therefore, Young's inequality implies

$$|A_{\varepsilon}^{\frac{1}{4}} u|^2 \leq 3 (|A_{\varepsilon}^{\frac{1}{4}} (v - \bar{v})|^2 + |A_{\varepsilon}^{\frac{1}{4}} w|^2 + |A_{\varepsilon}^{\frac{1}{4}} \bar{v}|^2).$$

Let us evaluate this at $t = \tau_N$ to obtain

$$\begin{aligned} 4K_0 \eta^{*-2} \eta^{-2} (B_1^2 + C_1^2) &= N D_4^2 \eta^{*-2} \eta^{-2} = |A_{\varepsilon}^{\frac{1}{4}} u|^2 \leq \\ &\leq 3 (|A_{\varepsilon}^{\frac{1}{4}} (v - \bar{v})|^2 + |A_{\varepsilon}^{\frac{1}{4}} w|^2 + |A_{\varepsilon}^{\frac{1}{4}} \bar{v}|^2) \leq \\ &\leq 3\Delta + 3C_1^2 \varepsilon + 3K_0 B_1^2 \eta^{*-2} \eta^{-2}. \end{aligned}$$

By our estimate on Δ this is

$$\leq \frac{1}{2} D_4^2 + 3C_1^2 + 3K_0 B_1^2 \eta^{*-2} \eta^{-2} \leq \left((3K_0 + \frac{1}{2}) B_1^2 + \frac{7}{2} C_1^2 \right) \eta^{*-2} \eta^{-2}$$

which contradicts the fact that $K_0 \geq 1$. It follows that $T_0 \leq \tau_N$.

Since $T_2 \leq T_0 \leq \tau_N$ it follows that

$$|A_{\varepsilon}^{\frac{1}{4}} w(T_0)|^2 \leq C_1^2 \varepsilon.$$

Finally, the above estimates imply

$$\begin{aligned} |A_{\varepsilon}^{\frac{1}{4}} v(T_0)|^2 &\leq 2 |A_{\varepsilon}^{\frac{1}{4}} (v - \bar{v})|_{t=T_0}^2 + 2 |A_{\varepsilon}^{\frac{1}{4}} \bar{v}(T_0)|^2 \leq 2\Delta + 2(L + \frac{1}{8} |A_{\varepsilon}^{\frac{1}{4}} \bar{v}_0|^2) \\ &\leq (\frac{1}{2} (B_1^2 \eta^{-2} - 4L) + 2L + \frac{1}{4} B_1^2 \eta^{-2}) = \frac{3}{4} B_1^2 \eta^{-2}. \end{aligned}$$

This ends the proof of the lemma.

It is now an easy matter to prove the Main Theorem 1.1, which follows by repeatedly applying Lemma 2.8. Notice that we are stating the result in terms of the original function U and the original gradient (H^1) norm in $L^2(\Omega_{\varepsilon})$. This is done by using the definition of the renormalized norm and the inequality

$$|U|_{H^1(\Omega_{\varepsilon})} \leq \varepsilon^{-1} \|u\|_{H^1(Q_2)}.$$

Existence of a local attractor

We recall that a set A is a *local attractor* for a nonlinear semigroup $\{S(t)\}$ if it is compact, invariant, and there exists a bounded neighborhood B of A such that A attracts B . Let us also recall the following result :

Lemma 2.10 ([H], Lemma 3.2.1)

Let $\{S(t), t \geq 0\}$ be an asymptotically smooth semigroup in a Banach space X and let B be a nonempty set in X such that the semiorbit $\gamma^+(B)$ is bounded. Then $\omega(B)$ is nonempty, compact, invariant, and it attracts B . If B is connected, so is $\omega(B)$.

By using this result for $B = B_\epsilon = \{U : \|U\|_{1/4} \leq R(\epsilon)\}$ we see that $A_\epsilon = \omega(B_\epsilon)$ is a local attractor whose basin of attraction contains at least the set B_ϵ .

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