**MAXIMAL SMOOTHNESS OF SOLUTIONS TO CERTAIN
EULER-LAGRANGE EQUATIONS FROM NONLINEAR
ELASTICITY**

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MAXIMAL SMOOTHNESS OF SOLUTIONS TO CERTAIN EULER-LAGRANGE EQUATIONS FROM NONLINEAR ELASTICITY

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Synopsis

We investigate the maximal smoothness of stationary states for the multiple integral $W(\mathcal{U}) = \int_{\Omega} \gamma(D\mathcal{U})dX$ where $\Omega \subset \mathbb{R}^n$, $\mathcal{U} \in W^{1,1}(\Omega; \mathbb{R}^n)$, $\gamma \geq 0$, $\gamma(P) = +\infty$ if $\det P \leq 0$, and $\lim_{\det P \rightarrow 0} \gamma(P) = +\infty$. Such variational problems are motivated by the study of nonlinear elasticity. Assuming certain structure conditions for γ and given a stationary state $\mathcal{U}$, we derive an a priori L^p estimate for $(\det D\mathcal{U})^{-1}$ for any $p < \infty$ in terms of $W(\mathcal{U})$ and $\|D\mathcal{U}\|_{L^{\bar{p}}}$ where $\bar{p} = \bar{p}(p, \gamma)$. As a consequence, we show that a $C^{1,\beta}$ stationary state necessarily satisfies $\det D\mathcal{U} > 0$ and is of class $C^{2,\beta}$ in Ω . Nevertheless, singular stationary states do exist: we construct a nonsmooth C^1 solution for a particular γ in two dimensions such that $\det D\mathcal{U} \geq 0$ in Ω and $\det D\mathcal{U}$ vanishes at precisely one point in Ω .

Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$ where $n \geq 2$. Let $M^{n \times n}$ denote the set of real $n \times n$ matrices and $M_+^{n \times n}$ the subset of matrices in $M^{n \times n}$ whose determinant is positive. In this paper we consider stationary states for the multiple integral

$$(0.1) \quad \mathcal{W}(\mathcal{U}) = \int_{\Omega} \gamma(D\mathcal{U}(X))dX,$$

where $\mathcal{U} \in W^{1,1}(\Omega; \mathbb{R}^n)$, $\gamma : M^{n \times n} \rightarrow [0, \infty]$, γ is quasiconvex, $\gamma = +\infty$ on $M^{n \times n} - M_+^{n \times n}$, and $\lim_{\det P \rightarrow 0} \gamma(P) = +\infty$. In particular, we are interested in the example given by

$$(0.2) \quad \gamma(P) = \begin{cases} F(P) + H(\det P) & \text{for } P \in M_+^{n \times n}, \\ +\infty & \text{for } P \in M^{n \times n} - M_+^{n \times n}, \end{cases}$$

where F is a quasiconvex function on $M^{n \times n}$ satisfying

$$c_1|P|^q \leq F(P) \leq c_2(1 + |P|^q) \text{ for } P \in M^{n \times n}$$

with $c_1, c_2 > 0$ and $q \geq n$. The function H is a nonnegative convex function on $(0, \infty)$, and $H(d)$ is proportional to d^{-s} for all sufficiently small positive values of d , where $s > 0$. Integrands of this type occur as stored energy densities for certain models from nonlinear elasticity. (See Ball [2] and Ogden [12].) An interesting feature of these models (which

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follows trivially from our assumptions on γ) is that if $\mathcal{U} \in W^{1,1}(\Omega; \mathbb{R}^n)$, then $\mathcal{W}(\mathcal{U}) < +\infty$ if and only if

$$\mathcal{U} \in \mathcal{A} \equiv \{\mathcal{U} \in W^{1,1}(\Omega; \mathbb{R}^n) : \mathcal{W}(\mathcal{U}) < +\infty \text{ and } \det D\mathcal{U} > 0 \text{ a.e. in } \Omega\}.$$

A state $\mathcal{U}$ in $\mathcal{A}$ is defined to be stationary relative to a given class of variations about it. We consider two classes of variations which correspond to two distinct systems of partial differential equations. Let $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$ and let $\mathcal{U}_\varepsilon(X) = \mathcal{U}(X) + \varepsilon\varphi(X)$ for all $X = (x_1 \dots x_n)$ in Ω where $\varepsilon \in \mathbb{R}$. We say that $\mathcal{U}$ satisfies the *Euler-Lagrange equations* corresponding to $\mathcal{W}$ in Ω if

$$(0.3) \quad \left(\frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \right)_{x_\alpha} = 0 \text{ in } \mathcal{D}'(\Omega)$$

for $1 \leq k \leq n$. Formally, this comes from the equation,

$$\frac{d}{d\varepsilon} \mathcal{W}(\mathcal{U}_\varepsilon) = 0 \text{ at } \varepsilon = 0 \text{ for all } \varphi \in C_c^1(\Omega; \mathbb{R}^n),$$

assuming $\gamma \in C^1(M_+^{n \times n})$.

A second notion of stationary comes from considering variations of the form $\tilde{\mathcal{U}}_\varepsilon(X) = \mathcal{U}(X + \varepsilon\varphi(X))$ for $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$. We say that $\mathcal{U}$ satisfies the *equilibrium equations* (i.e. $\mathcal{U}$ is an equilibrium solution) corresponding to $\mathcal{W}$ in Ω if it satisfies

$$(0.4) \quad \left[-\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right]_{x_\alpha} = 0 \text{ in } \mathcal{D}'(\Omega)$$

for $1 \leq k \leq n$, where $\mathcal{U} = (u^1, \dots, u^n)$. This system comes from the equation,

$$\frac{d}{d\varepsilon} \mathcal{W}(\tilde{\mathcal{U}}_\varepsilon) = 0 \text{ at } \varepsilon = 0 \text{ for all } \varphi \text{ in } C_c^1(\Omega; \mathbb{R}^n).$$

In both (0.3) and (0.4) we have used the convention that all terms with repeated indices are summed from 1 to n .

The equilibrium equations (0.4) occur in elasticity as follows. Assume $\mathcal{U}_0 \in \mathcal{A}$ and consider the variational problem:

$$(0.5) \quad \begin{aligned} &\text{Find } \mathcal{U} \text{ in } \mathcal{A}(\mathcal{U}_0) \equiv \{\mathcal{U} \in \mathcal{A} : \mathcal{U} = \mathcal{U}_0 \text{ on } \partial\Omega\} \\ &\text{such that } \mathcal{W}(\mathcal{U}) = \inf_{\mathcal{V} \in \mathcal{A}(\mathcal{U}_0)} \mathcal{W}(\mathcal{V}). \end{aligned}$$

Here γ is the stored energy density for an elastic material and $\mathcal{W}(\mathcal{U})$ is the total elastic energy of a deformation $\mathcal{U}$ in $\mathcal{A}$. Under certain conditions concerning quasiconvexity,

differentiability, and the growth of γ (which hold for γ as in (0.2)), $\mathcal{W}$ is weakly lower semicontinuous and the variational problem (0.5) has at least one solution. (See Ball and Murat [7] and Acerbi and Fusco [1].) Moreover, if $\mathcal{U}$ is any such solution and $\varphi \in C_c^1(\Omega; \mathbb{R}^n)$, then $\tilde{\mathcal{U}}_\varepsilon \in \mathcal{A}$, $\mathcal{W}(\tilde{\mathcal{U}}_\varepsilon)$ is C^1 on a neighbourhood of $\varepsilon = 0$, and $\mathcal{U}$ satisfies (0.4).

The systems (0.3) and (0.4) are equivalent for $\mathcal{U}$ in $C^2(\Omega)$ with $\det D\mathcal{U} > 0$ in Ω . (See Lemma 1.3.) However, it is not known whether solutions of (0.5) (which necessarily satisfy (0.4)) must also satisfy the Euler-Lagrange equations (0.3). The difficulty here is the singular behavior of $\gamma(P)$ near $\det P = 0$ and the fact that $\mathcal{W}(\mathcal{U})$ is typically nonconvex on $\mathcal{A}$. In fact for γ as in (0.2), it is an open problem to determine whether $\mathcal{W}(\mathcal{U}_\varepsilon) \equiv \mathcal{W}(\mathcal{U} + \varepsilon\varphi)$ is differentiable or even locally finite at $\varepsilon = 0$ when $\mathcal{U}$ minimizes $\mathcal{W}$ in $\mathcal{A}(\mathcal{U}_0)$.

In this paper we investigate maximal smoothness of solutions to (0.3) and (0.4) and point out some connections between regularity of solutions and positivity of $\det D\mathcal{U}$ when γ is given by (0.2). Our paper may be summarized as follows:

In Section 1 we state some preliminary results which we apply in Section 2 concerning minimizers of $\mathcal{W}$. We also show that the Euler-Lagrange and equilibrium systems, (0.3) and (0.4), are equivalent when $\mathcal{U} \in C^2(\Omega)$ and $\det D\mathcal{U} > 0$ in Ω .

In Section 2 we consider γ as in (0.2) assuming that F and H are sufficiently smooth. In this case there are examples of smooth solutions to (0.3) and (0.4) when the boundary data $\mathcal{U}_0$ is sufficiently near an affine function in $\mathcal{A}$. (See Ciarlet [9] and Valent [14].) We prove (based on results we obtained in [8]) that if $\mathcal{U} \in \mathcal{A}$ and $\mathcal{U}$ is an equilibrium solution, then $\|H(\det D\mathcal{U})\|_{L_{\text{loc}}^p(\Omega)}$ can be estimated in terms of $\mathcal{W}(\mathcal{U})$ and $\||D\mathcal{U}|^q\|_{L_{\text{loc}}^p(\Omega)}$ for $1 < p < \infty$. From this we show that if $\mathcal{U} \in \mathcal{A} \cap C^{1,\alpha}(\Omega)$ and $\mathcal{U}$ satisfies (0.4), then $\det D\mathcal{U}$ is strictly positive in Ω . Moreover, if F is strictly elliptic, then $\mathcal{U} \in C^{2,\alpha}(\Omega)$ and $\mathcal{U}$ satisfies the Euler-Lagrange equations (0.3). It follows from elliptic regularity theory that higher-order differentiability of F and H result in higher-order differentiability of $\mathcal{U}$. (See Giaquinta [11].) Thus we obtain maximal smoothness for $C^{1,\alpha}$ equilibrium solutions.

In Section 3 we show that the above results fail if we assume only that $\mathcal{U} \in \mathcal{A} \cap C^1(\Omega)$ and $\mathcal{U}$ satisfies (0.3) and/or (0.4). More precisely, we construct an example in which Ω is the unit ball centered at zero in $\mathbb{R}^2$, $\mathcal{U} \in \mathcal{A} \cap C^1(\overline{\Omega}; \mathbb{R}^2)$, $\gamma(P) = \frac{|P|^2}{2} + H(\det P)$ in $M_+^{n \times n}$, $\mathcal{U}$ satisfies both (0.3) and (0.4), and $\mathcal{U} \notin C^{1,\beta}(\Omega; \mathbb{R}^2)$ for any number β in $(0,1)$. Here $\det D\mathcal{U} > 0$ in $\Omega - \{0\}$ and $\det D\mathcal{U}(0) = 0$. The above equation for γ is used in nonlinear elasticity to model neo-Hookean materials in two dimensions. (See Ball [2] and Ogden [12].)

In the above example the restriction of $\mathcal{U}$ to $\partial\Omega$ is a smooth two-to-one function and $\mathcal{U}$ is two-to-one in $\Omega - \{0\}$. Thus our example demonstrates the existence of two-to-one C^1 solutions of (0.3) and (0.4) which are nonsmooth and satisfy $\det D\mathcal{U} \geq 0$ in Ω .

We remark that in nonlinear elasticity one is interested in *equilibrium deformations*, i.e. solutions of the equilibrium equations (0.4) which are in $\mathcal{A}$ and are one-to-one almost everywhere in Ω . It would be interesting to determine the maximal smoothness of these

solutions for γ as above.

1. Preliminaries. In this section we state some preliminary results which we shall use later. Let γ be a nonnegative function on $M^{n \times n}$ such that

$$(1.1) \quad \gamma \in C^1(M_+^{n \times n}) \text{ and } \gamma = +\infty \text{ on } M^{n \times n} - M_+^{n \times n}.$$

Recall that if $\mathcal{U} \in W^{1,1}(\Omega; \mathbb{R}^n)$, then $\mathcal{W}(\mathcal{U}) < +\infty$ if and only if $\mathcal{U} \in \mathcal{A}$.

The following result (due to Ball [6]) gives a criterion on γ under which variations of the type leading to the equilibrium system (0.4) are valid.

THEOREM 1.1 (Ball). *Assume that γ satisfies (1.1) and that for some positive constants N and θ , we have*

$$(1.2) \quad |P^T \cdot D\gamma(PC)| \leq N[1 + \gamma(P)]$$

for all P, C in $M_+^{n \times n}$ with $|C - I| < \theta$. For each $\mathcal{U}$ in $\mathcal{A}$ and φ in $C_c^1(\Omega; \mathbb{R}^n)$ there exists $\varepsilon_0 > 0$ such that $\tilde{\mathcal{U}}_\varepsilon(X) \equiv \mathcal{U}(X + \varepsilon\varphi(X)) \in \mathcal{A}$ for $|\varepsilon| < \varepsilon_0$, $\mathcal{W}(\tilde{\mathcal{U}}_\varepsilon) \in C^1((-\varepsilon_0, \varepsilon_0))$, and

$$\frac{d}{d\varepsilon} \mathcal{W}(\tilde{\mathcal{U}}_\varepsilon)|_{\varepsilon=0} = \int_{\Omega} \left[-\gamma(D\mathcal{U}) \cdot \delta_{\alpha}^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_{\alpha}^i}(D\mathcal{U}) \right] \cdot \varphi_{x_{\alpha}}^k dX,$$

where $\mathcal{U} = (u^1, \dots, u^n)$, $D\mathcal{U} = [u_{x_j}^i]$ and $X = (x_1, \dots, x_n)$. Moreover, for each k and α the term $\left[-\gamma(D\mathcal{U}) \cdot \delta_{\alpha}^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_{\alpha}^i}(D\mathcal{U}) \right]$ is in $L^1(\Omega)$.

A detailed proof of this theorem can be found in the appendix contained in [8]. An immediate consequence is:

COROLLARY 1.2. *Assume that $\mathcal{U}$ solves the variational problem (0.5) with γ as in Theorem 1.1. Then $\mathcal{U}$ satisfies the equilibrium equations (0.4).*

In Section 2 we verify that the hypotheses of Theorem 1.1 are valid for γ as in (0.2). Moreover the variational problem (0.5) has a solution in this case.

We conclude this section with a result concerning smooth solutions of (0.3) or (0.4) which satisfy $\det D\mathcal{U} \neq 0$ in Ω .

LEMMA 1.3. *Let $\mathcal{O}$ be an open subset of $M_+^{n \times n}$ and assume that $\gamma \in C^2(\mathcal{O})$. Set $A_{\alpha}^k(P) = \left[-\gamma(P) \cdot \delta_{\alpha}^k + P_{\alpha}^i \cdot \frac{\partial \gamma}{\partial P_{\alpha}^i}(P) \right]$ for $P \in \mathcal{O}$ and $1 \leq \alpha, k \leq n$. If $\mathcal{U} \in C^2(\Omega; \mathbb{R}^n)$ and $D\mathcal{U} \in \mathcal{O}$ in Ω , then*

$$(1.3) \quad \frac{\partial A_{\alpha}^k}{\partial P_{\beta}^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_{\alpha} \partial x_{\beta}} = [D\mathcal{U}]_k^i \cdot \frac{\partial^2 \gamma}{\partial P_{\alpha}^i \partial P_{\beta}^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_{\alpha} \partial x_{\beta}}$$

for $1 \leq k \leq n$ and $\mathcal{U}$ satisfies (0.3) if and only if $\mathcal{U}$ satisfies (0.4). Moreover for each $P \in \mathcal{O}$, we have

$$(1.4) \quad [P^{-1}]_\ell^k \cdot \frac{\partial A_\alpha^k}{\partial P_\beta^j}(P) \cdot \eta_\ell \eta_j \xi_\alpha \xi_\beta = \frac{\partial^2 \gamma}{\partial P_\alpha^\ell \partial P_\beta^j}(P) \cdot \eta_\ell \eta_j \xi_\alpha \xi_\beta$$

for all η, ξ in $\mathbb{R}^n$.

Proof. By definition of A_α^k we have

$$\begin{aligned} \frac{\partial}{\partial x_\alpha}(A_\alpha^k(D\mathcal{U})) &= \left[-\frac{\partial \gamma}{\partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} \cdot \delta_\alpha^k + \frac{\partial^2 u^i}{\partial x_k \partial x_\alpha} \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right] \\ &\quad + \frac{\partial u^i}{\partial x_k} \cdot \frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} \end{aligned}$$

for each k , where we sum on i, j, α and β . Noting that the term in brackets is zero, (1.3) follows immediately.

Now $\mathcal{U}$ satisfies (0.4) if and only if (1.3) vanishes for each k . Multiplying each equation in (1.3) by $[(D\mathcal{U})^{-1}]_\ell^k$ and summing on k gives the equivalent system of equations:

$$\begin{aligned} (1.5) \quad & [(D\mathcal{U})^{-1}]_\ell^k \cdot \frac{\partial A_\alpha^k}{\partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} \\ &= \delta_\ell^i \cdot \frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} \\ &= \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \gamma}{\partial P_\alpha^\ell}(D\mathcal{U}) \right) \text{ for } 1 \leq \ell \leq n. \end{aligned}$$

Clearly, (1.5) vanishes for each ℓ if and only if $\mathcal{U}$ satisfies (0.3). Thus $\mathcal{U}$ satisfies (0.3) if and only if $\mathcal{U}$ satisfies (0.4).

To prove (1.4), let $\mathcal{U}(X) = PX + \frac{\langle X, \xi \rangle^2}{2} \eta$ where $P \in \mathcal{O}$ and $\eta, \xi \in \mathbb{R}^n$. From (1.3) at $X = 0$ we have

$$\frac{\partial A_\alpha^k}{\partial P_\beta^j} \eta_j \xi_\alpha \xi_\beta = P_k^i \cdot \frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j} \cdot \eta_j \xi_\alpha \xi_\beta.$$

Multiplying by $[P^{-1}]_\ell^k$ and summing on k gives

$$[P^{-1}]_\ell^k \cdot \frac{\partial A_\alpha^k}{\partial P_\beta^j} \cdot \eta_j \xi_\alpha \xi_\beta = \delta_\ell^i \cdot \frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j} \cdot \eta_j \xi_\alpha \xi_\beta.$$

The result follows by multiplying by η_ℓ and summing over ℓ . $\square$

Section 2. Regularity of solutions and the behavior of $\det D\mathcal{U}$.

From now on we shall assume that γ is a function of the form (0.2), i.e.

$$(2.1) \quad \gamma(P) = F(P) + H(\det P) \text{ for } P \in M_+^{n \times n}$$

with $\gamma \equiv +\infty$ on $M^{n \times n} - M_+^{n \times n}$. We assume that $F \in C^1(M^{n \times n})$ and

$$(2.2) \quad c_1|P|^q \leq F(P) \leq c_2(1 + |P|^q) \text{ for all } P \text{ in } M^{n \times n},$$

where $c_1, c_2 > 0$ and $q \geq n$. We also assume that F is *quasiconvex*, i.e.

$$F(P) \leq \frac{1}{\text{meas } \mathcal{D}} \int_{\mathcal{D}} F(P + D\varphi(X)) dX$$

for all P in $M^{n \times n}$, any bounded domain $\mathcal{D}$ in $\mathbf{R}^n$, and for all $\varphi \in C_c^\infty(\mathcal{D}; \mathbf{R}^n)$. (Examples of quasiconvex functions can be found in Chapter 4 of [10]. In particular if F is convex (or polyconvex), then F is quasiconvex.)

The function H is assumed to satisfy:

(2.3)

$$(1) \quad H \geq 0.$$

$$(2) \quad H \in C^3((0, \infty)) \text{ and for some positive constants } s, c_1, c_2, \text{ and } d_0,$$

$$c_1 t^{-s-k} \leq (-1)^k \cdot \frac{d^k}{dt^k} H(t) \leq c_2 t^{-s-k} \text{ for } 0 < t < d_0 \text{ and } k = 0, 1, 2.$$

$$(3) \quad H(t) = +\infty \text{ for } t \leq 0.$$

$$(4) \quad \text{For some real number } \tau \text{ and positive constants } c_3, c_4 \text{ and } d_1,$$

$$c_3 t^\tau \leq \frac{d^2}{dt^2} H(t) \leq c_4 t^\tau \text{ for } t \geq d_1.$$

Models of this type occur in elasticity when $n = 2$ or $n = 3$. For example, when $n = 2$ and $F(P) = |P|^2$, the functional $\mathcal{W}(\mathcal{U}) \equiv \int_{\Omega} \gamma(D\mathcal{U}) dX$ represents the total elastic energy of a deformation $\mathcal{U}$ for a two-dimensional neo-Hookean material. (See [9] and [12] for other examples.)

In this section we prove the existence of minimizers for $\mathcal{W}$. These minimizers provide us with examples of equilibrium solutions. We also investigate the connection between the regularity of an equilibrium solution and the behavior of $\det D\mathcal{U}$.

Our first result states that the variational problem (0.5) has a solution for γ as in (2.1).

THEOREM 2.1 (Acerbi and Fusco [1], Ball and Murat [7]). *Assume that $\mathcal{U}_0 \in \mathcal{A}$ and let $\mathcal{A}(\mathcal{U}_0) = \{\mathcal{V} \in \mathcal{A} : \mathcal{V} - \mathcal{U}_0 \in W_0^{1,q}(\Omega; \mathbf{R}^n)\}$. There exists $\mathcal{U}$ in $\mathcal{A}(\mathcal{U}_0)$ such that $\mathcal{U}$ solves (0.5).*

Proof. Let $E_1(\mathcal{V}) = \int_{\Omega} F(D\mathcal{V}) dX$ and $E_2(\mathcal{V}) = \int_{\Omega} H(\det D\mathcal{V}) dX$. By [1], E_1 is weakly lower-semicontinuous on $W^{1,q}(\Omega; \mathbf{R}^n)$ and by [7] (since $q \geq n$), E_2 is weakly lower-semicontinuous on $W^{1,q}(\Omega; \mathbf{R}^n)$ as well. Since both integrands are bounded from below it

follows that $\mathcal{W} = E_1 + E_2$ is weakly lower semicontinuous with respect to the $W^{1,q}$ norm on $\mathcal{A}(\mathcal{U}_0)$. By (2.2), $\mathcal{W}$ is coercive. Hence, a minimizing sequence for $\mathcal{W}$ in $\mathcal{A}(\mathcal{U}_0)$ has a subsequence which converges weakly to a minimizer in $\mathcal{A}(\mathcal{U}_0)$. $\square$

Next we show that $\gamma(P)$ satisfies the hypotheses of Theorem 1.1. It follows (by Corollary 1.2) that every solution of (0.5) is an equilibrium solution.

LEMMA 2.2. *The integrand $\gamma(P) = F(P) + H(\det P)$ satisfies (1.1) and (1.2).*

Proof. Clearly γ satisfies (1.1). To verify (1.2), choose $\theta > 0$ so that $|C| + |C^{-1}| \leq 4$ and $\frac{1}{4} \leq \det C \leq 4$ whenever $C \in M_+^{n \times n}$ and $|C - I| < \theta$. We have

$$D\gamma(P) = DF(P) + H'(\det P) \cdot \det P \cdot (P^T)^{-1}$$

and

$$P^T \cdot D\gamma(P) = P^T \cdot DF(P) + \det P \cdot H'(\det P) \cdot I$$

for P in $M_+^{n \times n}$. Hence,

$$\begin{aligned} |P^T \cdot D\gamma(PC)| &= |(C^T)^{-1} \cdot (PC)^T \cdot D\gamma(PC)| \\ &\leq 4|(PC)^T \cdot D\gamma(PC)|. \end{aligned}$$

The result follows if we show that

$$(2.4) \quad |(PC)^T DF(PC)| \leq N_1 \cdot [1 + \gamma(P)]$$

and

$$(2.5) \quad |\det C \cdot \det P \cdot H'(\det C \cdot \det P)| \leq N_2 \cdot [1 + \gamma(P)],$$

for some constants N_1 and $N_2 > 0$ and for all C, P in $M_+^{n \times n}$ with $|C - I| \leq \theta$.

To verify (2.4) we note that since F is quasiconvex, F is convex in each variable separately. (See [10], Section 4.1.1.1.) This together with (2.2) imply that

$$(2.6) \quad |DF(P)| \leq a_1 \cdot [1 + |P|^{q-1}]$$

where a_1 is a constant depending on c_2 from (2.2). (See Lemma 2.2 of [10], Section 4.2.1.2. for a proof of (2.6).) Hence,

$$\begin{aligned} |(PC)^T \cdot DF(PC)| &\leq a_2 \cdot [1 + |CP|^q] \\ &\leq a_3 \cdot [1 + |P|^q] \end{aligned}$$

for $|C - I| \leq \theta$.

From the first inequality in (2.2) we obtain

$$|(PC)^T \cdot DF(PC)| \leq N_1 \cdot [1 + F(P)].$$

To prove (2.5) we note that

$$rd \cdot |H'(rd)| \leq a_4 \cdot [1 + d + H(d)]$$

for some constant a_4 whenever $\frac{1}{4} \leq r \leq 4$ and $d > 0$. Indeed, this follows directly from (2.3)₂ and (2.3)₄ for $0 < d < d_0$ and $d_1 < d < \infty$. But since $H \in C^3([d_0, d_1])$, the inequality holds for all d provided that a_4 is sufficiently large. Finally, since $\det P \leq |P|^n \leq 1 + |P|^q \leq a_5(1 + F(P))$ we have

$$|\det C \cdot \det P \cdot H'(\det C \cdot \det P)| \leq N_2 \cdot [1 + \gamma(P)]$$

for some constant N_2 when $|C - I| < \theta$. $\square$

From the above lemma and Theorem 1.1 we have:

COROLLARY 2.3. *Assume $\mathcal{U} \in \mathcal{A}$ and $\frac{d}{d\varepsilon} \mathcal{W}(\tilde{\mathcal{U}}_\varepsilon)|_{\varepsilon=0} = 0$ for every φ in $C_c^1(\Omega; \mathbb{R}^n)$ where $\tilde{\mathcal{U}}_\varepsilon(X) \equiv \mathcal{U}(X + \varepsilon\varphi(X))$. Then $\mathcal{U}$ satisfies the equilibrium equations (0.4). In particular, any solution of the variational problem (0.5) must satisfy (0.4).*

For each k and α the term in brackets in (0.4) is the k, α th entry of the matrix, $-\gamma(D\mathcal{U}) \cdot I + (D\mathcal{U})^T \cdot D\gamma(D\mathcal{U})$. From our formula for γ we can rewrite this term as

$$\begin{aligned} & [-F(D\mathcal{U}) \cdot I + (D\mathcal{U})^T \cdot DF(D\mathcal{U})] + \\ & [-H(\det D\mathcal{U}) + (\det D\mathcal{U}) \cdot H'(\det D\mathcal{U})] \cdot I \\ & \equiv G(D\mathcal{U}) + f(\det D\mathcal{U}) \cdot I, \end{aligned}$$

where $f(d) \equiv d \cdot H'(d) - H(d)$ for $d > 0$. Graphs of $H(d)$ and $f(d)$ are shown in Figures 1 and 2. Note that $f'(d) = d \cdot H''(d) > 0$ and that by (2.3) we have

$$(2.7) \quad -b_1 d^{-s} \leq f(d) \leq -b_2 \cdot d^{-s} \text{ for } 0 < d < b_3,$$

where b_1, b_2 , and b_3 are positive constants. Also by (2.2), (2.6), and the definition of $G(D\mathcal{U})$, we have

$$(2.8) \quad |G(D\mathcal{U})| \leq b_4 \cdot [1 + |D\mathcal{U}|^q].$$

Hence for $\mathcal{U}$ in $\mathcal{A}$ and γ as above, the system (0.4) is equivalent to the system

$$(2.9) \quad G_\alpha^k(D\mathcal{U})_{x_\alpha} + f(\det D\mathcal{U})_{x_k} = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for } 1 \leq k \leq n.$$

We use the system in this form to estimate $(\det D\mathcal{U})^{-s}$ a priori in terms of $|D\mathcal{U}|^q$.

LEMMA 2.4. Assume $\mathcal{U} \in \mathcal{A}$, $\mathcal{U}$ satisfies (0.4), and $|D\mathcal{U}|^q \in L^p(\Omega)$ for some p in $(1, \infty)$. Then for any $\Omega' \Subset \Omega$, we have

$$(2.10) \quad \|(\det D\mathcal{U})^{-s}\|_{L^p(\Omega')} \leq c \cdot (1 + \mathcal{W}(\mathcal{U}) + \| |D\mathcal{U}|^q \|_{L^p(\Omega)}),$$

where c is a positive constant depending only on Ω, Ω', p, F , and H .

Proof. Differentiate the equation (2.9) with respect to x_k for $1 \leq k \leq n$ and add the equations. We get

$$\Delta f(\det D\mathcal{U}) = - \sum_{k, \alpha=1}^n \frac{\partial^2}{\partial x_\alpha \partial x_k} G_\alpha^k(D\mathcal{U}) \text{ in } \mathcal{D}'(\Omega).$$

It follows from the Calderon-Zygmund inequality and duality that

$$(2.11) \quad \|f(\det D\mathcal{U})\|_{L^p(\Omega')} \leq c \cdot \left[\|f(\det D\mathcal{U})\|_{L^1(\Omega)} + \sum_{k, \alpha=1}^n \|G_\alpha^k(D\mathcal{U})\|_{L^p(\Omega)} \right]$$

for $c = c(n, p, \Omega, \Omega')$. A complete proof of this for $n = 2$ and $F(P) = |P|^2/2$ is given in [8], Theorem 2.2. The argument needed here is identical.

Next we note that

$$|f(d)| \leq c(1 + d + H(d)) \text{ for } d > 0$$

for some constant c . To see this, note that from (2.3)₂ and (2.3)₄ the inequality holds in a neighbourhood of $d = 0$ and $d = +\infty$. The inequality then extends to all positive d (for c sufficiently large) by the regularity of H . From this and (2.2), we obtain

$$\begin{aligned} & \|f(\det D\mathcal{U})\|_{L^1(\Omega)} \\ & \leq c \cdot \left(1 + \int_{\Omega} |D\mathcal{U}|^q dX + \int_{\Omega} H(\det D\mathcal{U}) dX \right) \\ & \leq c \cdot [1 + \mathcal{W}(\mathcal{U})]. \end{aligned}$$

Hence by (2.8) and (2.11),

$$(2.12) \quad \|f(\det D\mathcal{U})\|_{L^p(\Omega')} \leq c \cdot (1 + \| |D\mathcal{U}|^q \|_{L^p(\Omega)} + \mathcal{W}(\mathcal{U})).$$

Now by (1.9),

$$\begin{aligned} (\det D\mathcal{U})^{-s} & \leq c \cdot (1 + \det D\mathcal{U} + |f(\det D\mathcal{U})|) \\ & \leq c_1 \cdot (1 + |D\mathcal{U}|^q + |f(\det D\mathcal{U})|) \end{aligned}$$

in Ω . The assertion (2.10) follows from this and (2.12). $\square$

The central point of this section is the following observation relating the regularity of an equilibrium solution and the strict positivity of $\det D\mathcal{U}$.

THEOREM 2.5. Assume that $\mathcal{U} \in \mathcal{A} \cap C^{1,\beta}(\Omega)$ for some $\beta > 0$ and $\mathcal{U}$ satisfies (0.4). Then $\det D\mathcal{U} > 0$ in Ω .

Proof. Assume that $\det D\mathcal{U}(X_0) = 0$ for some X_0 in $B_R(X_0) \Subset \Omega$. Since $D\mathcal{U} \in C^\beta(\Omega)$ we have

$$\det D\mathcal{U}(Y) \leq c|Y - X_0|^\beta \quad \text{for } Y \text{ in } B_R(X_0).$$

But $\det D\mathcal{U} > 0$ almost everywhere in Ω . Hence

$$|Y - X_0|^{-\beta} \leq c|\det D\mathcal{U}(Y)|^{-1}$$

almost everywhere in $B_R(X_0)$. Since $\mathcal{U}$ is C^1 we deduce from Lemma 2.4 that $(\det D\mathcal{U})^{-1} \in L^p(B_R(X_0))$ for every p in $(1, \infty)$. Thus $|Y - X_0|^{-\beta} \in L^p(B_R(X_0))$ for all $p > 1$ which is false. $\square$

Assume that $F \in C^2(M^{n \times n})$. Then it follows from the quasiconvexity assumption that F satisfies the Legendre-Hadamard (or ellipticity) condition,

$$\frac{\partial^2 F}{\partial P_\alpha^i \partial P_\beta^j}(P) \cdot \xi^i \xi^j \eta_\alpha \eta_\beta \geq 0$$

for all P in $M^{n \times n}$ and for all ξ, η in $\mathbb{R}^n$. (See [10], Section 4.1.) Similarly since H is convex on $(0, \infty)$, we have

$$\frac{\partial^2}{\partial P_\alpha^i \partial P_\beta^j} H(\det P) \cdot \xi^i \xi^j \eta_\alpha \eta_\beta \geq 0$$

for all P in $M_+^{n \times n}$ and for all ξ, η in $\mathbb{R}^n$. If we further assume that F is strictly elliptic then there is a function $\mu(P) > 0$ for all P in $M_+^{n \times n}$ so that

$$(2.13) \quad \frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j}(P) \cdot \xi^i \xi^j \eta_\alpha \eta_\beta \geq \mu(P)|\xi|^2 \cdot |\eta|^2$$

for all P in $M_+^{n \times n}$ and for all ξ, η in $\mathbb{R}^n$.

Now consider $\Omega' \Subset \Omega$. If $\mathcal{U} \in \mathcal{A} \cap C^{1,\beta}(\Omega)$ and $\mathcal{U}$ is an equilibrium solution, then Theorem 2.5 implies that γ is bounded and C^1 on a neighbourhood of $\{P \in M_+^{n \times n} : P = D\mathcal{U}(X) \text{ for some } X \text{ in } \Omega'\}$. Hence for any φ in $C_c^1(\Omega; \mathbb{R}^n)$, the function $\mathcal{W}(\mathcal{U}_\varepsilon) \equiv \mathcal{W}(\mathcal{U} + \varepsilon\varphi)$ is well defined and differentiable for ε near zero. In particular, $\frac{d}{d\varepsilon} \mathcal{W}(\mathcal{U}_\varepsilon)|_{\varepsilon=0}$ is well defined. In the following theorem we show that if F is sufficiently smooth and strictly elliptic, then any $C^{1,\beta}$ equilibrium solution, $\mathcal{U}$, in $\mathcal{A}$ satisfies the Euler-Lagrange system (0.3). It follows from Theorem 2.5 and elliptic regularity theory for this system that $\mathcal{U} \in C^{2,\beta}(\Omega)$.

We shall use the notation $\Delta_{s,h} \mathcal{V}(X) = \left(\frac{\mathcal{V}(X + h e_s) - \mathcal{V}(X)}{h} \right)$ where e_s is the unit vector in the direction of the x_s coordinate axis.

THEOREM 2.6. Assume $F \in C^{2,\alpha}(M^{n \times n})$ and $\gamma(P) = F(P) + H(\det P)$ satisfies (2.13). Let $\mathcal{U} \in \mathcal{A} \cap C^{1,\beta}(\Omega; \mathbb{R}^n)$ for some β in $(0,1)$ and assume that $\mathcal{U}$ satisfies the equilibrium system (0.4). Then $\det D\mathcal{U} > 0$ in Ω , $\mathcal{U} \in C^{2,\alpha}(\Omega)$, and $\mathcal{U}$ is a solution of the Euler-Lagrange system (0.3). Moreover, if $\gamma \in C^{k,\alpha}(M_+^{n \times n})$ for $k \geq 3$, then $\mathcal{U} \in C^{k,\alpha}(\Omega)$.

Proof. Since $\mathcal{U}$ satisfies (0.4) only weakly we first derive a local L^2 -estimate on $D^2\mathcal{U}$. Note that from Theorem 2.5 we have $\det D\mathcal{U} > 0$ in Ω . Hence $\{D^j\gamma : j = 0, 1, 2, 3\}$ are bounded and continuous in a neighbourhood of $D\mathcal{U}(X_0)$ for any X_0 in Ω .

Fix X_0 in Ω with $B_{4R}(X_0) \Subset \Omega$. Let $\varphi \in C_c^1(B_{2R}(X_0))$ and set $A_\alpha^k(P) = [-\gamma(P) \cdot \delta_\alpha^k + P_k^i \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(P)]$. We have

$$(2.14) \quad \int_{\Omega} [A_\alpha^k(D\mathcal{U}(X + he_s)) - A_\alpha^k(D\mathcal{U}(X))] \cdot \varphi_{x_\alpha}^k dX = 0$$

for $1 \leq s \leq n$ and $h > 0$ sufficiently small. Given ε in $(0,1)$ we have

$$|D\mathcal{U}(X) - D\mathcal{U}(X_0)| < \varepsilon \text{ for } X \text{ in } B_{4R}(X_0)$$

provided R is sufficiently small (since $D\mathcal{U} \in C(\Omega)$). From (2.14) and the fact that $\{A_\alpha^k\}$ are of class C^2 near $D\mathcal{U}(X_0)$ we obtain

$$(2.15) \quad \begin{aligned} & \int_{\Omega} \frac{\partial A_\alpha^k}{\partial P_\beta^j}(D\mathcal{U}(X_0)) \cdot \Delta_{s,h}(u_{x_\beta}^j) \cdot \varphi_{x_\alpha}^k dX \\ & \leq \sup_{|P - D\mathcal{U}(X_0)| < \varepsilon} |DA(P) - DA(D\mathcal{U}(X_0))| \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}| \cdot |D\varphi| dX \\ & \leq c_1 \varepsilon^\alpha \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}| \cdot |D\varphi| dX. \end{aligned}$$

Let $\eta \in C_c^\infty(B_{2R}(X_0))$ with $0 \leq \eta \leq 1$, $|D\eta| \leq \frac{\varepsilon}{R}$ and $\eta \equiv 1$ on $B_R(X_0)$. Set

$$\varphi^k(X) = \eta^2(X) \cdot [D\mathcal{U}^{-1}(X_0)]_\ell^k \cdot \Delta_{s,h} u^\ell(X) \text{ for } 1 \leq k \leq n.$$

From (2.15) we obtain

$$\begin{aligned} & \int_{\Omega} \frac{\partial A_\alpha^k}{\partial P_\beta^j}(D\mathcal{U}(X_0)) \cdot [D\mathcal{U}^{-1}(X_0)]_\ell^k \cdot (\Delta_{s,h} u^j)_{x_\beta} \cdot (\Delta_{s,h} u^\ell)_{x_\alpha} \cdot \eta^2 dX \\ & \leq c_2 \varepsilon^\alpha \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}|^2 \cdot \eta^2 dX + \frac{c_3}{\varepsilon^\alpha R^2} \cdot \int_{B_{4R}(X_0)} |\Delta_{s,h} \mathcal{U}|^2 dX. \end{aligned}$$

Hence

$$(2.16) \quad \begin{aligned} & \int_{\Omega} \frac{\partial A_\alpha^k}{\partial P_\beta^j}(D\mathcal{U}(X_0)) \cdot [D\mathcal{U}^{-1}(X_0)]_\ell^k \cdot (\eta \Delta_{s,h} \mathcal{U})_{x_\beta}^j \cdot (\eta \Delta_{s,h} \mathcal{U})_{x_\alpha}^\ell dX \\ & \leq c_4 \varepsilon^\alpha \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}|^2 \cdot \eta^2 dX + \frac{c_5}{\varepsilon^\alpha R^2} \cdot \int_{B_{4R}(X_0)} |\Delta_{s,h} \mathcal{U}|^2 dX. \end{aligned}$$

By the Plancherel theorem and (1.4), the left hand side of inequality (2.16) is

$$\int_{\mathbf{R}^n} \frac{\partial^2 \gamma}{\partial P_\alpha^j \partial P_\beta^j} (D\mathcal{U}(X_0)) \cdot (\eta \widehat{\Delta_{s,h}} \mathcal{U})^j (\eta \widehat{\Delta_{s,h}} \mathcal{U})^\ell \cdot x_\alpha x_\beta dX.$$

From this, (2.13), and (2.16) we obtain

$$\begin{aligned} & \mu(D\mathcal{U}(X_0)) \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}|^2 \eta^2 dX \\ & \leq c_6 \varepsilon^\alpha \cdot \int_{\Omega} |\Delta_{s,h} D\mathcal{U}|^2 \eta^2 dX + \frac{c_7}{\varepsilon^\alpha R^2} \int_{B_{4R}(X_0)} |\Delta_{s,h} \mathcal{U}|^2 dX. \end{aligned}$$

Summing on s and choosing ε so small that $c_6 \varepsilon^\alpha < \frac{\mu(D\mathcal{U}(X_0))}{2}$ we conclude that $D^2 \mathcal{U} \in L_{\text{loc}}^2(\Omega)$.

Now we can carry out the differentiation in (0.4) to obtain

$$\frac{\partial A_\alpha^k}{\partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} = 0$$

almost everywhere in Ω for $1 \leq k \leq n$. By (1.5) it follows that

$$\frac{\partial^2 \gamma}{\partial P_\alpha^i \partial P_\beta^j}(D\mathcal{U}) \cdot \frac{\partial^2 u^j}{\partial x_\alpha \partial x_\beta} = 0$$

almost everywhere in Ω for $1 \leq i \leq n$. Thus

$$\left(\frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right)_{x_\alpha} = 0 \text{ in } \mathcal{D}'(\Omega) \text{ for } 1 \leq i \leq n.$$

Since $\mathcal{U} \in \mathcal{A} \cap C^1(\Omega)$, $\det D\mathcal{U} > 0$ in Ω , $\gamma \in C^{2,\alpha}(M_+^{n \times n})$, and γ satisfies (2.13) it follows from elliptic regularity theory (by differentiating the above equation with respect to any x_s) that $\mathcal{U} \in C^{2,\alpha}(\Omega)$. (See [11], Chapter III, Theorems 3.1 and 3.2.) From this it follows that if $\gamma \in C^{k,\alpha}(M_+^{n \times n})$ for $k \geq 3$ and α in $(0,1)$, then $\mathcal{U} \in C^{k,\alpha}(\Omega)$. (See [11], Chapter III, Theorem 3.3.) $\square$

Remark 2.7. The proof of Theorem 2.6 is valid for any equilibrium solution $\mathcal{U}$ which satisfies $\mathcal{U} \in C^1(\Omega)$ and $\det D\mathcal{U} > 0$ in Ω . Thus the conclusions of Theorem 2.6 hold under these assumptions on $\mathcal{U}$.

Section 3. Example: A two-to-one nonsmooth equilibrium solution in $\mathcal{A} \cap C^1(\overline{\Omega})$.

In the previous section we showed that for γ as in (2.1) an equilibrium solution, $\mathcal{U}$, in $\mathcal{A} \cap C^{1,\beta}(\Omega)$ satisfies $\det D\mathcal{U} > 0$ in Ω . Moreover, if γ is smooth and strictly elliptic on $M_+^{n \times n}$, then $\mathcal{U}$ is a classical solution to the Euler-Lagrange equations (0.3). In this

section we shall show that this result is optimal assuming only the hypothesis that $\mathcal{U}$ is an equilibrium solution in $\mathcal{A}$. In particular, we shall find a mapping $\mathcal{U} : B_1(0) \subset \mathbb{R}^2 \rightarrow B_1(0) \subset \mathbb{R}^2$ in $\mathcal{A}$ which satisfies (0.3) and (0.4) for

$$(3.1) \quad \gamma(P) = \frac{|P|^2}{2} + H(\det P), \quad P \in M_+^{2 \times 2},$$

where $H(d)$ satisfies (2.3). The mapping $\mathcal{U}$ in polar coordinates is of the form

$$\mathcal{U} : [R, \theta] \rightarrow [r(R), 2\theta]$$

where

$$(3.2) \quad \begin{aligned} (i) \quad & r \in C^1([0, 1]) \cap C^3((0, 1]), \\ (ii) \quad & r(0) = 0 \text{ and } r(1) = 1, \\ (iii) \quad & r'(R) > 0 \text{ for } 0 < R < 1 \text{ and } r'(0) = 0. \end{aligned}$$

It follows that

$$\begin{aligned} \mathcal{U} &\in \mathcal{A} \cap C^1(\overline{B}_1) \cap C^3(\overline{B}_1 - \{0\}), \\ \det D\mathcal{U}(X) &> 0 \text{ for } X \neq 0, \end{aligned}$$

and

$$\det D\mathcal{U}(0) = 0.$$

Thus $D\mathcal{U}$ cannot be Hölder continuous.

Remark 3.1. For applications to elasticity one is interested in equilibrium solutions in the class of *deformations*, i.e. mappings in $\mathcal{A}$ which are one-to-one almost everywhere. Here $\mathcal{U}$ is two-to-one in $\overline{B}_1 - \{0\}$. Thus our example does *not* provide evidence of singular equilibrium solutions corresponding to deformations of elastic solids in this class of materials. However, it does demonstrate that in studying regularity for deformations of elastic solids with stored energy density γ as above, one should consider more information than just the equilibrium system (0.3) or the Euler-Lagrange system (0.4). In particular, the properties developed by Ball [4] and Šverák [13] for one-to-one mappings in $\mathcal{A}$ may be relevant.

Let

$$\Omega = B_1 \equiv \{X \in \mathbb{R}^2 : |X| < 1\}.$$

We assume that γ is given by (3.1) so that $\mathcal{A} = \{\mathcal{U} \in W^{1,2}(\Omega; \mathbb{R}^2) : \mathcal{W}(\mathcal{U}) < +\infty \text{ and } \det D\mathcal{U} > 0 \text{ a.e. in } \Omega\}$. Consider mappings $\mathcal{V} : B_1 \rightarrow \mathbb{R}^2$ of the form

$$(3.3) \quad \mathcal{V}(X) = (r(R) \cdot \cos 2\theta, r(R) \cdot \sin 2\theta)$$

where $X = (R \cos \theta, R \sin \theta)$.

LEMMA 3.2. Let $\mathcal{V}$ be given by (3.3). Then $\mathcal{V} \in W^{1,2}(\Omega; \mathbb{R}^2)$ if and only if $r(\cdot)$ is absolutely continuous on all closed subintervals of $(0, 1]$ and

$$\int_0^1 \left[|r'(R)|^2 + \left| \frac{r(R)}{R} \right|^2 \right] R dR < +\infty.$$

Moreover, for almost every X in B_1 the weak derivatives of $\mathcal{V}$ are determined by

$$\partial_R \mathcal{V} = (r'(R) \cdot \cos 2\theta, \quad r'(R) \cdot \sin 2\theta)$$

and

$$\frac{1}{R} \partial_\theta \mathcal{V} = \left(\frac{-2r(R)}{R} \cdot \sin 2\theta, \quad \frac{2r(R)}{R} \cos 2\theta \right).$$

In particular,

$$(3.4) \quad |D\mathcal{V}|^2 = |r'(R)|^2 + 4 \cdot \left| \frac{r(R)}{R} \right|^2$$

and

$$\det D\mathcal{V} = \frac{2r'(R) \cdot r(R)}{R} \text{ for almost every } X \text{ in } B_1.$$

Set $\mathcal{A}_r = \{\mathcal{V} \in \mathcal{A} : \mathcal{V} \text{ satisfies (3.3) with } r(1) = 1\}$. Note that $\mathcal{A}_r \neq \emptyset$ since the function $\tilde{\mathcal{V}}(X)$ with $r(R) = R$ is in $\mathcal{A}_r$.

We shall find a minimizer, $\mathcal{U}$, of $\mathcal{W}$ in $\mathcal{A}_r$ and verify properties (3.2). Following this we shall prove that $\mathcal{U}$ satisfies (0.3) and (0.4) in $\Omega \equiv B_1$.

LEMMA 3.3. There exists a mapping $\mathcal{U}$ in $\mathcal{A}_r$ such that $\mathcal{W}(\mathcal{U}) = \inf_{\mathcal{V} \in \mathcal{A}_r} \mathcal{W}(\mathcal{V})$.

Proof. A weakly convergent minimizing sequence in $\mathcal{A}_r$ has its limit in $\mathcal{A}_r$. Thus the lemma follows as in the proof of Theorem 2.1. $\square$

For the remainder of the paper let $r(R)$ denote the radial component as in (3.3) for a minimizer $\mathcal{U}$ in $\mathcal{A}_r$.

LEMMA 3.4. The function $r(R)$ satisfies

$$(3.5) \quad \left(\frac{[r'(R)]^2}{2} + f \left(2 \frac{r'(R) \cdot r(R)}{R} \right) \right)' = 4 \frac{r(R) \cdot r'(R)}{R^2} - \frac{[r'(R)]^2}{R} \text{ in } \mathcal{D}'((0, 1)),$$

where $f(d) = dH'(d) - H(d)$. In particular $\left(\frac{[r']^2}{2} + f \left(2 \frac{r' \cdot r}{R} \right) \right)$ is essentially absolutely continuous on all closed subintervals of $(0, 1]$.

Proof. Let $g(R) \in C_c^\infty((0, 1))$. Set

$$(3.6) \quad \varphi(X) = (g(R) \cdot \cos \theta, g(R) \cdot \sin \theta).$$

Then for $|\varepsilon|$ small,

$$\begin{aligned}\tilde{\mathcal{U}}_\varepsilon(X) &\equiv \mathcal{U}(X + \varepsilon\varphi(X)) \\ &= \mathcal{U}([R + \varepsilon g(R)] \cos \theta, [R + \varepsilon g(R)] \sin \theta)\end{aligned}$$

is in $\mathcal{A}_r$ and it follows from Theorem 1.1 and Lemma 2.2 that $\frac{d}{d\varepsilon}\mathcal{W}(\tilde{\mathcal{U}}_\varepsilon)|_{\varepsilon=0} = 0$. Let $\mathcal{U} = (u, v)$ and $X = (x, y)$. As in the derivation of (2.9) with $F(P) = \frac{|P|^2}{2}$, we obtain

$$\begin{aligned}\int_{B_1} &\left\{ \left[\frac{1}{2}(u_x^2 + v_x^2 - u_y^2 - v_y^2) + f(\det D\mathcal{U}) \right] \cdot \varphi_x^1 \right. \\ &\quad + [u_x u_y + v_x v_y] \cdot \varphi_y^1 + [u_x u_y + v_x v_y] \cdot \varphi_x^2 \\ &\quad \left. + \left[\frac{1}{2}(u_y^2 + v_y^2 - u_x^2 - v_x^2) + f(\det D\mathcal{U}) \right] \cdot \varphi_y^2 \right\} dX \\ &= 0.\end{aligned}$$

Set

$$I_1(u, v) = \int_{B_1} \left[\frac{1}{2}(u_x^2 + v_x^2 - u_y^2 - v_y^2) \cdot (\varphi_x^1 - \varphi_y^2) + (u_x u_y + v_x v_y) \cdot (\varphi_y^1 + \varphi_x^2) \right] dX$$

and

$$I_2(u, v) = \int_{B_1} f(\det D\mathcal{U}) \cdot (\varphi_x^1 + \varphi_y^2) dX.$$

Then $I_1(u, v) + I_2(u, v) = 0$. From this we shall derive the ordinary differential equation (3.5).

To this end assume that $\text{supp } g \subset [R_1, R_2]$ with $0 < R_1 < R_2 < 1$. Let $\{r_\varepsilon(R)\}$ be a family of mollifications of $r(R)$ on a neighbourhood of $[R_1, R_2]$. Set

$$(3.7) \quad (u_\varepsilon, v_\varepsilon) = (r_\varepsilon(R) \cdot \cos 2\theta, r_\varepsilon(R) \cdot \sin 2\theta).$$

Then $(u_\varepsilon, v_\varepsilon) \rightarrow (u, v)$ in $W^{1,2}(B_{R_2} - \overline{B}_{R_1}; \mathbb{R}^2)$ and, since $\text{supp } \varphi \subset \overline{B}_{R_2} - B_{R_1}$, $I_1(u_\varepsilon, v_\varepsilon) \rightarrow I_1(u, v)$ as $\varepsilon \rightarrow 0$.

Integrating by parts in $I_1(u_\varepsilon, v_\varepsilon)$ we obtain:

$$\begin{aligned}I_1(u_\varepsilon, v_\varepsilon) &= - \int_{B_1} [(u_{\varepsilon x} \Delta u_\varepsilon + v_{\varepsilon x} \Delta v_\varepsilon) \cdot \varphi^1 + (u_{\varepsilon y} \Delta u_\varepsilon + v_{\varepsilon y} \Delta v_\varepsilon) \cdot \varphi^2] dX \\ &= - \int_{B_1} [\Delta u_\varepsilon \cdot (u_{\varepsilon x} \cdot \varphi^1 + u_{\varepsilon y} \cdot \varphi^2) + \Delta v_\varepsilon \cdot (v_{\varepsilon x} \cdot \varphi^1 + v_{\varepsilon y} \cdot \varphi^2)] dX.\end{aligned}$$

From (3.6) we have

$$\begin{aligned}u_{\varepsilon x} \cdot \varphi^1 + u_{\varepsilon y} \cdot \varphi^2 &= g(R) \cdot [u_{\varepsilon x} \cos \theta + u_{\varepsilon y} \sin \theta] \\ &= g(R) \cdot \partial_R u_\varepsilon\end{aligned}$$

and $v_{\varepsilon_x} \cdot \varphi^1 + v_{\varepsilon_y} \cdot \varphi^2 = g(R) \cdot \partial_R v_\varepsilon$. Hence

$$I_1(u_\varepsilon, v_\varepsilon) = - \int_{B_1} g(R) \cdot [\partial_R u_\varepsilon \cdot \Delta u_\varepsilon + \partial_R v_\varepsilon \cdot \Delta v_\varepsilon] dX.$$

Also from (3.7) we have

$$\partial_R u_\varepsilon \cdot \Delta u_\varepsilon = (r'_\varepsilon \cdot \cos 2\theta) \cdot \left(r''_\varepsilon + \frac{r'_\varepsilon}{R} - 4 \frac{r_\varepsilon}{R^2} \right) \cdot \cos 2\theta$$

and

$$\partial_R v_\varepsilon \cdot \Delta v_\varepsilon = (r'_\varepsilon \cdot \sin 2\theta) \cdot \left(r''_\varepsilon + \frac{r'_\varepsilon}{R} - 4 \frac{r_\varepsilon}{R^2} \right) \cdot \sin 2\theta.$$

Hence

$$\begin{aligned} I_1(u_\varepsilon, v_\varepsilon) &= - \int_{B_1} g(R) \cdot r'_\varepsilon \cdot \left(r''_\varepsilon + \frac{r'_\varepsilon}{R} - 4 \frac{r_\varepsilon}{R^2} \right) dX \\ &= -2\pi \cdot \int_0^1 [R \cdot g(R)] \cdot \left(\left(\frac{|r'_\varepsilon|^2}{2} \right)' + \frac{|r'_\varepsilon|^2}{R} - 4 \frac{r'_\varepsilon \cdot r_\varepsilon}{R^2} \right) dR \\ &= -2\pi \cdot \int_0^1 \left\{ [R \cdot g(R)] \cdot \left(\frac{|r'_\varepsilon|^2}{R} - 4 \frac{r'_\varepsilon \cdot r_\varepsilon}{R^2} \right) \right. \\ &\quad \left. - [R \cdot g(R)]' \cdot \frac{|r'_\varepsilon|^2}{2} \right\} dR. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ we obtain

$$I_1(u, v) = -2\pi \cdot \int_\Omega \left\{ [R \cdot g(R)] \cdot \left(\frac{|r'|^2}{R} - 4 \frac{r' \cdot r}{R^2} \right) - [R \cdot g(R)]' \cdot \frac{|r'|^2}{2} \right\} dR.$$

Now to simplify $I_2(u, v)$ we note that $\varphi_x^1 + \varphi_y^2 = g'(R) + \frac{g(R)}{R}$. Hence from (3.4), we have

$$\begin{aligned} I_2(u, v) &= 2\pi \cdot \int_0^1 f \left(2 \cdot \frac{r' \cdot r}{R} \right) \cdot \left[g'(R) + \frac{g(R)}{R} \right] \cdot R dR \\ &= 2\pi \cdot \int_0^1 f \left(2 \cdot \frac{r' \cdot r}{R} \right) \cdot [R \cdot g(R)]' dR. \end{aligned}$$

Thus

$$\begin{aligned} (3.8) \quad 0 &= I_1(u, v) + I_2(u, v) \\ &= 2\pi \cdot \int_0^1 \left[\frac{|r'|^2}{2} + f \left(2 \cdot \frac{r' \cdot r}{R} \right) \right] \cdot [R \cdot g(R)]' dR \\ &\quad - 2\pi \cdot \int_0^1 \left[\frac{|r'|^2}{R} - 4 \cdot \frac{r' \cdot r}{R^2} \right] \cdot R \cdot g(R) dR. \end{aligned}$$

This proves (3.5). $\square$

LEMMA 3.5. *The function $r(R) \in C([0, 1]) \cap C^3((0, 1])$ with $r(0) = 0$ and $r'(R) > 0$ for R in $(0, 1]$.*

Proof. Since $\mathcal{A}_r \subset \mathcal{A}$ we have $\det DU = \frac{2r'(R) \cdot r(R)}{R} > 0$ almost everywhere in $(0, 1]$. Hence $r^2(R)$ is strictly increasing. Note that $r(R)$ is continuous on $(0, 1]$. Thus if $r(R)$ is negative somewhere on $(0, 1)$ we must have $r(1) < 0$, a contradiction. Hence $r(R) > 0$ for R in $(0, 1]$ and $r' > 0$ almost everywhere on $(0, 1]$. Let $\ell = \lim_{R \rightarrow 0} r(R)$. If $\ell > 0$, this implies that $\int_0^1 \frac{r^2}{R} dR = \infty$. This is impossible by Lemma 3.2. Hence $r \in C([0, 1])$ with $r(0) = 0$.

To show that $r'(R)$ is continuous and strictly positive on $(0, 1]$, we use the fact that by Lemma 3.4, $\frac{|r'(R)|^2}{2} + f\left(2 \cdot \frac{r'(R) \cdot r(R)}{R}\right)$ is essentially absolutely continuous. First note that for each σ in $\mathbb{R}$ and R in $(0, 1]$, the equation

$$\frac{|r'|^2}{2} + f\left(2 \cdot \frac{r' \cdot r(R)}{R}\right) = \sigma$$

is uniquely solvable for $r' \in (0, \infty)$. To see this, set $q(w, a, b) = \frac{w^2}{2} + f\left(\frac{2wa}{b}\right)$ for $w, a, b > 0$. Then $\partial_w q = w + f'\left(\frac{2wa}{b}\right) \cdot \frac{2a}{b} > 0$ since $f'(d) = dH''(d) > 0$. Also, we have $\lim_{w \rightarrow 0^+} q(w, a, b) = -\infty$ and $\lim_{w \rightarrow +\infty} q(w, a, b) = +\infty$ for a and b fixed by (2.7) and (2.3), respectively. Thus for fixed a and $b > 0$, q is a homeomorphism of $(0, \infty)$ onto $\mathbb{R}$. It follows that we can redefine r' on a set of measure zero in $(0, 1]$ so that

$$q(r'(R), r(R), R) \equiv q(R)$$

is continuous on $(0, 1]$ and $r' > 0$ on $(0, 1]$.

We claim that r' is continuous on $(0, 1]$ and hence $r \in C^1((0, 1])$. To prove this, assume that $\{R_j\} \subset (0, 1]$, $R_j \rightarrow R_0 \in (0, 1]$, and $r'(R_j) \rightarrow +\infty$. Since $r(R)$ is continuous and $r(R_0) > 0$, we have $r'(R_j) \cdot r(R_j)/R_j \rightarrow +\infty$. From the formula for q we see that

$$q(r'(R_j), r(R_j), R_j) \rightarrow +\infty \text{ as } j \rightarrow \infty.$$

But $q(r'(R), r(R), R)$ is continuous on $(0, 1]$ and this is a contradiction. A similar argument implies that $\lim_{R \rightarrow R_0} r'(R) > 0$ for $R_0 \in (0, 1]$. Now assume that $R_j \rightarrow R_0 \in (0, 1]$ with $r'(R_j) \rightarrow \ell \in (0, \infty)$. From the definition of q , $\lim_{j \rightarrow \infty} q(r'(R_j), r(R_j), R_j) = q(\ell, r(R_0), R_0)$. On the other hand, since $q(R) = q(r'(R), r(R), R) \in C((0, 1])$, we have $q(r'(R_0), r(R_0), R_0) = q(\ell, r(R_0), R_0)$. By uniqueness, we conclude that $r'(R_0) = \ell$. Thus $r'(R) \in C((0, 1])$ and $r' > 0$ on $(0, 1]$.

Next we show that $r \in C^3((0, 1])$. From Lemma 3.4 we have

$$q(r'(R), r(R), R) = - \int_R^1 \left(4 \cdot \frac{r(s) \cdot r'(s)}{s^2} - \frac{(r'(s))^2}{s} \right) ds + c$$

for some constant c . By the preceding results, the right hand side is of class C^1 on $(0, 1]$. Since $q_w > 0$, the implicit function theorem implies that $r' \in C^1((0, 1])$. Since $H \in C^3((0, \infty))$, the implicit function theorem further implies that $r \in C^3((0, 1])$. $\square$

We remark that an argument similar to the one above was given by Ball in [5], Proposition 6.1. In the next lemma we show that $\mathcal{U}$ is a solution to the Euler-Lagrange system (0.3) in $B_1 - \{0\}$.

LEMMA 3.6. *Let $\mathcal{U}$ be a minimizer in $\mathcal{A}_r$. Then $\mathcal{U} = (u, v)$ satisfies*

$$(3.9) \quad \begin{aligned} \Delta u + [H'(\det D\mathcal{U})]_x \cdot v_y - [H'(\det D\mathcal{U})]_y \cdot v_x &= 0, \\ \Delta v + [H'(\det D\mathcal{U})]_y \cdot u_x - [H'(\det D\mathcal{U})]_x \cdot u_y &= 0 \text{ in } B_1 - \{0\}. \end{aligned}$$

This is precisely (0.3) in the special case $\gamma(P) = \frac{|P|^2}{2} + H(\det P)$ for P in $M_+^{n \times n}$ and $\Omega = B_1 - \{0\}$.

Proof. Since $\mathcal{U} \in C^2(B_1 - \{0\})$ and $\det D\mathcal{U} > 0$ in $B_1 - \{0\}$, the system (3.9) is equivalent to

$$\begin{aligned} u_x \Delta u + v_x \Delta v + \det D\mathcal{U} \cdot [H'(\det D\mathcal{U})]_x &= 0, \\ u_y \Delta u + v_y \Delta v + \det D\mathcal{U} \cdot [H'(\det D\mathcal{U})]_y &= 0 \quad \text{in } B_1 - \{0\}. \end{aligned}$$

Here the first equation is obtained by multiplying (3.9)₁ by u_x , (3.9)₂ by v_x , and then summing. The second equation is obtained similarly by multiplying (3.9)₁ by u_y and (3.9)₂ by v_y . Since $dH''(d) = f'(d)$, we have

$$\begin{aligned} u_x \Delta u + v_x \Delta v + [f(\det D\mathcal{U})]_x &= 0, \\ u_y \Delta u + v_y \Delta v + [f(\det D\mathcal{U})]_y &= 0 \quad \text{in } B_1 - \{0\}. \end{aligned}$$

This in turn is equivalent to

$$\begin{aligned} u_r \Delta u + v_r \Delta v + f(\det D\mathcal{U})_r &= 0, \\ u_\theta \Delta u + v_\theta \Delta v + f(\det D\mathcal{U})_\theta &= 0 \quad \text{for } 0 < r < 1, 0 \leq \theta < 2\pi. \end{aligned}$$

Substituting $\mathcal{U} = (r(R) \cos 2\theta, r(R) \sin 2\theta)$ into the first equation yields

$$r' \left(r'' + \frac{r'}{R} - \frac{4r}{R^2} \right) + f \left(2 \frac{r' \cdot r}{R} \right)' = 0 \quad \text{for } 0 < R < 1,$$

which is exactly (3.5). Substituting $\mathcal{U}$ into the second equation yields

$$2r \left(r'' + \frac{r'}{R} - \frac{4r}{R^2} \right) (\cos 2\theta \cdot \sin 2\theta - \cos 2\theta \cdot \sin 2\theta) + f \left(2 \frac{r' \cdot r}{R} \right)_\theta = 0 \quad \text{for } 0 < R < 1,$$

which holds identically. Thus $\mathcal{U}$ solves (3.9). $\square$

We wish to prove that $\lim_{R \rightarrow 0} r'(R) = 0$ and $\mathcal{U} \in C^1(\overline{B}_1)$. To do this, the following monotonicity results are needed.

LEMMA 3.7. Consider the function $d \equiv \det D\mathcal{U} = \frac{2 \cdot r'(R) \cdot r(R)}{R}$ and $z \equiv \frac{|D\mathcal{U}|^2}{2} + f(\det D\mathcal{U}) = \frac{|r'(R)|^2}{2} + 2 \cdot \left| \frac{r(R)}{R} \right|^2 + f\left(\frac{2 \cdot r'(R) \cdot r(R)}{R}\right)$. There exists a constant δ in $(0, 1]$ so that $d = d(R)$ and $z = z(R)$ are monotone on $(0, \delta)$.

Proof. We showed in [8], Theorems 3.3 and 3.7, that if $\mathcal{V}(X)$ is a C^3 solution of (3.9) on an open set, then $d \equiv \det D\mathcal{V}(x)$ cannot attain an interior minimum and $z \equiv \frac{|D\mathcal{V}(X)|^2}{2} + f(\det D\mathcal{V}(X))$ cannot attain an interior maximum unless these functions are constant. Since d and z are smooth functions of one variable here, it follows that $d'(R)$ and $z'(R)$ can change sign at most once in $(0, 1]$. Hence, $d(R)$ and $z(R)$ are monotone near $R = 0$. $\square$

COROLLARY 3.8. Let δ satisfy the conclusion of Lemma 3.7. Then one of the following holds identically in $(0, \delta)$:

$$(3.10) \quad \begin{aligned} (1) \quad & z'(R) \geq 0 \text{ and } 0 < 4 - 2\sqrt{3} \leq \frac{R \cdot r'(R)}{r(R)} \leq 4 + 2\sqrt{3}, \\ (2) \quad & z'(R) \leq 0 \text{ and } \left(\frac{r(R)}{R} \right)' > 0, \\ (3) \quad & z'(R) \leq 0 \text{ and } \left(\frac{r(R)}{R} \right)' < 0, \end{aligned}$$

Proof. Note that

$$\begin{aligned} z' &= \left[\frac{|r'|^2}{2} + \frac{2r^2}{R^2} + f\left(\frac{2r \cdot r'}{R}\right) \right]' \\ &= \left[\frac{|r'|^2}{2} + f\left(\frac{2r \cdot r'}{2}\right) \right]' + \frac{4r \cdot r'}{R^2} - \frac{4r^2}{R^3}. \end{aligned}$$

Using equation (3.5) we obtain

$$\begin{aligned} z' &= \frac{-|r'|^2}{R} + \frac{8r \cdot r'}{R^2} - \frac{4r^2}{R^3} \\ &= -\frac{r^2}{R^3} \left[\left(\frac{R \cdot r'}{r} \right)^2 - 8 \left(\frac{Rr'}{r} \right) + 4 \right]. \end{aligned}$$

The roots of the quadratic are $4 \pm 2\sqrt{3}$. Thus $z' \geq 0$ iff $4 - 2\sqrt{3} \leq \frac{R \cdot r'}{r} \leq 4 + 2\sqrt{3}$. This proves (3.10)₁. Since $r' > 0$ on $(0, 1]$ we have $z' \leq 0$ iff $1 < 4 + 2\sqrt{3} \leq \frac{R \cdot r'}{r}$ or $0 < \frac{R \cdot r'}{r} \leq 4 - 2\sqrt{3} < 1$. Since $\frac{R \cdot r'(R)}{r(R)}$ is continuous on $(0, \delta)$ only one of these possibilities can occur. The first implies (3.10)₂ and the second implies (3.10)₃. $\square$

THEOREM 3.9. *The function $r(R) \in C^1([0, 1])$ and $r'(0) = 0$.*

Proof. By monotonicity both $\lim_{R \rightarrow 0} d(R)$ and $\lim_{R \rightarrow 0} z(R)$ exist. By a process of elimination, we prove that the only situation which can occur is $\lim_{R \rightarrow 0} d(R) = 0$ and $\lim_{R \rightarrow 0} z(R) = -\infty$. The theorem then follows from this and the previous corollary.

Case 1. Assume that $\lim_{R \rightarrow 0^+} d(R) = +\infty$. If $z'(R) \geq 0$ for R sufficiently small then from (3.10)₁, r' and $\frac{r}{R}$ are proportional on $(0, \delta)$. Since $d = 2\frac{r}{R} \cdot r'$ this implies that $\lim_{R \rightarrow 0} r'(R) = +\infty = \lim_{R \rightarrow 0} \frac{r(R)}{R}$. Since f is increasing, the formula for z implies that $\lim_{R \rightarrow 0} z(R) = +\infty$ which is impossible.

If $z' \leq 0$ near $R = 0$ and (3.10)₂ holds, then $\frac{r(R)}{R}$ is increasing on $(0, \delta)$ and hence bounded on $(0, \delta)$ since $r \in C([0, 1])$. By the mean value theorem, there is a sequence $\{R_j\}$ with $R_j \rightarrow 0$ such that $\{r'(R_j)\}$ is bounded. But $\left\{\frac{r(R_j)}{R_j}\right\}$ is also bounded which implies that $\{d(R_j)\}$ is bounded, a contradiction.

Next assume that $z' \leq 0$ near $R = 0$ and (3.10)₃ holds. Thus $0 < r' < \frac{r}{R}$ on $(0, \delta)$. By hypothesis, $d'(R) < 0$ on $(0, \delta)$ so that $f(d)' < 0$. By (3.5), we have $r' \left(r'' + \frac{r'}{R} - \frac{4r}{R^2} \right) + f(d)' = 0$ on $(0, 1)$. Since $r' > 0$, we conclude that

$$r'' + \frac{r'}{R} - \frac{4r}{R^2} > 0 \quad \text{on } (0, \delta).$$

Let $M = r(\delta)/\delta^2$. Then $w(R) \equiv r(R) - MR^2$ satisfies $w(0) = w(\delta) = 0$ and $w'' - \frac{w'}{R} - 4\frac{w}{R^2} > 4M > 0$ on $(0, \delta)$. By the maximum principle, $w \leq 0$ on $(0, \delta)$. Hence $\frac{r(R)}{R} \leq MR$ on $(0, \delta)$ and we see that $\frac{r(R)}{R}$ and $r'(R) \rightarrow 0$ as $R \rightarrow 0^+$. This implies that $d(R) \rightarrow 0$ as $R \rightarrow 0^+$ which is a contradiction.

Case 2. Assume that $\lim_{R \rightarrow 0^+} d(R) = \ell \in (0, \infty)$. From the definition of z we see that in this case, $\lim_{R \rightarrow 0^+} z(R) > -\infty$. Suppose $\lim_{R \rightarrow 0^+} z(R) = +\infty$. Then $z' \leq 0$ near $R = 0$ and $\frac{r}{R}$ is monotone on $(0, \delta)$. By the mean value theorem, there exists a sequence $\{R_j\}$ with $R_j \rightarrow 0$ such that $\lim_{j \rightarrow \infty} r'(R_j) = \lim_{R \rightarrow 0^+} \frac{r(R)}{R} = \lim_{j \rightarrow \infty} \frac{r(R_j)}{R_j} \equiv m$. Since $d = 2\frac{r}{R} \cdot r'$ we see that $m = \sqrt{\frac{\ell}{2}}$. Hence $z(R_j) \rightarrow \frac{\ell}{4} + \ell + f(\ell) < \infty$, a contradiction.

It follows that in case 2 we must have $\lim_{R \rightarrow 0^+} z(R) = s \in (-\infty, +\infty)$. In order to get a contradiction, we first show that $\lim_{R \rightarrow 0^+} r'(R) = a$ exists and $a \in (0, +\infty)$.

Set $\nu_1(R) = r'(R)$ and $\nu_2(R) = 2\frac{r(R)}{R}$. Consider $d = d(\nu_1, \nu_2) = \nu_1 \cdot \nu_2$ and $z = \frac{\nu_1^2 + \nu_2^2}{2} + f(\nu_1 \nu_2)$ defined on $\mathcal{V} = \{(\nu_1, \nu_2) : \nu_1 > 0 \text{ and } \nu_2 > 0\}$. It is easy to check that the set $K \equiv \{(\nu_1, \nu_2) : \ell/2 \leq d \leq 2\ell\} \cap \{(\nu_1, \nu_2) : s-1 \leq z \leq s+1\}$ is a compact subset of $\mathcal{V}$. Also $\{d = \ell\} \cap \{z = s\}$ consists of at most two points $\{(a, b), (b, a)\} \subset \mathcal{V}$. From the continuity of z and d on K , it follows that for each $\varepsilon > 0$ there exists $R_0(\varepsilon) > 0$ so that $(\nu_1(R), \nu_2(R)) \in B_\varepsilon((a, b)) \cup B_\varepsilon((b, a)) \subset K$ for $0 < R < R_0$. If $a \neq b$ we

take ε so small that the balls are disjoint. In either case since $r \in C^3((0, 1])$, the curve $(\nu_1, \nu_2) = \left(r'(R), \frac{2r'(R) \cdot r(R)}{R}\right)$ for $0 < R < R_0$ is connected. As a result it must remain in one of these balls, say $B_\varepsilon((a, b))$. Hence $\lim_{R \rightarrow 0^+} r'(R) = a \in (0, \infty)$ exists. Whence $r(R) \in C^1([0, 1])$ and $r'(0) > 0$. We now show that this is impossible.

Consider $r_\varepsilon(R) \equiv \varepsilon^{-1}r(\varepsilon R)$. Since $r(R)$ satisfies equation (3.5) on $(0, 1)$, so does $r_\varepsilon(R)$ for $0 < \varepsilon < 1$. Now as $\varepsilon \rightarrow 0$, $r_\varepsilon(R) \rightarrow aR$ and $r'_\varepsilon(R) \rightarrow a$ uniformly on $[0, 1]$. Using (3.8), the weak form of equation (3.5), and letting $\varepsilon \rightarrow 0$, we see that $r_0(R) \equiv aR$ must satisfy (3.5) which is false.

Case 3. Assume that $\lim_{R \rightarrow 0^+} d(R) = 0$. If $z' \leq 0$ near $R = 0$, then as in Case 2 we can find a sequence $\{R_j\}$ with $R_j \rightarrow 0$ and $\lim_{j \rightarrow \infty} r'(R_j) = \lim_{R \rightarrow 0^+} \frac{r(R)}{R} = \lim_{j \rightarrow \infty} \frac{r(R_j)}{R_j} \equiv m$. Since $d = 2 \frac{r}{R} \cdot r'$, we see that $0 = \lim_{j \rightarrow \infty} d(R_j) = 2m^2$. This implies that $z(R_j) \rightarrow -\infty$ as $j \rightarrow \infty$, contradicting $z' \leq 0$ near $R = 0$.

The only remaining possibility is that $\lim_{R \rightarrow 0^+} d(R) = 0$ and $z'(R) \geq 0$ for R sufficiently near 0. By (3.10)₁ it follows that $r'(R)$ and $\frac{r(R)}{R}$ are proportional on $(0, \delta)$. Since $d(R)/2 = r'(R) \cdot \frac{r(R)}{R} \rightarrow 0$ as $R \rightarrow 0^+$, we must have $\lim_{R \rightarrow 0^+} r'(R) = 0 = \lim_{R \rightarrow 0^+} \frac{r(R)}{R}$. Thus $r(R) \in C^1([0, 1])$ and $r'(0) = 0$. $\square$

We are now in a position to prove:

THEOREM 3.10. *Let $\mathcal{U}$ be a minimizer of $\mathcal{W}$ in $\mathcal{A}_r$ with γ given by (3.1). Then $\mathcal{U} \in C^3(\overline{B}_1 - \{0\}) \cap C^1(\overline{B}_1)$, $\det D\mathcal{U} > 0$ in $\overline{B}_1 - \{0\}$, and $\det D\mathcal{U}(0) = 0$. Moreover, $\mathcal{U}$ satisfies the equilibrium equations (0.4) in $\Omega = B_1$.*

Proof. We have $\mathcal{U} = (r(R) \cdot \cos 2\theta, r(R) \cdot \sin 2\theta)$ where $X = (R \cos \theta, R \sin \theta)$. From Lemma 3.5 and Theorem 3.9, we have $r \in C^1([0, 1]) \cap C^3((0, 1])$, $r'(0) = 0$ and $r'(R) > 0$ for $R > 0$. Thus the regularity assertions follows directly. Now $\det D\mathcal{U} = 2 \cdot \frac{r'(R) \cdot r(R)}{R}$ for $X \neq 0$ and hence $\det D\mathcal{U} > 0$ in this case. Since $\mathcal{U}$ is C^1 and $r'(0) = 0$ we have $\det D\mathcal{U}(0) = 2(r'(0))^2 = 0$.

It remains to show that (0.4) holds. From Lemma 3.6, $\mathcal{U}$ solves the system (3.9) (and hence (0.3)) pointwise in $B_1 - \{0\}$. Thus by Lemma 1.3, $\mathcal{U}$ solves the system (0.4) pointwise in $B_1 - \{0\}$. Note that since $\mathcal{U} \in \mathcal{A}$, it follows from Theorem 1.1 and Lemma 2.2 that the terms in brackets in (0.4) are summable in B_1 .

Let $\varphi \in C_c^\infty(B_1; \mathbb{R}^2)$ and consider

$$\begin{aligned} & \int_{B_1} \left[-\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right] \cdot \varphi_{x_\alpha}^k dX \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{B_1 - B_\varepsilon} \left[-\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right] \cdot \varphi_{x_\alpha}^k dX \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(- \int_{\partial B_\varepsilon} \left[-\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial u_{x_\alpha}^i}(D\mathcal{U}) \right] \cdot n_\alpha \cdot \varphi^k dH^1 \right) \end{aligned}$$

where $n = (n_1, n_2)$ is the outer normal to ∂B_ε . Recall that

$$\begin{aligned} -\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial u_{x_\alpha}^i} &= [-\gamma(D\mathcal{U}) \cdot I + (D\mathcal{U})^T \cdot D\gamma(D\mathcal{U})]_\alpha^k \\ &= \left[-\frac{|D\mathcal{U}|^2}{2} \cdot I + D\mathcal{U}^T \cdot D\mathcal{U} \right]_\alpha^k + [f(\det D\mathcal{U}) \cdot I]_\alpha^k \\ &\equiv G_\alpha^k(D\mathcal{U}) + f(\det D\mathcal{U}) \cdot \delta_\alpha^k. \end{aligned}$$

Thus

$$\begin{aligned} (3.11) \quad & \int_{B_1} \left(-\gamma(D\mathcal{U}) \cdot \delta_\alpha^k + u_{x_k}^i \cdot \frac{\partial \gamma}{\partial P_\alpha^i}(D\mathcal{U}) \right) \cdot \varphi_{x_\alpha}^k dX \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(- \int_{\partial B_\varepsilon} G_\alpha^k(D\mathcal{U}) \cdot n_\alpha \cdot \varphi^k dH^1 - \int_{\partial B_\varepsilon} f(\det D\mathcal{U}) \cdot n_k \cdot \varphi^k dH^1 \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} \left(- \int_{\partial B_\varepsilon} f(\det D\mathcal{U}) \cdot n_k \cdot \varphi^k dH^1 \right), \end{aligned}$$

since $G_\alpha^k(D\mathcal{U}) \in C(B_1)$. Now $\det D\mathcal{U}$ is a radial function and hence

$$\begin{aligned} & - \int_{\partial B_\varepsilon} f(\det D\mathcal{U}) \cdot n_k \cdot \varphi^k dH^1 = \\ & - f(\det D\mathcal{U})(\varepsilon) \cdot \int_{\partial B_\varepsilon} n_k \varphi^k dH^1. \end{aligned}$$

Also $\int_{\partial B_\varepsilon} n_k dH^1 = 0$ for each k and hence

$$\left| \int_{\partial B_\varepsilon} n_k \cdot \varphi^k dH^1 \right| = \left| \int_{\partial B_\varepsilon} n_k (\varphi^k - \varphi^k(0)) dH^1 \right| \leq c \cdot \|D\varphi\|_{L^\infty(B_1)} \cdot \varepsilon^2,$$

where c is independent of ε . We claim that

$$(3.12) \quad \varliminf_{\varepsilon \rightarrow 0^+} \varepsilon^2 \cdot |f(\det D\mathcal{U})(\varepsilon)| = 0.$$

If so, the righthand side of (3.11) is zero and we obtain (0.4).

To prove this, assume that (3.12) fails. Then there exists $\varepsilon_0 > 0$ and $c_1 > 0$ such that

$$\varepsilon \cdot |f(\det D\mathcal{U})(\varepsilon)| \geq \frac{c_1}{\varepsilon} \quad \text{for } 0 < \varepsilon \leq \varepsilon_0.$$

But this contradicts the fact that $\int_{B_1} |f(\det D\mathcal{U})| dX < \infty$. Thus (3.12) holds and $\mathcal{U}$ satisfies (0.4). $\square$

From Theorem 2.6 it follows that since $\det D\mathcal{U}(0) = 0$, $\mathcal{U} \notin C^{1,\beta}(\Omega)$ for any $\beta > 0$. We conclude this section by showing that $\mathcal{U}$ satisfies the Euler-Lagrange equations (0.3) as well as the equilibrium equations (0.4) in B_1 .

THEOREM 3.11. *The mapping $\mathcal{U}$ as constructed above satisfies the Euler-Lagrange system (0.3) in $\Omega = B_1$.*

Proof. First we shall prove that $|D\gamma(D\mathcal{U})|$ is locally square integrable. Now

$$D\gamma(D\mathcal{U}) = D\mathcal{U} + H'(\det D\mathcal{U}) \cdot \begin{bmatrix} u_{x_2}^2 & -u_{x_1}^2 \\ -u_{x_2}^1 & u_{x_1}^1 \end{bmatrix}.$$

Since $|D\mathcal{U}|$ is bounded, we have

$$|D\gamma(D\mathcal{U})| \leq c_1(1 + |H'(\det D\mathcal{U})|)$$

and since $\det D\mathcal{U}$ is bounded, we use (2.3) to conclude that

$$|D\gamma(D\mathcal{U})| \leq c_2(1 + |\det D\mathcal{U}|^{-s-1}).$$

Now since $\mathcal{U} \in \mathcal{A}$ and $\mathcal{U}$ satisfies the equilibrium equations (0.4) in B_1 , Lemma 2.4 implies that

$$|\det D\mathcal{U}|^{-1} \in L_{\text{loc}}^p(B_1) \text{ for any } p \text{ in } (1, \infty).$$

Hence

$$(3.13) \quad |D\gamma(D\mathcal{U})| \in L^2(B_R) \text{ for any } R \text{ in } (0, 1).$$

Note that by Lemma 3.6, $\mathcal{U}$ satisfies

$$(3.14) \quad \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \cdot \varphi_{x_\alpha}^k dX = 0$$

for all $\varphi \in C_c^\infty(B_1; \mathbf{R}^2)$ such that $\varphi \equiv 0$ in a neighbourhood of $X = 0$.

Let $\eta_\varepsilon \in C^\infty(B_1)$ such that $\eta_\varepsilon \equiv 0$ in B_ε , $\eta_\varepsilon \equiv 1$ in $B_1 - B_{2\varepsilon}$, $0 \leq \eta_\varepsilon \leq 1$ and $|D\eta_\varepsilon| \leq c/\varepsilon$. Let $\psi \in C_c^\infty(B_1; \mathbf{R}^2)$ and set $\varphi = \eta_\varepsilon \psi$ in (3.14). Then

$$\begin{aligned} 0 &= \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \cdot (\eta_\varepsilon \psi)_{x_\alpha}^k dX \\ &= \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \cdot \psi_{x_\alpha}^k \eta_\varepsilon dX + \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \cdot \psi^k (\eta_\varepsilon)_{x_\alpha} dX \\ &\equiv I_1^\varepsilon + I_2^\varepsilon. \end{aligned}$$

By (3.13) and the dominated convergence theorem, $I_1^\varepsilon \rightarrow \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k}(D\mathcal{U}) \cdot \psi_{x_\alpha}^k dX$ as $\varepsilon \rightarrow 0^+$.

For each $\varepsilon > 0$ we have

$$\begin{aligned} |I_2^\varepsilon| &\leq \|\psi\|_{L^\infty(B_1)} \cdot \frac{c_1}{\varepsilon} \cdot \left(\int_{B_{2\varepsilon}} |D\gamma(D\mathcal{U})|^2 dX \right)^{1/2} \cdot |B_{2\varepsilon}|^{1/2}. \\ &\leq c_2 \cdot \|\psi\|_{L^\infty(B_1)} \cdot \left(\int_{B_{2\varepsilon}} |D\gamma(D\mathcal{U})|^2 dX \right)^{1/2} \end{aligned}$$

where c_i is independent of ε . From this and (3.13) we deduce that $I_2^\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Hence

$$0 = \int_{B_1} \frac{\partial \gamma}{\partial P_\alpha^k} (DU) \cdot \psi_{x_\alpha}^k dX \text{ for all } \psi \text{ in } C_c^\infty(B_1; \mathbb{R}^3)$$

which proves (0.3). $\square$

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figure 2.

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