**MAXIMUM PRINCIPLES AND A PRIORI
ESTIMATES FOR AN INCOMPRESSIBLE MATERIAL
IN NONLINEAR ELASTICITY**

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IN NONLINEAR ELASTICITY

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Introduction.

In this paper we derive several a priori estimates for classical solutions to a problem in two-dimensional incompressible nonlinear elasticity.

We consider a homogeneous, incompressible elastic body of neo-Hookean type which occupies a reference region Ω . Here Ω is an open bounded set in $\mathbb{R}^2$ with smooth boundary. Under a deformation of the body, a material point $X = (x, y)$ in Ω is displaced to a point $\mathcal{U}(X) = (u(x, y), v(x, y))$ in $\mathbb{R}^2$. Assuming incompressibility, the deformation gradient $D\mathcal{U}$, which is

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the matrix of partial derivatives of (u, v) , satisfies the volume preserving constraint,

$$(0.1) \quad \det D\mathcal{U} = u_x v_y - u_y v_x = 1 \text{ in } \Omega.$$

For an incompressible neo-Hookean material such as vulcanized rubber, the *Cauchy stress tensor* T is determined up to an indeterminate scalar pressure field p (cf. [8].) The stress tensor is given by

$$T = FF^T - pI,$$

where F is any 2×2 matrix with $\det F = 1$. The *Piola-Kirchhoff* stress tensor $S \equiv TF^{-T}$ is then

$$S = F - pF^{-T};$$

and the equilibrium equation for the deformation $\mathcal{U}$ is

$$\operatorname{div} S = \operatorname{div} (D\mathcal{U}(X) - p(X) \cdot [D\mathcal{U}(X)]^{-T}) = 0.$$

In terms of components we have the system:

$$(0.2) \quad \begin{aligned} \Delta u - v_y p_x + v_x p_y &= 0 & \text{in } \Omega, \\ \Delta v + u_y p_x - u_x p_y &= 0 & \text{in } \Omega. \end{aligned}$$

We define a *classical solution* of (0.1) and (0.2) to be a function $\mathcal{U} = (u, v)$ in $C^3(\Omega; \mathbb{R}^2) \cap C^1(\overline{\Omega}; \mathbb{R}^2)$ such that $\mathcal{U}$ satisfies (0.1) and, for some function p of class $C^2(\Omega; \mathbb{R}^2) \cap C(\overline{\Omega}; \mathbb{R}^2)$, $\mathcal{U}$ and p satisfy (0.2). While many examples of such solutions are known the problem of finding a classical solution taking on suitable prescribed boundary conditions is unsolved.

A variational approach to the above problem is to consider the total stored energy

$$J(\mathcal{U}) = \int_{\Omega} \frac{1}{2} |D\mathcal{U}|^2 dX$$

and find a minimizer for J among the set of all functions $\mathcal{U}$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ satisfying (0.1) almost everywhere and taking on prescribed boundary conditions. J.M. Ball (see [2]) has shown that when this set is nonempty a minimizer exists. Assuming that a given minimizer $\mathcal{U}$ is in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$ with $\mathcal{U}$ a bijection of Ω to $\mathcal{U}(\Omega)$, and assuming that Ω and $\mathcal{U}(\Omega)$ are C^2 domains, P. Le Tallec and J.T. Oden [7] have shown that there exists a hydrostatic pressure p in $W^{1,q}(\Omega)$ such that u, v , and p satisfy (0.1) and (0.2) pointwise almost everywhere in Ω . (Such a function $\mathcal{U}$ is called a *strong solution*.) However, there are no results which state that the minimizer obtained by Ball (under appropriate restrictions on $\Omega, \mathcal{U}(\Omega)$, and the boundary condition for $\mathcal{U}$) possesses any higher regularity.

One step in the study of regularity is to determine estimates which hold *a priori* for sufficiently smooth solutions. Let $\mathcal{U}$ be a classical solution and let $\nu_1(X)$ and $\nu_2(X)$ be the principle stretches of $D\mathcal{U}(X)$, i.e. the eigenvalues of $[D\mathcal{U}(X) \cdot D\mathcal{U}(X)^T]^{1/2}$, where $0 < \nu_1 \leq \nu_2 < \infty$; then

$$(0.3) \quad \begin{aligned} |D\mathcal{U}|^2 &= \nu_1^2 + \nu_2^2 \text{ and} \\ \det D\mathcal{U} &= \nu_1 \nu_2 = 1. \end{aligned}$$

In this paper we show that the pressure p and the function f defined by $f \equiv \frac{1}{2}|D\mathcal{U}|^2 - p$ are subsolutions to certain elliptic equations. As a result, we obtain a maximum principle for p and f , namely

$$\begin{aligned} \sup_{\Omega} p &\leq \sup_{\partial\Omega} p \text{ and} \\ \sup_{\Omega} f &\leq \sup_{\partial\Omega} f. \end{aligned}$$

We also obtain two pointwise estimates on ν_2 and p . The first is a global estimate which states that

$$\sup_{\Omega} \nu_2 + \sup_{\Omega} |p| \leq C_1,$$

where C_1 depends only on $\sup_{\partial\Omega} \nu_2$ and $\sup_{\partial\Omega} |p|$. (Bounds on $\inf_{\Omega} \nu_1$ then follow from (0.3).) The second is an interior estimate which states that

$$(0.4) \quad \sup_{\Omega'} \nu_2^2 + \sup_{\Omega'} |p| \leq C_2 \cdot [1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dX]^{1/2},$$

where $\Omega' \Subset \Omega$ and C_2 depends only on Ω and Ω' . The proof of (0.4) uses a local version of the Aleksandrov maximum principle due to N. Trudinger.

From (0.1), (0.2), and (0.4) we derive a Caccioppoli-type estimate on $D\mathcal{U}$ and p . From this it follows that $|D^2\mathcal{U}|$ and $|\nabla p|$ are in $L_{\text{loc}}^{2+\epsilon}(\Omega)$ for some $\epsilon > 0$ and hence (by Sobolev's theorem), $D\mathcal{U}$ is in $C^\alpha(\Omega')$ for some $\alpha > 0$. The bound on $\|D\mathcal{U}\|_{C^\alpha(\Omega')}$ depends only on $\|p\|_{L^1(\Omega)}$, $\int_{\Omega} |D\mathcal{U}|^{16} dX$, Ω and Ω' .

Next we show that a strong solution $\mathcal{U}$ in $W^{2,q}(\Omega; \mathbb{R}^2)$ with $q > 2$ is a classical solution with u, v and p of class $C^\infty(\Omega)$. This is based on two ideas: first, an observation of H. LeDret [6] that the three equations (0.1) and (0.2) form a nonlinear system in the sense of Agmon, Douglis and Nirenberg for the function (u, v, p) ; and second, an application of elliptic estimates for quasilinear systems. (LeDret proves a higher regularity result in his setting as well.) We prove that $\mathcal{U}$ and p satisfy an a priori estimate of the form

$$\|D\mathcal{U}\|_{C^k(\Omega'')} + \|p\|_{C^k(\Omega'')} \leq C,$$

where $\Omega'' \Subset \Omega' \Subset \Omega$, $k \geq 1$ and C depends on the solution only through the quantities $\|D\mathcal{U}\|_{W^{2,q}(\Omega')}$ and $\|p\|_{W^{1,q}(\Omega')}$. Thus with $q = 2 + \varepsilon$ we get from our previous estimates bounds on $\|D\mathcal{U}\|_{C^k(\Omega'')}$ and $\|p\|_{C^k(\Omega'')}$ in terms of $\|D\mathcal{U}\|_{L^{16}(\Omega)}$ and $\|p\|_{L^1(\Omega)}$. It is our hope that these estimates will help to produce a regularity theory for minimizers of J under suitable assumptions on the domain and boundary data.

The techniques used in this paper are based on those developed in [3] where the compressible problem was studied. The compressible problem involves an elliptic system of the same form as (0.2) but with p replaced by a function of $\det D\mathcal{U}$. Here we rely on the structure of (0.2) as well as the incompressible constraint (0.1).

Our paper is organized as follows: Assume that $\mathcal{U}$ is a classical solution of (0.1) and (0.2). In Section 1, we prove a preliminary L^q -estimate on p , for any $q \in (1, \infty)$, in terms of the L^1 -norm of p and the L^q -norm of $|D\mathcal{U}|^2$. In Section 2, we prove maximum principles for p and f and global estimates on ν_2 and $|p|$. In Section 3, we derive interior L^∞ -estimates on p and f , and hence on ν_2 and $|p|$. In Section 4, we prove a Caccioppoli-type estimate on $D\mathcal{U}$ and p from which we derive a Hölder estimate on $D\mathcal{U}$. Finally, in Section 5 we show that a strong solution $\mathcal{V}$ in $W^{2,q}(\Omega; \mathbb{R}^2)$ with $q > 2$ is of class C^∞ and we prove higher order estimates for such a solution.

Notation: We denote the open ball of radius r with center X_0 in $\mathbb{R}^2$ by $B_r(X_0)$, or, when there is no ambiguity, B_r . The usual inner product on $\mathbb{R}^2$ is denoted by $\langle \cdot, \cdot \rangle$ and the space of Hölder continuous functions on Ω' with exponent α is denoted by $C^\alpha(\Omega')$.

§1. A preliminary estimate.

Assume that $\mathcal{U}$ is a classical solution of (0.1) and (0.2). In this section we derive an L^q -estimate, with $1 < q < \infty$, for the hydrostatic pressure p in terms of $\|p\|_{L^1(\Omega)}$ and $\| |D\mathcal{U}|^2 \|_{L^q(\Omega)}$. To do this we first derive an equation for p . Solving (0.2) for p_x and p_y gives

$$\begin{aligned} p_x &= u_x \Delta u + v_x \Delta v, \\ p_y &= u_y \Delta u + v_y \Delta v, \end{aligned}$$

from which it follows that

$$\begin{aligned} (1.1) \quad \Delta p &= u_{xx} \Delta u + v_{xx} \Delta v + u_{yy} \Delta u + v_{yy} \Delta v \\ &\quad + u_x \Delta u_x + v_x \Delta v_x + u_y \Delta u_y + v_y \Delta v_y. \end{aligned}$$

Now

$$\begin{aligned}\Delta\left(\frac{1}{2}|D\mathcal{U}|^2\right) &= u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2 + v_{xx}^2 + 2v_{xy}^2 + v_{yy}^2 \\ &\quad + u_x\Delta u_x + u_y\Delta u_y + v_x\Delta v_x + v_y\Delta v_y.\end{aligned}$$

Substituting this into (1.1) and rearranging gives

$$(1.2) \quad \Delta p = \Delta\left(\frac{1}{2}|D\mathcal{U}|^2\right) - 2(u_{xy}^2 - u_{xx}u_{yy}) - 2(v_{xy}^2 - v_{xx}v_{yy}).$$

Finally, using the identity

$$2(u_{xy}^2 - u_{xx}u_{yy}) + 2(v_{xy}^2 - v_{xx}v_{yy}) = (u_x^2 + v_x^2)_{yy} - 2(u_xu_y + v_xv_y)_{xy} + (u_y^2 + v_y^2)_{xx},$$

we deduce that

$$(1.3) \quad \Delta p = \Delta\left(\frac{1}{2}|D\mathcal{U}|^2\right) - (u_y^2 + v_y^2)_{xx} + 2(u_xu_y + v_xv_y)_{xy} - (u_x^2 + v_x^2)_{yy}.$$

THEOREM 1.1. *Assume that $\mathcal{U}$ is a classical solution of (0.1) and (0.2) and $\Omega' \Subset \Omega$. For $1 < q < \infty$, there exists a constant $c = c(\Omega', q)$ such that*

$$\|p\|_{L^q(\Omega')} \leq C \cdot [\|p\|_{L^1(\Omega)} + \| |D\mathcal{U}|^2 \|_{L^q(\Omega)}].$$

The proof of this theorem follows closely that of Theorem 2.2 in [3]. However, for completeness we shall give the proof which is based on a duality argument.

PROOF: Fix $X_0 \in \Omega$ and let $r > 0$ be such that $B_{3r}(X_0) \Subset \Omega$. Equation (1.3) is of the form

$$(1.4) \quad \Delta p = \sum_{i,j=1}^2 (h_{ij})_{x_i x_j},$$

where $x_1 = x$, $x_2 = y$ and $|h_{ij}| \leq |D\mathcal{U}|^2$, $1 \leq i, j \leq 2$.

Let η be a standard cut-off function, i.e. $\eta \in C_0^\infty(B_{3r})$ with $0 \leq \eta \leq 1$, $\eta \equiv 1$ on B_{2r} , $|\nabla \eta| \leq \frac{M}{r}$, $|D^2 \eta| \leq \frac{M}{r^2}$, where M is independent of r . For $g \in C_0^\infty(B_r)$, set

$$w(X) = \frac{1}{2\pi} \int_{B_r} \log |X - Z| g(Z) dZ$$

so that $\Delta w = g$ in B_r and $\Delta w = 0$ in $\mathbb{R}^2 \setminus B_r$. Then

$$\begin{aligned}
\int_{B_r} pg \, dX &= \int_{\mathbb{R}^2} p\eta \Delta w \, dX = \int_{\mathbb{R}^2} [-pw\Delta\eta - 2p\langle \nabla\eta, \nabla w \rangle + p \cdot \Delta(\eta w)] dX \\
&= - \int_{B_{3r} \setminus B_{2r}} pw\Delta\eta \, dX - 2 \int_{B_{3r} \setminus B_{2r}} p\langle \nabla\eta, \nabla w \rangle dX \\
&\quad + \int_{B_{3r}} \left(\sum_{i,j=1}^2 h_{ij} \cdot (\eta w)_{x_i x_j} \right) dX \\
&= \int_{B_{3r} \setminus B_{2r}} \left\{ -pw\Delta\eta - 2p\langle \nabla\eta, \nabla w \rangle \right. \\
&\quad \left. + \sum_{i,j=1}^2 h_{ij}(w\eta_{x_i x_j} + w_{x_i} \eta_{x_j} + w_{x_j} \eta_{x_i}) \right\} dX \\
&\quad + \int_{B_{3r}} \eta \sum_{i,j=1}^2 h_{ij} w_{x_i x_j} dX \\
&\equiv I + II,
\end{aligned}$$

where we have used (1.4) in integrating by parts the term $p \cdot \Delta(\eta w)$.

By Hölder's inequality and the definition of w ,

$$|w| + |\nabla w| \leq c_1(r, q) \cdot \|g\|_{L^{q'}(B_{3r})} \text{ in } B_{3r} \setminus B_{2r},$$

where $\frac{1}{q} + \frac{1}{q'} = 1$. Combining this with (1.4) and the definition of I gives

$$(1.5) \quad |I| \leq c_2 \cdot [\|p\|_{L^1(B_{3r})} + \| |D\mathcal{U}|^2 \|_{L^1(B_{3r})}] \cdot \|g\|_{L^{q'}(B_r)},$$

where $c_2 = c_2(r, q)$. To estimate II , we use the Calderon–Zygmund inequality ([5]),

$$\|D^2 w\|_{L^{q'}(\mathbb{R}^2)} \leq c_3 \cdot \|g\|_{L^{q'}(\mathbb{R}^2)} = c_3 \cdot \|g\|_{L^{q'}(B_r)},$$

to give

$$\begin{aligned}
(1.6) \quad |II| &\leq \|\eta\|_{L^\infty(B_{3r})} \cdot \sum_{i,j=1}^2 \|h_{ij}\|_{L^q(B_{3r})} \cdot \|D^2 w\|_{L^{q'}(\mathbb{R}^2)} \\
&\leq c_4 \cdot \| |D\mathcal{U}|^2 \|_{L^q(B_{3r})} \cdot \|g\|_{L^{q'}(B_r)},
\end{aligned}$$

where $c_3 = c_3(q)$ and $c_4 = c_4(q)$.

From (1.5) and (1.6),

$$(1.7) \quad \left| \int_{B_r} pg \, dX \right| \leq c_5 \cdot [\|p\|_{L^1(B_{3r})} + \| |D\mathcal{U}|^2 \|_{L^q(B_{3r})}] \cdot \|g\|_{L^{q'}(B_r)},$$

where $c_5 = c_5(r, q)$. The result now follows since (1.7) implies that the mapping $g \mapsto \int_{B_r} pg \, dX$ is a bounded linear operator on $L^{q'}(B_r)$. $\blacksquare$

§2. Upper bounds for $|D\mathcal{U}|$ and $|p|$.

Here we show that the pressure $p(X)$ and the function

$$(2.1) \quad f(X) \equiv \frac{1}{2}|D\mathcal{U}(X)|^2 - p(X)$$

satisfy strong maximum principles. We begin by proving that f is a subsolution of the elliptic equation $\Delta f + \frac{1}{4}|D\mathcal{U}|^2|\nabla f|^2 = 0$.

PROPOSITION 2.1. *Assume that $\mathcal{U}$ is a classical solution. Then f , defined by (2.1), satisfies*

$$\Delta f + \frac{1}{4}|D\mathcal{U}|^2|\nabla f|^2 \geq 0.$$

PROOF: Fix $X_0 \in \Omega$ and choose r such that $B_r(X_0) \subset \Omega$. From (1.2),

$$(2.2) \quad \Delta f = 2(v_{xy}^2 - u_{xx}u_{yy}) + 2(u_{xy}^2 - v_{xx}v_{yy}).$$

By isotropy and the principle of frame-indifference, we may assume without loss of generality that $D\mathcal{U}(X_0)$ is diagonal, i.e.

$$(2.3) \quad D\mathcal{U}(X_0) = \begin{bmatrix} u_x(X_0) & 0 \\ 0 & v_y(X_0) \end{bmatrix},$$

with $v_y(X_0) \geq u_x(X_0) > 0$ and

$$(2.4) \quad u_x(X_0) \cdot v_y(X_0) = 1.$$

This follows from a change of variables since $\mathcal{U}$ is a smooth solution of (0.2), which is the Euler–Lagrange equation for the functional

$$I(\mathcal{U}) = I(\mathcal{U}; p) \equiv \int_{B_r(X_0)} \left\{ \frac{1}{2}|D\mathcal{U}|^2 - p(X) \cdot (\det D\mathcal{U} - 1) \right\} dX.$$

More precisely, observe that $I(\mathcal{U})$ is invariant under rotations. In fact, if P and Q are in $SO(2)$ with $\det P = \det Q = 1$ and $X - X_0 = Q(X' - X_0)$, then the function defined by

$$\tilde{\mathcal{U}}(X') \equiv P \cdot \mathcal{U}(X_0 + Q(X' - X_0)) = P \cdot \mathcal{U}(X) \text{ for } X' \in B_r(X_0)$$

satisfies $I(\mathcal{U}; p) = I(\tilde{\mathcal{U}}; \tilde{p})$ where $\tilde{p}(X') = p(X)$. Thus $(\tilde{\mathcal{U}}, \tilde{p})$ satisfies (0.1) and (0.2) in $B_r(X_0)$. For appropriate choices of P and Q , $D\tilde{\mathcal{U}}$ has the desired properties at X_0 . Indeed, we have a polar decomposition $D\mathcal{U}(X_0) = CR$ where C is symmetric and positive definite, and $R \in SO(2)$. Hence $D\mathcal{U}(X_0) = P^T \Lambda P R$ where $P \in SO(2)$, Λ is diagonal, and we may take $Q = R^T P^T$. (See [3], Section 3, for more details on a similar analysis.)

To estimate the first term on the right-hand side of (2.2) we first note that at X_0 ,

$$f_x = u_x u_{xx} + v_y v_{xy} - p_x,$$

which we rewrite as

$$u_x u_{xx} - p_x = f_x - v_y v_{xy}.$$

Multiplying both sides by $u_{xx} v_y$ and using (2.4), we obtain

$$(2.5) \quad u_{xx}^2 - u_{xx} v_y p_x = u_{xx} v_y f_x - v_y^2 u_{xx} v_{xy} \text{ at } X_0.$$

Differentiating (0.1) with respect to x and setting $(x, y) = X_0$ gives

$$(2.6) \quad u_{xx} v_y + u_x v_{xy} = 0,$$

while from (0.2),

$$(2.7) \quad u_{xx} + u_{yy} = p_x v_y \text{ at } X_0.$$

Hence from (2.5), (2.6), and (2.7) we obtain

$$\begin{aligned} v_{xy}^2 - u_{xx} u_{yy} &= v_{xy}^2 - u_{xx}(-u_{xx} + p_x v_y) \\ &= 2v_{xy}^2 - u_x f_x v_{xy} \\ &= 2(v_{xy} - \frac{1}{4} u_x f_x)^2 - \frac{1}{8} u_x^2 f_x^2 \\ (2.8) \quad &\geq -\frac{1}{8} |\nabla u|^2 f_x^2 \text{ at } X_0. \end{aligned}$$

By an analogous calculation,

$$(2.9) \quad u_{xy}^2 - v_{xx} v_{yy} = 2u_{xy}^2 - v_y f_y u_{xy} \geq -\frac{1}{8} |\nabla v|^2 f_y^2 \text{ at } X_0.$$

Thus, from (2.2), (2.8) and (2.9),

$$\Delta f \geq -\frac{1}{4} |\nabla u|^2 f_x^2 - \frac{1}{4} |\nabla v|^2 f_y^2 \geq -\frac{1}{4} |D\mathcal{U}|^2 |\nabla f|^2$$

at X_0 . ■

As an immediate consequence of Proposition 2.1 we have

THEOREM 2.2. Assume that $\mathcal{U}$ is a classical solution. Then the function f satisfies the strong maximum principle:

$$f(X) \leq \sup_{\partial\Omega} f \quad \forall X \in \Omega,$$

with equality holding if and only if $f \equiv \text{constant}$. In addition, if $f \equiv \text{constant}$, then $\mathcal{U}(X) = AX + b$ for some A and b , with $\det A = 1$.

PROOF: Since $\mathcal{U}$ is smooth, $|D\mathcal{U}|$ and $|\nabla f|$ are locally bounded in Ω and the strong maximum principle follows.

Suppose $f \equiv \text{constant}$ in Ω , so that $f_x \equiv f_y \equiv 0$. Fix $X_0 \in \Omega$ and assume, without loss of generality, that $D\mathcal{U}(X_0)$ is diagonal. From (2.8) we have

$$v_{xy}^2 - u_{xx}u_{yy} = 2v_{xy}^2 \text{ at } X_0.$$

Similarly,

$$u_{xy}^2 - v_{xx}v_{yy} = 2u_{xy}^2 \text{ at } X_0$$

(by (2.9)). Substituting these expressions into (2.2) gives $0 = \Delta f = 4u_{xy}^2 + 4v_{xy}^2$ at X_0 , so that $u_{xy} = v_{xy} = 0$ at X_0 . From (2.6), $u_{xx} = 0$ at X_0 , while differentiating (0.1) with respect to y at X_0 implies that $v_{yy} = 0$. Finally, the expressions for f_x and f_y imply that $p_x = p_y = 0$ at X_0 which, combined with (0.2), gives $u_{yy} = v_{xx} = 0$ at X_0 . Hence $D^2\mathcal{U}(X_0) = 0$. $\blacksquare$

Next we derive a maximum principle for $p(X)$. To do this we introduce two linear operators F_1 and F_2 defined by

$$\begin{aligned} F_1(w) &= (v_y w)_x - (v_x w)_y, \\ F_2(w) &= (u_x w)_y - (u_y w)_x. \end{aligned}$$

Applying F_1 to (0.2)₁, F_2 to (0.2)₂, and adding we obtain

$$\begin{aligned} &v_y \Delta u_x + u_x \Delta v_y - v_x \Delta u_y - u_y \Delta v_x = [(u_y^2 + v_y^2)p_x]_x \\ (2.10) \quad &-[(u_x u_y + v_x v_y)p_y]_x - [(u_x u_y + v_x v_y)p_x]_y + [(u_x^2 + v_x^2)p_y]_y. \end{aligned}$$

From (0.1),

$$\begin{aligned} 0 = \Delta(u_x v_y - v_x u_y) &= v_y \Delta u_x + u_x \Delta v_y - v_x \Delta u_y - u_y \Delta v_x \\ &\quad + 2\{\langle \nabla u_x, \nabla v_y \rangle - \langle \nabla u_y, \nabla v_x \rangle\}. \end{aligned}$$

Thus, (2.10) becomes

$$(2.11) \quad L_1(p) = 2\{\langle \nabla u_y, \nabla v_x \rangle - \langle \nabla u_x, \nabla v_y \rangle\},$$

where the operator $L_1(p)$ is defined by

$$(2.12) \quad L_1(p) = [(u_y^2 + v_y^2)p_x]_x - [(u_x u_y + v_x v_y)p_y]_x - [(u_x u_y + v_x v_y)p_x]_y + [(u_x^2 + v_x^2)p_y]_y.$$

It is straightforward to check that L_1 is a uniformly elliptic operator. Indeed, if we write $L_1(p) = \sum_{i,j=1}^2 (a_{ij}(X)p_{x_i})_{x_j}$, where $x_1 = x$, $x_2 = y$ and

$$(2.13) \quad [a_{ij}] = \begin{bmatrix} u_y^2 + v_y^2 & -(u_x u_y + v_x v_y) \\ -(u_x u_y + v_x v_y) & u_x^2 + v_x^2 \end{bmatrix},$$

then the eigenvalues of $[a_{ij}]$ are ν_1^2 and ν_2^2 , where ν_1 and ν_2 are the principal stretches (cf. (0.3)).

PROPOSITION 2.3. *Assume that $\mathcal{U}$ is a classical solution. Then $p(X)$ satisfies*

$$L_1(p) + 18\nu_2^4 \cdot |\nabla p|^2 \geq \frac{\nu_1^2}{4} |D^2 \mathcal{U}|^2$$

(i.e. p is a subsolution of the equation $L_1(p) + 18\nu_2^4 \cdot |\nabla p|^2 = 0$).

PROOF: Fix $X_0 \in \Omega$ and choose r so that $B_r(X_0) \subset \Omega$. Let I denote the right-hand side of (2.11). Then

$$L_1(p) = I \equiv 2\{\langle \nabla u_y, \nabla v_x \rangle - \langle \nabla u_x, \nabla v_y \rangle\}.$$

In [3] it is shown that for $\tilde{\mathcal{U}}$ as defined in the proof of Proposition (2.1) we have

$$\langle \nabla u_y, \nabla v_x \rangle - \langle \nabla u_x, \nabla v_y \rangle = \langle \nabla \tilde{u}_{y'}, \nabla \tilde{v}_{x'} \rangle - \langle \nabla \tilde{u}_{x'}, \nabla \tilde{v}_{y'} \rangle$$

where $X' = (x', y')$. Moreover, $|D^2 \mathcal{U}|^2 = |D^2 \tilde{\mathcal{U}}|^2$ and, by definition of $\tilde{p}$, $|\nabla \tilde{p}| = |\nabla p|$. Thus to obtain the desired inequality we may assume without loss of generality that $D\mathcal{U}(X_0)$ is a diagonal matrix of the form (2.3). From this assumption we have

$$(2.14) \quad \begin{aligned} I &= 2\{\langle \nabla u_y, \nabla v_x \rangle + \langle \nabla u_x, \nabla v_y \rangle\} - 4\langle \nabla u_x, \nabla v_y \rangle \\ &= 2\{u_{xy}v_{xx} + u_{yy}v_{xy} + u_{xx}v_{xy} + u_{xy}v_{yy}\} - 4\langle \nabla u_x, \nabla v_y \rangle \\ &= 2 \cdot \{v_{xy}\Delta u + u_{xy}\Delta v\} - 4\langle \nabla u_x, \nabla v_y \rangle \\ &= 2\{v_{xy}v_y p_x + u_{xy}u_x p_y\} - 4\langle \nabla u_x, \nabla v_y \rangle \end{aligned}$$

at X_0 , by (0.2).

To estimate this term from below we first take the gradient of (0.1):

$$u_x \nabla v_y + v_y \nabla u_x - v_x \nabla u_y - u_y \nabla v_x = 0.$$

At X_0 , this reduces to

$$(2.15) \quad u_x \nabla v_y = -v_y \nabla u_x.$$

Taking the scalar product of (2.15) with $\nabla u_x(X_0)$ and using (2.3), we obtain

$$\langle \nabla v_y, \nabla u_x \rangle = -\frac{v_y}{u_x} |\nabla u_x|^2 = -v_y^2 |\nabla u_x|^2 \text{ at } X_0.$$

Similarly, by taking the scalar product of (2.15) with $\nabla v_y(X_0)$, we obtain

$$\langle \nabla v_y, \nabla u_x \rangle = -u_x^2 |\nabla v_y|^2 \text{ at } X_0.$$

Thus,

$$(2.16) \quad 2\langle \nabla v_y, \nabla u_x \rangle = -u_x^2 |\nabla v_y|^2 - v_y^2 |\nabla u_x|^2 \text{ at } X_0.$$

Next, we note that at X_0 , (0.2)₁ implies that

$$u_{yy} = -u_{xx} + p_x v_y,$$

and therefore

$$u_{yy}^2 \leq 2u_{xx}^2 + 2p_x^2 v_y^2.$$

Thus,

$$(2.17) \quad \begin{aligned} |\nabla u_x|^2 &\geq \frac{|\nabla u_x|^2}{2} + \frac{u_{xx}^2}{2} \\ &\geq \frac{|\nabla u_x|^2}{2} + \frac{u_{yy}^2}{4} - \frac{p_x^2 v_y^2}{2} \\ &\geq \frac{|D^2 u|^2}{4} - |\nabla p|^2 \frac{v_y^2}{2} \end{aligned}$$

at X_0 . By a similar calculation,

$$(2.18) \quad |\nabla v_y|^2 \geq \frac{|D^2 v|^2}{4} - |\nabla p|^2 \frac{u_x^2}{2}$$

at X_0 .

From (2.14), (2.16), (2.17) and (2.18), and since $u_x(X_0) \leq v_y(X_0)$, we obtain

$$\begin{aligned} I &\geq 2\{v_{xy}v_y p_x + u_{xy}u_x p_y\} + u_x^2 \frac{|D^2 v|^2}{2} + v_y^2 \frac{|D^2 u|^2}{2} - (u_x^4 + v_y^4) |\nabla p|^2 \\ &\geq 2\{v_{xy}v_y p_x + u_{xy}u_x p_y\} + u_x^2 \frac{|D^2 \mathcal{U}|^2}{2} - 2v_y^4 |\nabla p|^2 \text{ at } X_0. \end{aligned}$$

The term in brackets is dominated by $v_y |D^2 v| |\nabla p| + u_x |D^2 u| |\nabla p|$, so that

$$\begin{aligned}
 I &\geq -2(v_y |D^2 v| + u_x |D^2 u|) |\nabla p| + u_x^2 \frac{|D^2 \mathcal{U}|^2}{2} - 2v_y^4 |\nabla p|^2 \\
 (2.19) \quad &\geq -4v_y |D^2 \mathcal{U}| |\nabla p| + u_x^2 \frac{|D^2 \mathcal{U}|^2}{2} - 2v_y^4 |\nabla p|^2 \text{ at } X_0.
 \end{aligned}$$

By Young's inequality,

$$4v_y |D^2 \mathcal{U}| |\nabla p| \leq 2\{\epsilon |D^2 \mathcal{U}|^2 + \epsilon^{-1} v_y^2 |\nabla p|^2\}$$

for any $\epsilon > 0$. Setting $\epsilon = \frac{u_x^2}{8}$ and substituting into (2.19) gives

$$\begin{aligned}
 I &\geq -u_x^2 \frac{|D^2 \mathcal{U}|^2}{4} - \frac{16}{u_x^2} v_y^2 |\nabla p|^2 + u_x^2 \frac{|D^2 \mathcal{U}|^2}{2} - 2v_y^4 |\nabla p|^2 \\
 &\geq \frac{1}{4} u_x^2 |D^2 \mathcal{U}|^2 - 18v_y^4 |\nabla p|^2
 \end{aligned}$$

at X_0 , by (2.3).

To complete the proof we note that $\nu_1 = u_x$, $\nu_2 = v_y$ at X_0 . ■

As an immediate consequence of Proposition 2.3 we have:

THEOREM 2.4. *Assume that $\mathcal{U}$ is a classical solution. Then p satisfies the strong maximum principle,*

$$p(X) \leq \sup_{\partial\Omega} p \quad \forall X \in \Omega,$$

with equality if and only if $p \equiv \text{constant}$. In addition, if $p \equiv \text{constant}$, then $\mathcal{U} = AX + b$, for some A and b , with $\det A = 1$.

We may use Theorems 2.2 and 2.4 to obtain bounds on ν_2 and p in terms of their values on the boundary:

THEOREM 2.5. *Assume that $\mathcal{U}$ is a classical solution. There exists a positive constant θ , depending only on $\sup_{\partial\Omega} \nu_2$ and $\sup_{\partial\Omega} |p|$, such that*

$$(2.20) \quad \sup_{\Omega} \nu_2 \leq \theta, \quad \sup_{\Omega} |p| \leq \theta.$$

PROOF: Let $\hat{p} = \sup_{\Omega} |p|$ and $\hat{f} = \sup_{\Omega} f$. By (0.3)₁ and (2.1),

$$\frac{\nu_2^2}{2} \leq \frac{|D\mathcal{U}|^2}{2} = f + p \leq \hat{f} + p,$$

and from Theorem 2.2,

$$\hat{f} \leq \left(\sup_{\partial\Omega} \nu_2 \right)^2 + \sup_{\partial\Omega} |p|.$$

Therefore,

$$\begin{aligned} \frac{\nu_2^2}{2} &\leq \left(\sup_{\partial\Omega} \nu_2 \right)^2 + \sup_{\partial\Omega} |p| + p \\ &\leq \left(\sup_{\partial\Omega} \nu_2 \right)^2 + \sup_{\partial\Omega} |p| + \sup_{\partial\Omega} p \end{aligned}$$

by Theorem 2.4, which proves (2.20)₁.

Theorem 2.4 gives an upper bound on p . For a lower bound we note that by Theorem 2.2,

$$\frac{1}{2}|D\mathcal{U}|^2 - p \leq \sup_{\partial\Omega} \left(\frac{1}{2}|D\mathcal{U}|^2 - p \right) \leq \sup_{\partial\Omega} \left(\frac{1}{2}|D\mathcal{U}|^2 \right) + \sup_{\partial\Omega} |p|,$$

which implies that

$$-p \leq \sup_{\partial\Omega} \left(\frac{1}{2}|D\mathcal{U}|^2 \right) + \sup_{\partial\Omega} |p| \leq \left(\sup_{\partial\Omega} \nu_2 \right)^2 + \sup_{\partial\Omega} |p|.$$

Hence (2.20)₂ is proved. ■

§3. Interior estimates for f and p .

The aim of this section is to establish L^∞ -estimates for f and p on domains $\Omega' \Subset \Omega$. To do this we use a result by Trudinger [9], which is an extension of the Aleksandrov maximum principle ([5]) to local estimates for nonlinear elliptic equations.

We begin by stating an application of Trudinger's result. Let $[b_{ij}(X)]$ be a symmetric, positive definite, 2×2 matrix defined on a domain $B_{2r} \equiv B_{2r}(X_0) \subset \mathbb{R}^2$. Let $\mathcal{B}(X) = \det[b_{ij}(X)]$ and let $b(X)$ be the largest eigenvalue of $[b_{ij}(X)]$. Suppose $w \in C^2(B_{2r}) \cap C(\overline{B_{2r}})$ and w satisfies

$$(3.1) \quad \sum_{i,j=1}^2 b_{ij} w_{x_i x_j} \geq g(X) \cdot |\nabla w|^2$$

in B_{2r} . Let $\varphi = \eta w$ where $\eta(X) = \left(4 - \left| \frac{X - X_0}{r} \right|^2 \right)^2$. Then $\varphi \in C^2(B_{2r}) \cap C_0(\overline{B_{2r}})$ and

$$\sum_{i,j=1}^n b_{ij} \varphi_{x_i x_j} \geq \eta g |\nabla w|^2 + w \sum_{i,j=1}^2 b_{ij} \eta_{x_i x_j} + 2 \sum_{i,j=1}^2 b_{ij} \eta_{x_i} w_{x_j}$$

in B_{2r} . Now

$$|\nabla \eta| \leq \frac{c}{r} \eta^{1/2} \text{ and } |D^2 \eta| \leq \frac{c}{r^2}$$

in B_{2r} where c is a universal constant. Moreover, Trudinger observed that

$$|\nabla w| \leq \frac{c}{r} \cdot \eta^{-1/2} w \text{ on } \Gamma(\varphi),$$

where $\Gamma(\varphi)$ is the *upper contact set* of φ in B_{2r} defined by

$$\begin{aligned} \Gamma(\varphi) = \{Y \in B_{2r} : \varphi(X) \leq \varphi(Y) + \langle P, X - Y \rangle \\ \text{for all } X \in B_{2r} \text{ and some } P = P(Y) \in \mathbb{R}^2\}. \end{aligned}$$

(See inequality (10) of [9].) By (3.1) and the above inequalities, we have

$$\sum_{i,j=1}^2 b_{ij} \varphi_{x_i x_j} \geq -\frac{c}{r^2} (|g|w^2 + bw) \text{ on } \Gamma(\varphi).$$

Hence by the Aleksandrov maximum principle,

$$\sup_{B_{2r}} \varphi \leq \frac{c}{r} \left(\int_{\Gamma(\varphi)} \frac{(|g|w^2 + bw)^2}{\mathcal{B}} dX \right)^{1/2}.$$

Since $\varphi = \eta w$, we obtain:

$$\begin{aligned} (3.2) \quad \sup_{B_r} w &\leq \frac{c}{r} \left(\int_{\Gamma(\varphi)} \frac{(|g|w^2 + bw)^2}{\mathcal{B}} dX \right)^{1/2} \\ &\leq \frac{c}{r} \left(\int_{\Gamma(\varphi)} \frac{(g^2 w^4 + b^2 w^2)}{\mathcal{B}} dX \right)^{1/2} \end{aligned}$$

where c is a universal constant.

We can apply (3.2) with $w = f$ since, by Proposition 2.1,

$$\Delta f \geq -\frac{1}{4} |D\mathcal{U}|^2 |\nabla f|^2$$

which is of the form (3.1) with $[b_{ij}] = [\delta_{ij}]$ and $g = -\frac{1}{4} |D\mathcal{U}|^2$. Hence

$$(3.3) \quad \sup_{B_r} f \leq \frac{c}{r} \left[\int_{B_{2r}} \left(\frac{1}{16} |D\mathcal{U}|^4 f^4 + f^2 \right) dX \right]^{\frac{1}{2}}$$

by (3.2). Combining this inequality with the estimate in Theorem 1.1, we obtain:

PROPOSITION 3.1. Assume that $\mathcal{U}$ is a classical solution and $\Omega' \Subset \Omega$. Then

$$\sup_{\Omega'} f \leq C_1 \left[1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dX \right]^{\frac{1}{2}},$$

where $C_1 = C_1(\Omega, \Omega')$.

PROOF: Fix $X_0 \in \Omega'$ and choose r such that $B_{6r}(X_0) \subset \Omega$. Let $f^+ = \max\{f, 0\}$. From (3.3),

$$\begin{aligned} \left(\sup_{B_r} f^+ \right)^2 &\leq \frac{c_2}{r^2} \int_{B_{2r}} \left[\frac{1}{16} |D\mathcal{U}|^4 \left(\frac{1}{2} |D\mathcal{U}|^2 - p \right)^4 + \left(\frac{1}{2} |D\mathcal{U}|^2 - p \right)^2 \right] dX \\ &\leq \frac{c_3}{r^2} \int_{B_{2r}} [|D\mathcal{U}|^{12} + |D\mathcal{U}|^4 p^4 + |D\mathcal{U}|^4 + p^2] dX \\ &\leq \frac{c_4}{r^2} \int_{B_{2r}} [1 + |D\mathcal{U}|^{12} + p^8] dX, \end{aligned}$$

where c_2, c_3 and c_4 are universal constants. From Theorem 1.1,

$$\|p\|_{L^8(B_{2r})}^8 \leq c \left\{ 1 + \|p\|_{L^1(B_{6r})}^8 + \int_{B_{6r}} |D\mathcal{U}|^{16} dX \right\},$$

and the result follows. ■

We use the same technique to estimate p from above.

PROPOSITION 3.2. Assume that $\mathcal{U}$ is a classical solution and $\Omega' \Subset \Omega$. Then

$$\sup_{\Omega'} p \leq C \left[1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dX \right]^{\frac{1}{2}},$$

where $C = C(\Omega, \Omega')$.

PROOF: Define an elliptic operator L_2 by

$$L_2(p) = \sum_{i,j=1}^2 a_{ij}(X) p_{x_i x_j},$$

where $[a_{ij}]$ is given by (2.13). Then

$$\begin{aligned} |L_1(p) - L_2(p)| &= \left| \sum_{i,j=1}^2 (a_{ij})_{x_j} p_{x_i} \right| \\ &\leq c |D\mathcal{U}| \cdot |D^2\mathcal{U}| \cdot |\nabla p| \\ &\leq c \left(\frac{\epsilon |D^2\mathcal{U}|^2}{2} + \frac{|D\mathcal{U}|^2 |\nabla p|^2}{2\epsilon} \right) \end{aligned}$$

for any $\epsilon > 0$ by Young's inequality. Setting $\epsilon = \frac{\nu_1^2}{2c}$ we obtain

$$|L_1(p) - L_2(p)| \leq \frac{\nu_1^2}{4} |D^2\mathcal{U}|^2 + c^2 \nu_2^2 |D\mathcal{U}|^2 |\nabla p|^2.$$

It follows from Proposition 2.3 that

$$\begin{aligned} L_2(p) &\geq L_1(p) - \frac{\nu_1^2}{4} |D^2\mathcal{U}|^2 - c^2 \nu_2^2 |D\mathcal{U}|^2 |\nabla p|^2 \\ &\geq -18\nu_2^4 |\nabla p|^2 - c^2 \nu_2^2 |D\mathcal{U}|^2 |\nabla p|^2 \\ &\geq -M |D\mathcal{U}|^4 |\nabla p|^2 \end{aligned}$$

for some constant $M > 0$. Thus,

$$\sum_{i,j=1}^2 a_{ij} p_{x_i x_j} \geq -M |D\mathcal{U}|^4 |\nabla p|^2,$$

which is in the form (3.1) with $[b_{ij}] = [a_{ij}]$ and $g = -M |D\mathcal{U}|^4$. Consequently $\mathcal{B} = 1$, $b = \nu_2^2$, and from (3.2),

$$\sup_{B_r} p \leq \frac{c}{r} \left[\int_{B_r} (M^2 |D\mathcal{U}|^8 p^4 + |D\mathcal{U}|^4 p^2) dX \right]^{\frac{1}{2}}.$$

Let $p^+ = \max(p, 0)$. Then

$$\left(\sup_{B_r} p^+ \right)^2 \leq \frac{c'}{r^2} \int_{B_{2r}} (1 + |D\mathcal{U}|^{16} + p^8) dX. \quad \blacksquare$$

From these two propositions we get

THEOREM 3.3. *Assume that $\mathcal{U}$ is a classical solution and $\Omega' \Subset \Omega$. Then*

$$\sup_{\Omega'} \nu_2^2 + \sup_{\Omega'} |p| \leq C \left[1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dx \right]^{\frac{1}{2}}$$

where $C = C(\Omega, \Omega')$.

PROOF: Let $a = \left[1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dX \right]^{1/2}$. By Propositions 3.1 and 3.2, there is a constant c , depending only on Ω and Ω' , such that

$$\sup_{\Omega'} f \leq ac \text{ and } \sup_{\Omega'} p \leq ac.$$

Hence

$$\frac{1}{2} |D\mathcal{U}|^2 = f + p \leq 2ac$$

in Ω' . Furthermore,

$$-p \leq f \leq ac$$

in Ω' and the assertion follows. $\blacksquare$

§4. Hölder estimates.

In this section we prove the following Hölder estimate:

THEOREM 4.1. *Assume that $\mathcal{U}$ is a classical solution and $\Omega' \Subset \Omega$. Then*

$$\|D\mathcal{U}\|_{C^\alpha(\Omega')} + \|p\|_{C^\alpha(\Omega')} \leq C,$$

where $0 < \alpha < 1$ and $C > 0$ are constants depending only on Ω , Ω' , $\|p\|_{L^1(\Omega)}$ and $\|D\mathcal{U}\|_{L^{16}(\Omega)}$.

The proof of this theorem is based on the following *Caccioppoli inequality* for $D\mathcal{U}$ and p . Let $\Omega' \Subset \Omega'' \Subset \Omega$ and let $\hat{\nu} = \sup_{\Omega''} \nu_2$, $\hat{p} = \sup_{\Omega''} |p|$, and $a = [1 + \|p\|_{L^1(\Omega)}^8 + \int_{\Omega} |D\mathcal{U}|^{16} dX]^{1/2}$. Let $\bar{F}_{X_0, r}$ be the average of F over B_r , i.e.

$$\bar{F}_{X_0, r} = \fint_{B_r(X_0)} F dX = \frac{1}{|B_r(X_0)|} \int_{B_r(X_0)} F dX.$$

LEMMA 4.2. (*Caccioppoli inequality*) *Fix $X_0 \in \Omega'$ and choose r such that $B_{2r}(X_0) \subset \Omega''$. There is a constant $c = c_1(\hat{\nu}, \hat{p}) = c_2(a, \Omega'', \Omega)$ such that*

$$\int_{B_r} \{ |D^2\mathcal{U}|^2 + |\nabla p|^2 \} dX \leq \frac{c}{r^2} \int_{B_{2r}} \{ |D\mathcal{U} - \overline{D\mathcal{U}}_{X_0, 2r}|^2 + |p - \bar{p}_{X_0, 2r}|^2 \} dX.$$

PROOF OF THEOREM 4.1: Without loss of generality, we assume that Ω' is a Lipschitz domain. Choose Ω'' such that $\Omega' \Subset \Omega'' \Subset \Omega$. By the Sobolev–Poincaré inequality,

$$\int_{B_{2r}} \{ |D\mathcal{U} - \overline{D\mathcal{U}}_{X_0, 2r}|^2 + |p - \bar{p}_{X_0, 2r}|^2 \} dX \leq c_1 \left(\int_{B_{2r}} \{ |D^2\mathcal{U}| + |\nabla p| \} dX \right)^2$$

if $B_{2r} \subset \Omega$ where c_1 is independent of r . (For ease of notation, we write $\overline{D\mathcal{U}}$ and $\bar{p}$ for $\overline{D\mathcal{U}}_{X_0, 2r}$ and $\bar{p}_{X_0, 2r}$, respectively.) From Lemma 4.2 we obtain the “reverse–Hölder”–type inequality,

$$(4.1) \quad \left[\fint_{B_r} |D^2\mathcal{U}|^2 + |\nabla p|^2 dX \right]^{1/2} \leq c_2 \fint_{B_{2r}} (|D^2\mathcal{U}| + |\nabla p|) dX$$

for any $X_0 \in \Omega'$ and $B_{2r} = B_{2r}(X_0) \Subset \Omega''$ where c_2 depends only on $\hat{\nu}$ and $\hat{p}$ (and hence only on a , Ω' and Ω'').

By Chapter V, Proposition 1.1 of [4] and Lemma 4.2, inequality (4.1) implies that there exist constants $q > 2$, $c_3 > 0$ and $c_4 > 0$, such that

$$\left(\fint_{\Omega'} \{ |D^2\mathcal{U}|^q + |\nabla p|^q \} dX \right)^{1/q} \leq c_3 \left(\fint_{\Omega''} \{ |D^2\mathcal{U}|^2 + |\nabla p|^2 \} dX \right)^{1/2} \leq c_4.$$

Here q and c_3 depend only on Ω' , Ω'' , Ω , $\hat{\nu}$, and $\hat{p}$; c_4 depends on these parameters and $\|p\|_{L^1(\Omega)} + \|D\mathcal{U}\|_{L^{16}(\Omega)}$ (and hence by Theorem 3.3 only on a , Ω' , and Ω). Therefore

$$\|D^2\mathcal{U}\|_{L^q(\Omega')} \leq c_4, \quad \|\nabla p\|_{L^q(\Omega')} \leq c_4,$$

and by the Sobolev imbedding theorem,

$$\begin{aligned} \|D\mathcal{U}\|_{C^\alpha(\Omega')} &\leq c\|D\mathcal{U}\|_{W^{1,q}(\Omega')} \leq c_5, \\ \|p\|_{C^\alpha(\Omega')} &\leq c\|p\|_{W^{1,q}(\Omega)} \leq c_5, \end{aligned}$$

where $\alpha = 1 - \frac{2}{q}$ and $c_5 = c_5(a, \Omega', \Omega)$. The result now follows. $\blacksquare$

PROOF OF LEMMA 4.2: Since the proof is rather complicated, we split it into several steps.

Step 1: Let $\phi = \eta^2(e^{k\bar{p}} - e^{kp})$, where $k > 0$ is a constant to be determined, $\bar{p} = \bar{p}_{X_0, 2r}$, and η is a cut-off function satisfying $\eta \in C_0^1(B_{2r})$, with $\eta \equiv 1$ on B_r and $|\nabla \eta| \leq \frac{M}{r}$. Multiplying the operator $L_1(p)$, defined in Section 2, by ϕ and integrating by parts, we obtain

$$\begin{aligned} (4.2) \quad \int_{B_{2r}} L_1(p)\phi dX &= k \int_{B_{2r}} a_{ij}p_{x_i}p_{x_j}\eta^2 e^{kp} dX \\ &\quad - \int_{B_{2r}} 2a_{ij}p_{x_i}\eta_{x_j}\eta(e^{k\bar{p}} - e^{kp}) dX \\ &= I - II. \end{aligned}$$

Step 2: We estimate $\int L_1(p)\phi dX$ from below. By (0.2),

$$\begin{aligned} (\Delta u)^2 + (\Delta v)^2 &= (u_y^2 + v_y^2)p_x^2 - 2(u_x u_y + v_x v_y)p_x p_y + (u_x^2 + v_x^2)p_y^2 \\ &= a_{ij}p_{x_i}p_{x_j}, \end{aligned}$$

where we have used the summation convention over repeated indices. Also, since $[a_{ij}]$ is positive-definite with eigenvalues ν_1^2, ν_2^2 ,

$$a_{ij}p_{x_i}p_{x_j} \geq \nu_1^2 |\nabla p|^2.$$

Consequently,

$$\begin{aligned} (4.3) \quad I &= \frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] \eta^2 e^{kp} dX + \frac{k}{2} \int_{B_{2r}} a_{ij}p_{x_i}p_{x_j}\eta^2 e^{kp} dX \\ &\geq \frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] \eta^2 e^{kp} dX + \frac{k}{2} c_1 \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX, \end{aligned}$$

where $c_1 = c_1(\hat{\nu})$.

To bound II we use Young's inequality:

$$(4.4) \quad |II| \leq \int_{B_{2r}} |\nabla p|^2 \eta^2 e^{kp} dX + \frac{c_2}{r^2} \int_{B_{2r}} |e^{k\bar{p}} - e^{kp}|^2 e^{-kp} dX,$$

where $c_2 = c_2(\hat{\nu})$.

Combining (4.2)–(4.4), we obtain

$$(4.5) \quad \begin{aligned} \int_{B_{2r}} L_1(p) \phi dX &\geq \frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] e^{kp} \eta^2 dX + \frac{k}{2} c_1 \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX \\ &\quad - \int_{B_{2r}} |\nabla p|^2 \eta^2 e^{kp} dX - \frac{c_2}{r^2} \int_{B_{2r}} |e^{k\bar{p}} - e^{kp}|^2 e^{-kp} dX. \end{aligned}$$

Step 3: We estimate $\int L_1(p) \phi dX$ from above. From the definition of ϕ ,

$$(4.6) \quad \begin{aligned} \int_{B_{2r}} L_1(p) \phi dX &= \int_{B_{2r}} L_1(p) \eta^2 e^{k\bar{p}} dX - \int_{B_{2r}} L_1(p) \eta^2 e^{kp} dX \\ &= III + IV. \end{aligned}$$

By Proposition 2.3,

$$L_1(p) \geq -18\nu_2^4 |\nabla p|^2$$

and hence

$$(4.7) \quad IV \leq c_3 \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX,$$

where $c_3 = c_3(\hat{\nu})$.

To bound III , we use (2.11):

$$(4.8) \quad \begin{aligned} L_1(p) &= 2\{\langle \nabla u_y, \nabla v_x \rangle - \langle \nabla u_x, \nabla v_y \rangle\} \\ &= 2\{\langle (\nabla u - \overline{\nabla u}_{X_0, 2r}), \nabla v_x \rangle_y - \langle (\nabla u - \overline{\nabla u}_{X_0, 2r}), \nabla v_y \rangle_x\}. \end{aligned}$$

Substituting (4.8) into III , integrating by parts and applying Young's inequality gives

$$\begin{aligned}
|III| &= \left| 4e^{k\bar{p}} \int_{B_{2r}} \left\{ \langle (\nabla u - \overline{\nabla u}_{X_0, 2r}), \nabla v_x \rangle \eta \eta_y \right. \right. \\
&\quad \left. \left. - \langle (\nabla u - \overline{\nabla u}_{X_0, 2r}), \nabla v_y \rangle \eta \eta_x \right\} dX \right| \\
&\leq 4e^{k\bar{p}} \int_{B_{2r}} \left\{ |\nabla u - \overline{\nabla u}_{X_0, 2r}| \cdot |\nabla v_x| \cdot |\eta| \cdot |\eta_y| \right. \\
&\quad \left. + |\nabla u - \overline{\nabla u}_{X_0, 2r}| \cdot |\nabla v_y| \cdot |\eta| \cdot |\eta_x| \right\} dX \\
&\leq \int_{B_{2r}} 8e^{k\bar{p}} |D^2 \mathcal{U}| \cdot |\eta| \cdot |D\mathcal{U} - \overline{D\mathcal{U}}_{X_0, 2r}| \frac{M}{r} dX \\
(4.9) \quad &\leq \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX + \frac{16M^2}{\epsilon r^2} e^{2k\bar{p}} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}_{X_0, 2r}|^2 dX
\end{aligned}$$

for any $\epsilon > 0$.

Hence from (4.6), (4.7) and (4.9),

$$\begin{aligned}
(4.10) \quad \int_{B_{2r}} L_1(p) \phi dX &\leq c_3 \int_{B_{2r}} e^{kp} |\nabla p|^2 \eta^2 dX + \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX \\
&\quad + \frac{16M^2}{\epsilon r^2} e^{2k\bar{p}} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX
\end{aligned}$$

where $\overline{D\mathcal{U}} = \overline{D\mathcal{U}}_{X_0, 2r}$.

Step 4: The estimates (4.5) and (4.10) imply that

$$\begin{aligned}
\frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] e^{kp} \eta^2 dX &\leq \left(1 + c_3 - \frac{kc_1}{2} \right) \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX \\
&\quad + \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX + \frac{16M^2}{\epsilon r^2} e^{2k\bar{p}} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX \\
&\quad + \frac{c_2}{r^2} \int_{B_{2r}} |e^{k\bar{p}} - e^{kp}|^2 e^{-kp} dX.
\end{aligned}$$

Let $k = \frac{2(2+c_3)}{c_1}$ so that $\frac{kc_1}{2} = c_3 + 2$. Then

$$\begin{aligned}
(4.11) \quad \frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] e^{kp} \eta^2 dX &+ \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX \leq \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX \\
&\quad + \frac{16M^2}{\epsilon r^2} e^{2k\bar{p}} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX + \frac{c_2}{r^2} \int_{B_{2r}} |e^{k\bar{p}} - e^{kp}|^2 e^{-kp} dX.
\end{aligned}$$

Since k depends only on $\hat{v}$, there is a constant $c_4 = c_4(\hat{v}, \hat{p})$, such that $e^{2k\bar{p}} + e^{-kp} \leq c_4$. Thus, from (4.11)

$$\begin{aligned}
(4.12) \quad & \frac{k}{2} \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] e^{kp} \eta^2 dX + \int_{B_{2r}} |\nabla p|^2 e^{kp} \eta^2 dX \\
& \leq \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX + \frac{c_5}{\epsilon r^2} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX \\
& \quad + \frac{c_5}{r^2} \int_{B_{2r}} |p - \bar{p}|^2 dX,
\end{aligned}$$

where $c_5 = c_5(\hat{v}, \hat{p})$.

In [2, Appendix B], the following elementary inequality is derived:

$$\begin{aligned}
(4.13) \quad & \int_{B_{2r}} [(\Delta u)^2 + (\Delta v)^2] \eta^2 dX \geq \frac{1}{2} \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX \\
& \quad - \frac{c}{r^2} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX,
\end{aligned}$$

where c is a universal constant. Let $\theta = \theta(\hat{v}, \hat{p}) = \frac{k}{4} e^{-k\hat{p}}$. Then from (4.12) and (4.13), we obtain

$$\begin{aligned}
\theta \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX + \frac{4\theta}{k} \int_{B_{2r}} |\nabla p|^2 \eta^2 dX & \leq \epsilon \int_{B_{2r}} |D^2 \mathcal{U}|^2 \eta^2 dX \\
& \quad + \left(\frac{c_5}{\epsilon} + 2c\theta \right) \frac{1}{r^2} \int_{B_{2r}} |D\mathcal{U} - \overline{D\mathcal{U}}|^2 dX + \frac{c_5}{r^2} \int_{B_{2r}} |p - \bar{p}|^2 dX.
\end{aligned}$$

The proof is completed by setting $\epsilon = \frac{\theta}{2}$ and recalling that $\eta \equiv 1$ on B_r . ■

§5. High-order Regularity and estimates.

In this section we show that the nonlinear system of three equations in (0.1) and (0.2) is elliptic in the sense of Agmon, Douglis and Nirenberg [1] and has a quasilinear structure. As a consequence, if $\mathcal{U}$ is a strong solution in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$, then $\mathcal{U}$ and p are of class C^∞ in Ω . Furthermore, from the estimates in section 4 and standard elliptic estimates, we show that the C^k norms of $D\mathcal{U}$ and p in $\Omega' \Subset \Omega$ are bounded in terms of appropriate L^q norms of $D\mathcal{U}$ and p .

Consider the system of linear partial differential equations defined on an open set $\mathcal{D}$ in $\mathbb{R}^n$:

$$(5.1) \quad L_i(Z) = \sum_{j=1}^N l_{ij} z_j = f_i(X) \text{ for } i = 1, \dots, N \text{ and } X \in \mathcal{D},$$

where $l_{ij} = l_{ij}(D)$ are constant coefficient linear differential operators, $Z = (z_1, \dots, z_N)$, $X = (x_1, \dots, x_n)$ and $D = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right)$. Assume that there are integers $s_1, \dots, s_N, t_1, \dots, t_N$ so that the order of l_{ij} is not greater than $s_i + t_j$. Let $l'_{ij}(D)$ be the sum of terms in l_{ij} of exactly order $s_i + t_j$. For $\xi \in \mathbb{R}^n$ define the $N \times N$ matrix $[l'_{i,j}(\xi)]$ and its determinant $L(\xi)$.

DEFINITION 5.1. The system (5.1) is *elliptic* if there are constants s_i and t_i such that $L(\xi) \neq 0$ for $\xi \neq 0$.

Let $\mathcal{U}$ be a strong solution in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$. By Sobolev's theorem and the equations (0.1) and (0.2), it follows that $D\mathcal{U}$ and p are well defined pointwise as continuous functions in Ω . Let $X_0 \in \Omega$, $z_1 = u$, $z_2 = v$ and $z_3 = p$. Using the notation

$$\check{g}(X) \equiv g(X) - g(X_0),$$

we can rewrite the system (0.1) and (0.2) as

$$(5.2) \quad \begin{cases} L_1(Z) & \equiv \Delta u - v_y(X_0)p_x + v_x(X_0)p_y = (\check{v}_y\check{p})_x - (\check{v}_x\check{p})_y \equiv f_1, \\ L_2(Z) & \equiv \Delta v - u_x(X_0)p_y + u_y(X_0)p_x = (\check{u}_x\check{p})_y - (\check{u}_y\check{p})_x \equiv f_2, \\ L_3(Z) & \equiv v_y(X_0)u_x + u_x(X_0)v_y - v_x(X_0)u_y - u_y(X_0)v_x = \\ & -\check{v}_y u_x - \check{u}_x v_y + \check{v}_x u_y + \check{u}_y v_x + 2 \equiv f_3. \end{cases}$$

The constant coefficient system $\{L_i\}$ is just the principal part of the linearization of (0.1) and (0.2) about the state $Z^0 = (\nabla u(X_0) \cdot X, \nabla v(X_0) \cdot X, p(X_0))$. Indeed, if we denote (0.1) and (0.2) as $F_i(Z) = 0$ for $i = 1, 2, 3$ and set $l_{ij} \equiv \sum_{\alpha} \frac{\partial F_i(Z^0)}{\partial D^{\alpha} Z_j} D^{\alpha}$, then we shall choose s_i and t_i in Lemma 5.2 so that (5.2) is given by $L_i = \sum_j l'_{ij}$.

To say that (0.1) and (0.2) make up an *elliptic system* at $Z = (u, v, p)$ is equivalent to saying that for each $X_0 \in \Omega$, the linear system (5.2) is elliptic and the corresponding integers, $s_1, \dots, s_N, t_1, \dots, t_N$, may be chosen independent of X_0 .

LEMMA 5.2. The system (5.2) is elliptic. In fact,

$$(5.3) \quad L(\xi) \geq \nu_1^2(X_0) \cdot |\xi|^4 \text{ for all } \xi \in \mathbb{R}^2$$

and the coefficients of L_i are bounded by $\nu_2(X_0)$, where $\nu_1(X_0)$ and $\nu_2(X_0)$ are the singular values for $D\mathcal{U}(X_0)$.

PROOF: We have $l_{ij}(\xi) = |\xi|^2 \delta_{ij}$ for $1 \leq i, j \leq 2$, $l_{31}(\xi) = -l_{13}(\xi) = \langle \nabla v(X_0), (-\xi_2, \xi_1) \rangle$, $l_{32}(\xi) = -l_{23}(\xi) = \langle \nabla u(X_0), (-\xi_2, \xi_1) \rangle$ and $l_{33}(\xi) = 0$.

Selecting $s_i = 0$, $t_i = 2$ for $i = 1, 2$, $s_3 = -1$ and $t_3 = 1$ we have the order of $l_{ij} \leq s_i + t_j$ as required and $l'_{ij} = l_{ij}$. Thus,

$$\begin{aligned} L(\xi) &= \det[l'_{ij}(\xi)] = |\xi|^2 \cdot (\langle \nabla u(X_0), (-\xi_2, \xi_1) \rangle^2 + \langle \nabla v(X_0), (-\xi_2, \xi_1) \rangle^2) \\ &= |\xi|^2 \cdot \left| D\mathcal{U}(X_0) \begin{pmatrix} -\xi_2 \\ \xi_1 \end{pmatrix} \right|^2 \end{aligned}$$

and

$$L(\xi) \geq \nu_1^2(X_0) \cdot |\xi|^4 \text{ for all } \xi \in \mathbb{R}^2. \quad \blacksquare$$

The analysis in this section is based on two ideas: First, we note that the linearized system (5.2) has the same structure as Stokes equation for which the elliptic regularity theory of Agmon, Douglis and Nirenberg applies. (This observation was made by H. LeDret in [6] where it was used to obtain local existence results.) Second, the system (0.1) and (0.2) is quasilinear in the sense that the three equations are linear in the derivatives $\{D^{t_1}z_1, D^{t_2}z_2, D^{t_3}z_3\} = \{D^2u, D^2v, \nabla p\}$. Thus, if $\mathcal{U}$ is a strong solution we get higher order estimates by using a perturbation argument from elliptic theory.

We use the following estimate for functions Z with compact support in $\mathbb{R}^2$:

$$(5.4) \quad \sum_{j=1}^3 \|D^{t_j-1}z_j\|_{L^q(\mathbb{R}^2)} \leq c \cdot \sum_{j=1}^3 \|L_j(Z)\|_{W^{-s_j-1,q}(\mathbb{R}^2)}.$$

Here $1 < q < \infty$ and c depends on q , the ellipticity bounds for the system (which are $\nu_2(X_0)$ and $\nu_1(X_0)$ from Lemma 5.2), and $\text{diam}(\text{sppt } Z)$.

Specifically we have

$$(5.5) \quad \begin{aligned} (\|\nabla z_1\|_{L^q(\mathbb{R}^2)} + \|\nabla z_2\|_{L^q(\mathbb{R}^2)} + \|z_3\|_{L^q(\mathbb{R}^2)}) &\leq c \cdot (\|L_1(Z)\|_{W^{-1,q}(\mathbb{R}^2)} \\ &\quad + \|L_2(Z)\|_{W^{-1,q}(\mathbb{R}^2)} + \|L_3(Z)\|_{L^q(\mathbb{R}^2)}). \end{aligned}$$

The estimate (5.4) follows from the representation for Z given in the proof of Theorem 10.1 in [1]. We first show higher integrability for $D^2\mathcal{U}$ and ∇p .

LEMMA 5.3. *Let $\mathcal{U}$ be a strong solution in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$. Then for any $\Omega' \Subset \Omega$ and $s > q$, we have $\mathcal{U} \in W^{2,s}(\Omega'; \mathbb{R}^2)$ and*

$$\|p\|_{W^{1,s}(\Omega')} + \|D\mathcal{U}\|_{W^{1,s}(\Omega')} \leq C,$$

where C depends on $\Omega, \Omega', s, \|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$.

PROOF: We show that there is a positive constant $r < \min(\frac{1}{4} \text{ dist. } (\Omega', \partial\Omega), 1)$, depending on $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$, $\|p\|_{W^{1,q}(\Omega)}$ and s , so that

$$(5.6) \quad \|\nabla p\|_{L^s(B_r(X_0))} + \|D^2\mathcal{U}\|_{L^s(B_r(X_0))} \leq c = c(\|D\mathcal{U}\|_{W^{1,q}(\Omega)}, \|p\|_{W^{1,q}(\Omega)})$$

for all $X_0 \in \Omega'$. The lemma then follows by using a cover for Ω' .

For $X_0 \in \Omega'$, we assume without loss of generality that $D\mathcal{U}(X_0)$ is diagonal. Then (5.2) becomes

$$\begin{aligned}
(5.7) \quad L_1(Z) &= \Delta u - v_y(X_0)p_x = (\check{v}_y\check{p})_x - (\check{v}_x\check{p})_y = f_1, \\
L_2(Z) &= \Delta v - u_x(X_0)p_y = (\check{u}_x\check{p})_y - (\check{u}_y\check{p})_x = f_2, \\
L_3(Z) &= v_y(X_0)u_x + u_x(X_0)v_y = -\check{v}_y u_x - \check{u}_x v_y + 2u_y v_x + 2 = f_3.
\end{aligned}$$

Let $0 < r < \min(\frac{1}{4} \text{dist.}(\Omega', \partial\Omega), 1)$ and select a cut-off function $\varphi \in C_0^\infty(B_{2r}(X_0))$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on $B_r(X_0)$, $|D\varphi| \leq \frac{M}{r}$ and $|D^2\varphi| \leq \frac{M}{r^2}$. Let e_i be a unit vector parallel to the x_i coordinate axis and let h be a constant with $0 < |h| < r$. Set

$$\delta_{h,i}g \equiv \frac{1}{h}(g(X + he_i) - g(X))$$

and let $\overline{Z} \equiv \varphi\delta_{h,i}Z$ where $Z = (u, v, p)$ and (u, v) is the strong solution. Placing $\overline{Z}$ into the system (5.7) and using the estimate (5.5), we get

$$\begin{aligned}
(5.8) \quad & \|D(\varphi\delta_{h,i}u)\|_{L^s(B_{2r})} + \|D(\varphi\delta_{h,i}v)\|_{L^s(B_{2r})} + \|\varphi\delta_{h,i}p\|_{L^s(B_{2r})} \leq \\
& c_1 \cdot (\|\varphi\delta_{h,i}f_1\|_{W^{-1,s}(B_{2r})} + \|\varphi\delta_{h,i}f_2\|_{W^{-1,s}(B_{2r})} + \|\varphi\delta_{h,i}f_3\|_{L^s(B_{2r})}) \\
& + \frac{c_2}{r^2} \cdot (\|D\mathcal{U}\|_{L^s(B_{4r})} + \|p\|_{L^s(B_{4r})}),
\end{aligned}$$

where c_1 and c_2 depend on $\nu_1(X_0)$, $\nu_2(X_0)$ and s . Note that $\nu_1\nu_2 = 1$, so we obtain a positive lower bound on ν_1 from a bound on ν_2 from above.

Since $q > n$, we have

$$(5.9) \quad \nu_2(X_0) \leq |D\mathcal{U}(X_0)| \leq c\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$$

by Sobolev's theorem. Thus, c_1 and c_2 depend only on s and $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$.

We now estimate the terms on the right in (5.8). First,

$$\begin{aligned}
\varphi\delta_{h,i}f_1 &= \varphi[\delta_{h,i}(\check{v}_y\check{p})]_x - \varphi[\delta_{h,i}(\check{v}_x\check{p})]_y \\
&= \{[\varphi\delta_{h,i}(\check{v}_y\check{p})]_x - [\varphi\delta_{h,i}(\check{v}_x\check{p})]_y\} \\
&\quad - \{\varphi_x\delta_{h,i}(\check{v}_y\check{p}) - \varphi_y\delta_{h,i}(\check{v}_x\check{p})\} \\
&\equiv g_1 - g_2.
\end{aligned}$$

Set

$$(5.10) \quad \omega(d) \equiv \sup_{B_d(X_0)} |\check{p}| + \sup_{B_d(X_0)} |\check{D}\mathcal{U}|, \text{ for } 0 < d < 4r.$$

By definition, $\|g_1\|_{W^{-1,s}(B_{2r})} = \sup \left\{ \int g_1 \eta dX / \|\eta\|_{W^{1,s'}(B_{2r})} : \eta \in C_c^\infty(B_{2r}) \right\}$, where s' is such that $\frac{1}{s} + \frac{1}{s'} = 1$. Hence

$$\|g_1\|_{W^{-1,s}(B_{2r})} \leq \omega(2r) \cdot (\|\varphi \delta_{h,i} D\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi \delta_{h,i} p\|_{L^s(B_{2r})}).$$

To estimate g_2 let $\eta \in C_c^\infty(B_{2r})$. Then

$$\begin{aligned} \int g_2 \eta dX &= - \int \eta(\check{v}_y \check{p}) \cdot \delta_{-h,i} \varphi_x dX - \int \varphi_x(x + he_i) \cdot (\check{v}_y \check{p}) \cdot \delta_{-h,i} \eta dX \\ &\quad + \int \eta(\check{v}_x \check{p}) \delta_{-h,i} \varphi_y dX + \int \varphi_y(x + he_i) \cdot (\check{v}_x \check{p}) \cdot \delta_{-h,i} \eta dX. \end{aligned}$$

Since $\|\delta_{h,i} \eta\|_{L^{s'}(B_{2r})} \leq \|D\eta\|_{L^{s'}(B_{2r})}$, we get

$$\|g_2\|_{W^{-1,s}(B_{2r})} \leq \frac{c}{r^2} \| |\check{D}\mathcal{U}| \cdot |\check{p}| \|_{L^s(B_{2r})} \leq \frac{c}{r^2} \omega^2(2r).$$

By Sobolev's Theorem (since $q > 2$),

$$\omega(2r) \leq C$$

where C depends on $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$. Thus

$$\begin{aligned} (5.11) \quad \|\varphi \delta_{h,i} f_1\|_{W^{-1,s}(B_{2r})} &= \|g_1 - g_2\|_{W^{-1,s}(B_{2r})} \\ &\leq \omega(2r) \cdot (\|\varphi \delta_{h,i} D\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi \delta_{h,i} p\|_{L^s(B_{2r})}) + \frac{\overline{C}}{r^2}. \end{aligned}$$

Similarly we obtain the same bound for $\|\varphi \delta_{h,i} f_2\|_{W^{-1,s}(B_{2r})}$.

Finally, consider

$$\delta_{h,i} f_3 = -\delta_{h,i}(\check{v}_y u_x) - \delta_{h,i}(\check{u}_x v_y) + 2\delta_{h,i}(u_y v_x).$$

We add and subtract $\frac{\check{v}_y(X+he_i)u_x(X)}{h}$ to the first term on the right and $\frac{\check{u}_x(X)v_y(X+he_i)}{h}$ to the second. This yields

$$\begin{aligned} \delta_{h,i} f_3(X) &= -\check{v}_y(X+he_i) \cdot \delta_{h,i} u_x(X) - u_x(X) \cdot \delta_{h,i} v_y(X) \\ &\quad - v_y(X+he_i) \cdot \delta_{h,i} u_x(X) - \check{u}_x(X) \cdot \delta_{h,i} v_y(X) \\ &\quad + 2\delta_{h,i}(u_y v_x)(X). \end{aligned}$$

Since $\delta_{h,i}(u_y v_x) = \delta_{h,i}(u_x v_y)$ by (0.1), we have:

$$-u_x(X) \cdot \delta_{h,i} v_y(X) - v_y(X+he_i) \cdot \delta_{h,i} u_x(X) + \delta_{h,i}(u_y v_x)(X) = 0.$$

Now, by adding and subtracting $u_y(X + he_i) \cdot v_x(X)$, we have

$$\begin{aligned}\delta_{h,i}(u_y v_x)(X) &= u_y(X + he_i) \cdot \delta_{h,i} v_x(X) + v_x(X) \cdot \delta_{h,i} u_y(X) \\ &= \check{u}_y(X + he_i) \cdot \delta_{h,i} v_x(X) + \check{v}_x(X) \cdot \delta_{h,i} u_y(X)\end{aligned}$$

since $u_y(X_0) = v_x(X_0) = 0$. Thus

$$\begin{aligned}\delta_{h,i} f_3(X) &= -\check{v}_y(X + he_i) \delta_{h,i} u_x(X) - \check{u}_x(X) \delta_{h,i} v_y(X) \\ &\quad + \check{u}_y(X + he_i) \delta_{h,i} v_x(X) + \check{v}_x(X) \delta_{h,i} u_x(X)\end{aligned}$$

and

$$(5.12) \quad \|\varphi \delta_{h,i} f_3\|_{L^s(B_{2r})} \leq 4\omega(3r) \|\varphi \delta_{h,i} D\mathcal{U}\|_{L^s(B_{2r})}.$$

From (5.8), (5.9), (5.11), and (5.12), we have

$$\begin{aligned}(5.13) \quad &\|\varphi \delta_{h,i} D\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi \delta_{h,i} p\|_{L^s(B_{2r})} \leq 6c_1 \omega(3r) \cdot \{\|\varphi \delta_{h,i} D\mathcal{U}\|_{L^s(B_{2r})} \\ &\quad + \|\varphi \delta_{h,i} p\|_{L^s(B_{2r})}\} + \frac{\overline{C}}{r^2},\end{aligned}$$

where $\overline{C}$ depends on $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$. By the definition of $\omega(r)$ (see (5.10)) and Sobolev's theorem, $\omega(d) \leq c \cdot (\|D\mathcal{U}\|_{W^{1,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)}) \cdot d^\alpha$ where $\alpha = 1 - \frac{2}{q}$. Thus we can choose r sufficiently small, depending on $\mathcal{U}$ and p through the quantities $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$, so that $c_1 \omega(3r) < \frac{1}{12}$. Hence (5.6) follows from (5.13). $\blacksquare$

THEOREM 5.4. *Let $\mathcal{U}$ be a strong solution in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$. Then $\mathcal{U}$ and p are of class C^∞ in Ω . Moreover for any integer $k \geq 2$, any constant s in $(1, \infty)$, and any subdomain $\Omega' \Subset \Omega$,*

$$(*) \quad \|D^k \mathcal{U}\|_{L^s(\Omega')} + \|D^{k-1} p\|_{L^s(\Omega')} \leq C,$$

where C depends on $s, k, \Omega', \Omega, \|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$.

PROOF: Assume that $\mathcal{U}$ is a strong solution in $W^{2,q}(\Omega; \mathbb{R}^2)$ for some $q > 2$. We shall prove by induction that for each integer $k \geq 2$, $D\mathcal{U}$ and p are in $W_{\text{loc}}^{k-1,s}(\Omega)$ and $(*)$ holds for all s in $(1, \infty)$. Note that for $k = 2$ and s in $(1, \infty)$, this follows from Lemma 5.3.

Now assume that $k > 2$, $D\mathcal{U}$ and p are in $W_{\text{loc}}^{k-2,s}(\Omega)$ for all s in $(1, \infty)$, and $(*)$ holds for all s in $(1, \infty)$ with k replaced by $k - 1$. Fix X_0 in Ω and let φ be a cut-off function with $\text{sppt } \varphi \subset B_{2r}(X_0)$ and $B_{4r}(X_0) \Subset \Omega$. Assume without loss of generality that $D\mathcal{U}(X_0)$ is diagonal, and let D^{k-2}

denote an arbitrary partial derivative of order $k-2$. Applying D^{k-2} to the constant coefficient equations in (5.7) gives $L_i(D^{k-2}Z) = D^{k-2}f_i$ for $i = 1, 2, 3$. Setting $\bar{Z} = \varphi\delta_{h,i}D^{k-2}Z$ and proceeding as in the previous lemma we get

$$(5.14) \quad \begin{aligned} & \|\varphi\delta_{h,i}D^{k-1}\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi\delta_{h,i}D^{k-2}p\|_{L^s(B_{2r})} \leq \\ & c_1 \left(\|\varphi\delta_{h,i}D^{k-2}f_1\|_{W^{-1,s}(B_{2r})} + \|\varphi\delta_{h,i}D^{k-2}f_2\|_{W^{-1,s}(B_{2r})} + \|\varphi D^{k-2}f_3\|_{L^s(B_{2r})} \right) \\ & \quad + \frac{c_2}{r^2} \left(\|D^{k-1}\mathcal{U}\|_{L^s(B_{4r})} + \|D^{k-2}p\|_{L^s(B_{4r})} \right), \end{aligned}$$

where c_1 and c_2 depend on $\nu_1(X_0), \nu_2(X_0), s$ and k . The right hand side of (5.14) can then be estimated by

$$(5.15) \quad \begin{aligned} & \bar{c}_1\omega(3r) \cdot \left(\|\varphi\delta_{h,i}D^{k-1}\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi\delta_{h,i}D^{k-2}p\|_{L^s(B_{2r})} \right) \\ & + \bar{c}_2 \left(1 + \frac{1 + \omega(4r)}{r^2} \right) \cdot \left(1 + \|D\mathcal{U}\|_{W^{k-2,2s}(B_{4r})}^2 + \|p\|_{W^{k-2,2s}(B_{4r})}^2 \right). \end{aligned}$$

To see this, note that

$$\delta_{h,i}D^{k-2}f_1 = (\delta_{h,i}D^{k-2}(\check{v}_y\check{p}))_x - (\delta_{h,i}D^{k-2}(\check{v}_x\check{p}))_y.$$

Hence

$$\begin{aligned} \|\varphi\delta_{h,i}D^{k-2}f_1\|_{W^{-1,s}(B_{2r})} & \leq \|\varphi\delta_{h,i}D^{k-2}(\check{v}_y\check{p})\|_{L^s(B_{2r})} + \\ & \|\varphi\delta_{h,i}D^{k-2}(\check{v}_x\check{p})\|_{L^s(B_{2r})} + \frac{c_4}{r^2} (\|D^{k-2}(\check{v}_y\check{p})\|_{L^s(B_{4r})} + \|D^{k-2}(\check{v}_x\check{p})\|_{L^s(B_{4r})}). \end{aligned}$$

It is a straight forward argument (using the Leibniz formula and the elementary inequality, $|ab| \leq a^2 + b^2$) to bound the right hand side by (5.15). Similarly one can bound $\|\varphi\delta_{h,i}D^{k-2}f_2\|_{W^{-1,s}(B_{2r})}$ by (5.15).

Next, consider the norm of $\varphi D^{k-2}f_3$ in $L^s(B_{2r})$. Since $u_y(X_0) = 0 = v_x(X_0)$ and $u_xv_y - u_yv_x = 1$, we have

$$\begin{aligned} Df_3 & = -\check{v}_yDu_x - \check{u}_xDv_y + D(-u_xv_y + v_xu_y) + D(v_xu_y) \\ & = -\check{v}_yDu_x - \check{u}_xDv_y + \check{v}_xDu_y + \check{u}_yDv_x. \end{aligned}$$

Thus, we see that

$$\|\varphi\delta_{h,i}D^{k-2}f_3\|_{L^s(B_{2r})} = \|\varphi\delta_{h,i}D^{k-3}(Df_3)\|_{L^s(B_{2r})}$$

can be estimated by (5.15) as well.

The first term in (5.15) is a multiple of the left hand side of (5.14). By choosing r_0 sufficiently small (depending only on $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$ and $\|p\|_{W^{1,q}(\Omega)}$), we can ensure that $\bar{c}_1\omega(3r) < \frac{1}{2}$ provided $0 < r \leq r_0$. Thus

$$(5.16) \quad \begin{aligned} & \|\varphi\delta_{h,i}D^{k-1}\mathcal{U}\|_{L^s(B_{2r})} + \|\varphi\delta_{h,i}D^{k-2}p\|_{L^s(B_{2r})} \\ & \leq 2\bar{c}_2 \left(1 + \frac{1 + \omega(4r)}{r^2}\right) \cdot \left(1 + \|D\mathcal{U}\|_{W^{k-2,2s}(B_{4r})}^2 + \|p\|_{W^{k-2,2s}(B_{4r})}\right) \end{aligned}$$

for $0 < r \leq r_0$, where $\bar{c}_2$ depends on $\nu_1(X_0), \nu_2(X_0)$, s and k (and hence only on $\|D\mathcal{U}\|_{W^{1,q}(\Omega)}$, s , and k). By (5.16), our inductive hypothesis, and Sobolev's theorem, we conclude that $D\mathcal{U}$ and p are in $W_{\text{loc}}^{k-1,s}(\Omega)$ for all s in $(1, \infty)$ and if $\Omega' \Subset \Omega$, we have

$$\|D^k\mathcal{U}\|_{L^s(\Omega')} + \|D^{k-1}p\|_{L^s(\Omega')} \leq C,$$

where $C = C(s, k, \Omega', \Omega, \|D\mathcal{U}\|_{W^{1,q}(\Omega)}, \|p\|_{W^{1,q}(\Omega)})$ and s is any constant in $(1, \infty)$. This proves (*) for all integers $k \geq 2$. $\blacksquare$

Finally, using Sobolev's theorem and the results from Section 4 we obtain:

COROLLARY 5.5. *Assume that $q > 2$ and $\mathcal{U}$ is a strong solution in $W_{\text{loc}}^{2,q}(\Omega)$. Then for any integer $k \geq 2$ and any subdomains $\Omega' \Subset \Omega'' \Subset \Omega$, we have*

$$\|D\mathcal{U}\|_{C^k(\Omega')} + \|p\|_{C^k(\Omega')} \leq C,$$

where C depends only on $\Omega', \Omega'', k, \|p\|_{L^1(\Omega'')}$ and $\|D\mathcal{U}\|_{L^{16}(\Omega'')}$.

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