ON BOUNDING THE EFFECTIVE CONDUCTIVITY
OF ANISOTROPIC COMPOSITES

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ON BOUNDING THE EFFECTIVE CONDUCTIVITY
OF ANISOTROPIC COMPOSITES

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1. Introduction

There has recently been a renewal of interest in bounding the effective moduli of composite materials. Several factors are responsible, including attention to applications in structural optimization [1,18,20,28]. The developments of the past several years include new ways of applying old variational principles such as those of Hashin and Shtrikman. They also include two entirely new methods for proving bounds: one based on compensated compactness, and the other on explicit representation formulas.

One result of this activity has been the specification of optimal bounds for anisotropic composites made by mixing two isotropic conducting materials in specified proportions. First established using the compensated compactness method [21,22,33], the bounds have also been proved using representation formulas [24], and a third proof is possible using the Hashin-Shtrikman variational principle [17]. Their applications to structural optimization are discussed in [19] and [28].

This paper presents, in self-contained form, each of the three approaches to proving these bounds. Our goal is to popularize the methods, and to highlight their similarities and differences, by explaining them in the context of this simple but fundamental example.

2. The Bounds

From the physical viewpoint, we are interested in composites made by mixing two isotropic materials with conductivities α and β , $0 < \alpha < \beta$, in specified proportions θ_α and $\theta_\beta = 1 - \theta_\alpha$. The effective conductivity A is a symmetric second-order tensor relating the (local) average electric field to the (local) average current flow. In the case of a spatially periodic composite the averaging is over a period cell [2,30]; for a random one it is over the statistical ensemble [29]. There is also a third, more general approach, which replaces averaging by weak convergence [6,25,38]. For simplicity, we shall only discuss the spatially periodic case. This represents no loss of generality, since the effective conductivity of any composite can be approximated by spatially periodic ones with increasingly complicated microstructures. (This is established for random composites in [10]; a proof using the theory of G-convergence will appear in [16].)

Mathematically, then, our composites will be described on the microscopic scale by a periodic function $a(y)$, with period cell $Q \subset \mathbb{R}^n$, taking only the values α and β . The effective conductivity A is defined by the variational principle

$$\langle A\xi, \xi \rangle = \inf_{\phi} \int_Q a |\xi + \nabla \phi|^2 dy$$

for any $\xi \in \mathbb{R}^n$, where ϕ ranges over Q -periodic functions and $\int_Q$ denotes the average over Q . If $\chi_\alpha(y)$ and $\chi_\beta(y)$ denote the characteristic functions of the sets where a equals α and β , respectively, then

$$a(y) = \alpha \chi_\alpha + \beta \chi_\beta$$

and the volume fractions are

$$\theta_\alpha = \int_Q \chi_\alpha dy, \quad \theta_\beta = \int_Q \chi_\beta dy.$$

An elementary bound on A has been known early in this century:

$$(EB) \quad hI \leq A \leq mI,$$

where h and m denote the harmonic and arithmetic means of A , respectively,

$$h = (f_Q a^{-1} dy)^{-1} = (\alpha^{-1} \theta_\alpha + \beta^{-1} \theta_\beta)^{-1}$$

$$m = \int_Q a \, dy = \alpha \theta_\alpha + \beta \theta_\beta.$$

This bound is sharp, in the sense that for a laminar microstructure the largest eigenvalue of A equals h . However, it is not optimal: not every matrix A which satisfies (EB) is the effective conductivity of a composite made by mixing α and β in proportions θ_α and θ_β .

The optimal bounds were first established by Murat and Tartar [33] and by Lurie and Cherkhev [21,22]. In addition to (EB), A must satisfy the lower bound

$$(LB) \quad \text{tr}[(A - \alpha I)^{-1}] \leq \frac{n-1}{m-\alpha} + \frac{1}{h-\alpha}$$

and the upper bound

$$(UB) \quad \text{tr}[(\beta I - A)^{-1}] \leq \frac{n-1}{\beta-m} + \frac{1}{\beta-h}.$$

If A satisfies these three bounds, then a composite with effective conductivity A and volume fractions $\theta_\alpha, \theta_\beta$ can be constructed using coated ellipsoids, a generalization of the well-known coated sphere construction that works in the isotropic case [33]. There is also a second, entirely different construction using successive lamination [21,22,33] (c.f. the contributions of Milton and Tartar). The details of these constructions will not be repeated here; our attention is focused instead on the proofs of (LB) and (UB).

Some remarks are in order about the relation of these results to earlier work. Bounds equivalent to (LB) and (UB) were established by Hashin and Shtrikman in 1962, for the special case of random, statistically isotropic composites [13]. A mathematically more rigorous proof was subsequently given by Willis [34]. An extensive literature has developed from this pioneering work; it is reviewed, for example, in [4,12,35,36]. Much of it addresses the specification of better bounds for the effective conductivity A in terms of additional statistical information, such as the two-point correlation function of the (randomly varying) microscopic conductivity. The task of improving (EB) for anisotropic composites, in terms of the volume fractions alone, seems simply to

have been ignored for many years. It was rediscovered in the 1970's by specialists in optimal control theory, who realized that optimal bounds such as (EB), (LB), (UB) determine the relaxation of a class of distributed parameter control problems.

3. Variational Principles

We have defined the effective conductivity by means of a variational principle

$$(3.1) \quad \langle A\xi, \xi \rangle = \inf_{\phi} \int_Q a(y) |\xi + \nabla \phi|^2 dy.$$

The minimizer ϕ^ξ solves

$$(3.2) \quad \nabla \cdot (a(y)(\xi + \nabla \phi^\xi)) = 0,$$

and integration by parts in (3.1) gives

$$(3.3) \quad \langle A\xi, \xi \rangle = \int_Q \langle a(y)(\xi + \nabla \phi^\xi), \xi \rangle dy.$$

Since ϕ^ξ depends linearly on ξ , it follows by polarization that

$$(3.4) \quad A\xi = \int_Q a(y)(\xi + \nabla \phi^\xi) dy.$$

3A. Dual variational principles.

A second characterization of A is obtained by taking the convex dual of (3.1). We have

$$\frac{1}{2} \langle E\xi, \xi \rangle \geq \langle \xi, \sigma \rangle - \frac{1}{2} \langle E^{-1}\sigma, \sigma \rangle$$

for any $\xi, \sigma \in R^n$ and any positive, symmetric matrix E , with equality precisely when $\sigma = E\xi$. Therefore (3.1) can be written

$$(3.5) \quad \frac{1}{2} \langle A\xi, \xi \rangle = \inf_{\phi} \sup_{\sigma} \int_Q [\langle \xi + \nabla \phi, \sigma \rangle - \frac{1}{2} a(y)^{-1} |\sigma|^2] dy,$$

in which σ ranges over Q -periodic vector fields. Interchanging the inf and sup

in (3.5) yields

$$(3.6) \quad \frac{1}{Z} \langle A\xi, \xi \rangle = \sup_{\sigma} \inf_{\phi} \int_Q [\langle \xi + \nabla \phi, \sigma \rangle - \frac{1}{Z} a(y)^{-1} |\sigma|^2] dy$$

If $\operatorname{div} \sigma \neq 0$ then the infimum over ϕ in (3.6) is $-\infty$, not of interest since we are maximizing over σ . If $\operatorname{div} \sigma = 0$ then $\int_Q \langle \nabla \phi, \sigma \rangle dy = 0$ and so

$$(3.7) \quad \frac{1}{Z} \langle A\xi, \xi \rangle = \sup_{\operatorname{div} \sigma = 0} \int_Q [\langle \xi, \sigma \rangle - \frac{1}{Z} a(y)^{-1} |\sigma|^2] dy.$$

This is the variational principle dual to (3.1).

A word about the passage from (3.5) to (3.6). The fact that $\inf \sup = \sup \inf$ follows from a general saddle-point theorem [7, Prop. 2.2, page 173]. For quadratic variational problems there is also an easy, direct proof of duality: for any periodic σ with $\operatorname{div} \sigma = 0$,

$$\begin{aligned} \frac{1}{Z} \langle A\xi, \xi \rangle &= \int_Q \frac{1}{Z} \langle a(y)(\xi + \nabla \phi^\xi), (\xi + \nabla \phi^\xi) \rangle dy \\ (3.8) \quad &> \int_Q [\langle \sigma, \xi + \nabla \phi^\xi \rangle - \frac{1}{Z} \langle a(y)^{-1} \sigma, \sigma \rangle] dy \\ &= \int_Q [\langle \sigma, \xi \rangle - \frac{1}{Z} \langle a(y)^{-1} \sigma, \sigma \rangle] dy \end{aligned}$$

with equality in (3.8) when σ is

$$(3.9) \quad \sigma^\xi = a(y)(\xi + \nabla \phi^\xi).$$

There is a modification of the dual principle (3.7) which is more symmetric to (3.1):

$$(3.10) \quad \langle A^{-1} \bar{\sigma}, \bar{\sigma} \rangle = \inf_{\substack{\operatorname{div} \sigma = 0 \\ f\sigma = \bar{\sigma}}} \int_Q a(y)^{-1} |\sigma|^2 dy,$$

for any $\bar{\sigma} \in \mathbb{R}^n$. To prove it, we observe that the extremal σ^ξ of (3.7) has mean value

$$\bar{\sigma}^\xi = \int_Q \sigma^\xi dy = \int_Q a(y)(\xi + \nabla \phi^\xi) dy = A\xi.$$

This can therefore be imposed as a constraint in (3.7) without changing the value of the supremum. Hence

$$\begin{aligned} \frac{1}{2} \langle A\xi, \xi \rangle &= \sup_{\substack{\operatorname{div} \sigma = 0 \\ f\sigma = A\xi}} \int_Q \left[\langle \xi, \sigma \rangle - \frac{1}{2} a(y)^{-1} |\sigma|^2 \right] dy \\ &= \langle A\xi, \xi \rangle - \frac{1}{2} \inf_{\substack{\operatorname{div} \sigma = 0 \\ f\sigma = A\xi}} \int_Q a(y)^{-1} |\sigma|^2 dy. \end{aligned}$$

Rearrangement and the substitution $\bar{\sigma} = A\xi$ leads to (3.10).

The elementary bounds (EB) are an immediate consequence of these variational principles: The choice $\phi=0$ in (3.1) gives $A \leq mI$, while $\sigma \equiv \bar{\sigma}$ in (3.10) gives $A \geq hI$.

3B. Hashin-Shtrikman variational principles.

These principles were introduced by Hashin and Shtrikman in [13], and clarified by Hill [14] and Willis [34]. The key idea is to dualize not the full energy $a(y)|\xi + \nabla\phi|^2$, but only a part of it. We shall present two versions, one giving a lower bound and the other an upper bound for $\langle A\xi, \xi \rangle$.

We begin by adding and subtracting $\gamma|\xi + \nabla\phi|^2$ — the energy of a "reference medium" with constant conductivity γ — on the right of (3.1):

$$(3.11) \quad \frac{1}{2} \langle A\xi, \xi \rangle = \inf_{\phi} \frac{1}{2} \int_Q [(a(y) - \gamma)|\xi + \nabla\phi|^2 + \gamma|\xi + \nabla\phi|^2] dy.$$

If γ is restricted to the range $0 < \gamma < \alpha$, then $a(y) - \gamma > 0$, and the first term on the right can be written as a supremum over Q -periodic vector fields σ :

$$(3.12) \quad \frac{1}{2} \langle A\xi, \xi \rangle = \inf_{\phi} \sup_{\sigma} \int_Q \left[\langle \sigma, \xi + \nabla\phi \rangle - \frac{1}{2} (a(y) - \gamma)^{-1} |\sigma|^2 + \frac{\gamma}{2} |\xi + \nabla\phi|^2 \right] dy.$$

We interchange the inf and sup — appealing again to convex analysis [7] or arguing as section 3A — and do the minimization on ϕ directly. The optimal choice solves

$$(3.13) \quad \Delta \phi = -\frac{1}{\gamma} \operatorname{div} \sigma ;$$

it is admissible, because the Laplacian is an isomorphism from periodic H^1 functions to periodic H^{-1} functions with mean value zero. Substitution in (3.12) and integration by parts gives

$$(3.14) \quad \frac{1}{Z} \langle (A-\gamma)\xi, \xi \rangle = \sup_{\sigma} \int_Q [\langle \sigma, \xi \rangle - \frac{1}{Z} (a(y)-\gamma)^{-1} |\sigma|^2 - \frac{1}{2\gamma} \langle \sigma, \nabla \Delta^{-1} \operatorname{div} \sigma \rangle] dy.$$

This is the variational principle of Hashin and Shtrikman for bounding A below.

For the second principle, we return to (3.11). If γ is restricted to the range $\beta < \gamma < \infty$, then $a(y) - \gamma$ is negative and the first term on the right becomes an infimum:

$$(3.15) \quad \frac{1}{Z} \langle A\xi, \xi \rangle = \inf_{\phi} \inf_{\sigma} \int_Q [\langle \sigma, \xi + \nabla \phi \rangle - \frac{1}{Z} (a(y)-\gamma)^{-1} |\sigma|^2 + \frac{\gamma}{Z} |\xi + \nabla \phi|^2] dy.$$

We interchange the order of minimization - no convex duality is required here - and evaluate the infimum over ϕ . The optimal choice is again (3.13), and substitution into (3.15) gives

$$(3.16) \quad \frac{1}{Z} \langle (A-\gamma)\xi, \xi \rangle = \inf_{\sigma} \int_Q [\langle \sigma, \xi \rangle - \frac{1}{Z} (a(y)-\gamma)^{-1} |\sigma|^2 - \frac{1}{2\gamma} \langle \sigma, \nabla \Delta^{-1} \operatorname{div} \sigma \rangle] dy.$$

This is the Hashin-Shtrikman variational principle for bounding A above. Note that the integrands in (3.14) and (3.16) are the same, except that γ must be less than α for (3.14) and greater than β for (3.16).

4. Bounds via HS Variational Principles

We shall make the customary substitution of a piecewise constant σ into the Hashin-Shtrikman variational principle. This leads to a bound for the effective conductivity A in terms of a tensor F of "geometric parameters" defined by

$$(4.1) \quad \langle F n, n \rangle = \int_Q \langle n \chi, \nabla \Delta^{-1} \operatorname{div}(n \chi) \rangle dy,$$

where $n \in \mathbb{R}^n$ and χ is either χ_{α} or χ_{β} , the characteristic functions of the

two materials. The traditional approach for random composites is to write the expected value of F in terms of the kernel for $(\nabla \Delta^{-1} \text{div})$ and the two-point correlation function of the random field χ . With a hypothesis of statistical isotropy, Hashin and Shtrikman used the symmetries of the kernel to evaluate the resulting expression in terms of the volume fractions alone. We shall proceed differently, by deriving a relation on the trace of F irrespective of the symmetry of the composite. Algebraic manipulation then leads to the desired bound.

4A. Constraints on the geometric parameters F_{ij} .

The key to our approach is this simple result.

Lemma 4.1: Let χ be any Q - periodic characteristic function with mean value $\bar{\chi} = \int_Q \chi dy$. Then the matrix F defined by (4.1) is symmetric and nonnegative, with trace $\bar{\chi}(1-\bar{\chi})$.

Proof: Since $\chi - \bar{\chi}$ has mean value zero, there is a periodic solution of

$$\Delta \psi = \chi - \bar{\chi}.$$

Differentiating the equation gives

$$\frac{\partial \psi}{\partial y_i} = \Delta^{-1} \frac{\partial \chi}{\partial y_i}, \quad 1 \leq i \leq n,$$

and so

$$(4.2) \quad \frac{\partial}{\partial y_i} \Delta^{-1} \frac{\partial}{\partial y_j} \chi = \frac{\partial^2 \psi}{\partial y_i \partial y_j}.$$

By definition

$$F_{ij} = \int_Q \chi \frac{\partial}{\partial y_i} \Delta^{-1} \frac{\partial}{\partial y_j} \chi \, dy,$$

and substitution of (4.2) into this gives

$$\begin{aligned} F_{ij} &= \int_Q \chi \frac{\partial^2 \psi}{\partial y_i \partial y_j} \, dy = \int_Q (\chi - \bar{\chi}) \frac{\partial^2 \psi}{\partial y_i \partial y_j} \, dy \\ (4.3) \quad &= \int_Q \Delta \psi \cdot \frac{\partial^2 \psi}{\partial y_i \partial y_j} \, dy \end{aligned}$$

$$(4.4) \quad = f_Q \langle \nabla \frac{\partial \psi}{\partial y_i}, \nabla \frac{\partial \psi}{\partial y_j} \rangle dy.$$

Symmetry is clear from (4.3), and positivity is a consequence of (4.4), since it yields

$$\langle F\eta, \eta \rangle = f_Q |\nabla(\eta \cdot \nabla \psi)|^2 dy \quad \forall \eta \in R^n.$$

The trace is easily computed from (4.3):

$$\text{tr } F = f_Q \Delta \psi \cdot \Delta \psi dy = f_Q (\chi - \bar{\chi})^2 dy = \bar{\chi}(1 - \bar{\chi}).$$

4B. The lower bound.

We begin with the Hashin-Shtrikman lower bound (3.14). The test field σ is chosen to be a constant vector η where $a(y) = \beta$, and zero elsewhere:

$$\sigma = \eta \chi_\beta.$$

Substitution into (3.14) and passage to the limit $\gamma \uparrow \alpha$ gives

$$(4.5) \quad \frac{1}{2} \langle (A - \alpha I) \xi, \xi \rangle - \theta_\beta \langle \eta, \xi \rangle \geq -\frac{1}{2} \theta_\beta (\beta - \alpha)^{-1} |\eta|^2 - \frac{1}{2\alpha} \langle F^\beta \eta, \eta \rangle,$$

where F^β is defined by (4.1) with $\chi = \chi_\beta$.

The relation (4.5) holds for all vectors $\xi, \eta \in E^n$. Fixing η , we choose ξ to minimize the left hand side:

$$\xi = \theta_\beta (A - \alpha I)^{-1} \eta.$$

Substitution into (4.5) gives

$$\frac{1}{2} \theta_\beta^2 \langle (A - \alpha I)^{-1} \eta, \eta \rangle \leq \frac{\theta_\beta}{2(\beta - \alpha)} |\eta|^2 + \frac{1}{2\alpha} \langle F^\beta \eta, \eta \rangle.$$

This is a relation between symmetric matrices,

$$(A - \alpha I)^{-1} \leq \frac{1}{\theta_\beta (\beta - \alpha)} I + \frac{1}{\alpha \theta_\beta^2} F^\beta.$$

Taking the trace and using Lemma 4.1, we conclude that

$$(4.6) \quad \text{tr}[(A - \alpha I)^{-1}] \leq \frac{n}{\theta_\beta(\beta - \alpha)} + \frac{(1 - \theta_\beta)}{\alpha \theta_\beta}.$$

The right hand side of (4.6) is easily seen to agree with that of (LB).

4C. The upper bound.

This time we substitute

$$\sigma = \eta \chi_\alpha$$

into the Hashin-Shtrikman upper bound (3.16). After passage to the limit $\gamma \rightarrow \beta$ this yields

$$(4.7) \quad \frac{1}{2} \langle (A - \beta I) \xi, \xi \rangle - \theta_\alpha \langle \eta, \xi \rangle \leq \frac{1}{2} \theta_\alpha (\beta - \alpha)^{-1} - \frac{1}{2\beta} \langle F^\alpha \eta, \eta \rangle,$$

where F^α is defined by (4.1) with $\chi = \chi_\alpha$. We choose ξ to maximize the left hand side:

$$\xi = \theta_\alpha (A - \beta I)^{-1} \eta.$$

Substitution into (4.7) leads to

$$\frac{1}{2} \theta^2 \langle (\beta I - A)^{-1} \eta, \eta \rangle \leq \frac{1}{2} \theta_\alpha (\beta - \alpha)^{-1} |\eta|^2 - \frac{1}{2\beta} \langle F^\alpha \eta, \eta \rangle,$$

or in other words

$$(\beta I - A)^{-1} \leq \frac{1}{\theta_\alpha (\beta - \alpha)} I - \frac{1}{\beta \theta_\alpha^2} F^\alpha.$$

Taking the trace and applying Lemma 4.1, it follows that

$$(4.8) \quad \text{tr}[(\beta I - A)^{-1}] \leq \frac{n}{\theta_\alpha (\beta - \alpha)} - \frac{(1 - \theta_\alpha)}{\beta \theta_\alpha}.$$

The right hand side of (4.8) is easily seen to agree with that of (UB).

5. Bounds via Compensated Compactness

The arguments in this section are drawn from [21,22,33]. Since the desired bounds are relations among the eigenvalues of A , it is natural to use the basic variational principles (3.1) or (3.10) simultaneously for n different choices of ξ or $\bar{\sigma}$. This leads to variational principles for $\langle A\xi, \xi \rangle$ and $\langle A^{-1}\bar{\sigma}, \bar{\sigma} \rangle$ when ξ and $\bar{\sigma}$ are $n \times n$ matrices. Whereas the method of Hashin and Shtrikman adds and subtracts the energy of a reference medium, the compensated compactness method adds an indefinite quadratic form whose integral is controlled. The bound is obtained by applying convex duality to the resulting expression.

5A. More variational principles.

We shall use $\langle \cdot, \cdot \rangle$ for the Hilbert-Schmidt inner product on $n \times n$ matrices

$$\langle X, Y \rangle = \text{tr}(X^t Y)$$

as well as for the inner product on $\mathbb{R}^n$.

The basic variational principle (3.1) applied to n different vectors $\xi^1, \dots, \xi^n$ gives

$$\sum_{j=1}^n \langle A\xi^j, \xi^j \rangle = \inf_{\phi} \int_Q a(y) \sum_{j=1}^n |\xi^j + \nabla \phi_j|^2 dy,$$

where $\phi = (\phi_1, \dots, \phi_n)$ ranges over periodic, $\mathbb{R}^n$ -valued functions. Denoting by ξ the $n \times n$ matrix with columns $\{\xi^j\}$, this can be written

$$(5.1) \quad \langle A\xi, \xi \rangle = \inf_{\phi} \int_Q a(y) |\xi + \nabla \phi|^2 dy.$$

It is identical to (3.1), except that ξ is now an $n \times n$ matrix and ϕ a vector-valued function.

Similarly, applying (3.10) to n different vectors $\bar{\sigma}^1, \dots, \bar{\sigma}^n$ yields

$$\sum_{j=1}^n \langle A^{-1}\bar{\sigma}^j, \bar{\sigma}^j \rangle = \inf_{\substack{\text{div } \sigma^j = 0 \\ f_Q \sigma^j = \bar{\sigma}^j}} \int_Q a(y)^{-1} \sum_{j=1}^n |\sigma^j|^2 dy,$$

Writing $\bar{\sigma}$ for the $n \times n$ matrix with columns $\{\bar{\sigma}^j\}$, this becomes

$$(5.2) \quad \langle A^{-1} \bar{\sigma}, \bar{\sigma} \rangle = \inf_{\substack{\text{div} \sigma = 0 \\ f_Q \sigma = \bar{\sigma}}} \int_Q a(y)^{-1} \sum |\sigma|^2 dy.$$

It is identical to (3.10), except that $\bar{\sigma}$ is an $n \times n$ matrix and σ a periodic, matrix-valued function whose columns are divergence-free.

5B. Special quadratic functions.

We shall specify two quadratic functions of $n \times n$ matrices, R and S , such that

$$(5.3) \quad R(\xi) \leq \int_Q R(\xi + \nabla \phi) dy$$

$$(5.4) \quad S(\bar{\sigma}) \leq \int_Q S(\sigma) dy$$

whenever ϕ and σ are admissible for (5.1) and (5.2). Any convex function would have these properties, but the useful examples are nonconvex. They are provided by the following two lemmas.

Lemmas 5.1: The choice

$$(5.5) \quad R(\xi) = |\xi|^2 - (\text{tr} \xi)^2$$

satisfies (5.3) for any $n \times n$ matrix ξ and any periodic, R^n -valued, H^1 function ϕ .

Proof: The difference between the right and the left sides of (5.3) equals

$$(5.6) \quad \sum_{i,j=1}^n \int_Q [(\nabla_j \phi_i)^2 - (\nabla_i \phi_i)(\nabla_j \phi_j)] dy,$$

since the terms which are linear in $\nabla \phi$ vanish. Integration by parts gives

$$\int_Q [(\nabla_j \phi_i)(\nabla_i \phi_j) - (\nabla_i \phi_i)(\nabla_j \phi_j)] dy = 0$$

for each i and j . Therefore (5.6) equals

$$\sum_{i,j=1}^n f_Q [(\nabla_j \phi_i)^2 - (\nabla_j \phi_i)(\nabla_i \phi_j)] dy = \frac{1}{2} \sum_{i,j=1}^n f_Q |\nabla_i \phi_j - \nabla_j \phi_i|^2 dy,$$

which is clearly nonnegative. □

Lemma 5.2: The function

$$(5.7) \quad S(\bar{\sigma}) = |\bar{\sigma}|^2 - \frac{1}{n-1} (\text{tr } \bar{\sigma})^2$$

satisfies (5.4) for any $n \times n$ matrix $\bar{\sigma}$ and any periodic σ with divergence-free columns and mean value $\bar{\sigma}$.

Proof: Setting $\eta = \sigma - \bar{\sigma}$, the difference between the right and left sides of (5.4) equals

$$(5.8) \quad f_Q [|\eta|^2 - \frac{1}{n-1} (\text{tr } \eta)^2] dy.$$

Taking $Q = [0, 2\pi]^n$ for convenience, we consider the Fourier expansion of η :

$$\eta(y) = \sum_k \hat{\eta}(k) e^{ik \cdot y}.$$

Its coefficients $\hat{\eta}_s^r(k)$ are complex $n \times n$ matrices, and k ranges over $\mathbb{Z}^n$.

Moreover, $\eta(0) = 0$ since η has mean value zero, and

$$(5.9) \quad \sum_{s=1}^n \hat{\eta}_s^r(k) k_s = 0 \quad \text{for each } k,$$

since $\text{div } \eta = 0$.

For each $k \neq 0$, (5.9) shows that the matrix $\hat{\eta}(k)$ is singular. Therefore 0 is an eigenvalue, and

$$(5.10) \quad |\hat{\eta}(k)|^2 - \frac{1}{n-1} |\text{tr } \hat{\eta}(k)|^2 = \sum_{j=1}^{n-1} |\lambda_j|^2 - \frac{1}{n-1} \left| \sum_{j=1}^{n-1} \lambda_j \right|^2$$

in terms of the remaining $n-1$ (complex) eigenvalues, with the convention that $|X|^2 = \text{tr}(X^t \bar{X})$ when X is a complex matrix. The inequalities

$$\left| \sum_{j=1}^{n-1} \lambda_j \right|^2 \leq \left(\sum_{j=1}^{n-1} |\lambda_j| \right)^2 \leq (n-1) \sum_{j=1}^{n-1} |\lambda_j|^2$$

show that (5.10) is nonnegative. It follows by Plancherel's theorem that

$$\int_Q (|n|^2 - \frac{1}{n-1} |\text{tr} n|^2) dy = \sum_k |\hat{n}(k)|^2 - \frac{1}{n-1} |\text{tr} \hat{n}(k)|^2 \geq 0,$$

which completes the proof.

□

The preceding lemmas are special cases of a phenomenon called compensated compactness [5,26,32]. The general theory provides necessary and sufficient conditions for a quadratic function to satisfy (5.3) or (5.4): R must be rank-one convex, and S rank- $(n-1)$ convex. The choices discussed here are by no means the only examples, but they are among the few which are orthogonally invariant.

5C. The lower bound.

Given a matrix ξ , let ϕ^ξ be the minimizer of (5.1). Then

$$\begin{aligned} \langle A\xi, \xi \rangle &= \int_Q a(y) |\xi + \nabla \phi^\xi|^2 dy \\ &\geq \int_Q [a(y) |\xi + \nabla \phi^\xi|^2 - \gamma R(\xi + \nabla \phi^\xi)] dy + \gamma R(\xi) \end{aligned}$$

for any $\gamma > 0$, by (5.3), whence

$$\langle A\xi, \xi \rangle - \gamma R(\xi) \geq \inf_{\phi} \int_Q [a(y) |\xi + \nabla \phi|^2 - \gamma R(\xi + \nabla \phi)] dy.$$

This can be rewritten as

$$(5.11) \quad \Phi_A(\xi) \geq \inf_{\phi} \int_Q \Phi_a(y) I(\xi + \nabla \phi) dy,$$

where $\Phi_B(\xi)$ is the quadratic function of $n \times n$ matrices defined by

$$\Phi_B(\xi) = \frac{1}{2} \langle \xi, (B - \gamma) \xi \rangle + \frac{\gamma}{2} (\text{tr} \xi)^2$$

for any symmetric matrix B .

We shall apply duality to (5.11). If $B > \gamma I$ then Φ_B is convex, and can be written as a supremum of linear functions

$$(5.12) \quad \Phi_B(\xi) = \sup_{\eta} \langle \xi, \eta \rangle - \Phi_B^*(\eta).$$

The Legendre transformation Φ_B^* is again quadratic, and it is characterized by

$$(5.13) \quad \Phi_B^*(\eta) = \sup_{\xi} \langle \xi, \eta \rangle - \Phi_B(\xi).$$

We now restrict γ to the range $0 < \gamma < \alpha$, so that $a(y) > \gamma$ for each y . Then (5.11) and (5.12) yield

$$\Phi_A(\xi) \geq \inf_{\phi} \sup_{\eta} \int_Q \langle \xi + \nabla \phi, \eta \rangle - \Phi_a^*(y) I(\eta) dy,$$

in which η ranges over periodic, square-integrable, matrix-valued functions.

Interchanging the inf and sup - appealing as usual to [7] or to a direct argument - and evaluating the infimum over ϕ , we get

$$(5.14) \quad \Phi_A(\xi) \geq \sup_{\text{div } \eta = 0} \int_Q [\langle \xi, \eta \rangle - \Phi_a^*(y) I(\eta)] dy,$$

where " $\text{div } \eta = 0$ " means that the columns of η are divergence-free.

Restricting attention to constant test fields in (5.14), it follows that

$$\Phi_A(\xi) - \langle \xi, \eta \rangle \geq - \int_Q \Phi_a^*(y) I(\eta) dy$$

for any $n \times n$ matrices ξ and η . Minimizing over ξ on the left, evaluating the integral on the right, and rearranging, we finally conclude that

$$(5.15) \quad \Phi_A^*(\eta) \leq \theta_{\alpha} \Phi_{\alpha I}^*(\eta) + \theta_{\beta} \Phi_{\beta I}^*(\eta).$$

This gives infinitely many bounds on A , one for each η . To make them explicit we must calculate $\Phi_A^*(\eta)$. The optimal ξ for (5.13) (with B replaced by A) satisfies

$$(5.16) \quad (A-\gamma)\xi + \gamma(\text{tr } \xi)I = \eta,$$

and the extremal value is half the linear term

$$(5.17) \quad \Phi_A^*(\eta) = \frac{1}{2} \langle \xi, \eta \rangle.$$

The general inversion of (5.16) is messy, but the special case $\eta = I$ is easy: then (5.16) yields

$$\xi + \gamma(\text{tr } \xi)(A-\gamma I)^{-1} = (A - \gamma I)^{-1}$$

whence

$$\text{tr } \xi \cdot (1 + \gamma \text{tr}[(A-\gamma I)^{-1}]) = \text{tr}[(A-\gamma I)^{-1}],$$

and (5.17) leads to

$$(5.18) \quad \Phi_A^*(I) = \frac{1}{2} \text{tr } \xi = \frac{1}{2} \frac{\text{tr}[(A-\gamma I)^{-1}]}{1+\gamma \text{tr}[(A-\gamma I)^{-1}]}.$$

We need not calculate $\Phi_A^*(\eta)$ for other choices of η , because $\eta = I$ gives the desired lower bound: substitution of (5.18) into (5.15) yields

$$\frac{\text{tr}[(A-\alpha I)^{-1}]}{1+\alpha \text{tr}[(A-\alpha I)^{-1}]} < \theta_\alpha \cdot \frac{1}{\alpha} + \theta_\beta \cdot \frac{n}{\beta-\alpha + \alpha n},$$

after passage to the limit $\gamma \uparrow \alpha$. This is equivalent to (LB) by straightforward algebraic manipulation.

5D. The upper bound.

The argument is parallel to that for the lower bound. Given a matrix $\bar{\sigma}$, let $\hat{\sigma}$ be the minimizer of (5.2). Then

$$\begin{aligned} \langle A^{-1}\bar{\sigma}, \bar{\sigma} \rangle &= \int_Q a(y)^{-1} |\hat{\sigma}|^2 dy \\ &> \int_Q [a(y)^{-1} |\hat{\sigma}|^2 - \gamma S(\hat{\sigma})] dy + \gamma S(\bar{\sigma}) \end{aligned}$$

for any $\gamma > 0$, by (5.4), whence

$$\langle A^{-1} \bar{\sigma}, \bar{\sigma} \rangle - \gamma S(\bar{\sigma}) \geq \inf_{\substack{\text{div } \sigma=0 \\ \int_Q \sigma = \bar{\sigma}}} \int_Q [a(y)^{-1} |\sigma|^2 - \gamma S(\sigma)] dy.$$

This can be rewritten as

$$(5.19) \quad \Psi_A(\bar{\sigma}) \geq \inf_{\substack{\text{div } \sigma=0 \\ \int_Q \sigma = \bar{\sigma}}} \int_Q \Psi_a(y) I(\sigma) dy,$$

with

$$\Psi_B(\eta) = \frac{1}{2} \langle B^{-1} - \gamma \rangle \eta, \eta \rangle + \frac{\gamma}{2(n-1)} (\text{tr } \eta)^2$$

for any $n \times n$ matrix η and any symmetric B .

We shall dualize (5.19). First rewrite it as

$$(5.20) \quad \Psi_A(\bar{\sigma}) \geq \inf_{\sigma} \sup_{\xi, \phi} \int_Q [\Psi_a(y) I(\sigma) - \langle \sigma - \bar{\sigma}, \xi \rangle - \langle \sigma, \nabla \phi \rangle] dy,$$

in which σ ranges over periodic, matrix valued functions; ϕ over periodic, vector-valued functions; and ξ over constant matrices. Next write $\Psi_a(y) I(\sigma)$ as a supremum of linear functions: we have

$$(5.21) \quad \Psi_B(\sigma) = \sup_{\eta} \langle \eta, \sigma \rangle - \Psi_B^*(\eta)$$

with

$$(5.22) \quad \Psi_B^*(\xi) = \sup_{\sigma} \langle \xi, \sigma \rangle - \Psi_B(\sigma)$$

whenever $B^{-1} \geq \gamma I$. Restricting γ to the range $0 < \gamma < \beta^{-1}$, and using (5.21) with $B = a(y)I$, the right side of (5.20) becomes

$$\inf_{\sigma} \sup_{\xi, \phi, \eta} \int_Q [\langle \eta - \xi - \nabla \phi, \sigma \rangle + \langle \bar{\sigma}, \xi \rangle - \Psi_a^*(y) I(\eta)] dy.$$

Duality lets us switch the inf and the sup, and the infimum over σ is $-\infty$ unless $\eta = \xi + \nabla \phi$, so

$$(5.23) \quad \Psi_A(\bar{\sigma}) \geq \sup_{\xi, \phi} \int_Q [\langle \bar{\sigma}, \xi \rangle - \Psi_a^*(y) I(\xi + \nabla \phi)] dy.$$

We take $\phi = 0$ in (5.23) to get

$$\Psi_A(\bar{\sigma}) - \langle \bar{\sigma}, \xi \rangle \geq - \int_Q \Psi_a^*(y) I(\xi) dy$$

for each pair of matrices $\bar{\sigma}, \xi$. Minimizing over $\bar{\sigma}$ on the left, evaluating the integral on the right, and rearranging, it follows finally that

$$(5.24) \quad \Psi_A^*(\xi) \leq \theta_\alpha \Psi_{\alpha I}^*(\xi) + \theta_\beta \Psi_{\beta I}^*(\xi).$$

Like (5.15), this gives infinitely many bounds on A , one for each ξ ; and as before, the one we want is obtained when $\xi = I$. Indeed, for this ξ the minimizer σ of (5.22) (with B replaced by A) satisfies

$$I = (A^{-1} - \gamma I)\sigma + \frac{\gamma}{n-1} (\text{tr} \sigma) I,$$

whence

$$(1 - \frac{\gamma}{n-1} \text{tr} \sigma) \cdot \text{tr} ([A^{-1} - \gamma I]^{-1}) = \text{tr} \sigma.$$

The extremal value of (5.22) is half the linear term,

$$(5.25) \quad \Psi_A^*(I) = \frac{1}{2} \langle I, \sigma \rangle = \frac{1}{2} \frac{(n-1) \text{tr}[A(I-\gamma A)^{-1}]}{(n-1) + \gamma \text{tr}[A(I-\gamma A)^{-1}]}.$$

Substitution in (5.24) gives, after passage to the limit $\gamma \uparrow \beta^{-1}$,

$$\frac{(n-1)\beta \text{tr}[A(\beta I - A)^{-1}]}{(n-1) + \text{tr}[A(\beta I - A)^{-1}]} \leq \theta_\alpha \frac{(n-1)n\alpha\beta}{(n-1)\beta + \alpha} + \theta_\beta \frac{(n-1)n\beta}{n}$$

This is equivalent to (UB) by algebraic manipulation, making use of the identity

$$\text{tr}[A(\beta I - A)^{-1}] = \beta \text{tr}([\beta I - A]^{-1}) - n.$$

§6. Bounds via Representation Formulas

This method, first introduced by D. Bergman [3], has been developed and extended by many authors, a recent review is [24]. Our exposition draws primarily from [9] and [10]. The starting point is a pair of representation formulas for the effective conductivity A ,

$$(6.1a) \quad A - \alpha I = \alpha(\beta - \alpha) \int_0^1 \frac{d\mu(z)}{\alpha + (\beta - \alpha)z},$$

$$(6.1b) \quad \beta I - A = \beta(\beta - \alpha) \int_0^1 \frac{dv(z)}{\beta + (\alpha - \beta)z},$$

in terms of two nonnegative, symmetric matrix-valued measures μ and ν .

Perturbation theory around the case $\alpha = \beta$ leads to constraints on the measures:

$$(6.2a) \quad \int_0^1 d\mu = \theta_\beta I, \quad \text{tr} \left(\int_0^1 z d\mu \right) = \theta_\beta \theta_\alpha,$$

$$(6.2b) \quad \int_0^1 dv = \theta_\alpha I, \quad \text{tr} \left(\int_0^1 z dv \right) = \theta_\beta \theta_\alpha.$$

These, in turn, lead by algebraic manipulation to the desired bounds.

6A. Representation formulas.

We shall establish (6.1a) for any $\alpha, \beta > 0$ (without assuming that $\alpha < \beta$). There is no need to prove (6.1b) separately, since it follows from (6.1a) on interchanging the roles of α and β .

The starting point is the definition (3.4):

$$A\xi = \int_Q a(y)(\xi + \nabla\phi^\xi)dy,$$

where ϕ^ξ is the periodic solution of

$$\text{div}(a(y)(\xi + \nabla\phi^\xi)) = 0.$$

Since $a(y) = \alpha\chi_\alpha + \beta\chi_\beta$, the formula for A can be written

$$(6.3) \quad (A - \alpha I)\xi = (\beta - \alpha) \int_Q \chi_\beta (\xi + \nabla\phi^\xi) dy.$$

The equation for ϕ^ξ becomes

$$(\beta - \alpha) \operatorname{div}(\chi_\beta(\xi + \nabla \phi^\xi)) = -\alpha \Delta \phi^\xi,$$

or again

$$(6.4) \quad (I + (\frac{\beta}{\alpha} - 1) \nabla \Delta^{-1} \operatorname{div} \chi_\beta)(\xi + \nabla \phi^\xi) = \xi.$$

A combination of (6.3) and (6.4) gives

$$(6.5) \quad (A - \alpha I)\xi = (\beta - \alpha) \int_Q \chi_\beta (I + (\frac{\beta}{\alpha} - 1) \Lambda)^{-1} \xi \, dy,$$

where Λ is the linear operator

$$(6.6) \quad \Lambda \eta = \nabla \Delta^{-1} \operatorname{div}(\chi_\beta \eta),$$

acting on the space L^2_{per} of Q -periodic, square-integrable vector fields. One verifies easily that Λ is self-adjoint with respect to the inner product

$$[\eta, \lambda] = \int_Q \chi_\beta \langle \eta(y), \lambda(y) \rangle \, dy,$$

and that

$$0 \leq [\eta, \Lambda \eta] \leq [\eta, \eta].$$

Therefore the spectrum of Λ lies in $[0, 1]$. The spectral theorem provides a family of projection-valued measures $dP(z)$ on $[0, 1]$ such that

$$[f(\Lambda)\xi, \eta] = \int_0^1 f(z) [dP(z)\xi, \eta]$$

for any $\xi, \eta \in L^2_{\text{per}}$ and any bounded, continuous function f on $[0, 1]$.

Restricting attention to constant ξ and η , and choosing

$f(z) = (1 + (\frac{\beta}{\alpha} - 1)z)^{-1}$, it follows that

$$\int_Q \chi_\beta \langle (I + (\frac{\beta}{\alpha} - 1)\Lambda)^{-1} \xi, \eta \rangle \, dy = \int_0^1 \frac{[dP(z)\xi, \eta]}{1 + (\frac{\beta}{\alpha} - 1)z}.$$

In view of (6.5), this gives the desired representation formula with the measure μ defined by

$$\langle d\mu(z)\xi, \eta \rangle = [dP(z)\xi, \eta], \quad \xi, \eta \in R^n.$$

It is clear from the construction that μ takes its value in the space of non-negative, symmetric matrices, and that it depends on the geometry (through χ_β) but not on the microscopic conductivities α and β .

6B. Perturbation theory.

We need only verify (6.2a), since (6.2b) is obtained by switching the roles of α and β . The series

$$(1+\epsilon z)^{-1} = 1 - \epsilon z + \epsilon^2 z^2 + \dots$$

converges uniformly for $z \in [0, 1]$ when $|\epsilon| < 1$. With $\epsilon = \frac{\beta}{\alpha} - 1$, substitution into (6.1a) yields

$$(6.7) \quad \frac{1}{\alpha} A - I = \left(\frac{\beta}{\alpha} - 1\right) \int_0^1 d\mu - \left(\frac{\beta}{\alpha} - 1\right)^2 \int_0^1 z d\mu + \dots$$

On the other hand, the series

$$(I + \epsilon \Lambda)^{-1} = I - \epsilon \Lambda + \epsilon^2 \Lambda^2 + \dots$$

converges when $|\epsilon| < 1$, and substitution into (6.5) with $\epsilon = \frac{\beta}{\alpha} - 1$ gives

$$(6.8) \quad \frac{1}{\alpha} A - I = \left(\frac{\beta}{\alpha} - 1\right) \int_Q \chi_\beta I dy - \left(\frac{\beta}{\alpha} - 1\right)^2 \int_Q \chi_\beta \Lambda dy + \dots$$

Equating the coefficients of $\left(\frac{\beta}{\alpha} - 1\right)$ in (6.7) and (6.8) gives the first part of (6.2a):

$$\int_0^1 d\mu = \int_Q \chi_\beta I dy = \theta_\beta I.$$

Equating the coefficients of $\left(\frac{\beta}{\alpha} - 1\right)^2$ gives, for any $\xi \in R^n$,

$$\begin{aligned} \int_0^1 z \langle d\mu(z)\xi, \xi \rangle &= \int_Q \chi_\beta \langle \xi, \Lambda \xi \rangle dy \\ &= \int_Q \langle \chi_\beta \xi, \nabla \Delta^{-1} \operatorname{div} \chi_\beta \xi \rangle dy, \end{aligned}$$

using the definition (6.6) of Λ . This last expression is just $\langle F\xi, \xi \rangle$, the quadratic form of geometric parameters (4.1), with $x = x_B$. Its trace is $\theta_\alpha \theta_\beta$, by Lemma 4.1, and this gives the second part of (6.2a).

6C. An inequality on positive measures.

Relations (6.2a,b) constrain the masses and first moments of μ and ν . The main tool for drawing conclusions about Λ is the following lemma concerning real-valued measures.

Lemma 6.1. Let m be a positive measure on $[0,1]$ with

$$(6.9) \quad \int_0^1 dm = m_0, \quad \int_0^1 z dm = m_1.$$

If $s > 0$ and $s + t > 0$ then

$$\int_0^1 \frac{dm(z)}{s+tz} > m_0 \left(s + \frac{tm_1}{m_0} \right)^{-1}.$$

Proof: We may suppose $t \neq 0$, since the case $t = 0$ is trivial. Clearly

$$(6.10) \quad \int_0^1 \frac{dm(z)}{s+tz} = \int_0^1 \left(\frac{1}{s+tz} + \lambda z \right) dm - \lambda m_1$$

for any λ , by (6.9). If

$$(6.11) \quad \lambda = t^{-1} \left(\frac{s}{t} + \frac{m_1}{m_0} \right)^{-2}$$

then

$$\inf_{0 \leq z \leq 1} \frac{1}{s+tz} + \lambda z = t^{-1} \left(\frac{s}{t} + \frac{m_1}{m_0} \right)^{-1} \left(1 + \frac{m_1}{m_0} \left(\frac{s}{t} + \frac{m_1}{m_0} \right)^{-1} \right),$$

achieved exactly when $z = m_1/m_0$. Since m has mass m_0 ,

$$\begin{aligned} \int_0^1 \frac{dm}{s+tz} &\geq m_0 \cdot \inf_{0 \leq z \leq 1} \left(\frac{1}{s+tz} + \lambda z \right) - \lambda m_1 \\ &= m_0 t^{-1} \left(\frac{s}{t} + \frac{m_1}{m_0} \right)^{-1}. \end{aligned}$$

□

The choice of λ in the preceding proof can be motivated by convex duality. Indeed, we seek

$$\inf_{\int_0^1 dm = m_0} \sup_{\lambda} \int_0^1 \frac{dm}{s+tz} + \lambda \left(\int_0^1 z dm - m_1 \right),$$

where λ serves as a Lagrange multiplier for the constraint $\int_0^1 z dm = m_1$.

Interchanging the inf and sup leads to the dual problem

$$\sup_{\lambda} \left\{ \inf_{\int_0^1 dm = m_0} \int_0^1 \left(\frac{1}{s+tz} + \lambda z \right) dm \right\} - \lambda m_1.$$

Calculating the infimum explicitly then maximizing the result over λ leads to (6.11).

6D. The bounds.

We now establish the lower bound (LB), as a consequence of (6.1a) and (6.2a). Fixing α and β , let us work in a basis of eigenvectors of A , denoting by A_i the i^{th} eigenvalue of A . We shall write μ_i instead of μ_{ij} for the i^{th} diagonal element of μ . By (6.1a),

$$A_i - \alpha = \alpha(\beta - \alpha) \int_0^1 \frac{d\mu_i}{\alpha + (\beta - \alpha)z} \quad ;$$

also, by (6.2a),

$$\int_0^1 d\mu_i = \theta_{\beta} \quad , \quad \int_0^1 z d\mu_i = F_i$$

with

$$(6.12) \quad \sum_{i=1}^n F_i = \theta_{\alpha} \theta_{\beta}$$

Lemma 6.1 gives

$$A_i - \alpha \geq \alpha \theta_{\beta} \left(\frac{\alpha}{\beta - \alpha} + \frac{F_i}{\theta_{\beta}} \right)^{-1}$$

whence

$$(A_i - \alpha)^{-1} \leq \alpha^{-1} \theta_\beta^{-1} \left(\frac{\alpha}{\beta - \alpha} + \frac{F_i}{\theta_\beta} \right).$$

Summation over i gives

$$(6.13) \quad \text{tr}[(A - \alpha I)^{-1}] \leq \frac{n}{\theta_\beta(\beta - \alpha)} + \frac{\theta_\alpha}{\alpha \theta_\beta},$$

which is identical to (4.6) and equivalent to (LB).

The upper bound is obtained similarly as a consequence of (6.1b) and (6.2b):

$$\beta - A_i = \beta(\beta - \alpha) \int_0^1 \frac{dv_i}{\beta + (\alpha - \beta)z}$$

with

$$\int_0^1 dv_i = \theta_\alpha, \quad \int_0^1 z dv_i = G_i, \quad \sum_{i=1}^n G_i = \theta_\alpha \theta_\beta.$$

Lemma 6.1 gives

$$(\beta - A_i) \geq \beta \theta_\alpha \left(\frac{\beta}{\beta - \alpha} - \frac{G_i}{\theta_\alpha} \right)^{-1},$$

whence

$$(\beta - A_i)^{-1} \leq \beta^{-1} \theta_\alpha^{-1} \left(\frac{\beta}{\beta - \alpha} - \frac{G_i}{\alpha} \right)$$

and so

$$(6.14) \quad \text{tr}[(\beta I - A)^{-1}] \leq \frac{n}{\theta_\alpha(\beta - \alpha)} - \frac{\theta_\beta}{\beta \theta_\alpha},$$

the same as (4.8) and equivalent to (UB).

This proof explains the symmetry between the right hand sides of (LB) and (UB). Each becomes minus the other when the roles of α and β are exchanged (c.f. (6.13) and (6.14)). It must be so, because they follow from (6.1a) - (6.2a) and (6.1b) - (6.2b), which are related similarly.

§7. Comparison of the Methods.

We have presented three different methods for bounding the effective conductivity of a (generally anisotropic) composite. The first combines the Hashin-Shtrikman variational principles with linear relations on a tensor F of geometric parameters; the second uses lower semicontinuous, nonconvex quadratic forms; and the third combines representation formulas with perturbation theory. In the case discussed here - mixtures of two isotropic conductors in R^n - all three approaches, different as they are, lead to the same (optimal) result. In their generalizations, however, the methods diverge.

The first one, using the Hashin-Shtrikman variational principles, extends directly to a large class of problems; [17] discusses its use for multicomponent composites with anisotropic components, in the context of both conductivity and linear elasticity. Except for the choice of a reference medium there is very little flexibility - which makes the method easy to execute (an advantage), but restricts the range of conclusions (possibly a disadvantage). When both the reference medium and the composite itself are isotropic, this approach coincides with that executed by Hashin and Shtrikman in the 1960's.

The second method, using compensated compactness, is also easily extended to multicomponent composites with anisotropic components, and to systems such as linear elasticity. Being new, it has been applied as yet to just a few situations. In linear elasticity, it has given a generalization of the Hashin-Shtrikman bound on the bulk modulus of an isotropic mixture of isotropic materials [27]. It has also been used successfully in plate theory [8]. In general, it is less automatic than the variational approach: the class of lower semicontinuous, quadratic functionals is larger than the class of reference media. This presents a problem of choice: the method is flexible (an advantage) but allows a plethora of possibilities (a disadvantage). Perhaps more experience will provide better guidance on its use.

The situation is quite different as concerns our third method, using representation formulas. Unlike the other two, it makes essential use of the

hypothesis that there are only two components. However, again unlike the others, it yields information on complex conductivities, by analytic continuation of the representation formulas into the complex plane. This has been thoroughly explored; see [9,24,39] and the references cited there. A similar method has been applied to linear elasticity in [15], and a more sophisticated representation formula has been derived for use in the multicomponent case [11]. These remain areas of current research activity.

Since the three methods lead in many cases to the same bounds, it is natural to look for direct relations between them. An equivalence can indeed be drawn between the variational and representation formula methods [23], at least for two-component composites in the context of conductivity. A similar connection to the compensated compactness method has yet to be found.

Nonlinear problems present an important, largely uncharted territory for future exploration. If each component material is characterized by a convex variational principle, then the composite will behave so as to minimize a convex effective energy. Physical applications include both nonlinear conductors and Hencky plasticity. The Hashin-Shtrikman variational method is applied to such a problem in [31,37], yielding upper and lower bounds for the effective energy. It would be interesting to know what the compensated compactness approach gives for the same problem.

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