THE RIEMANN PROBLEM FOR SYSTEMS OF CONSERVATION LAWS OF MIXED TYPE

By

Haitao Fan

and

Marshall Slemrod

IMA Preprint Series # 766

February 1991

Equations (0.1a.b,d) consist of a system of conservation laws of hyperbolic-elliptic mixed type. Hyperbolic regions $w < \alpha$ and $(w > \beta)$ correspond, for van der Waals fluids, liquid and vapor phase region respectively. The Riemann problem (0.1) serves as a prototype model for the dynamics of phase transitions in van der Waals like materials.

It is well known that (0.1) generally admits many solutions and not everyone of them is physically relevant. Therefore, we need some admissibility criteria to single out the "physically correct" solutions, or better yet, to lead to the well posedness for Cauchy problems of the system (0.1a,b,d).

One admissibility criterion we can borrow from the theory of scalar conservation laws is that a solution of (0.1) is admissible if each shock of the solution can be connected by a travelling wave solution of the system

$$(0.2) \qquad \begin{aligned} u_t + p(w)_x &= \varepsilon u_{xx}, \\ w_t - u_x &= 0. \end{aligned}$$

More precisely, a shock solution of (0.1a,b,d) (u_1, w_1), (u_2, w_2) is admissible by viscosity travelling wave criterion if

$$(0.3) \qquad \begin{aligned} &\frac{dw}{d\zeta} = -s^2(w(\zeta) - w_1) - p(w(\zeta)) + p(w_1). \\ &w(-\infty) = w_1, \ w(+\infty) = w_2. \end{aligned}$$

has a solution. System (0.2) arises from the theory of viscoelastic bar [cf. Dafermos [11], Pago [56]). Slemrod [66] observes that this viscosity admissibility criterion is equivalent to the "chord condition" which states that

$$(0.4a) \qquad \frac{p(w) - p(w_1)}{w - w_1} \geq + \frac{p(w_2) - p(w_1)}{w_2 - w_1} \quad \text{if} \quad s \geq 0\,, \quad \text{or}$$

$$(0.4b) \qquad \frac{p(w) - p(w_1)}{w - w_1} \leq \frac{p(w_2) - p(w_1)}{w_2 - w_1} \quad \text{if} \quad s \leq 0$$

for any w between w_1 and w_2, where

$$s^2 = -\frac{p(w_2) - p(w_1)}{w_2 - w_1}\,.$$

Condition (0.4) is analogues to Oleinik's Condition E for scalar conservation laws. Epitomizing the experiences accumulated from detailed studies of special systems, Liu [52, 53] proposed Liu's criterion which can be viewed as a generalization of the chord condition. Liu's criterion yields a satisfactory solution of the Riemann problem for strictly hyperbolic systems when the waves are of moderate strength. Attempts for solving the Riemann problem for mixed type system (0.1) by using viscosity travelling wave criterion or variations of the chord conditions were carried out by James [46] and later by Shearer [61] as well as Hsiao [44, 45].

A satisfactory admissibility criterion for mixed type system (0.1a.b.d) should satisfy physical principles governing phenomenons under study. For materials which are viscously dominated, e.g., viscoelastic fluids and solids exhibiting phase transitions, a reasonable model may be the viscosity admissibility criterion [56]. For van der Waals fluids, the interfacial energy and hence the capillarity which corresponds to it is not negligible. Therefore, the viscosity criterion or chord conditions which only count viscosity are inadequate for van der Waals fluids. For example, the chord condition and viscosity criterion fail to comply with the well known Maxwell equal area rule, which states that stationary shock connecting m and M is the only possible stationary phase boundary. In order to take capillarity into consideration, we see that we need more high gradiant terms. Based on Korteweg's theory of capillarity [50], Slemrod [67] proposed the viscosity-capillarity criterion which states that a solution of (0.1) is admissible if it is an $\varepsilon \to 0+$ limit of solutions of

$$\begin{aligned} u_t + p(w)_x &= -\varepsilon^2 A w_{xxx} + \varepsilon u_{xx} \\ w_t - u_x &= 0 \end{aligned} \tag{0.5}$$

where $A > 0$ is constant. A localized version of this criterion is the viscosity-capillarity travelling wave criterion which says a shock solution $(u_1, w_1), (u_2, w_2)$ of (0.1a, b,d) is admissible if the boundary value for the travelling wave equations of (0.5)

$$\begin{aligned} &A\frac{d^2\hat{w}}{d\zeta^2} = -s\frac{d\hat{w}}{d\zeta} - s^2(\hat{w} - w_1) + p(\hat{w}) - p(w_1) \\ &\hat{w}(-\infty) = w_1, \ \hat{w}(+\infty) = w_2, \quad \hat{w}'(\pm\infty) = 0 \end{aligned} \tag{0.6}$$

A solution of (0.1) is said admissible according to viscosity-capillarity travelling wave criterion if each shock in the solution is admissible by this criterion.

The viscosity-capillarity travelling wave criterion not only admits the Maxwell line as the only admissible stationary phase boundary but also presents shock splitting phenomena [40] observed in experiments [49,72-76]. For shocks with two sides in hyperbolic region $w < \alpha$ (or $w > \beta$), the viscosity-capillarity travelling wave criterion admits classical compressive shocks.

There are also some investigation of the appropriateness of the entropy rate admissibility criterion, which was proposed by Dafermos [12], for the mixed type system (0.1.a,b) (cf. Hattori [41-43], Abeyaratne and Knowles [1] and Pence [57]). The entropy rate admissibility criterion, which dubs admissible those solutions that maximize the rate of entropy production, also admits the Maxwell line as an admissible shock solutions (cf [41]) and leads to the unique solution to the Riemann problem (cf [1] and [56]).

The program of this survey is follows: In §1, we shall derive the equations of motion of a viscous, heat conducting fluid possessing a Korteweg-van der Waals contribution to the stress [50]. We also derive the travelling wave equations corresponding to these equations. In §2, we review some results on the boundary value problem of these travelling wave equations. In §3, we shall first list some results on the existence of solutions of Riemann problem (0.1) admissible by the travelling wave criterion via the wave and shock curve construction approach. Then, we shall, in §3.0, §3.1 and §3.2, give details of the proof of the existence of solutions for (0.1) by the similarity viscosity approach, As a consequence of this, we get better results, in §3.3, on the existence of solutions satisfying the travelling wave criterion. In §3.4, we state results on the uniqueness and stability of the solution of (1.1) which is admissible according to the travelling wave criterion.

§1. **The equations of motion.** We consider the one dimensional motion of fluid processing a free energy

$$f(w,\theta) = f_0(w,\theta) + \frac{\varepsilon^2 A}{2}\left(\frac{\partial w}{\partial x}\right)^2 . \tag{1.1}$$

Here w is the specific volume, θ the absolute temperature, $A > 0$ a constant, and x the Lagrangian coordinate. The term

$$\frac{\varepsilon^2 A}{2}\left(\frac{\partial w}{\partial x}\right)^2 ,$$

where $\varepsilon > 0$ is a small parameter, is the specific interfacial energy. The graph of f_0 as a function of w for fixed θ will vary smoothly from a single well potential for $\theta > \theta_{\text{crit}}$ to double well potential for $\theta < \theta_{\text{crit}}$. The θ_{crit} is called the critical temperature. Discussions of such free energy formulations may be found in [2-10, 20, 28-32, 59-60, 78]

The stress corresponding to the free energy (1.1) is given by

$$T = \frac{\partial f}{\partial w} = \frac{\partial f_0}{\partial w}(w,\theta) - \varepsilon^2 A \frac{\partial^2 w}{\partial x^2} . \tag{1.2}$$

Note that there is no viscous forces in (1.2). Adding a viscous stress term gives us the stress of the form

$$T = -p(w,\theta) + \epsilon\frac{\partial u}{\partial x} - \varepsilon^2 A \frac{\partial^2 w}{\partial x^2} \tag{1.3}$$

suggested by Korteweg's theory of capillarity [50]. In (1.3), $u(x,t)$ denotes the velocity of the fluid, $\epsilon > 0$ is the viscosity and $p = \frac{\partial f_0}{\partial w}$ is the pressure.

The one dimensional balance laws of mass and linear momentum are easily written down:

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial x} \quad \text{(mass balance)} , \tag{1.4a}$$

$$\frac{\partial u}{\partial t} = \frac{\partial T}{\partial x} \quad \text{(linear momentum balance)} . \tag{1.4b}$$

The equation for balance of energy is more subtle. While a thorough examination of the energy equation appears in Dunn & Serrin [20] it is the conceptually simple approach of Felderhof [26] we recall here. Let $e(w,\theta)$ denote the internal energy. Felderhof's postulate is that the internal energy is influenced only by the component of internal stress $\tau = -p(w,\theta) + \epsilon\frac{\partial u}{\partial x}$, i.e., the balance of energy is given by

$$\frac{\partial e}{\partial t} = \tau\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x}\ , \tag{1.4c}$$

where h is the heat flux. Unlike equations (1.4a, b), equation (1.4c) is not in divergence form. To alleviate this difficulty we consider the specific total energy

$$E = \frac{u^2}{2} + e(w,\theta) + \frac{\varepsilon^2 A}{2}\left(\frac{\partial w}{\partial x}\right)^2$$

made up the specific kinetic, internal, and interfacial energy. Now compute the time rate of change of E:

$$\frac{\partial E}{\partial t} = u\frac{\partial u}{\partial t} + \varepsilon^2 A\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial t} = u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x} + \varepsilon^2 A\frac{\partial^2 w}{\partial x^2}\frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} + \varepsilon^2 A\frac{\partial w \partial^2 u}{\partial x}\frac{\partial^2 u}{\partial x^2}$$

where we have used the relation $T = \tau - \varepsilon^2 A\frac{\partial^2 w}{\partial x^2}$. We easily see that the balance of energy can be written as

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial x}(uT) + \varepsilon^2 A\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right) + \frac{\partial h}{\partial x}\ . \tag{1.5}$$

The term $\varepsilon^2 A\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}$ represents the "interstitial working" [20]. For simplicity we constitute h by Fourier's law: $h = \kappa\epsilon\frac{\partial\theta}{\partial x}$ where $\kappa\epsilon > 0$ is the (assumed constant) thermal conductivity. Then we may collect the balance laws and write them as

(1.6a)

$$\frac{\partial w}{\partial t} = \frac{\partial u}{\partial x} \qquad \text{(mass)}\ ,$$

(1.6b)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left\{-p(w,\theta) + \epsilon\frac{\partial u}{\partial x} - \varepsilon^2 A\frac{\partial^2 w}{\partial x^2}\right\} \qquad \text{(linear momentum)}\ ,$$

(1.6c)

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial x}\left\{u(-p + \epsilon\frac{\partial u}{\partial x} - \varepsilon^2 A\frac{\partial^2 w}{\partial x^2} + \varepsilon^2 A\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right) + \kappa\epsilon\frac{\partial\theta}{\partial x}\right\} \qquad \text{(energy)}\ .$$

The travelling wave equations corresponding to (1.6) with $\xi = (x - st)/\epsilon$, $w = w(\xi)$,

$u = u(\xi)$, $\theta = \theta(\xi)$ are

$$\frac{dw}{d\zeta} = v \ , \tag{1.7a}$$

$$A\frac{dv}{d\zeta} = -s^2(w - w_1) - p(w,\theta) + p(w_1,\theta_1) - sv \ , \tag{1.7b}$$

$$\begin{aligned}\kappa\frac{d\theta}{d\zeta} &= -s\{(e(w,\theta) - e(w_1,\theta_1)) \\ &\quad - \frac{s^2}{2}(w - w_1)^2 - \frac{Asv^2}{2} - p(w_1,\theta_1)(w - w_1)\}\end{aligned} \tag{1.7c}$$

where s is the speed of the travelling wave.

The most up to date results on the full system (1.7) with boundary values

$$\begin{aligned}(w,u,\theta)(-\infty) &= (w_1,u_1,\theta_1) \ , \\ (w,u,\theta)(+\infty) &= (w_2,u_2,\theta_2)\end{aligned} \tag{1.8}$$

were given by Grinfeld [33- 35] and Mischiakow [55]. In this survey, however, we shall confine ourself to the isothermal case, i.e., $\theta = \text{const}$, for (1.7):

$$\frac{dw}{d\zeta} = v \ , \tag{1.9a}$$

$$A\frac{dv}{d\zeta} = -s^2(w - w_1) + p(w_1) - p(w) - sv \ . \tag{1.9b}$$

The solvability and qualitative behavior of the boundary value problem of (1.9a,b) with

$$(w,u)(-\infty) = (w_1,u_1) \ , \quad (w,u)(+\infty) = (w_2,u_2) \tag{1.9c}$$

were studied by Slemrod [66-68, 70] and Hagan & Slemrod [40], Hagan & Serrin [39] and Shearer [62-64].

§2. The Viscosity-Capillarity Travelling Wave Criterion. We first review some results about the boundary value problem of the travelling wave equation (1.9). We are particularly interested in the case $w_1 < \alpha$, $w_2 > \beta$. The study of the solvability and qualitative behavior of (1.9) with $w_1 < \alpha$ and $w_2 > \beta$ is not only important, but also inevitable in the sense that each solution of (1.9) must have a shock which jumps over the spinodal region (α, β), at least for the case $A \geq 1/4$, (cf Fan [23]).

Let $w_1 \leq \alpha$ and $s \geq 0$. For simplicity, we assume the ray starting from $(w_1, p(w_1))$, with slope $-s^2$, to the right can intersect the graph of p at most at three points (cf Figure 2).

Figure 2

We denote the w-coordinates of these points by

$$w_2(w_1, s)\ ,\ w_3(w_1, s) \quad \text{and} \quad w_4(w_1, s)$$

respectively. w_1 and $w_k(w_1, S)$, $k = 2, 3, 4$, are equilibrium points of (1.9). w_1 and $w_3(w_1, s)$ are saddle points of (1.9) while $w_4(w_1, s)$ is a node of (1.9). By Fan [23], there is no travelling wave connecting w_1 and $w_2(w_1, s)$, at least for the case $A \geq 1/4$.

Now we consider the existence of a travelling wave solution of (1.9) connecting w_1 and $w_3(w_1, s)$. i.e. $\hat{w}(-\infty) = w_1, \hat{w}(+\infty) = w_3(w_1, s)$. For $w_1 \in [\gamma, m]$, if there is a $\overline{s} \geq 0$ such that the signed area between the graph of p and the chord connecting $(w_1, p(w_1))$ and $(w_3(w_1, \overline{s}), p(w_3(w_1, \overline{s})))$ is 0 (cf Figure 2), then there is a speed $s^* \geq 0$ such that

$$0 \leq s^* \leq \overline{s}$$

and the problem (1.9) with $s = s^*, w_2 = w_3(w_1, s^*)$ has a solution, which satisfies $\hat{w}'(\zeta) > 0$ and is a saddle-saddle connection, i.e.

$$\text{(2.2)} \qquad 0 \leq s^* < \sqrt{-p'(w_1)}, \quad s^* < \sqrt{-p'(w_3(w_1, s^*))}.$$

In (2.1), equality hold if and only if $\overline{s} = 0$. Furthermore, for any $0 < s < s^*$ the trajectory of (1.9) emanating from $(w_1, 0)$ will overshoot $w_3(w_1, s)$ and flow to $(w_4(w_1, s), 0)$ as $\zeta \to \infty$. In other words, for all $w_2 > w_4(w_1, s^*)$, there is a travelling wave solution of (1.9). Furthermore, this travelling wave solution is a sadddle-node connection, i.e.

$$\text{(2.3)} \qquad \sqrt{p'(w_1)} > s > \sqrt{-p'(w_4(w_1, s))}.$$

These statements were proved in Hagan & Slemrod's paper [40].

If $p(w)$ further satisfies

$$\text{(2.5)} \qquad p''(w)(w - w_0) > 0 \quad \text{for} \quad w \neq w_0,$$

for some $w_0 \in (\alpha, \beta)$ then, for $\gamma \leq w_1 \leq m$, there is a unique speed $s^* \geq 0$ such that w_1 can be connected to $w_3(w_1, s^*)$ by a travelling wave solution of (1.9) with $w_2 = w_3(w_1, s^*)$, which is a saddle-saddle connection (We notice that when (2.5) holds there is no $w_4(w_1, s)$ for $w_1 \in (\gamma, m]$.)). This fact was shown by Shearer [62, 63]. In fact, by arguments similar to that of Lemma 4.4 of Hagan & Slemrod [40], we can show more:

THEOREM 2.1. *Let (2.4) hold and $A > 1/8$. Then, for each fixed $w_1 < \alpha$, (1.9) can have at most one solution with $s \geq 0$ satisfying $w_2 = w_3(w_1, s)$, which is a saddle-saddle connection.*

Proof. Let $w(\zeta)$ be solution of (1.9) with $s \geq 0$ and $w(+\infty) = w_3(w_1, s)$ which is a saddle-saddle connection. Then, $w'(\zeta) > 0$. We rewrite (1.9a,b) as

$$\text{(2.5)} \qquad Av\frac{dv}{dw} = -sv - s^2(w - w_1) - p(w) + p(w_1),$$

where $v = w'(\zeta)$. We can parametrize by $v(w,s)$ the trejectory of $w(\zeta)$ in the $v > 0$ half-plane which connects to $(w_1, 0)$. and $(w_3(w-1,s)$. Assume, for contradiction, that there is a $0 < \bar{s} \neq s$ such that (1.9) with s replaced by $\bar{s}$ has a solution $\bar{w}(\zeta)$ which is also a saddle-saddle connection, i.e. $\bar{w}(+\infty) = w_3(w_1, \bar{s})$. Without loss of generality, we assume that

$$\bar{s} > s. \tag{2.6}$$

We parametrize the trejectory of $\bar{w}(\zeta)$ in the upper (w,v)-plane as $v(w,\bar{s})$. Clearly, $v(w,\bar{s})$ satisfies

$$Av(w,\bar{s})\frac{dv(w,\bar{s})}{dw} = -\bar{s}v(w,\bar{s}) - \bar{s}^2(w-w_1) - p(w) + p(w_1), \tag{2.7}$$

Fig.3

A calculation shows that

$$\frac{dv(w,s)}{dw}\mid_{w=w_1+} = \frac{1}{2A}\left(-s + \sqrt{(1-4A)s^2 - 4Ap'(w_1)}\right) > 0$$

which decreases as s increases when $A > 1/8$. Thus, for $w - w_1 > 0$ and small,

$$v(w,s) > v(w,\bar{s}). \tag{2.8}$$

Since $w_3(w_1,s) < w_3(w_1,\bar{s})$ and the trejectory $v(w,s)$ connects w_1 and $w_3(w_1,s)$ on the $v > 0$ half-plane, (2.8) implies that the trajectory $v(w,\bar{s})$ has to intersect the trajectory $v(w,s)$ at some $w \in (w_1, w_3(w_1,s)$. In other words,

$$v(w^*,s) = v(w^*,\bar{s}), \quad \frac{dv(w,s)}{dw}\mid_{w=w^*} \leq \frac{dv(w,\bar{s})}{dw}\mid_{w=w^*} \tag{2.9}$$

for some $w^* \in (w_1, w_3(w_1,s))$. Subtracting (2.6) from (2.7) yields

$$Av(w,\bar{s})\frac{dv(w,\bar{s})}{dw} - Av(w,s)\frac{dv(w,s)}{dw} = sv(w,s) - \bar{s}v(w,\bar{s}) + (s^2 - \bar{s}^2)(w-w_1).$$

Applying (2.9) to above equation, we obtain

$$0 \leq Av(w,s)\left(\frac{dv(w,\bar{s})}{dw} - \frac{dv(w,s)}{dw}\right) = v(w,s)(s-\bar{s}) + (s^2-\bar{s}^2)(w-w_1) < 0$$

which is a contradiction. □

We note when $p(w)$ is a cubic polynomial, we can have explicit solution for (1.9). Let

$$p(w) = p_0 - p_1(w-m)(w-M)\left(w - \frac{m+M}{2}\right) \tag{2.10}$$

where m and M are Maxwell constants. Then a solution of (1.9) is (cf Truskinovskii [77, 78])

$$w(\zeta) = \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} \tanh\left(\sqrt{\frac{p_1}{2A}}\frac{w_+ - w_-}{2}(\zeta - \zeta_0)\right). \tag{2.11}$$

For each w_- fixed, w_+ in (2.11) is determined by equations:

$$\begin{aligned} &3(1-6A)(2y - z + 1)^2 + z^2 = 1 , \\ &y = (M - w_+)/(M - m) , \\ &z = (w_+ - w_-)/(M - m) . \end{aligned} \tag{2.12}$$

The number of solutions of (2.12) ranges from zero to two. When (2.12) has two solutions, we get two solutions of (1.9) of the form (2.11); one of them has positive speed and the other negative. This is, of cause, consistent with Theorem 2.1.

§3. The Riemann problem.

Now, we return to the Riemann problem (0.1). One method for solving (0.1) is to construct wave and shock curves that are admissible according to some criteria and then construct a wave fan of waves and shocks that matches the initial data. This approach has been pursued by James [46], Shearer [61-64], Hsiao [45] and Fan [23]. Shearer [64] proved that solutions of (0.1) satisfying viscosity-capillarity travelling wave criterion exist for Riemann datum close to the Maxwell line, i.e.,

$$\begin{aligned} &|w_- - m| < \delta , \quad |w_+ - M| < \delta , \\ &|u_+ - u_-| < \delta \quad \text{for some} \quad \delta > 0 . \end{aligned}$$

To establish the existence of solutions of (0.1). Slemrod [69] applied the vanishing similarity viscosity approach to the system (0.1) which will be reviewed in next section. Later, Fan [22] proved, under the assumption

$$p(w) \to -\infty \text{ as } w \to \infty, \quad p(w) \to -\infty \text{ as } w \to \infty , \tag{3.1}$$

that there are solutions of (0.1) satisfying the viscosity-capillarity travelling wave criterion. In this section, we shall give details of their proofs. We shall also discuss the uniqueness and stability the solution of (0.1) admissible according to viscosity-capillarity travelling wave criterion.

§3.0. Riemann problems – the similarity viscosity approach. Admissibility criteria we reviewed above are shock admissibility criteria, which are local restrictions on points of jump discontinuity. We hope these local restrictions can characterize completely admissible solutions of (0.1). At this time, it is known only that this is true for scalar conservation laws (cf Volpert [82). There are, however, strong indications that shock admissibility criteria may be inadequate for systems that are not strictly hyperbolic (cf Glimm [27]) or for systems that change type (cf Shearer [63]). For example, in the context of (0.1) (cf Shearer [63]), it is not obvious, even if shock admissibility conditions are known $a - priori$, in what manner full solution which is a composition of admissible shocks and rarefaction waves is admissible. Thus it is important to experiment with various criteria with global authority, especially those motivated by physics.

One example of such criteria is to dub admissible those solutions which are $\varepsilon \to 0+$ limit of solutions of

$$\begin{aligned} u_t + p(w)_x &= \varepsilon u_{xx} - \varepsilon^2 A w_{xxx}, \\ w_t - u_x &= 0 \end{aligned} \tag{3.0.1}$$

as proposed in Slemrod [67]. We can also employ the conventional form of viscosity

$$\begin{aligned} u_t + p(w)_x &= \varepsilon u_{xx} \\ w_t - u_x &= \varepsilon w_{xx} \end{aligned} \tag{3.0.2}$$

to which (3.1) reduces when $A = 1/4$ (cf Slemrod [68]). Such an approach seems like an extremely difficult task at present. For the system (3.0.2) written in Eulerian coordinates with $p(\rho) = C\rho^\gamma$, for which the system is hyperbolic, DiPerna [19] and Ding, Chen and Luo [18] constructed solutions of the Cauchy problem, when the states at $\pm\infty$ are the same, by limits of viscous regularization and finite difference schemes. For hyperbolic-elliptic mixed type system (3.0.1), we only have some numerical shemes (cf. Shu [65] and references cited therein) which are at least visually good for Riemann problems.

Reasonable admissibility criteria should comply with the irreversibility of solutions and thus may impose restrictions on wave sources but not on wave sinks. Also admissibility criteria should be compatible with translations and dilations of coordinates, under which the system is invariant. Based on these requirements, Dafermos [16] argued that admissibility should be tested in the frame work of Riemann problem (Riemann [58], Lax [51], Liu [52]), i.e. in the context of solutions of the form $U(x,t) = V(x/t)$ which represent wave fans emanating from the origin at time $t = 0$.

Thus we utilize the similarity viscosity to handle the Riemann problem (0.1). The idea is to replace (0.1 a,b) by

$$\begin{aligned} u_t + p(w)_x &= \varepsilon t u_{xx} \ , \\ w_t - u_x &= \varepsilon t w_{xx} \ , \end{aligned} \tag{3.0.3}$$

which is invariant under dilatation of coordinates, and construct weak solutions of (0.1) as $\varepsilon \to 0+$ limits of solutions of (3.0.3). For convenience, we shall call solutions of (0.1) constructed in this way as solutions of (0.1) admissible according to similarity viscosity criterion. This approach has been pursued by Kalasinikov [49], Dafermos [13, 14], Tupciev [79, 80], Dafermos and DiPerna [17], Slemrod [70], Slemrod & Tzavaras [71] and Fan [21, 22, 24, 25].

To take the advantage of the invariance of (0.1) under dilatation of coordinates, we make variable change $\xi = x/t$ in (3.0.3). A simple computation shows that (3.0.3) reduces to the following system

$$\varepsilon u'' = -\xi u' + p(w)' , \tag{3.0.4a}$$

$$\varepsilon w'' = -\xi w' - w' , \tag{3.0.4b}$$

$$(u, w)(-\infty) = (u_-, w_-) , \quad (u, w)(+\infty) = (u_+, w_+) . \tag{3.0.4c}$$

Slemrod proved that (3.0.4) has a solution $(u(\xi), w(\xi)$ satisfying

$$w'(\xi) > 0 , \quad \text{when} \quad \alpha \le w(\xi) \le \beta , \tag{3.0.5}$$

by using Leray–Schauder type fixed point theory. Later, Fan [22] proved that the total variation of solutions of (3.0.4) is bounded uniformly in ε and hence established the existence of weak solutions of (0.1) satisfying the similarity viscosity criterion under the assumption

$$p(w) \to \infty \text{ as } w \to -\infty, \quad p(w) \to -\infty \text{ as } w \to \infty. \tag{3.0.6}$$

(In fact, we can prove the existence for (0.1) written in Eulerian coordinate under a weaker assumption on p which includes the van der Waals state function (cf. Fan [25]) We will give details of the proof for the existence of these solutions in the following sections §3.1, §3.2. In §3.3, we shall show that weak solutions of (0.1) constructed by the similarity viscosity approach in §3.1 and §3.2 are also admissible according to the viscosity-capillarity travelling wave criterion when $A = 1/4$. As a consequence, solutions of (0.1), admissible by the viscosity-capillarity travelling wave criterion with $A = 1/4$ exist if (3.0.5) holds. It is interesting to see that the travelling wave equation of the common form for most artificial viscosity (3.0.2) is the same as the equation of the shock profile of solutions of (0.1) we constructed by similarity viscosity approach. This is not surprising if we recall that (3.0.1) reduces to (3.0.2) when $A = 1/4$. From the above and the uniqueness and stability results of Fan [23] which we shall state in §3.4, we can see clearly that the $\varepsilon \to 0$ limit of solutions (3.0.4) with (3.0.5) is unique and stable in Lebesgue measure with respect to changes in Riemann datum if $p(w)$ satisfies

$$p'' < 0 \text{ if } w < \alpha, \quad p'' > 0 \text{ if } w > \beta. \tag{3.0.7}$$

The structure of solutions of (0.1) constructed in Slemrod [70] and Fan [22] by similarity viscosity approach is as follows: Each of these solutions can be imbeded on a continuous curve in (u, w) phase plan. Solutions must have a phase boundary, i.e. $w(\xi) \notin (\alpha, \beta)$ for any $\xi \in \mathbf{R}$. Solutions consist of two wave fans: $\xi < 0$ the first kind wave fan and $\xi > 0$ the second kind wave fan. A first (second) kind wave fan consists of 1-shocks and (2-shocks) and 1-simple waves (2-simple waves) and possibly the phase boundary and constant states. $\xi = 0$ is either a constant state or the phase boundary (cf. Fan [22]).

§**3.1. The existence of solutions to (3.0.4).** Here, we present the proof, given by Slemrod (1989), of the existence of solutions for the system (3.0.4):

We consider, instead of (3.0.4), the following altered system

$$\varepsilon u'' = -\xi u' + \mu p(w)' \tag{3.1.1a}$$

$$\varepsilon w'' = -\xi w' - \mu u' \tag{3.1.1b}$$

$$(u(\pm L),\ w(\pm L)) = (u_\pm,\ w_\pm) \tag{3.1.1c}$$

where $L > 1$, $0 \leq \mu \leq 1$.

For the system (3.1.1) we have the following lemmas:

LEMMA 3.1.1. *Let $(u(\xi),\ w(\xi))$ be a solution of (3.1.1). Then on any interval $(l_1, l_2) \subset (-L, L)$ for which $p'((w(\xi)) < 0$, one of the following holds.*

(i) $u(\xi)$ is strictly increasing (or decreasing) with no critical point in (l_1, l_2) while $w(\xi)$ has at most one critical point in (l_1, l_2) which must be a maximum (or minimum).

(ii) $w(\xi)$ is strictly increasing (or decreasing) with no critical point in (l_1, l_2) while $u(\xi)$ has at most one critical point which must be a maximum (or minimum).

LEMMA 3.1.2. *Let $(u(\xi), w(\xi))$ be a solution of (3.1.1) with $\mu > 0$. Then on any interval $(l_1, l_2) \subset (-L, L)$ for which $p'(w(\xi)) > 0$ the graph of $u(\xi)$ versus $w(\xi)$ is convex at points where $w'(\xi) > 0$ and concave at points where $w'(\xi) < 0$.*

THEOREM 3.1.3. *If $(u(\xi).w(\xi))$ is a solution of (3.1.1) with $w'(\xi) > 0$ when $\alpha \leq w(\xi) \leq \beta$, then*

$$\sup_{-L \leq \xi \leq -L} (|u(\xi)| + |w(\xi)| + |u'(\xi)| + |w'(\xi)| \leq M_0 \tag{3.1.2}$$

where M_0 is a constant independent of $\mu \in [0, 1]$ and $L > 1$.

With above preparations, we are ready to prove the existence theorem for (3.4):

THEOREM 3.1.4. *There are solutions of (3.4) which satisfy the constraints $w'(\xi) > 0$ when $\alpha \leq w(\xi) \leq \beta$. In other words, the one phase change data connecting orbit problem (3.4) possesses a one phase change solution.*

Proof. First notice that when $\mu = 0$, (3.1.1) possesses a unique solution

$$u_0(\xi) = \frac{(u_+ - u_-) \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon)d\zeta}{\int_{-L}^{L} \exp(-\xi^2/2\varepsilon)d\xi} + u_-,$$

$$w_0(\xi) = \frac{(w_+ - w_-) \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon)d\zeta}{\int_{-L}^{L} \exp(-\xi^2/2\varepsilon)d\xi} + w_-.$$

Also note that $w_0'(\xi) > 0, \ \xi \in [-L, L]$.

Now set $U(\xi) = u(\xi) - u_0(\xi), \ W(\xi) = w(\xi) - w_0(\xi)$ and impose boundary conditions

$$U(-L) = U(L) = W(-L) = W(L) = 0. \tag{3.1.3}$$

If u, w are to solve (3.1.1) we see that U, W must satisfy (1.4) and

$$\begin{aligned} \varepsilon U'' &= \mu p(w_0 + W)' - \xi U', \\ \varepsilon W'' &= -U' - \mu u_0' - \xi W'. \end{aligned} \tag{3.1.4}$$

Define the vectors

$$\mathbf{y}(\xi) = \begin{pmatrix} U(\xi) \\ W(\xi) \end{pmatrix}, \quad f(\xi, y) = \begin{pmatrix} p(w_0 + W) \\ -U(\xi) - u_0 \end{pmatrix}.$$

Then the system (3.1.4) takes the form

$$\varepsilon \mathbf{y}''(\xi) = \mu \mathbf{f}(\xi, \mathbf{y})'(\xi), \tag{3.1.5}$$

$$\mathbf{y}(-L) = 0, \quad \mathbf{y}(L) = 0. \tag{3.1.6}$$

Let $\mathbf{v} \in C^1([-L, L]; \mathbb{R}^2)$. Define $\mathbf{T}$ to be the solution map that carries $\mathbf{v}$ into $\mathbf{y}$ where $\mathbf{y}$ solves

$$\varepsilon \mathbf{y}''(\xi) = \mathbf{f}(\xi, \mathbf{v})' - \xi \mathbf{y}'(\xi), \tag{3.1.7}$$

$$\mathbf{y}(-L) = 0, \quad \mathbf{y}(L) = 0. \tag{3.1.8}$$

A straightforward computation shows that $\mathbf{y}(\xi)$ is given by the formula

$$\begin{aligned} \mathbf{y}(\xi) = z \int_{-L}^{\xi} \exp(-\zeta^2/2\varepsilon)d\zeta + \frac{1}{\varepsilon} \int_{-L}^{\xi} \mathbf{f}(\zeta, \mathbf{v}(\zeta))d\zeta \\ - \frac{1}{\varepsilon^2} \int_{-L}^{\xi} \int_{0}^{\zeta} \tau \mathbf{f}(\tau, \mathbf{v}(\tau)) \exp\left(\frac{\tau^2 - \zeta^2}{2\varepsilon}\right) d\tau \, d\zeta \end{aligned} \tag{3.1.9}$$

where

$$(3.1.10)\qquad z\int_{-L}^{L}\exp(-\zeta^2/2\varepsilon)d\zeta = -\frac{1}{\varepsilon}\int_{-L}^{L}\mathbf{f}(\zeta,\mathbf{v}(\zeta))d\zeta + \frac{1}{\varepsilon^2}\int_{-L}^{L}\int_{0}^{\zeta}\tau\mathbf{f}(\tau,\mathbf{v}(t))\exp\left(\frac{\tau^2-\zeta^2}{2\varepsilon}\right)d\tau\ d\zeta.$$

Notice the fixed points of $\mu\mathbf{T}$ are solutions of (3.1.5), (3.1.6) which in turn yield solutions of (3.1.1).

It is clear that $\mathbf{T}$ maps $C^0([-L,L];\mathbf{R}^2)$ continuously into $C^0([-L,L];\mathbf{R}^2)$. Of course this implies that $\mathbf{T}$ maps $C^1([-L,l];\mathbf{R}^2)$ continuously into $C^0([-L,L];\mathbf{R}^2)$. We now show $\mathbf{T}$ maps $C^1([-L,L];\mathbf{R}^2)$ continuously into $C^1([-L,L];\mathbf{R}^2)$.

For this purpose let $\mathbf{v}_1,\mathbf{v}_2 \in C^1([-L,L];\mathbf{R}^2)$, $\mathbf{v}_1 = (U_1,W_1), \mathbf{v}_2 = (U_2,W_2)$, and $y_1 = \mu\mathbf{T}\mathbf{v}_1, y_2 = \mu\mathbf{T}\mathbf{v}_2$. Differentiation of (1.10) shows

$$(3.1.11)\qquad y_1'(\xi) - y_2'(\xi) = (z_1-z_2)\exp(-\xi^2/2\varepsilon) + \frac{\mathbf{f}(\xi,\mathbf{v}_1(\xi))}{\varepsilon} - \frac{\mathbf{f}(\xi,\mathbf{v}_2(\xi))}{\varepsilon} - \frac{1}{\varepsilon^2}\int_0^{\xi}\tau(\mathbf{f}(\tau,\mathbf{v}_1(\tau)) - \mathbf{f}(\tau,\mathbf{v}_2(\tau))\exp\left(\frac{\tau^2-\xi^2}{2\varepsilon}\right)d\tau,$$

where z_1, z_2 are defined in the obvious manner.

Now let $\mathbf{v}_1,\mathbf{v}_2$ be in a finite ball B in $C^1([-L,L;]\mathbf{R}^2)$. In particular for $\mathbf{v} = (U,W)$ in B, $w_0 + W$ is uniformly bounded in $\mathbf{R}$ and hence p is a uniformly continuous function of the argument $w_0 + w$. But for $\delta > 0$ arbitrary we know from uniform continuity of p that there is $l(\delta) > 0$ such that $p(w_0(\xi)+W_1(\xi)) - p(w_0(\xi)+W_2(\xi))| < \delta$ if $|(W_1(\xi)+w_0(\xi)) - (W_2(\xi)+w_0(\xi))| < l(\delta)$, i.e. if $|W_1(\xi)-W_2(\xi)| < l(\delta)$. Hence $\sup\limits_{-L<\xi<L} |p(w_0(\xi)+W_1(\xi)) - p(w_0(\xi)+W_2(\xi))| < \delta$ if $\sup\limits_{-L<\xi<L} |W_1(\xi)-W_2(\xi)| < l(\delta)$ and so $\sup\limits_{-L<\xi<L} |p(w_0(\xi)+W_1(\xi)) - p(w_0(\xi)+W_2(\xi))| \to 0$ as $\sup\limits_{-L<\xi<L} |W_1(\xi)-W_2(\xi)| \to 0$. But this argument implies by the special nature of $\mathbf{f}(\xi,\mathbf{v}(\xi))$ that $\sup\limits_{-L<\xi<L} |\mathbf{f}(\tau,\mathbf{v}_1(\tau)) - \mathbf{f}(\tau,\mathbf{v}_2(\tau))| \to 0$ as $\sup\limits_{-L<\xi<L} |\mathbf{v}_1(\tau) - \mathbf{v}_2(\tau)| \to 0$. From (3.1.10), (3.1.11) we see then that $\sup\limits_{-L<\xi<L} |\mathbf{y}_1'(\xi) - \mathbf{y}_2'(\xi)| \to 0$ as $\sup\limits_{-L<\xi<L} |\mathbf{v}_1(\xi) - \mathbf{v}_2(\xi)| \to 0$, and so $\mathbf{T}$ is a continuous map of $C^1([-L,L];\mathbf{R}^2)$ into itself.

Now note that (1.8) implies that if $\mathbf{v}$ is in a bounded set of $C^1([-L,L];\mathbf{R}^2)$, $\mathbf{y}$ will be in a bounded set of $C^2([-L,l];\mathbf{R}^2)$. This is because $\mathbf{f}(\xi,\mathbf{v}(\xi))'$ is uniformly bounded.

Hence $\mathbf{T}$ *is a continuous compact map of* $C^1([-L,L];\mathbf{R}^2)$ *into itself.*

Now define $\Omega = \{U,W \in C^1([-L,L];\mathbf{R}^2)$ such that $W(-L)+w_0(-L) < \alpha, W(L) + w_0(L) > \beta$; $W'(\xi) + w_0'(\xi) > 0$ if $\alpha \geq W(\xi) + w_0(\xi) \leq \beta$; and $\sup\limits_{L<\xi<L} [|U(\xi)+u_0(\xi)| +$

$|W(\xi) + w_0(\xi)| + |U'(\xi) + u_0'(\xi)| + |W'(\xi) + w_0'(\xi)|\} < M_0 + 1\}$. *$\Omega$ is a bounded set* in $C^1([-L, L]; \mathbf{R}^2)$.

In addition Ω is open. To see this let $U, W \in \Omega$. Note the definition of Ω implies the set $A \overset{\text{define}}{=} \{\xi \in |[-L, L]; \alpha \leq w_0(\xi) + W(\xi) \leq \beta\}$ is a closed interval $[\xi_1, \xi_2]$. For if $A \neq [\xi_1, \xi_2]$ for some $\xi_1, \xi_2 \in [-L, L]$ means by the monotonicity of $w_0 + W$ on $[\xi_1, \xi_2]$ that for some $\overline{\xi} \notin [\xi_1, \xi_2]$ either $w_0(\overline{\xi}) + W(\overline{\xi}) = \alpha$ or $w_0(\overline{\xi}) + W(\overline{\xi}) = \beta$, with $w_0'(\overline{\xi}) + W'(\overline{\xi}) \leq 0$ in either case. Of course this would imply $U, W \notin \Omega$, contradiction.

Thus we have $A = [\xi_1, \xi_2]$, and we set $\bar{m} = \min\limits_{\xi \in A}(w_0'(\xi) + W'(\xi))$ which is positive. Since $w_0 + W \in C^1[-L, L]$ there is a larger interval $A_\delta \subseteq [-L, L], A_\delta \supset A, A_\delta = [\xi_1 - \delta, \xi_2 + \delta]$ for some small $\delta > 0$, so that $w_0'(\xi) + W'(\xi) \geq \frac{1}{2}\bar{m}$ for $\xi \in A_\delta$.

Let $D = \min \min\limits_{\xi_2 - \delta \leq \xi \leq L}(w_0(\xi) - W(\xi) - \beta), \min\limits_{-L \leq \xi \leq \xi_1 - \delta}(\alpha - w_0(\xi) - W(\xi))$. Since $w_0(\xi) - W(\xi) - \beta > 0$ on $[\xi_2 + \delta, L]$ and $\alpha - w_0(\xi) - W(\xi) > 0$ on $[-L, \xi_1 - \delta]$ we see that $D > 0$.

Now let $\overline{U}, \overline{W}$ be such that

$$\sup -L \leq \xi \leq L(|\overline{U}(\xi)| + |\overline{U}'(\xi)| + |\overline{W}(\xi)| + \overline{W}'(\xi) < v\ ,$$

where $v = \min(\frac{1}{2}\ D, \frac{1}{4}\bar{m})$. Consider $\xi \in [-L, L]$ for which $\alpha \leq w_0(\xi) + W(\xi) + \overline{W}(\xi) \leq \beta$. If we can show that $w_0'(\xi) + \overline{W}'(\xi) + W'(\xi) > 0$, we will have proven Ω is open. But we see in this case that

$$w_0(\xi) + W(\xi) - \beta \leq -\overline{W}(\xi), \quad \alpha - w_0(\xi) - W(\xi) \leq \overline{W}(\xi),$$

and hence $w_0(\xi) + W(\xi) - \beta \leq D/2,\ \alpha - w_0(\xi) - W(\xi) \leq D/2$. But this implies by the definition of D that $\xi \in (\xi_1 - \delta, \xi_2 + \delta)$. Thus we have shown that $\alpha \leq w_0(\xi) + W(\xi) + \overline{W}(\xi) \leq \beta$ implies $\xi \in A_\delta$. Now we compute at this value ξ:

$$w_0'(\xi) + W'(\xi) + \overline{W}'(\xi) \geq \frac{1}{2}\ \bar{m} + \overline{W}'(\xi) \geq \frac{1}{4}\ \bar{m} > 0.$$

Hence Ω is open.

Now we recall a well known theorem of Leray-Schauder type (see for example Mawhin [54] , Corollary IV.7).

PROPOSITION 3.1.5. *Let X be a real normed vector space, Ω an open bounded subset of X, and $\mathbf{T}$ a compact map of X into itself. If zero is an interior point of Ω and $\phi + \mu\mathbf{T}\phi$ for all $\phi \in \partial\Omega$, $0 < \mu < 1$, then $\mathbf{T}$ has at least one fixed point in $\overline{\Omega}$.*

In our problem we take $X = C^1([-L, L]; \mathbf{R}^2)$ and $\mathbf{T}, \Omega$ is as defined above. The origin is an interior point of Ω since the constraint $w_0'(\xi) + \overline{W}'(\xi) > 0$ is satisfied for all $\xi \in [-L, L]$ if $(\overline{U}, \overline{W})$ is a small $C^1([-L, L]; \mathbf{R}^2)$ perturbation. Note $\phi \in \partial\Omega,\ \phi = \mu\mathbf{T}\phi,\ \mu \in (0, 1)$, means that there is a solution $(u(\xi), w(\xi))$ of (1.1), (1.2), (1.3) which satisfies $w'(\xi) \geq 0$ if $\alpha \leq w(\beta) \leq \beta$ and either

(i) $w'(\xi_0) = 0,\ \alpha \le w(\xi_0) \le \beta$ for some $\xi_0 \in (-L, L)$

or

(ii) $\sup\limits_{-L<\xi<L} \{|u(\xi)| + |w(\xi)| + |u'(\xi)| + |w'(\xi)|\} = M_0 + 1$ or both (i) and (ii).

Let us first consider possibility (i). In this case either $\alpha < w(\xi_0) < \beta$, $w(\xi_0) = \alpha$, or $w(\xi_0) = \beta$. We consider these cases separately.

Case 1. $\alpha < w(\xi_0) < \beta$, $w'(\xi_0) = 0$. In this case there are three possibilities, either $w''(\xi_0) < 0$, $w''(\xi_0) > 0$, or $w''(\xi_0) = 0$. If $w''(\xi_0) < 0$ then $w(\xi_0)$ is a local maximum which implies $w'(\xi) < 0$ for some $\xi < \xi_0$, $|\xi - \xi_0|$ small. But this implies $\alpha < w(\xi) < \beta$ and violates the requirement that $w'(\xi) \ge 0$. An analogous statement holds if $w''(\xi_0) < 0$ and now $w(\xi_0)$ is a local minimum. The case $w''(\xi_0) = 0$ is excluded since $w''(\xi_0) = 0$, $w'(\xi_0) = 0$ implies because of (3.1.1b) that $u'(\xi_0) = 0$. But in this case uniqueness of solutions for (3.1.1a.b) as an initial value problem (see [7], Lemma 4.1) $u'(\xi_0) = 0$, $w'(\xi_0) = 0$ implies $u(\xi) = u(\xi_0)$, $w(\xi) = w(\xi_0)$ for all $\xi \in [-L, L]$ and hence we cannot satisfy (3.1.1c) $w_- < \alpha$, $w_+ > \beta$.

Case 2. $w(\xi_0) = \alpha$, $w'(\xi_0) = 0$. In this case there are again the three canonical possibilities, $w''(\xi_0) < 0$, $w''(\xi_0) > 0$, or $w''(\xi_0) = 0$. We can immediately dismiss $w''(\xi_0) > 0$ and $w''(\xi_0) = 0$ for the same reasons as in Case 1. So we need only consider $w''(\xi_0) < 0$. In this case $w(\xi_0) = \alpha$ is a local maximum. Hence if we are to satisfy $w(L) = w_+ > \beta$ we must proceed through a local minimum at $\xi_1 > \xi_0$, i.e. $w(\xi_1) < \alpha$, $w'(\xi_1) = 0$, $w''(\xi_1) \ge 0$; $w(\xi) < \alpha$, $w'(\xi) < 0, \xi_0 < \xi \le \xi_1$. Again $w''(\xi_1) = 0$ is impossible since that forces $u'(\xi_1) = 0$ and the uniqueness theorem [13], Lemma 4.1) is contradicted. Thus we need only consider $w''(\xi_1) > 0$. From (1.2) (3.1.1b) we see $u'(\xi_1) > 0$, $u'(\xi) < 0$ which implies u has a local maximum at a point $\xi_0 < \zeta < \xi_1$, $u'(\zeta) = 0$, $u''(\zeta) \le 0$, and again Lemma 4.1 of [13] tells us $u''(\xi) < 0$. Since $p'(w) < 0$ for $w < \alpha$ this implies by use of (3.1.1a) that $w'(\zeta) > 0$ which contradicts the fact that w is decreasing on ξ_0, ξ_1. Hence $w''(\xi_0) < 0$ is excluded as well.

Case 3. $w(\xi_0) = \beta$, $w'(\xi_0) = 0$. Here again we see we can exclude $w''(\xi_0) < 0$ and $w''(\xi_0) = 0$ immediately. If $w''(\xi_0) > 0$ it follows that $w(\xi_0) = \beta$ is a local minimum to satisfy $w(-L) = w_- < \alpha$ there must be $\xi_1 < \xi_0$ where $w(\xi_1) > \beta$ and w has a local maximum, $w(\xi) > \beta$ on (ξ_1, ξ_0). But the same reasoning as in Case 2 yields a contradiction.

From Cases 1, 2, 3 of (i) we see there is no solution of (3.1.1), $\mu \in (0, 1)$, $(u(\xi) - u_0(\xi), w(\xi) - w_0(\xi))$ in Ω for which (i) can hold. Thus all solutions of (3.1.1), $\mu \in (0, 1)$ in $\overline{\Omega}$ must satisfy $w'(\xi) > 0$ in $\alpha \le w(\xi) \le \beta$. But now Theorem 3.1.3 says (ii) cannot hold either. Thus we conclude from Prop. 3.1.5 that (3.1.1) possesses a solution *for which* $u(\xi) - u_0(\xi), w(\xi) - w_0(\xi)$ is in $\overline{\Omega}$.

To complete the proof we follow Dafermos [13] and extend the domains of u, w: Set

$$u(\xi; L) = u_+, \qquad w(\xi; L) = w_+, \qquad \xi > L,$$
$$u(\xi; L) = u_-, \qquad w(\xi; L) = w_-, \qquad \xi < -L.$$

The extended pair $\{u(\cdot, L), w(\cdot, L)\}$ form a sequence in $C^0((-\infty, \infty); \mathbf{R}^2)$ and by virtue of the hypothesis of theorem we know $\sup_{-L<\xi<L} \{|u'(\xi; L)|+|w'(\xi; L)|\} \leq M$. Thus the sequence $\{(u(\cdot; L),\ w(\cdot; L)\}$ is precompact in $C^0((-\infty, \infty); \mathbf{R}^2))$ and so there is a subsequence $L_n \to \infty$ as $n \to \infty$ since that $(u(\xi; L),\ w(\xi; L)) \to (u(\xi),\ w(\xi))$ uniformly as $n \to \infty$ on $(-\infty, \infty)$. As in Dafermos (1973) $u(\xi)\ w(\xi)$ is a solution of (3.0.4) and by its construction $w'(\xi) \geq 0$ if $\alpha \leq w(\xi) \leq \beta$. But by the same reasoning used in Cases 2, 3, this connecting orbit must satisfy the more restrictive requirement $w'(\xi) > 0$ if $\alpha \leq w(\xi \leq \beta$. This completes the proof of Theorem 3.1.4 □

§3.2. Solutions for the Riemann problem (0.1) exist. In this section we continue the program carried out in §3.1 to prove the existence for the Riemann problem (0.1). Our strategy is to prove $(u_\varepsilon(\xi),\ w_\varepsilon(\xi))$, the solutions of (3.0.4), have total variation bounded uniformly in ε and then to employ Helly's theorem to prove the convergence of these solutions, as $\varepsilon \to 0+$, to a weak solution of (0.1). To this end, it suffices, by Lemmas 3.1.1 and 3.1.2, that $(u_\varepsilon(\xi),\ w_\varepsilon(\xi))$ is bounded uniformly in ε. This is done by the following theorems supplied by Fan [22].

In this section, we assume $p(w)$ satisfies (3.0.6).

We shall prove, under (3.0.6), that

$$\sup_{\xi \in \mathbf{R}}(|u_\epsilon(\xi)|, |w_\epsilon(\xi)|) \leq C \tag{3.2.1}$$

where C is independent of ϵ.

THEOREM 3.2.1. *$u_\epsilon(\xi)$ are bounded from above, uniformly in ϵ:*

$$u_\epsilon(\xi) \leq \max(u_-, u_+) + \max(w_+ - \beta, \alpha - w_-) \max_{w \in [w_-, \alpha] \cup [\beta, w_+]} (\sqrt{-p'(w)}). \tag{3.2.1}$$

Proof. Without loss of generality, we assume that each $u_\epsilon(\xi)$ has a local maximum point $\xi = \theta_\epsilon$ with $w_\epsilon(\theta_\epsilon) \geq \beta$. The proof in the other case is similar. From (1.7) and the chain rule, we have

$$\epsilon \frac{d}{d\xi}\left(\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}\right) = \left(\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)} - \sqrt{-p'(w_\epsilon(\xi))}\right)\left(\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)} + \sqrt{-p'(w_\epsilon(\xi))}\right). \tag{3.2.2a}$$

This implies that, as ξ increases, $\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}$ is decreasing if $|\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}| \leq \sqrt{-p'(w_\epsilon(\xi))}$ and is increasing if $|\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}| \geq \sqrt{-p'(w_\epsilon(\xi))}$. Thus the "initial" condition

$$\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}\Big|_{\xi=\theta_\epsilon} = 0 \tag{3.2.2b}$$

leads to

$$\left|\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}\right| \le \max_{w_+\ge w\ge\beta}(\sqrt{-p'(w)}). \tag{3.2.3}$$

as long as $w_+ \ge w_\epsilon(\xi) \ge \beta$. By Lemma 2.1, $w_\epsilon(\xi)$ is increasing when $w_\epsilon(\xi) \ge \beta$. Thus (3.2.1) follows easily. □

THEOREM 3.2.2. *$u_\epsilon(\xi)$ are bounded from below, uniformly in ϵ.*

Proof. Assume the contrary. Then there exists a sequence of $\{\epsilon_n\}$ such that each $u_{\epsilon_n}(\xi)$ has a local minimum point τ_n with

$$u_{\epsilon_n}(\tau_n) \to -\infty \quad \text{as } n \to \infty, \tag{3.2.4}$$

$$w_{\epsilon_n}(\tau_n) \in (\alpha, \beta). \tag{3.2.5}$$

Without loss of generality, we assume that

$$\tau_n \ge 0, \quad \text{for } n = 1, 2, ... \tag{3.2.6}$$

The proof for the case $\tau_n \le 0$ is similar. We denote the maximum point of $u_{\epsilon_n}(\xi)$ in the region where $w_{\epsilon_n}(\xi) \ge \beta$ by ζ_n. By Lemma 3.1.1 and 3.1.2, we know that

$$u'_{\epsilon_n}(\xi) > 0 \quad \text{for } \xi \in (\tau_n, \zeta_n). \tag{3.2.7}$$

By integrating (3.4a) on (τ_n, θ) where $\theta \in (\tau_n, \zeta_n)$, we obtain

$$\begin{aligned} 0 \le \epsilon u'_{\epsilon_n}(\theta) = &\int_{\tau_n}^{\theta} -\xi u'_{\epsilon_n}(\xi) d\xi \\ &+ p(w_{\epsilon_n}(\theta)) - p(w_{\epsilon_n}(\tau_n)). \end{aligned} \tag{3.2.8}$$

It follows from (3.2.6) and (3.2.7) that $-\xi u'_{\epsilon_n}(\xi) < 0$ for $\xi \in (\tau_n, \zeta_n)$. Thus, in view of (3.2.5), we have

$$\begin{aligned} 0 \le \epsilon u'_{\epsilon_n}(\theta) &\le p(w_{\epsilon_n}(\theta)) - p(w_{\epsilon_n}(\tau_n)) \\ &\le p(w_{\epsilon_n}(\theta)) - p(\alpha). \end{aligned} \tag{3.2.9}$$

Therefore,

$$\alpha < w_{\epsilon_n}(\theta) \le \delta \quad \text{for } \theta \in (\tau_n, \zeta_n]. \tag{3.2.10}$$

Equation (3.2.9) also yields the useful inequality

$$0 \le \epsilon u'_{\epsilon_n}(\theta) \le p(\beta) - p(\alpha) \tag{3.2.11}$$

for $\theta \in [\tau_n, \zeta_n]$.

CLAIM. *There exist* $\eta_n \in (\tau_n, \zeta_n]$ *(cf. Fig.4) such that*

$$u_{\epsilon_n}(\eta_n) \geq u_+ - 2(\delta-\gamma) \max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}) - \frac{2(p(\beta)-p(\alpha))}{\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}} \tag{3.2.12}$$

and

$$\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\Big|_{\xi=\eta_n} \geq \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})}. \tag{3.2.13}$$

*Figure*4

By (3.2.4), we take ξ_n such that

$$\begin{aligned} u_{\epsilon_n}(\xi_n) =& u_{\epsilon_n}(\zeta_n) - 2(\delta-\gamma)\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}) \\ &\geq u_+ - 2(\delta-\gamma)\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}). \end{aligned} \tag{3.2.14}$$

For each n, there is a $\theta \in [\xi_n, \zeta_n]$ such that $u_{\epsilon_n}(\theta) \geq u_{\epsilon_n}(\xi_n)$ and

$$\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\Big|_{\xi=\theta} = \frac{w_{\epsilon_n}(\zeta_n) - w_{\epsilon_n}(\xi_n)}{u_{\epsilon_n}(\zeta_n) - u_{\epsilon_n}(\xi_n)}.$$

Substituting the denominator of the above by equation (3.2.14) and noticing that

$$|w_{\epsilon_n}(\zeta_n) - w_{\epsilon_n}(\xi_n)| \leq \delta - \gamma,$$

we obtain

$$\left|\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\Big|_{\xi=\theta}\right| \leq \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}}.$$

Thus, the following set is nonempty:

$$A := \left\{ \eta \in [\tau_n, \theta] \;\Big|\; \left|\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\right| \leq \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})} \text{ for } \xi \in [\eta, \theta] \right\}. \tag{3.2.15}$$

A straightforward computation based on (3.4) shows that

$$\frac{d^2 w_{\epsilon_n}}{du^2_{\epsilon_n}}(\xi) = \frac{-1}{\epsilon u'_{\epsilon_n}(\xi)}\left[1 + p'(w_{\epsilon_n}(\xi))\left(\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\right)^2\right]. \tag{3.2.16}$$

Hence, if $\xi \in A$, then

$$\left|\frac{d^2 w_{\epsilon_n}}{du_{\epsilon_n}^2}(\xi)\right| \geq \frac{1}{2\epsilon u'_{\epsilon_n}(\xi)} \geq \frac{1}{2(p(\beta)-p(\alpha))} \tag{3.2.17}$$

where we have used (3.2.11).

Now, we show that (3.2.12) and (3.2.13) hold at $\eta_n = \inf A$ and hence complete the proof of our claim. Indeed, by the definition of the set A,

$$\left|\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\,\Big|_{\xi=\inf A}\right| = \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})}.$$

In fact, we can show more i.e.

$$\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\,\Big|_{\xi=\inf A} = \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})}. \tag{3.2.18}$$

To see this, we note, from (3.2.16), that as long as $\xi \in A$ or $\xi = \inf A$,

$$\frac{d}{d\xi}\left(\frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)}\right) < 0. \tag{3.2.19}$$

So, if

$$\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\,\Big|_{\xi=\inf A} < 0,$$

then (3.2.19) implies that, for $\xi \in (\inf A - \mu, \inf A]$ for some $\mu > 0$,

$$\begin{aligned} 0 > \frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)} &> \frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)}\,\Big|_{\xi=\inf A} \\ &= -\frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})}, \end{aligned}$$

or in other words,

$$\left|\frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)}\right| \leq \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})},$$

for $\xi \in (\inf A - \mu, \inf A]$. This simply says that $(\inf A - \mu. \inf A] \subset A$ which is impossible. Thus (3.28) holds.

By using (3.2.18) and (3.2.17), we obtain

$$\begin{aligned} \frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})} &= \frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)}\,\Big|_{\xi=\inf A} \\ &= \frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)}\,\Big|_{\xi=\theta} + \int_\theta^{\inf A} \frac{d^2 w_{\epsilon_n}}{du_{\epsilon_n}^2}(\xi)\frac{du_\epsilon(\xi)}{dw_\epsilon(\xi)}d\xi \\ &\geq -\left|\frac{dw_\epsilon(\xi)}{du_\epsilon(\xi)}\,\Big|_{\xi=\theta}\right| + \left|\int_{u_\epsilon(\theta)}^{u_\epsilon(\inf A)} \left|\frac{d^2 w_{\epsilon_n}}{du_{\epsilon_n}^2}(\xi)\right| du_\epsilon\right| \\ &\geq -\frac{1}{2\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})} + |u_\epsilon(\theta) - u_\epsilon(\inf A)|\frac{1}{2(p(\beta)-p(\alpha))}. \end{aligned}$$

By virtue of (3.2.7), the above inequalities imply

$$
\begin{aligned}
u_{\epsilon_n}(\inf A) &\geq u_{\epsilon_n}(\theta) - \frac{2(p(\beta)-p(\alpha))}{\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})} \\
&\geq u_{\epsilon_n}(\xi_n) - \frac{2(p(\beta)-p(\alpha))}{\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})} \\
&\geq u_+ - 2(\delta-\gamma)\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}) \\
&\quad - \frac{2(p(\beta)-p(\alpha))}{\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})}.
\end{aligned} \tag{3.2.20}
$$

(3.2.18) and (3.2.20) prove our claim.

Similar analysis on (3.2.16) yields that

$$
\frac{dw_{\epsilon_n}(\xi)}{du_{\epsilon_n}(\xi)} \geq \frac{1}{2\max_{w\in[\gamma,\delta]}\sqrt{|p'(w)|}}
$$

or

$$
\frac{du_{\epsilon_n}(\xi)}{dw_{\epsilon_n}(\xi)} \leq 2\max_{w\in[\gamma,\delta]}\sqrt{|p'(w)|} \tag{3.2.21}
$$

for $\xi \in [\tau_n, \eta_n]$. Thus an estimation on the equation

$$
u_{\epsilon_n}(\eta_n) - u_{\epsilon_n}(\tau_n) = \int_{w_{\epsilon_n}(\tau_n)}^{w_{\epsilon_n}(\eta_n)} \frac{du_{\epsilon_n}(\xi)}{dw_{\epsilon_n}(\xi)} dw_{\epsilon_n}(\xi)
$$

based on (3.2.21), (3.2.5) and (3.2.10) leads to

$$
\begin{aligned}
u_{\epsilon_n}(\tau_n) &\geq u_{\epsilon_n}(\eta_n) - 2(\delta-\gamma)\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}) \\
&\geq u_+ - 4(\delta-\gamma)\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|}) - \frac{2(p(\beta)-p(\alpha))}{\max_{w\in[\gamma,\delta]}(\sqrt{|p'(w)|})},
\end{aligned}
$$

which is a contradiction to (3.2.4). □

Remark. Above proof for Theorem 3.2.2 only needs the existence of numbers δ and γ (cf. Fig. 1).

With the help of Theorem 3.1.1, 3.1.2, we can prove the following theorem by slightly modifying the idea of Dafermos [13].

THEOREM 3.2.3. *If $p(w)$ satisfies (3.0.6), then $w_\epsilon(\xi)$ are bounded uniformly in ϵ.*

Proof. We only prove that $w_\epsilon(\xi)$ are bounded from below uniformly in ϵ. The uniform boundedness of $w_\epsilon(\xi)$ from above can be proved similarly.

Assume the contrary. Then there is a sequence of $\{\epsilon_n\}$ such that each $w_{\epsilon_n}(\xi)$ has a local minimum point at $\xi = \tau_n$ with

$$w_{\epsilon_n}(\tau_n) \to -\infty \quad \text{as } n \to \infty. \tag{3.2.22}$$

Without loss of generality, we assume that

$$\tau_n \leq 0. \tag{3.2.23}$$

We know from Lemma 2.1 that $w_{\epsilon_n}(\xi)$ and $u_{\epsilon_n}(\xi)$ are decreasing on $(-\infty, \tau_n)$. By integrating (3.4b), we obtain

$$0 \leq -\epsilon w'_{\epsilon_n}(-\infty) = -\int_{-\infty}^{\tau_n} \xi w'_{\epsilon_n}(\xi) d\xi + u_- - u(\tau_n) \tag{3.2.25}$$

It is easy to see from (3.2.23) that $\xi w'_{\epsilon_n}(\xi) \geq 0$ on $(-\infty, \tau_n)$ and hence

$$0 \leq \int_{-\infty}^{\tau_n} \xi w'_{\epsilon_n}(\xi) d\xi \leq u_- - u(\tau_n) \leq u_- - u_*. \tag{3.2.25}$$

For any $\theta \leq \min(-1, \tau_n)$, we have

$$\int_{-\infty}^{\theta} \xi w'_{\epsilon_n}(\xi) d\xi \geq -\int_{-\infty}^{\theta} w'_{\epsilon_n}(\xi) d\xi = w_- - w_{\epsilon_n}(\theta)$$

Thus (3.2.25) leads to

$$w_{\epsilon_n}(\theta) \geq w_- + u_* - u_-. \tag{3.2.26}$$

It now remains to consider the case $\tau_n \in (-1, 0)$. Using the mean value theorem, we can choose, for each n, a $\theta \in (-2, -1)$ such that

$$|u'_{\epsilon_n}(\theta)| \leq u^* - u_*. \tag{3.2.27}$$

By integrating (3.4a) on $[\theta, \tau_n]$, we obtain

(3.2.28)

$$\begin{aligned} p(w_{\epsilon_n}(\tau_n)) &= \epsilon u'_{\epsilon_n}(\tau_n) - \epsilon u'_{\epsilon_n}(\theta) + p(w_{\epsilon_n}(\theta)) - \int_{\theta}^{\tau_n} \xi u'_{\epsilon_n}(\xi) d\xi \\ &\leq -\epsilon u'_{\epsilon_n}(\theta) + p(w_{\epsilon_n}(\theta)) - \int_{\theta}^{\tau_n} \xi u'_{\epsilon_n}(\xi) d\xi \\ &\leq -|\epsilon u'_{\epsilon_n}(\theta)| + p(w_{\epsilon_n}(\theta)) - \tau_n u_{\epsilon_n}(\tau_n) + \theta u_{\epsilon_n}(\theta) + \int_{\theta}^{\tau_n} u_{\epsilon_n}(\xi) d\xi. \end{aligned}$$

In view of (3.2.26), (3.2.27) and the uniform boundedness of $u_\epsilon(\xi)$ and τ_n, θ, it follows that the right hand side of (3.2.28) is bounded uniformly in ϵ. Thus, by virtue of Assumption 1, $w_{\epsilon_n}(\tau_n)$ are bounded from below uniformly in ϵ which contradictes (3.2.22). □

THEOREM 3.2.4. *If (3.0.6) holds, then there exist solutions of (0.1) which are admissible according to the similarity viscosity admissibility criterion.*

Proof. In view of Lemmas 3.1.1, 3.1.2, Theorem 3.2.1, 3.2.2, 3.2.3. $(u_\epsilon(\xi), w_\epsilon(\xi))$ have total variation bounded independently of ϵ. Helly's theorem implies that $(u_\epsilon(\xi), w_\epsilon(\xi))$ possesses a subsequence which converges a.e. on $(-\infty, \infty)$ to a function $(u(\xi), w(\xi))$ of bounded variation, By Theorem 3.2 of Dafermos [13] or Proposition 3.2 of Slemrod [70], $(u(x/t), w(x/t))$ is a weak solution of (1.1). □

§3.3. Solutions Constructed in §3.0, §3.1, §3.2 are also Admissible by Viscosity-Capillarity Travelling Wave Criterion.

In this section, we study the relation between the similarity viscosity admissibility criterion and the viscosity-capillarity travelling wave criterion. Results in this section are contained in Fan [22, 24].

Consider $u_{\epsilon_n}(\xi)$ (or $w_{\epsilon_n}(\xi)$) as a multivalued function of w (or u). Denote these functions by $U_{\epsilon_n}(w)$ (or $W_{\epsilon_n}(u)$). An analysis based on Lemma 3.1.1, 3.1.2 shows that $U_{\epsilon_n}(w)$ may be a two-valued function. For details about this, see Lemma 2.4 of Slemrod [70].

LEMMA 3.3.1 [22]. *$\{U_{\epsilon_n}(w)\}$ has a subsequence which converges to a continuous curve $U(w)$. Furthermore, $(u(\xi), w(\xi))$ lies on the curve $U(w)$ for every $\xi \in \mathbb{R}$.*

Remark. $U(w)$, like $U_{\epsilon_n}(w)$, may be a two valued function.

For convenience, we parametrize the curve $u = U(w)$ by (U(s),W(s)) where s is the length of the arc of $u = U(w)$ joining (u_-, w_-) and the point $(U(s), W(s))$. Since the curve $u = U(w)$ does not intersect itself, the parametrization is bijective. In this kind of parametrization, s increases when ξ increases. We call the curve (U(s),W(s)) the *base curve* of the solution $(u(\xi), w(\xi))$.

Now, we study the discontinuities of $(u(\xi), w(\xi))$. Let ξ_0 be a point of discontinuity of $(u(\xi), w(\xi))$. We use C_{ξ_0} to denote the portion of the base curve in the (u, w)-plane that connects points $(u(\xi_0-), w(\xi_0-))$ and $(u(\xi_0+), w(\xi_0+))$. We fix $(\bar{u}, \bar{w}) \in C_{\xi_0}$. We define, for n large, $\xi_{\epsilon_n}(w; \bar{u}, \bar{w})$ to be the branch of the inverse function of $w = w_{\epsilon_n}(\xi)$ for which

$$u_{\epsilon_n}(\xi_{\epsilon_n}(\bar{w}; \bar{u}, \bar{w})) \to \bar{u} \tag{3.3.1}$$

as $n \to \infty$. We further define, for n large, $\xi_{\epsilon_n}, \hat{u}_{\epsilon_n}, \hat{w}_{\epsilon_n}$ by the relations

$$\xi_{\epsilon_n} := \xi_{\epsilon_n}(\bar{w}) + \epsilon\zeta, \tag{3.3.2}$$

$$\hat{u}_{\epsilon_n}(\zeta) := u_{\epsilon_n}(\xi_{\epsilon_n}), \tag{3.3.3}$$

$$\hat{w}_{\epsilon_n}(\zeta) := w_{\epsilon_n}(\xi_{\epsilon_n}). \tag{3.3.4}$$

After a modification of the proof of Proposition 3.4 of Dafermos [14], we obtain the following lemma:

LEMMA 3.3.2. FAN [22]. *Let ξ_0 be a point of discontinuity of $(u(\xi), w(\xi))$. For $(\hat{u}_{\epsilon_n}(\zeta), \hat{w}_{\epsilon_n}(\zeta))$ defined above, there is a subsequence of $\{\epsilon_n\}$, also denoted by $\{\epsilon_n\}$, such that*

$$(3.3.5) \qquad (\hat{u}_{\epsilon_n}(\zeta), \hat{w}_{\epsilon_n}(\zeta)) \to (\hat{u}(\zeta), \hat{w}(\zeta)) \in C^1(\mathbf{R}; \mathbf{R}^2) \quad \text{as } n \to \infty$$

uniformly for ζ in a compact subset of $\mathbf{R}$. $(\hat{u}(\zeta), \hat{w}(\zeta))$ satisfies the following initial value problem:

$$(3.3.6a) \qquad \frac{d\hat{u}(\zeta)}{d\zeta} = -\xi_0(\hat{u}(\zeta) - u(\xi_0-)) + p(\hat{w}(\zeta)) - p(w(\xi_0-))$$

$$(3.3.6b) \qquad \frac{d\hat{w}(\zeta)}{d\zeta} = -\xi_0(\hat{w}(\zeta) - w(\xi_0-)) - [\hat{u}(\zeta) - u(\xi_0-)],$$

$$(3.3.6c) \qquad \hat{u}(0) = \bar{u} \quad \hat{w}(0) = \bar{w}.$$

Furthermore, $(\hat{u}(\zeta), \hat{w}(\zeta))$ lies on C_{ξ_0}.

Remark. By the definitions (3.3.1-3.3.4), we can see easily that as ζ increases, $(\hat{u}(\zeta), \hat{w}(\zeta))$ moves along C_{ξ_0} in the direction from $(u(\xi_0-), w(\xi_0-))$ to $(u(\xi_0+), w(\xi_0+))$.

Remark. The travelling wave equations of the most common form of artificial viscosity (3.2) for shocks with speed ξ_0 is the same as (3.3.6a,b).

LEMMA 3.3.3 [24]. *The boundary value problem of (3.3.6a,b) and*

$$(3.3.7a) \qquad (\hat{u}(-\infty), \hat{w}(-\infty)) = (u_1, w_1),$$

$$(3.3.7b) \qquad (\hat{u}(+\infty), \hat{w}(+\infty)) = (u_2, w_2)$$

has a solution if and only if

$$(3.3.8) \qquad u_2 = u_1 - \xi_0(w_2 - w_1)$$

and the following boundary value problem has a solution:

$$(3.3.9a) \qquad \frac{d^2\hat{w}}{d\zeta^2} = -2\xi_0\frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0^2(\hat{w}(\zeta) - w(\xi_0-)) - (p(\hat{w}(\zeta)) - p(w(\xi_0-))).$$

$$(3.3.9b) \qquad (\hat{w}(-\infty) = w_1, \quad \hat{w}(+\infty) = w_2.$$

Proof. Suppose w_1 and w_2 can be connected by a shock of speed ξ_0. Then (3.3.8), which is one of the the Rankine-Hugoniot conditions, is satisfied. Eliminating $\hat{u}(\zeta)$ in (3.3.6a, b), we obtain (3.3.9a).

Suppose (3.3.9) has a solution $\hat{w}(\zeta)$. We define

$$\hat{u}(\zeta) = -\frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0(\hat{w}(\zeta) - w_1) + u_1. \tag{3.3.10}$$

(3.3.9) and (3.3.10) imply (3.3.6a,b) and (3.3.7). □

Comparing (3.3.6) with the traveling wave equation (0.6):

$$A\frac{d^2\hat{w}}{d\zeta^2} = -\xi_0\frac{d\hat{w}(\zeta)}{d\zeta} - \xi_0^2(\hat{w}(\zeta) - w_1) - (p(\hat{w}(\zeta)) - p(w_1)). \tag{3.3.11}$$

we can see that (3.3.6a) is a special case of (3.3.11) when $A = 1/4$, In[Slemrod (1983), Slemrod proposed the viscosity-capillarity travelling wave criterion. We state his criterion in a more general setting as follows:

Definition 3.3.4. (i) We say w_1 and w_2 can be connected by a shock with speed ξ_0 if (3.3.11) and (3.3.7) has a solution.

(ii) A shock of speed ξ_0 with (u_1, w_1) and (u_2, w_2) on its sides, where $(u_1, w_1), (u_2, w_2)$ satisfy the Rankine-Hugoniot conditions at ξ_0, is admissible by traveling wave criterion if there are $v_k, k = 1, 2, ..., n \in \mathbf{N}$, and $w_1 = v_1, v_n = w_2$ such that v_k can be connected by a shock with speed ξ_0 to v_{k+1}, k=1,2,...,n-1.

(iii) We say a solution $(u(\xi), w(\xi))$ of (1.1) is admissible by the traveling wave criterion if every discontinuity of $(u(\xi), w(\xi))$ is admissible in the sense of (ii).

COROLLARY 3.3.5 [24]. *If $p(w)$ has the property that any straight line in (w,p)-plane intersects the graph of $p(w)$ at finite many points, then the solutions of (0.1) given by Theorem 3.2.4, which are admissible by the similarity viscosity criterion, are also admissible by the traveling wave criterion with $A = 1/4$. Hence, solutions of (0.1) admissible by the traveling wave criterion with $A = 1/4$ always exist.*

Proof. Let ξ_0 be a point of discontinuity of $(u(\xi), w(\xi))$ given in Theorem 3.2.4. Without loss of generality, we assume $w(\xi_0-) < w(\xi_0+)$. By Lemma 3.3.2, C_{ξ_0} satisfies (3.3.6). By the property of $p(w)$ assumed in this theorem, we know that there are only finitely many points (u, w) satisfy the Rankine-Hugoniot conditions at ξ_0

$$-\xi_0(u(\xi_0-) - u) + p(w(\xi_0-)) - p(w) = 0, \tag{3.3.11a}$$

$$-\xi_0(w(\xi_0-) - w) - u(\xi_0-) + u = 0. \tag{3.3.11b}$$

Since C_{ξ_0} can be oriented in the direction from $(u(\xi_0-), w(\xi_0-))$ to $(u(\xi_0+), w(\xi_0+))$, we can assume that the points on C_{ξ_0} satisfying (3.3.11) are (u_1, w_1), $(u_2, w_2), ..., (u_n, w_n)$. which are ordered in the direction of C_{ξ_0}. Let $(\bar{u}_1, \bar{w}_1) \in C_{\xi_0}$ in (3.3.6c) be on the portion of C_{ξ_0} between (u_1, w_1) and (u_2, w_2) and $(\hat{u}_1(\zeta), \hat{w}_1(\zeta))$ to be the corresponding solution of (3.3.6). By Lemma 3.3.2, 3.3.3, we can see that w_1 is connected to some w_{j_1}, $1 < j_1 \leq n$. If $j_1 = n$ then our theorem is proved. If otherwise, we repeat above procedure to see that w_{j_1} is connected to w_{j_2}, $j_1 < j_2 \leq n$. Repeating this process finite times, we can prove that w_{j_k} can be connected to $w_{j_{k+1}}$, $k = 0, 1, ..., m \leq n$, where $w_{j_0} = w(\xi_0-)$ and $w_{j_m} = w(\xi_0+)$. Thus, the first statement is proved. The last statement is a consequence of Theorem 3.2.4 and the first statement of this theorem. □

§**3.4. The uniqueness and stability of the solution of (0.1).** In §3.0-§3.3, we established the existence of solutions of the Riemann problem (0.1) which are admissible according to the viscosity-capillarity travelling wave criterion. Now, it is natural to ask questions about the uniqueness and stability of these solutions i.e. solutions of (0.1) which are admissible by viscosity-capillarity travelling wave criterion. Shearer [63] proved that for each (u_-, w_-) with $w_- < m$ $(w_- > M)$, there are (u_+, w_+) with $w_+ < \alpha$ $(w_+ > \beta)$ such that (0.1) has two admissible centered solutions. The assumptions he used in [63] is

$$p'' < 0 \text{ if } w < \alpha \text{ }, p'' > 0 \text{ if } w > \beta \text{ }. \tag{3.4.1}$$

Note that the initial data in Shearer's results is in the same phase. If the initial datum are in different phase i.e. $w_- < \alpha$, $w_+ > \beta$, this kind of nonuniqueness does not hold, as claimed by the following theorem supplied by Fan [23]:

THEOREM 3.4.1. *Suppose $p(w)$ satisfies (3.4.1) and $w_- < \alpha < \beta < w_+$. Then (i) (0.1) has a unique solution within the class of centered wave solutions satisfying the viscosity-capillarity travelling wave criterion.*

(ii) Let $(u(\xi), w(\xi))$ be the solution of (0.1) satisfying the viscosity-capillarity travelling wave criterion. For any $\epsilon > 0$ and $\gamma > 0$, there is a $\delta > 0$ such that if

$$|u_- - \bar{u}_-| + |u_+ - \bar{u}_+| + |w_- - \bar{w}_-| + |w_+ - \bar{w}_+| < \delta$$

then

$$meas\{\xi \in \mathbb{R} \mid |u(\xi) - \bar{u}(\xi)| + |w(\xi) - \bar{w}(\xi)| \geq \epsilon\} < \gamma$$

where $(\bar{u}(\xi), \bar{w}(\xi))$ is the solution of (0.1a, b), admissible by the viscosity-capillarity travelling wave criterion, with Riemann initial values $(\bar{u}_-, \bar{w}_-)$ and $(\bar{u}_+, \bar{w}_+)$, and 'meas' denotes the Lebesgue measure.

In §3.0-3.2, we constructed solutions of (0.1) as the $\epsilon_n \to 0+$ limits of solutions of (3.0.4). Is it possible that different sequences ϵ_n give us different limits and hence different solutions of (0.1)? Since solutions we constructed in §3.0-3.3 satisfy the viscosity-capillarity travelling wave criterion also, we can see from Theorem 3.4.1 that the answer to above question is almost negative: The $\epsilon \to 0+$ limit of solutions of (3.0.4) with (3.0.5) is unique, if $p(w)$ satisfies (3.4.1). In this case, of course, the (ii) of Theorem 3.4.1 also holds for this solution.

§4. Mixture State Solutions of (0.1). For Van der Waals fluids, the region $w < \alpha$ and $w > \beta$ corresponds to the liquid and vapor phase respectively. We know that the only stationary phase boundary admissible according to the viscosity-capillarity travelling wave criterior is (u_-, m), (u_-, M), where m and M are the Maxwell constants and u_- is arbitrary. Based on this fact, we shall construct a weak solution of (0.1) with a region of mixture of liquid and vapor as foolows: Choose an arbitrary subset A of $\mathbb{R}_- := \{x \in \mathbb{R} : x < 0\}$. We define

$$(u(x,t), w(x,t)) := \begin{cases} (u_-, m) & \text{if } x \in A, \\ (u_-, M) & \text{if } x \in R \backslash A, \\ (u_-, M) & \text{if } 0 < x/t < s, \\ (u_+, w_+) & \text{if } s < x/t, \end{cases} \tag{4.1}$$

where (u_-, M) and (u_+, w_+) satisfy the Rankine-Hugoniot conditions with speed $s > 0$. We claim that $(u(x,t), w(x,t))$ defined by (4.1) is a weak solution of (0.1). We assume that $p(w)$ is convex for $w \geq M$. Then, by Theorem 3.3 of [40], the jump discontinuity at $\xi = s > 0$ is admissible by the viscosity-capillarity travelling wave criterion For $x < 0$, $u(x,t)$ and $p(w(x,t))$ are constants, and $w_t(x,t) = 0$. Thus, $(u(x,t), w(x,t))$ defined by (4.1) is indeed a weak solution of (0.1). We notice that the region $x < 0$ consists of mixture of liquid and vapor. Similarly, we can construct a weak solution of (0.1) with $x > 0$ being the region of mixture.

REFERENCES

[1] R. Abeyaratne and J. Knowles, *Kinetic relations and the propagation of phase boundaries in solids*, to appear in Archive for Rational Mechanics and Analysis..

[2] Aifantis, E.C. and J. Serrin, *The mechanical theory of fluid interfaces and Maxwell's rule*, J. Colloidal Interface Science 96 (1983), 517–529.

[3] Aifantis, E.C. and J. Serrin, *Equilibrium solutions in the mechanical theory of fluid microstructures*, J. Colloidal Interface Science 96 (1983), 530–547.

[4] Berdichevskii, V. and L. Truskinovskii, *Energy structure of localization*, In Studies in Applied Mechanics 12, Local effects in the analysis of structures, ed. P. Ladereze, Elsevier (1985) 127–158.

[5] J. Carr, M. Gurtin, and M. Slemrod, *Structured phase transitions on a finite interval*, Archive for Rational Mechanics and Analysis 86 (1984) 317–351.

[6] P. Casal,, J. de mecanique, Paris 5 No. 2 (1966),149.

[7] P. Casal, and H. Gouin, Comptes Rendus Acad. Sci. Paris (306) II (1988), 99–104.

[8] P. CASAL, AND H. GOUIN, *Sur les interfaces liquide-vapor nonisothermes*, J.de mecanique théorique et appliquée, Vol 7, No. 6 (1988), 1–43.

[9] P. CASAL AND H. GOUIN, *A representation of liquid vapour interfaces by using fluids of grade n*, Annales de Physique, Special issue No. 2 (1988).

[10] P. CASAL, AND H. GOUIN, *Invariance properties of fluids of grade n*, Lecture Notes in Math., Springer, Berlin (1988).

[11] C.M. DAFERMOS, *The mixed initial-boundary value problem for the equations of nonlinear one-dimensional visco-elasticity*, D. Diff. 61 (1969), 71–86.

[12] C.M.DAFERMOS, *The entropy rate admissibility criterion for solutions of hyperbolic conservation laws*, J. Diff. Eqs., 14, 202-212 (1973).

[13] C.M. DAFERMOS, *Solution of the Riemann problem for a class of hyperbolic conservation laws by the viscosity method*, Arch. Rational Mech. Analysis **52** (1973), 1–9.

[14] C.M. DAFERMOS, *Structure of solutions of the Riemann problem for hyperbolic systems of conservation laws*, Arch. Rational Mech. Analysis **53** (1974), 203–217.

[15] C.M. DAFERMOS, *Hyperbolic systems of conservation laws*, in: "Systems of Non-linear Partial Differential Equations", J.M. Ball, ed.; Boston: Reidel, 1983, pp. 25–70.

[16] C.M. DAFERMOS, *Admissible wave fans in nonlinear hyperbolic systems*, Arch. Rational Mech. Analysis **106** (1989), 243–260.

[17] C.M. DAFERMOS & R.J. DIPERNA, *The Riemann problem for certain classes of hyperbolic systems of conservation laws*, J. Diff. Eqs. **20** (1976), 90–114.

[18] DING XIAXI, CHEN GUI–QIANG AND LUO PEIZHU, *Convergence of the fractional step Lax–Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics*, Comm. Math. Phys. 121 (1989), 63–84.

[19] R.J. DIPERNA, *Convergence of the viscosity method for isentropic gas dynamics*, Comm. Math. Phys. 91 (1983), 1–30.

[20] J. E. DUNN, AND J. SERRIN, *On the thermodynamics of interstitial working*, Archive for Rational Mechanics and Analysis 88 (1985), 95–133.

[21] H.–T. FAN, *The structure of solutions of the gas dynamics equations and the formation of the vacuum state*, to appear in Quarterly of Appl. Math..

[22] H.–T. FAN, *A limiting "viscosity" approach to the Riemann problem for materials exhibiting changes of phase (II)*, to appear in Arch. Rational Mech. Anal..

[23] H.–T. FAN, *The uniqueness and stability of the solution of the Riemann problem for a system of conservation laws of mixed type. preprint (1990)*..

[24] H.–T. FAN, *The existence, uniqueness and stability of the solutions of the Riemann problem of a system of conservation laws of mixed type , Ph.D thesis, Univ. of Wisconsin-Madison, (1990).*

[25] H.–T. FAN,, *A vanishing viscosity approach on the dynamics of phase transitions in van der Waals fluids*, Forthcoming.

[26] FELDERHOF, B.U, *Dynamics of the diffuse gas-liquid interface near the critical point*, Physica 48 (1970), 514–560.

[27] J. GLIMM, *The interactions of nonlinear hyperbolic waves*, Comm. Pure Appl. Math. 41 (1988), 569–590.

[28] H. GOUIN,, Comptes rendus Acad. Sci Paris (305), II (1987), 833.

[29] H. GOUIN,, Comptes rendus Acad. Sci. Paris (306), II (1988), 755–759.

[30] H. GOUIN,, J. de mécanique, Paris, 20, No. 2 (1981) 273.

[31] H. GOUIN,, Research *Notes in Mathematics*, 46 Pitman, London (1981), 128.

[32] H. GOUIN,, Mech. Res. Comm. 3 (1976) 151.

[33] M. GRINFELD, *Topological techniques in dynamic phase transitions*, Ph.D. thesis, Rensseler Polytechnic Institute, Troy, NY (1986).

[34] M. GRINFELD, *Isothermal dynamic phase transitions: existence of "cavitation waves"*, Proc. Royal Society of Edinburgh Sect. A 107 A (1987) 153–163.

[35] M. GRINFELD M., *Nonisothermal dynamic phase transitions*, Quarterly of Applied Math. 47 (1989), 71–84.

[36] M. E. GURTIN,, *On a theory of phase transitions with interfacial energy*, Archive for Rational Mechanics and Analysis 87 (1984), 187–212.

[37] M. E. GURTIN,, *On phase transitions with bulk, interfacial and boundary energy*, Archive for Rational Mechanics and Analysis 96 (1986), 243–264.

[38] M. E. GURTIN,, *Some results and conjectures in the gradient theory of phase transitions*, In Metastability and Incompletely Posed Problems, S. antman, J.E. Ericksen, D. Kinderlehrer, I. Muller, eds., Springer (1987), 135–146.

[39] R. HAGAN & J. SERRIN, *Dynamic changes of phase in a van der Waals fluid*, In New Perspective in Thermodynamics, ed. J. Serrin, Springer–Verlag (1985).

[40] R. HAGAN AND M. SLEMROD, *The viscosity-capillarity admissibility criterion for shocks and phase transitions*, Arch. Rational Mech. Anal. 83 (1984), 333–361.

[41] H. HATTORI, *The Riemann problem for a van der Waals fluid with entropy rate admissibility criterion*, Isothermal case, Arch. Rational Mech. Anal. 92 (1986), 247–263.

[42] H. HATTORI, *The Rieman problem for a van der Waals fluid with entropy rate admissibility criterion*, Nonisothermal case. J. Diff. Eqs. 65 (1986), 158–174.

[43] H. HATTORI, *The entropy rate admissibility criterion and the double phase boundary problem*, Contemporary Math. 60 (1987), 51–65.

[44] L.HSIAO, *Admissibility criterion and admissible weak solutions of Riemann problem for conservation laws of mixed type,*, workshop proceedings on nonlinear evolution equations that change type, to appear in IMA volumes in mathematics and its applications..

[45] L.HSIAO, *Uniqueness of admissible solutions of Riemann problem of system of conservation laws of mixed type*, J. Diff. Eqs. 86 (1990) 197-233.

[46] R.D. JAMES, *The propagation of phase boundaries in elastic bans*, Arch. Rational Mech. Anal. 73 (1980), 125–158.

[47] A.S. KALASNIKOV, *Construction of generalized solutions of quasi-linear equations of first order without convexity conditions as limits of solutions of parabolic equations with a small parameter*, Dokl. Akad. Nauk. SSSR 127 (1959, 27–30 (in Russian).

[48] B.L. KEYFITZ & H.C. KRANZER, *A system of non-strictly hyperbolic conservation laws arising in elasticity theory*, Arch. Rational Mech. Anal. 72 (1980), 219–241.

[49] Y.-G. KIM, *Shock splitting with phase change in a liquid-vapor system*, Ph.D. thesis. Renselaer Polytechnic Institute (1987).

[50] D.J. KORTEWEG, *Sur la forme que prennent les equations du mouvement des fluides si L'on tient compte des forces capillarires par des variations de densité*, Archives Neerlandaises des Sciences Exactes et Naturells (1901).

[51] P.D.LAX, *Hyperbolic systems of conservation laws*, Comm. Pure Appl. Math. 10 , 537-566 (1957)..

[52] T.-P.LIU, *The Riemann problem for general system of conservation laws*, J. Diff. Eqs. 18, 218-234 (1975)..

[53] T.-P.LIU, *Admissible solutions of hyperbolic system of conservation laws*, Memoirs AM. Math. Soc. 240(1981) 1-78.

[54] M. MAWHIN, *Topological degree methods in nonlinear boundary value problems*, Conferences Board of Mathematical Sciences Regional Conference Series in Mathematics No. 40, American Math. Society (1979).

[55] K. MISCHIAKOW, *Dynamic phase transitions: a connection matrix approach*, To appear Proc. March 1989, Conference on Equations that change type, IMA, Minneapolis, ed. M. Shearer, Springer (1989).

[56] R. PEGO, *Phase transitions in one dimensional nonlinear viscoelasticity: admissibility and stability*, Arch. Rational Mech. Anal. 97 (1987), 353–394.

[57] T.J. PENCE, *On the mechanical dissipation of solutions to the Riemann problem for impact involving a two-phase elastic material.*

[58] B. RIEMANN, *Ueber die Fortpflanziung ebener Luftwellen von endlicher Schwingungs-weite Gŏtt.pin Abh. Math. CI. 8 (1960) 43–65.*

[59] J. SERRIN,, *Phase transitions and interfacial layers for van der Waals fluids*, SAFA IV Conference, Recent Methods in Nonlinear Analysis and Applications, Naples, eds. A. Canfora, S. Rionero, C. Sbordone, C. Trombetti, Liguori, Naples (1980), 169–176.

[60] J. SERRIN,,, *The form of interfacial surfaces in Korteweg's theory of phase equilibria*, Q. Applied Math. 41 (1983), 351–364.

[61] M.SHEARER, *Riemann problem for a class of conservation laws of mixed type*, J.Diff.Eqs. 46 (1982) 426-443..

[62] M.SHEARER, *Admissibility criteria for shock wave solutions of a system of conservation laws of mixed type*, Proceedings Royal Society of Edinburgh 93 (1983) 233-244..

[63] M.SHEARER, *Nonuniqueness of admissible solutions of Riemann initial value problem for a system of conservation laws of mixed type*, Arch.Rational Mech.Anal. 93 (1986) 45-59..

[64] M.SHEARER, *Dynamic phase transitions in a van der Waals gas*, Quarterly of Applied Math., 46(1988) 631-636..

[65] C.-W. SHU, *Numerical methods for systems of conservation laws of mixed type using flux splitting*, preprint (1990).

[66] M.SLEMROD, *Admissibility criterion for propagating phase boundaries in a van der Waals fluid*, Archive for Rational Mechanics and Analysis, 81, 301-315 (1983)..

[67] M. SLEMROD, *Dynamic phase transitions in a van der Waals fluid*, J. Differential Equations 52 (1984), 1–23.

[68] M. SLEMROD, *Dynamics of first order phase transitions*, Phase transitions and Material Instabilities in Solids, ed. (1984), 163–203.

[69] M. SLEMROD, *A limiting "viscosity" approach to the Riemann problem for materials exhibiting change of phase*, Arch.Rational Mech.Anal. 105(1989) 327-365..

[70] M.SLEMROD, *Remarks on the travelling wave theory of the dynamics of phase trasitions*, preprint, 1989..

[71] M. SLEMROD AND A. TZAVARAS, *A limiting viscosity approach for the Riemann problem in isentropic gas dynamics*, Indiana Univ. Math. J. 38 (1989), 1047–1073.

[72] P.A. THOMPSON, G.C. CAROFANO, Y.-G. KIM, *Shock waves and phase changes in a large-heat-capacity fluid emerging from a tube*, J. Fluid Mechanics 166 (1966) 57–92.

[73] P.A. THOMPSON, H. CHAVES, G.E.A. MEIER, Y.-G. KIM AND H.-D. SPECKMANN, *Wave splitting in a fluid of large heat capacity*, J. Fluid Mechanics 185 (1987) 385–414.

[74] P.A. THOMPSON, Y.-G. KIM, *direct observation of shock splitting in a vapor-liquid system*, Physics of Fluids 26 (1983) 3211–3215.

[75] P.A. THOMPSON, Y.-G. KIM AND G.E.A. MEIER, Proc. 14th Int. Symposium on Shock tubes and Waves, *Shock tube studies with incident liquefaction shocks*, R.D. Archer and B.E. Milton, eds., New South Wales Univ. Press, Sydney (1984) 413–420.

[76] P.A. THOMPSON AND D.A. SULLIVAN, *On the possibility of complete condensation shock waves in retrograde fluids*, J. Fluid Mechanics 70 (1975) 639–650.

[77] L.M. TRUSKINOVSKII, *Dynamics of non-equilibrium phase boundaries in a heat conducting non-linearly elastic medium*, PMM USSR 51 (1987) 777–784 (English transl. of Prikl. Matem. Mekhan. 51 (1987), 1009–1019.

[78] L.M. TRUSKINOVSKII, *Structure of an isothermal phase jump*, Dokl. Akad. Nauk SSSR, 285 (1985) 2.

[79] V.A.TUPCIEV, *The asymptotic behavior of the solutions of Cauchy problem for the equation* $\epsilon^2 t u_{xx} = u_t + [\phi(u)]_x$ *that degenerates for* $\xi = 0$ *into the problem of the decay of an arbitrary discontinuity for the case of a rarefaction wave,*, Z.Vycisl.Mat.Fiz. 12(1972) 770-775; English translation in USSR Commput. Math. and Phys. 12..

[80] V.A.TUPCIEV, *On the method of introducing viscosity in the study of problems involving the decay of discontinuity*, Dokl.Akad.Nauk.SSSR, 211 (1973) 55-58..

[81] J.D. VAN DER WAALS,, Veshandel. Konik. Akad. Weten. Amsterdam Vol. 1 No. 8 (1893); Translation of J.D. van der Waals' "The thermodynamic theory of capillarity under the hypothesis of a continuous variation of density" by S. Rowlinson, J. Statistical Physics 20 (1979) 197–244.

[82] A.I. VOLPERT, *The space BV and the quasilinear equations*, Mat Sbornik 73 (115) (1967), 355–302. English translation, Math USSR Sbornik 2, (1967) 225–267.

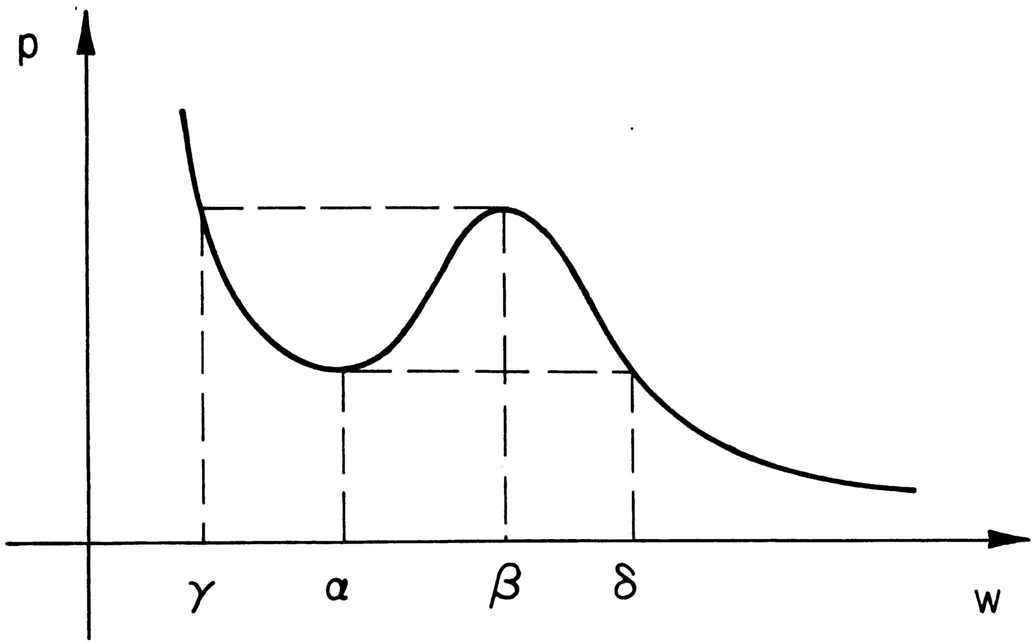

FIG. 1

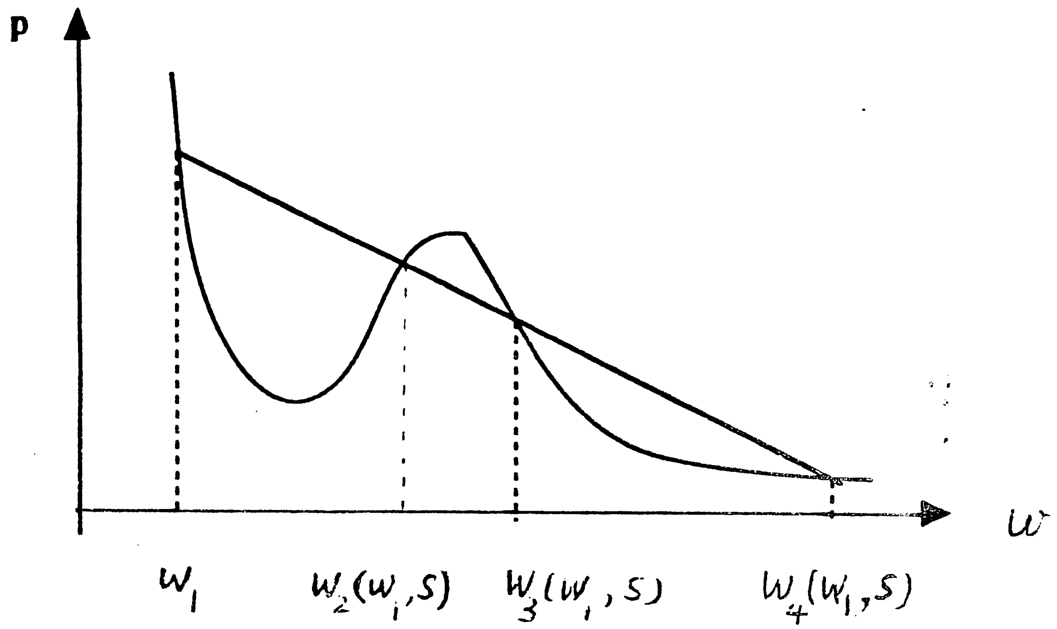

Figure 2

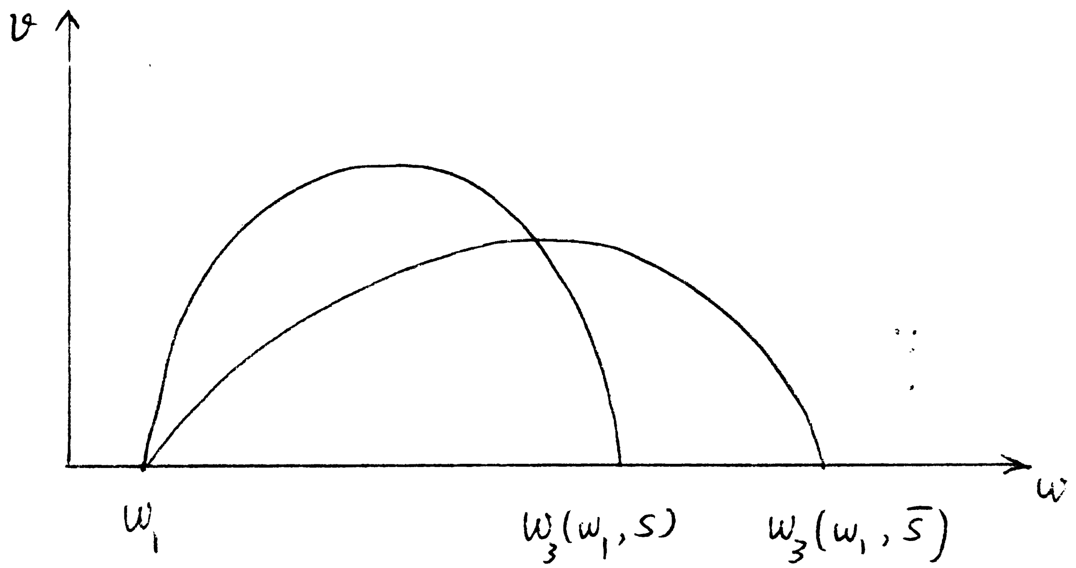

Figure 3

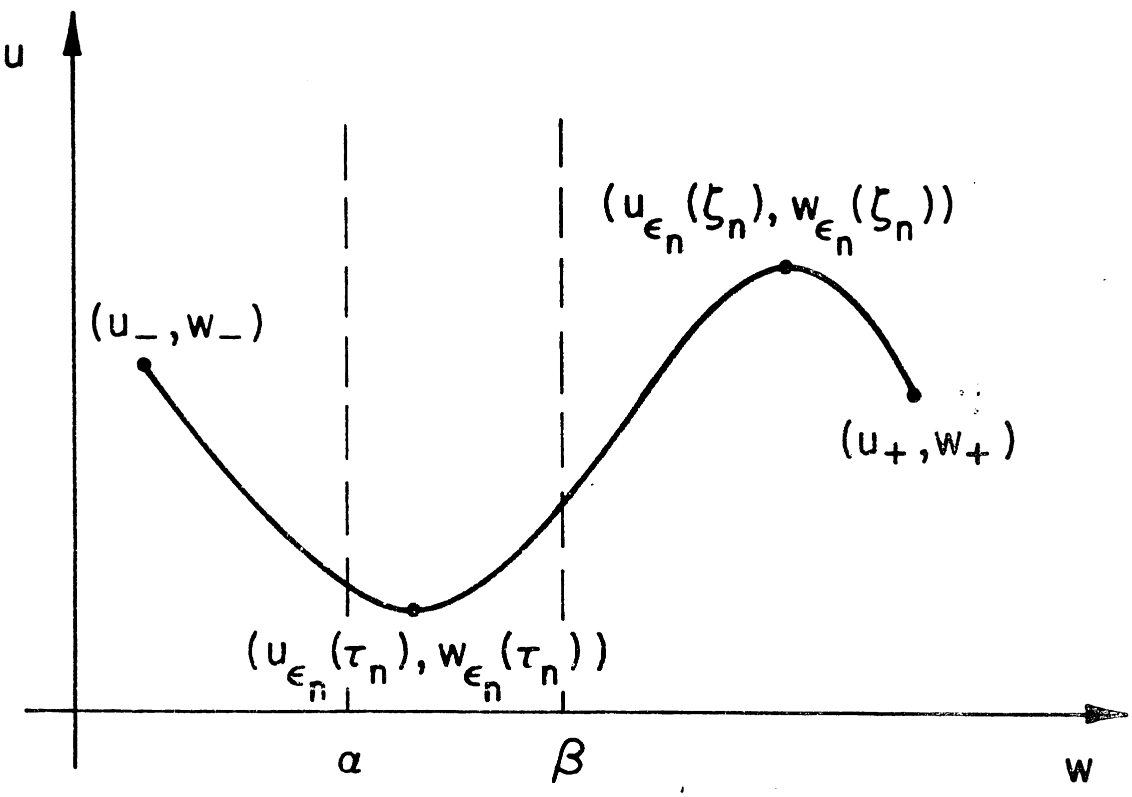
u
(u_-, w_-)
$(u_{\epsilon_n}(\zeta_n), w_{\epsilon_n}(\zeta_n))$
(u_+, w_+)
$(u_{\epsilon_n}(\tau_n), w_{\epsilon_n}(\tau_n))$
α
β
w

FIG. 4

Recent IMA Preprints

#	Author/s	Title
693	**A. Eden, A.J. Milani and B. Nicolaenko**, Finite dimensional exponential attractors for semilinear wave equations with damping	
694	**A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Inertial sets for dissipative evolution equations	
695	**A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Hölder continuity for the inverse of Mañé's projection	
696	**Michel Chipot and Charles Collins**, Numerical approximations in variational problems with potential wells	
697	**Huanan Yang**, Nonlinear wave analysis and convergence of MUSCL schemes	
698	**László Gerencsér and Zsuzsanna Vágó**, A strong approximation theorem for estimator processes in continuous time	
699	**László Gerencsér**, Multiple integrals with respect to L-mixing processes	
700	**David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation	
701	**Bo Deng**, Symbolic dynamics for chaotic systems	
702	**P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses	
703	**Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem	
704	**Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces	
705	**A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation	
706	**H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging	
707	**M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems	
708	**Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems	
709	**Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films	
710	**Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels	
711	**Wenxiong Liu**, Blow-up behavior for semilinear heat equations:multi-dimensional case	
712	**Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems	
713	**Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems	
714	**Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation	
715	**Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling	
716	**C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter rings	
717	**W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations	
718	**Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations	
719	**Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II	
720	**Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts	
721	**Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries	
722	**C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I	
723	**Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations	
724	**Adelina Georgescu**, Models of asymptotic approximation	
725	**A. Makagon and H. Salehi**, On bounded and harmonizable solutions on infinite order arma systems	
726	**San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws	
727	**J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure	
728	**KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves	
729	**J.L. Ericksen**, Local bifurcation theory for thermoelastic Bravais lattices	
730	**Mario Taboada and Yuncheng You**, Some stability results for perturbed semilinear parabolic equations	
731	**A.J. Lawrance**, Local and deletion influence	
732	**Bogdan Vernescu**, Convergence results for the homogenization of flow in fractured porous media	
733	**Xinfu Chen and Avner Friedman**, Mathematical modeling of semiconductor lasers	
734	**Yongzhi Xu**, Scattering of acoustic wave by obstacle in stratified medium	
735	**Songmu Zheng**, Global existence for a thermodynamically consistent model of phase field type	
736	**Heinrich Freistühler and E. Bruce Pitman**, A numerical study of a rotationally degenerate hyperbolic	

system part I: the Riemann problem
737 **Epifanio G. Virga**, New variational problems in the statics of liquid crystals
738 **Yoshikazu Giga and Shun'ichi Goto**, Geometric evolution of phase-boundaries
739 **Ling Ma**, Large time study of finite element methods for 2D Navier–Stokes equations
740 **Mitchell Luskin and Ling Ma**, Analysis of the finite element approximation of microstructure in micromagnetics
741 **M. Chipot**, Numerical analysis of oscillations in nonconvex problems
742 **J. Carrillo and M. Chipot**, The dam problem with leaky boundary conditions
743 **Eduard Harabetian and Robert Pego**, Efficient hybrid shock capturing schemes
744 **B.L.J. Braaksma**, Multisummability and Stokes multipliers of linear meromorphic differential equations
745 **Tae Il Jeon and Tze-Chien Sun**, A central limit theorem for non-linear vector functionals of vector Gaussian processes
746 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
747 **Mario Taboada and Yuncheng You**, Invariant manifolds for retarded semilinear wave equations
748 **Peter Rejto and Mario Taboada**, Unique solvability of nonlinear Volterra equations in weighted spaces
749 **Hi Jun Choe**, Holder regularity for the gradient of solutions of certain singular parabolic equations
750 **Jack D. Dockery**, Existence of standing pulse solutions for an excitable activator-inhibitory system
751 **Jack D. Dockery and Roger Lui**, Existence of travelling wave solutions for a bistable evolutionary ecology model
752 **Giovanni Alberti, Luigi Ambrosio and Giuseppe Buttazzo**, Singular perturbation problems with a compact support semilinear term
753 **Emad A. Fatemi**, Numerical schemes for constrained minimization problems
754 **Y. Kuang and H.L. Smith**, Slowly oscillating periodic solutions of autonomous state-dependent delay equations
755 **Emad A. Fatemi**, A new splitting method for scaler conservation laws with stiff source terms
756 **Hi Jun Choe**, A regularity theory for a more general class of quasilinear parabolic partial differential equations and variational inequalities
757 **Haitao Fan**, A vanishing viscosity approach on the dynamics of phase transitions in Van Der Waals fluids
758 **T.A. Osborn and F.H. Molzahn**, The Wigner–Weyl transform on tori and connected graph propagator representations
759 **Avner Friedman and Bei Hu**, A free boundary problem arising in superconductor modeling
760 **Avner Friedman and Wenxiong Liu**, An augmented drift-diffusion model in semiconductor device
761 **Avner Friedman and Miguel A. Herrero**, Extinction and positivity for a system of semilinear parabolic variational inequalities
762 **David Dobson and Avner Friedman**, The time-harmonic Maxwell equations in a doubly periodic structure
763 **Hi Jun Choe**, Interior behaviour of minimizers for certain functionals with nonstandard growth
764 **Vincenzo M. Tortorelli and Epifanio G. Virga**, Axis-symmetric boundary-value problems for nematic liquid crystals with variable degree of orientation
765 **Nikan B. Firoozye and Robert V. Kohn**, Geometric parameters and the relaxation of multiwell energies
766 **Haitao Fan and Marshall Slemrod**, The Riemann problem for systems of conservation laws of mixed type
767 **Joseph D. Fehribach**, Analysis and application of a continuation method for a self-similar coupled Stefan system
768 **C. Foias, M.S. Jolly, I.G. Kevrekidis and E.S. Titi**, Dissipativity of numerical schemes
769 **D.D. Joseph, T.Y.J. Liao and J.-C. Saut**, Kelvin–Helmholtz mechanism for side branching in the displacement of light with heavy fluid under gravity
770 **Chris Grant**, Solutions to evolution equations with near-equilibrium initial values
771 **B. Cockburn, F. Coquel, Ph. LeFloch and C.W. Shu**, Convergence of finite volume methods
772 **N.G. Lloyd and J.M. Pearson**, Computing centre conditions for certain cubic systems
773 **João Palhoto Matos**, Young measures and the absence of fine microstructures in the $\alpha - \beta$ quartz phase transition
774 **L.A. Peletier & W.C. Troy**, Self-similar solutions for infiltration of dopant into semiconductors
775 **H. Scott Dumas and James A. Ellison**, Nekhoroshev's theorem, ergodicity, and the motion of energetic charged particles in crystals
776 **Stathis Filippas and Robert V. Kohn**, Refined asymptotics for the blowup of $u_t - \Delta u = u^p$.
777 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximum principles and a priori estimates for an incompressible material in nonlinear elasticity
778 **Patricia Bauman, Nicholas C. Owen and Daniel Phillips**, Maximal smoothness of solutions to certain Euler–Lagrange equations from nonlinear elasticity
779 **Jack Carr and Robert Pego**, Self-similarity in a coarsening model in one dimension
780 **J.M. Greenberg**, The shock generation problem for a discrete gas with short range repulsive forces
781 **George R. Sell and Mario Taboada**, Local dissipativity and attractors for the Kuramoto–Sivashinsky equation in thin 2D domains
782 **T. Subba Rao**, Analysis of nonlinear time series (and chaos) by bispectral methods