**INTERIOR BEHAVIOUR OF MINIMIZERS FOR CERTAIN
FUNCTIONALS WITH NONSTANDARD GROWTH**

By

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1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in W^{1,p}(\Omega \rightarrow \mathbb{R}^N)$ is a minimizer of

$$(1) \quad I[u] = \int_{\Omega} f(|\nabla u|^2) dx,$$

where f is a C^2 function and satisfies the following ellipticity conditions

$$(2) \quad c_1 |Q|^{p-2} |\xi|^2 \leq f_{Q_\alpha^i Q_\beta^j} (|Q|^2) \xi_\alpha^i \xi_\beta^j \leq c_2 (|Q|^{p-2} + |Q|^{q-2}) |\xi|^2$$

for some $c_1, c_2 > 0$ and for all $Q, \xi \in M^{nN}$, where we assume that

$$1 < p \leq q < \infty.$$

From the ellipticity condition (2) we note that

$$\frac{c_1}{2} t^{\frac{p-2}{2}} \leq f'(t) + 2f''(t)t \leq \frac{c_2}{2} \left(t^{\frac{p-2}{2}} + t^{\frac{q-2}{2}} \right)$$

for all $t \in [0, \infty)$ and

$$\frac{c_1}{2} t^{\frac{p-2}{2}} \leq f'(t) \leq \frac{c_2}{2} \left(t^{\frac{p-2}{2}} + t^{\frac{q-2}{2}} \right)$$

for all $t \in [0, \infty)$.

For example we see that

$$f(t^2) = at^p + bt^q, \quad a > 0, b > 0, 1 < p < q < \infty$$

and

$$f(t^2) = t^p \log(1+t), \quad p > 1,$$

satisfy the ellipticity condition (2).

Recently a number of authors have considered regularity questions of minimizers for I under nonstandard growth conditions on f (see [1], [2], [3], [6], [7], [13] and [14]). Indeed in the scalar cases Choe[3] has already obtained $C^{1,\alpha}$ regularity result for minimizers of I when $f(t^2) = t^p \log(1+t)$, $p > 1$, using Lewis' method[12]. In this paper we prove that a locally bounded minimizer u is $C_{loc}^{1,\alpha}$ under the assumption that

$$(3) \quad 1 < p \leq q < p+1.$$

In fact if $p = q > 1$, then $C^{1,\alpha}$ regularity for minimizers of I is well known now even for systems (see [16] and [17]). The following theorem is a main result in this paper.

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THEOREM 1. Suppose $u \in W^{1,p}(\Omega)$ is a locally bounded minimizer of I . Moreover suppose that

$$1 < p \leq q < p + 1,$$

then $u \in C_{loc}^{1,\alpha}(\Omega)$ for some $\alpha > 0$.

By Sobolev theorem we know that if $p > n$, then $W^{1,p} \subset L_{loc}^\infty(\Omega)$. Hence we have the following natural application of theorem 1 when $p > n$.

LEMMA 1. Suppose $u \in W^{1,p}(\Omega)$, $p > n$, is a minimizer of I , and p and q satisfy (3), then

$$\nabla u \in C_{loc}^\alpha$$

for some $\alpha > 0$.

Moreover we recall the following lemma due to Fusco and Sobordone[6]. A similar theorem also can be found in [1].

LEMMA 2. Suppose that

$$1 < p \leq q < \frac{np}{n-p}.$$

Moreover we assume that f satisfies Δ_2 condition, that is,

$$f((2t)^2) \leq kf(t^2) \text{ for all } t > 0 \text{ and for some } k,$$

and $\frac{f(t^2)}{t^p}$ is increasing and $\frac{f(t^2)}{t^q}$ is decreasing. Then a minimizer $u \in W^{1,p}(\Omega)$ of I is in $W_{loc}^{1,r}(\Omega)$ for some $r > p$.

Lemma 2 follows essentially from the reverse Hölder inequality by Gehring and a Cacciopoli type inequality. Consequently combining theorem 1 and lemma 2 we also have the following lemma.

LEMMA 3. Suppose f satisfies the assumptions in lemma 2, and p and q satisfy (3). Then there exists $\delta > 0$ depending on n, N and p such that a minimizer $u \in W^{1,p}(\Omega)$, $p > n - \delta$, of I is locally bounded and

$$\nabla u \in C_{loc}^\alpha(\Omega)$$

for some $\alpha > 0$.

On the other hand we prove in the next section that if $1 < p \leq q < p + \frac{p}{n}$, then a minimizer $u \in W^{1,p}(\Omega)$ of I is locally bounded. Hence we also have a lemma without the boundedness assumption on u .

LEMMA 4. Suppose $u \in W^{1,p}(\Omega)$ is a minimizer of I , and p and q satisfy

$$1 < p \leq q < p + \frac{p}{n}.$$

Then u is locally bounded and

$$\nabla u \in C_{loc}^\alpha(\Omega)$$

for some $\alpha > 0$.

Theorem 1 is rather striking compared to the known results, that is, 1 is independent of n, N and p . Indeed in the scalar cases with anisotropic growth it is necessary to assume that $q \leq \frac{np}{n-p}$ to prove boundedness of u (see [2], [7] and [14]). We have also counterexamples showing that if $q > \frac{np}{n-p}$, then there are unbounded minimizers in the scalar cases with anisotropic growth (see [9]).

We can't expect everywhere regularity for systems even if the functional satisfy a standard growth condition, that is, we can expect only partial regularity. Hence the dependence of modulus of gradient is essential for everywhere regularity for systems. Indeed the proof for $C^{1,\alpha}$ regularity depends on the fact that $|\nabla u|^k$ for some k is a subsolution to a certain elliptic equation.

We write c a constant depending only on exterior data. We also assume that double indices means summation up to n or N according to situations. We use the following notation

$$\int_A u dx = \frac{1}{|A|} \int_A u dx,$$

where $|A|$ means the n -dimensional Lebesgue measure of A . Finally we write $B_R(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < R\}$ and drop x_0 from various symbols if there is no confusion.

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2. L_{loc}^∞ regularity. In this section we assume a weaker ellipticity condition than (2). Suppose that f is a C^2 function and satisfies the following ellipticity condition

$$(4) \quad c_1 |Q|^{p-2} |\xi|^2 \leq f_{Q_\alpha^i Q_\beta^j} (|Q|^2) \xi_\alpha^i \xi_\beta^j \leq c_2 (1 + |Q|)^{q-2} |\xi|^2$$

for some $c_1, c_2 > 0$ and for all $Q, \xi \in M^{nN}$ when $p \geq 2$ and

$$(5) \quad c_1 (1 + |Q|)^{p-2} |\xi|^2 \leq f_{Q_\alpha^i Q_\beta^j} (|Q|^2) \xi_\alpha^i \xi_\beta^j \leq c_2 (1 + |Q|^{q-2}) |\xi|^2$$

when $p \leq 2$. We also assume f attains its minimum at 0 and $f(0) = 0$. Note that the ellipticity condition (4) and (5) are weaker than (2). Indeed $f(t^2) = t^2 + t^p + t^q$ for some large p and q satisfies (4), but it doesn't satisfies (2).

First we prove that a minimizer u of I is bounded under the assumption that

$$(6) \quad 1 < p \leq q < p + \frac{p}{n}.$$

In fact we use a variation of Moser iteration method. Because of asymmetry of growth condition the iteration is complicated.

THEOREM 2. *Suppose $u \in W^{1,p}(\Omega)$ is a minimizer of I . Assume that*

$$(7) \quad 1 < p \leq q < p + \frac{p}{n},$$

then $u \in L_{loc}^\infty(\Omega)$. Furthermore suppose that $B_{R_0} \subset \Omega$, then we have the following inequality for u

$$(8) \quad \sup_{B_{R_0/2}} |u| \leq \frac{c}{R_0^{\frac{p(n-p)(q-p)}{ns_0(p+1-q)}}} \left[\int_{B_{R_0}} |u|^{\frac{np}{n-p}} dx \right]^{\frac{p}{ns_0}} + cR_0^{\frac{n^2 s_0}{p(n-p)} \left[p + \frac{(p-q)(n-p)}{p+1-q} \right]}$$

for some c depending only on n, N and p , where

$$(9) \quad s_0 = \frac{np}{n-p} - \frac{p}{p+1-q}.$$

proof. From Sobolev imbedding theorem if $p > n$, u is bounded and there is nothing to prove. If $p = n$, the proof is relatively easy. Hence we assume that $p < n$ from now on.

We know that u satisfies the Euler-Lagrange equation

$$(10) \quad (f'(|\nabla u|^2)u_{x_\alpha}^i)_{x_\alpha} = 0, \quad i = 1, \dots, N.$$

Since we are assuming that

$$q < p + \frac{p}{n},$$

we see immediately that

$$s_0 > 0.$$

Let $\eta \in C_0^\infty(B_R)$ be a standard cutoff function such that

$$\begin{aligned} 0 &\leq \eta \leq 1 \\ \eta &\equiv 1 \quad \text{in } B(\rho) \\ |\nabla \eta| &\leq \frac{c}{R-\rho} \end{aligned}$$

for some c and we apply

$$u^i |u|^s \eta^k$$

as a test function to (10), where we assume that

$$s \geq s_0$$

and k is a large integer decided later. Hence from direct computation we have

$$(11) \quad \begin{aligned} & \int_{B_R} f'(|\nabla u|^2) u_{x_\alpha}^i (u_{x_\alpha}^i |u|^s + s u^i u^j u_{x_\alpha}^j |u|^{s-2}) \eta^k dx \\ & + k \int_{B_R} f'(|\nabla u|^2) u_{x_\alpha}^i u^i |u|^s \eta_{x_\alpha} \eta^{k-1} dx = 0. \end{aligned}$$

Now assume $p \geq 2$. In this case using the ellipticity condition (4) we have that

$$(12) \quad \begin{aligned} & \int_{B_R} |\nabla u|^p |u|^s \eta^k dx \\ & \leq c \int_{B_R} (1 + |\nabla u|)^{q-2} |\nabla u| |u|^{s+1} |\nabla \eta| \eta^{k-1} dx \\ & \leq c \int_{B_R} (1 + |\nabla u|^{q-1}) |u|^{s+1} |\nabla \eta| \eta^{k-1} dx \end{aligned}$$

for some c depending on k and s . Using Young's inequality on the right hand side of (12) we have

$$(13) \quad \begin{aligned} & \int_{B_R} |\nabla u|^{q-1} |u|^{s+1} |\nabla \eta| \eta^{k-1} dx \\ & \leq \varepsilon \int_{B_R} |\nabla u|^p |u|^s \eta^k dx + c \int_{B_R} |u|^{s+\frac{p}{p+1-q}} |\nabla \eta|^{\frac{p}{p+1-q}} \eta^{k-\frac{p}{p+1-q}} dx \end{aligned}$$

for some c and small ε , and

$$(14) \quad \int_{B_R} |u|^{s+1} |\nabla \eta| \eta^{k-1} dx \leq c \int_{B_R} |u|^{s+\frac{p}{p+1-q}} \left(1 + |\nabla \eta|^{\frac{p}{p+1-q}}\right) dx + c|B_R|$$

for some c . If we take k so large that

$$k \geq \frac{p}{p+1-q}$$

and ε small enough, then we have

$$(15) \quad \int_{B_R} \left| \nabla \left(u^{1+\frac{s}{p}} \eta^{\frac{k}{p}} \right) \right|^p dx \leq c \int_{B_R} |u|^{s+\frac{p}{p+1-q}} \left(1 + |\nabla \eta|^{\frac{p}{p+1-q}}\right) dx + c|B_R|$$

for some c . Applying Sobolev's inequality on the left hand side of (15) we have

$$(16) \quad \left[\int_{B_R} |u|^{\frac{n(p+s)}{n-p}} \eta^{\frac{nk}{n-p}} dx \right]^{\frac{n-p}{n}} \leq c(s) \int_{B_R} |u|^{s+\frac{p}{p+1-q}} \left(1 + |\nabla \eta|^{\frac{p}{p+1-q}} \right) dx + c(s)|B_R|$$

for some $c(s)$. Note that

$$(17) \quad c(s) \leq c(1 + s^l)$$

for some c and l depending only on n, N and p .

We define ρ_ν , $\nu = 0, 1, 2, \dots$, by

$$\rho_\nu = (1 + 2^{-\nu})R_0/2.$$

We also define s_ν by

$$s_{\nu+1} = \theta s_\nu + s_0, \quad \nu = 0, 1, 2, \dots,$$

where $\theta = \frac{n}{n-p}$ and $s_0 = \frac{np}{n-p} - \frac{p}{p+1-q}$. Hence we see that

$$s_\nu = \frac{\theta^{\nu+1} - 1}{\theta - 1} s_0$$

for all $\nu = 0, 1, 2, \dots$. Finally define

$$(18) \quad \Phi_\nu = \int_{B_{\rho_\nu}} |u|^{s_\nu + \frac{p}{p+1-q}} dx$$

for all $\nu = 0, 1, 2, \dots$. Note that

$$\Phi_0 = \int_{B_{\rho_0}} |u|^{s_0 + \frac{p}{p+1-q}} dx = \int_{B_{R_0}} |u|^{\frac{np}{n-p}} dx.$$

We set $\rho = \rho_{\nu+1}$, $R = \rho_\nu$ and $s = s_\nu$, then we can write (16) as follows

$$(19) \quad \Phi_{\nu+1} \leq c^\nu R_0^p + c^\nu R_0^{\frac{p(p-q)}{p+1-q}} \Phi_\nu^\theta.$$

for some c depending only on n, N and p . Note that

$$(20) \quad \lim_{\nu \rightarrow \infty} \frac{\theta^\nu}{s_\nu} = \frac{p}{ns_0}.$$

Therefore iterating (19) we have

$$(21) \quad \sup_{B_{R_0}} |u| \leq \frac{c}{R_0^{\frac{p(n-p)(q-p)}{ns_0(p+1-q)}}} \left[\int_{B_{R_0}} |u|^{\frac{np}{n-p}} dx \right]^{\frac{p}{ns_0}} + cR_0^{\frac{n^2 s_0}{p(n-p)} \left[p + \frac{(p-q)(n-p)}{p+1-q} \right]}$$

for some c depending only on n, N and p . Finally we know that from Sobolev imbedding theorem $u \in L_{loc}^{\frac{np}{n-p}}(\Omega)$ and complete the proof when $p \geq 2$.

Now we assume $1 < p \leq 2$. From (11) and the ellipticity condition (5) we have

$$(22) \quad \begin{aligned} & \int_{B_R} (1 + |\nabla u|)^{p-2} |\nabla u|^2 |u|^s \eta^k dx \\ & \leq c \int_{B_R} (|\nabla u|^{p-1} + |\nabla u|^{q-1}) |u|^{s+1} |\nabla \eta| \eta^{k-1} dx \end{aligned}$$

for some c . Since $p \leq 2$ we see that

$$(23) \quad \int_{B_R} (1 + |\nabla u|)^{p-2} |u|^s \eta^k dx \leq \int_{B_R} |u|^s \eta^k dx.$$

Hence combining (22) and (23) we have

$$(24) \quad \begin{aligned} & \int_{B_R} (1 + |\nabla u|)^p |u|^s \eta^k dx \\ & \leq c \int_{B_R} (1 + |\nabla u|^{q-1}) |u|^{s+1} |\nabla \eta| \eta^{k-1} dx + c \int_{B_R} |u|^s \eta^k dx \end{aligned}$$

for some c . Once we have (24), applying Young's inequality on the right hand side of (24) we have

$$(25) \quad \int_{B_R} \left| \nabla \left(u^{1+\frac{s}{p}} \eta^{\frac{k}{p}} \right) \right|^p dx \leq c \int_{B_R} |u|^{s+\frac{p}{p+1-q}} \left(1 + |\nabla \eta|^{\frac{p}{p+1-q}} \right) dx + c|B_R|$$

for some c . Observe that inequality (25) is precisely the same as (15). Hence the following computations are the same as in the case $p \geq 2$ and we complete the proof in this case too. $\square$

3. $C_{loc}^{1,\alpha}$ regularity. In this section we assume that u is locally bounded and f satisfies the ellipticity condition (2). We also assume that p and q satisfy (3).

By standard approximation argument we can assume u has second derivative. First we prove that $u \in L^s$ for all $s \in [p, \infty)$ using integration by parts.

THEOREM 3. Suppose $u \in L_{loc}^\infty(\Omega)$, then $\nabla u \in L_{loc}^s(\Omega)$ for all $s \in [p, \infty)$. Furthermore suppose that $B_{R_0} \subset \Omega$, then we have

$$(26) \quad \int_{B_{R_0/2}} |\nabla u|^s dx \leq c \int_{B_{R_0}} |\nabla u|^p + c$$

for some c depending on $\|u\|_{L^\infty(B_{R_0})}$, R_0, n, N, p and q .

Proof. As usual we know that u satisfies the Euler-Lagrange equation

$$(27) \quad (f'(|\nabla u|^2)u_{x_\alpha}^i)_{x_\alpha} = 0, \quad i = 1, \dots, N.$$

By taking x_k -th derivative we note that

$$(28) \quad (A_{\alpha\beta}^{ij}u_{x_k x_\alpha}^i)_{x_\beta} = 0, \quad i = 1, \dots, N,$$

where

$$A_{\alpha\beta}^{ij} = f'(|\nabla u|^2)\delta^{ij}\delta_{\alpha\beta} + 2f''(|\nabla u|^2)u_{x_\alpha}^i u_{x_\beta}^j$$

From condition (2) we note that

$$c_1 |\nabla u|^{p-2} |\xi|^2 \leq A_{\alpha\beta}^{ij} \xi_\alpha^i \xi_\beta^j \leq c_2 (|\nabla u|^{p-2} + |\nabla u|^{q-2}) |\xi|^2.$$

Suppose $\eta \in C_0^\infty(B(R))$ is a standard cutoff function such that

$$\begin{aligned} 0 &\leq \eta \leq 1 \\ \eta &\equiv 1 \quad \text{in } B(\rho) \\ |\nabla \eta| &\leq \frac{c}{R - \rho} \end{aligned}$$

for some c . We take $|\nabla u|^s u_{x_k}^i \eta^2$, $s > -1$ as a test function to (28). Hence we get

$$(29) \quad \begin{aligned} &\int_{\Omega} A_{\alpha\beta}^{ij} u_{x_k x_\alpha}^i \left(u_{x_k x_\beta}^j |\nabla u|^s + s u_{x_k}^j |\nabla u|^{s-2} u_{x_\ell}^m u_{x_\ell x_\beta}^n \right) \eta^2 dx \\ &= - \int_{\Omega} A_{\alpha\beta}^{ij} u_{x_k x_\alpha}^i |\nabla u|^s u_{x_k}^j \eta_{x_\beta} \eta dx. \end{aligned}$$

Since $s > -1$, we have

$$(30) \quad \begin{aligned} &A_{\alpha\beta}^{ij} u_{x_k x_\alpha}^i \left(u_{x_k x_\beta}^j |\nabla u|^s + s |\nabla u|^{s-2} u_{x_k}^j u_{x_\ell}^m u_{x_\ell x_\beta}^n \right) \\ &= f'(|\nabla u|^2) |\nabla u|^s \left(u_{x_k x_\beta}^j u_{x_k x_\beta}^j + s \frac{u_{x_k}^j u_{x_k x_\beta}^j u_{x_\ell}^m u_{x_\ell x_\beta}^n}{|\nabla u|^2} \right) \\ &\quad + 2f''(|\nabla u|^2) |\nabla u|^{s+2} \left(\frac{u_{x_\alpha}^i u_{x_k x_\alpha}^i u_{x_\beta}^j u_{x_\beta x_k}^j}{|\nabla u|^2} + s \frac{u_{x_\alpha}^i u_{x_k x_\alpha}^i u_{x_k}^j u_{x_\beta}^j u_{x_\ell}^m u_{x_\ell x_\beta}^n}{|\nabla u|^4} \right) \\ &\geq (1-s) f'(|\nabla u|^2) |\nabla u|^s |\nabla^2 u|^2. \end{aligned}$$

Hence considering the ellipticity condition (2) and (30) in (29), we have

$$\begin{aligned}
(31) \quad & \int_{\Omega} |\nabla u|^{p+s-2} |\nabla^2 u|^2 \eta^2 dx \\
& \leq c \int_{\Omega} (|\nabla u|^{p-2} + |\nabla u|^{q-2}) |\nabla u|^{s+1} |\nabla^2 u| |\nabla \eta| \eta dx
\end{aligned}$$

for some c . By Young's inequality on the right hand side of (31) we have

$$\begin{aligned}
(32) \quad & \int_{\Omega} |\nabla u|^{q+s-1} |\nabla^2 u| |\nabla \eta| \eta dx \\
& \leq \varepsilon \int_{\Omega} |\nabla u|^{p+s-2} |\nabla^2 u|^2 \eta^2 dx + c \int_{\Omega} |\nabla u|^{2q+s-p} |\nabla \eta|^2 dx
\end{aligned}$$

and

$$\begin{aligned}
(33) \quad & \int_{\Omega} |\nabla u|^{p+s-1} |\nabla^2 u| |\nabla \eta| \eta dx \\
& \leq \varepsilon \int_{\Omega} |\nabla u|^{p+s-2} |\nabla^2 u|^2 \eta^2 dx + c \int_{\Omega} |\nabla u|^{p+s} |\nabla \eta|^2 dx
\end{aligned}$$

for some c and small ε . Thus combining (31), (32) and (33) we have the following Cacciopoli type inequality for $\nabla^2 u$

$$\begin{aligned}
(34) \quad & \int_{\Omega} |\nabla u|^{p+s-2} |\nabla^2 u|^2 \eta^2 dx \\
& \leq c \int_{\Omega} |\nabla u|^{p+s} |\nabla \eta|^2 dx + c \int_{\Omega} |\nabla u|^{2q+s-p} |\nabla \eta|^2 dx
\end{aligned}$$

for some c depending only on n, N, p and q .

Now we prove our theorem. Let k be a large integer decided later. Using integration by parts we have

$$\begin{aligned}
(35) \quad & \int |\nabla u|^{p+\alpha+2} \eta^{2k} dx \\
& = - \int u^i \nabla \cdot (|\nabla u|^{p+\alpha} \nabla u^i \eta^{2k}) dx \\
& \leq c \|u\|_{L^\infty(B_R)} \int |\nabla u|^{p+\alpha} |\nabla^2 u| \eta^{2k} dx \\
& \quad + c \|u\|_{L^\infty(B_R)} \int |\nabla u|^{p+\alpha+1} |\nabla \eta| \eta^{2k-1} dx \\
& = I + II.
\end{aligned}$$

By Young's inequality we have

$$(36) \quad II \leq \varepsilon \int |\nabla u|^{p+\alpha+2} \eta^{2k} dx + \frac{c}{(R-\rho)^2} \int_{B_R} |\nabla u|^{p+\alpha} \eta^{2k-2} dx$$

for some c and small ε . On the other hand we also have

$$(37) \quad \begin{aligned} I &\leq \varepsilon \int |\nabla u|^{p+\alpha+\beta-2} |\nabla^2 u|^2 \eta^{2k+2} dx \\ &\quad + c(\varepsilon) \int_{B_R} |\nabla u|^{p+\alpha+2-\beta} \eta^{2k-2} dx \end{aligned}$$

for some small ε and for some β which is decided later.

We take $s = \alpha + \beta$ in (34), then we have

$$(38) \quad \begin{aligned} &\int |\nabla u|^{p+\alpha+\beta-2} |\nabla^2 u|^2 \eta^{2k+2} dx \\ &\leq \frac{c}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+\beta+2q-p} \eta^{2k} dx + \frac{c}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+\beta+p} \eta^{2k} dx \end{aligned}$$

for some c as long as $\alpha + \beta > -1$. Combining (37) and (38) we have that

$$(39) \quad \begin{aligned} I &\leq c \int_{B_R} |\nabla u|^{p+\alpha+2-\beta} \eta^{2k-2} dx \\ &\quad + \frac{c\varepsilon}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+\beta+2q-p} \eta^{2k} dx + \frac{c\varepsilon}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+\beta+p} \eta^{2k} dx \end{aligned}$$

for some c and small ε .

If we choose $\beta = 2 - 2(q - p) > 0$, then we note that

$$\alpha + \beta + 2q - p = p + \alpha + 2$$

and

$$p + \alpha + \beta < p + \alpha + 2.$$

Since $\alpha + \beta + p \leq \alpha + p + 2$, using Young's inequality on the right hand side of (39) we have

$$(40) \quad \begin{aligned} &\int_{B_R} |\nabla u|^{\alpha+\beta+p} \eta^{2k} dx \\ &\leq \varepsilon \int_{B_R} |\nabla u|^{\alpha+2+p} \eta^{2k} dx + c|B_R| \end{aligned}$$

for some c depending on R and small ε . On the other hand since $\alpha + \beta + 2q - p = \alpha + p + 2$, we see that

$$(41) \quad \int_{B_R} |\nabla u|^{\alpha+\beta+2q-p} \eta^{2k} dx = \int_{B_R} |\nabla u|^{\alpha+2+p} \eta^{2k} dx$$

for some c depending on R and small ε . Now choose k so large that

$$k \geq \frac{p + \alpha + 2}{\beta},$$

then we see that

$$(42) \quad \begin{aligned} & \int_{B_R} |\nabla u|^{p+\alpha+2-\beta} \eta^{2k-2} dx \\ & \leq \varepsilon \int_{B_R} |\nabla u|^{p+\alpha+2} \eta^{2k} dx + c|B_R| \end{aligned}$$

for some c and small ε . Hence from (39) through (42) we have

$$(43) \quad I \leq \frac{c\varepsilon}{(R-\rho)^2} \int_{B_R} |\nabla u|^{p+\alpha+2} \eta^{2k} dx + c$$

for small ε . Hence taking ε very small and combining (35), (36) and (43), we have

$$(44) \quad \begin{aligned} & \int |\nabla u|^{p+\alpha+2} \eta^{2k} dx \\ & \leq c \int_{B_R} |\nabla u|^{p+\alpha} dx + c \end{aligned}$$

for some c depending on $\|u\|_{L^\infty}$, R, ρ, n and N .

We define ρ_ν and α_ν , $\nu = 0, 1, 2, \dots$, by

$$\rho_\nu = (1 + 2^{-\nu})R_0/2$$

and

$$\alpha_{\nu+1} = \alpha_\nu + 2, \quad \alpha_0 = 0.$$

So $\alpha_\nu = 2\nu$, $\nu = 0, 1, 2, \dots$.

Now take $\rho = \rho_{\nu+1}$, $R = \rho_\nu$ and $\alpha = \alpha_\nu$.

Hence we can rewrite (44) as follows

$$(45) \quad \int_{B_{\rho_{\nu+1}}} |\nabla u|^{p+2(\nu+1)} dx \leq c \int_{B_{\rho_\nu}} |\nabla u|^{p+2\nu} dx + c$$

for some c depending on R_0, ν, n, N and p .

Note that since $q - p < 1$,

$$\alpha_0 + \beta > -1.$$

So we can start iteration. Hence iterating (45) we get

$$(46) \quad \int_{B_{R_0/2}} |\nabla u|^s dx \leq c \int_{B_{R_0}} |\nabla u|^p + c$$

for all $s \in [p, \infty)$ for some c depending on $\|u\|_{L^\infty}, R_0, s, p, n$ and N . $\square$

Once we know that ∇u lies in high L^s space we can apply a Moser type iteration. Because of asymmetric growth condition of f we need the fact that $\nabla u \in L^s_{loc}(\Omega)$ for some large s to start iteration. Consequently we have the following theorem.

THEOREM 4. *We have $\nabla u \in L^\infty_{loc}$. Suppose that $B_{R_0} \subset \Omega$, then we have the following inequality*

$$(47) \quad \sup_{B_{R_0/2}} |\nabla u| \leq c \left[\int_{B_{2R_0}} |\nabla u|^p dx \right]^{\frac{1}{p}} + c.$$

for some c depending on R_0 .

Proof. We assume that $n \geq 3$. Indeed the case $n = 2$ is relatively easy compared to the cases $n \geq 3$. Now assume $\alpha > -1$. From (34) we have

$$(48) \quad \begin{aligned} & \int |\nabla (|\nabla u|^{\frac{p+\alpha}{2}} \eta)|^2 dx \\ & \leq c(\alpha) \int |\nabla u|^{\alpha+2q-p} |\nabla \eta|^2 dx + c(\alpha) \int |\nabla u|^{\alpha+p} |\nabla \eta|^2 dx \end{aligned}$$

for some $c(\alpha)$ depending on α . But we note that

$$c(\alpha) \leq c(|\alpha|^l + 1)$$

for some large l and for some large c independent of α .

From Sobolev's imbedding theorem and (48) we have

$$(49) \quad \begin{aligned} & \left[\int |\nabla u|^{(p+\alpha)\frac{N}{N-2}} \eta^{\frac{2N}{N-2}} dx \right]^{\frac{N-2}{N}} \\ & \leq \frac{c(\alpha)}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+2q-p} dx + \frac{c(\alpha)}{(R-\rho)^2} \int_{B_R} |\nabla u|^{\alpha+p} dx \end{aligned}$$

As usual we define ρ_ν as follows

$$\rho_\nu = \frac{R_0}{2}(1 + 2^{-\nu}), \quad \nu = 0, 1, 2, \dots$$

We also define α_ν by

$$(50) \quad (p + \alpha_\nu) \frac{n}{n-2} = \alpha_{\nu+1} + 2q - p, \quad \nu = 0, 1, 2, 3, \dots$$

Hence we see that

$$\alpha_{\nu+1} = \theta \alpha_\nu + \xi,$$

where we defined

$$\theta = \frac{n}{n-2} \text{ and } \xi = 2p \frac{n-1}{n-2} - 2q.$$

Hence we can write α_ν by

$$\alpha_\nu = \alpha_0 \theta^\nu + \xi \frac{\theta^\nu - 1}{\theta - 1}$$

with appropriate α_0 . Note that if

$$\alpha_0 + \frac{\xi}{\theta - 1} = \alpha_0 + p(n-1) - q(n-2) > 0,$$

then

$$\lim_{\nu \rightarrow \infty} \alpha_\nu = \infty.$$

Take α_0 so large that

$$\alpha_0 > q(n-2) - p(n-1).$$

Hence taking $\alpha = \alpha_\nu$, $\rho = \rho_{\nu+1}$ and $R = \rho_\nu$, we have from (49)

$$(51) \quad \begin{aligned} & \int_{B_{\rho_{\nu+1}}} |\nabla u|^{\alpha_{\nu+1}+2q-p} dx \\ & \leq c^\nu \left[\int_{B_{\rho_\nu}} |\nabla u|^{\alpha_\nu+2q-p} dx + 1 \right]^\theta \end{aligned}$$

for some c independent ν .

Define $\phi(\nu)$ by

$$(52) \quad \phi(\nu) = \int_{B_{\rho_\nu}} |\nabla u|^{\alpha_\nu+2q-p} dx,$$

then we can write (51) using $\phi(\nu)$

$$(53) \quad \phi(\nu) \leq c^{\nu-1} \phi^\theta(\nu-1) + c^{\nu-1}$$

for some c independent of ν with $\nu = 1, 2, 3, \dots$. Iterating (53) we have

$$\phi(\nu) \leq c^{(\nu-1)+(\nu-2)\theta+\dots+\theta^{\nu-1}} \phi^{\theta^\nu}(0).$$

Also we see immediately that

$$(54) \quad \lim_{\nu \rightarrow \infty} \frac{\theta^\nu}{\alpha_\nu + 2q - p} = \frac{1}{\alpha_0 + \frac{\xi}{\theta-1}} = \frac{1}{\alpha_0 + p(n-1) - q(n-2)}.$$

and

$$(55) \quad \left[c^{(\nu-1)+(\nu-1)\theta+\dots+\theta^{\nu-1}} \right]^{\frac{1}{\alpha_\nu + 2q - p}} \leq c < \infty$$

for some c depending only on p, q, n and N . Therefore we have

$$(56) \quad \begin{aligned} \sup_{B_{R/2}} |\nabla u| &= \lim_{\nu \rightarrow \infty} [\phi(\nu)]^{\frac{1}{\alpha_\nu + 2q - p}} \\ &\leq c \left[\int_{B_R} |\nabla u|^{\alpha_0 + 2q - p} dx \right]^{\frac{1}{\alpha_0 + p(n-1) - q(n-2)}} \end{aligned}$$

for some c depending only on p, q, n and N .

Therefore using theorem 3 we know that

$$\nabla u \in L_{loc}^{\alpha_0}$$

with sufficiently large α_0 and we prove that

$$(57) \quad \begin{aligned} \sup_{B_{R/2}} |\nabla u| &\leq c \left[\int_{B_R} |\nabla u|^{\alpha_0 + 2q - p} dx \right]^{\frac{1}{\alpha_0 + p(n-1) - q(n-2)}} \\ &\leq c \left[\int_{B_{2R}} |\nabla u|^p dx \right]^{\frac{1}{p}} + c \end{aligned}$$

for some c depending on R . $\square$

REMARK. In fact under the assumption that $u \in L^s_{loc}$ for some large s we can prove theorem 4.

REMARK. Suppose that

$$1 < p < q < \infty.$$

Here we don't assume $q < p + 1$. If we follow the proof of theorem 4 carefully, then it is enough to assume $\nabla u \in L^s_{loc}$ with $s > n(q - p)$ for L^∞_{loc} estimate for ∇u .

proof of theorem 1. For each $M > 0$ we can find $f^M : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f^M(t) = f(t) \text{ if } 0 \leq t \leq M$$

and

$$c|Q|^{p-2}|\xi|^2 \leq f^M_{Q^\alpha_i Q^\beta_j}(|Q|^2)\xi^\alpha_i \xi^\beta_j \leq \frac{1}{c}|Q|^{p-2}|\xi|^2$$

for some small c and for all $Q, \xi \in M^{nN}$. Hence we prove theorem 1 from the fact that ∇u is locally bounded and standard regularity theory for p-Laplacian functions(see [16]). $\square$

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