**A FREE BOUNDARY PROBLEM ARISING IN
SUPERCONDUCTOR MODELING**

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IMA Preprint Series # 759

February 1991

A FREE BOUNDARY PROBLEM ARISING IN SUPERCONDUCTOR MODELING*

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Abstract. By gradually decreasing the ambient magnetic field, under isothermal conditions, a superconductor material (of “type I”) will develop two phases separated by a thin interface $\Gamma(t)$. In the “normal” conducting phase the magnetic field $\vec{H}$ is divergence free and satisfies the heat equation, whereas on the interface $\Gamma(t)$, $\text{curl } \vec{H} \times n = -V_n \vec{H}$ where n is the normal and V_n the velocity of $\Gamma(t)$; further, $|\vec{H}| = H_c$ (constant) on $\Gamma(t)$. This free boundary problem is studied in the present paper. Existence, uniqueness and asymptotic behavior are established under assumptions which enable us to reduce the 3-dimensional problem to a problem depending on essentially one-dimensional space variable.

Keywords and phrases: Superconductor, Maxwell’s equations, Stefan problem

AMS(MOS) subject classifications. 35R35, 81J05, 82A25

§0. Introduction. Superconductors are materials which are capable of changing phase from being “normal” conductors to a phase where there is no resistance to the motion of free electrons. In normal conductor the normalized Maxwell equations (neglecting displacement currents) are satisfied

$$(0.1) \quad \text{div } \vec{E} = 0, \text{div } \vec{H} = 0, \text{curl } \vec{H} = \vec{j}, \text{curl } \vec{E} + \frac{\partial \vec{H}}{\partial t} = 0$$

together with Ohm’s law $\vec{j} = \sigma \vec{E}$ where σ is the electric conductivity. In a superconducting phase Ohm’s law is no longer valid, and Maxwell’s equations are supplemented by the Ginzburg–Landau field equations [4].

Under isothermal conditions, the change of phase from “normal” to superconducting occurs as the ambient magnetic strength is decreased through a prescribed value H_c . Furthermore, the superconducting phase is devoid of magnetic field except at very short distance from the phase boundary (the Meissner effect).

Starting with the Ginzburg–Landau field equations for superconductors and exploiting the above observations, Chapman [2] has recently derived, by asymptotic analysis, equations of motion of the interface $\Gamma(t)$ separating the “normal” and superconducting phases, provided the superconducting phase is collapsing in time:

$$(0.2) \quad \text{curl } \vec{H} \times n = -V_n \vec{H} \quad \text{on } \Gamma(t)$$

*This work is partially supported by National Science Foundation Grant DMS-86-12880.

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where n is the normal to $\Gamma(t)$ and V_n is the velocity of the interface in the direction n . There also holds:

$$(0.3) \quad |\vec{H}| = H_c \quad \text{on} \quad \Gamma(t) .$$

By (0.1), in the “normal” region $\Omega(t)$, $\vec{H}$ satisfies

$$(0.4) \quad \operatorname{div} \vec{H} = 0, \quad \Delta \vec{H} = \frac{\partial \vec{H}}{\partial t} \quad (x \in \Omega(t), \quad t > 0) .$$

The above equations are appropriate only for “type I” superconductor under the isothermal conditions; it is also assumed, as mentioned above, that the “normal” region is continuously increasing in time, that is,

$$(0.5) \quad \Omega(t) \quad \text{is increasing in} \quad t .$$

For complicated magnetic fields steady “intermediate” configurations are possible in which normal and superconducting phases are both present (this is quantified by “order parameter”) as long as the material is of “type I.” Abrikosov [1] discovered “type II” superconductors whose morphology is quite different. We refer to the book of Kittel [7] and the references given there for the basic theory of superconductors.

In the present paper we deal with “type I” superconductors in which the two phases are separated, and use the modeling (0.2)–(0.5). In a special case of a planar domains with magnetic field perpendicular to the domain, Keller [5] suggested a modeling which reduces to solving a classical Stefan problem. The problems (0.2)–(0.5) may be viewed as a “vector Stefan problem,” with the coupling taking place on the free boundary $\Gamma(t)$.

Let us denote the fixed portion of the boundary of $\Omega(t)$ by $\partial_0 \Omega$. Thus $\partial \Omega(t) = \partial_0 \Omega \cup \Gamma(t)$. We complement (0.2)–(0.5) with initial conditions

$$(0.6) \quad \vec{H}(x, 0) = \vec{H}_0(x) \quad , \quad x \in \Omega(0)$$

and with boundary conditions, such as

$$(0.7_a) \quad \vec{H} = \vec{g} \quad \text{on} \quad \partial_0 \Omega \times (0 < t < \infty) ,$$

or

$$(0.7_b) \quad \frac{\partial \vec{H}}{\partial n} = \vec{g} \quad \text{on} \quad \partial_0 \Omega \times (0 < t < \infty) .$$

Of course the data must be consistent with the equation $\operatorname{div} \vec{H} = 0$.

In this paper we study the systems (0.2)–(0.7) only in special cases when it can be reduced to a system with essentially one spatial dimension.

In §§1–5 we consider the case when the geometry of $\Omega(t)$ and the data are independent of (x_2, x_3) . Seeking a solution $\vec{H}$ we find (see §1) that $\vec{H} = \vec{H}(x_1, t)$; $H_1 \equiv 0$ so that $\text{div } \vec{H} = 0$, whereas H_2, H_3 and the free boundary $x_1 = s(t)$ satisfy:

$$\begin{aligned} \frac{\partial H_j}{\partial x_1} + s'(t)H_j &= 0 \quad \text{on } x_1 = s(t), \\ H_2^2 + H_3^2 &= H_c^2 \quad \text{on } x_1 = s(t). \end{aligned}$$

In §2 we establish existence of a classical solution with $s'(t) > 0$, and in §3 we prove uniqueness. Asymptotic behavior, as $t \rightarrow \infty$, is studied in §§4,5.

In §6 we consider the case where the geometry of $\Omega(t)$ and the data are independent of x_3 , and depend on (x_1, x_2) only through $r = \sqrt{x_1^2 + x_2^2}$. We show that there exists a unique solution, that it has the form

$$\begin{aligned} \vec{H}(r, t) &= H_1(r, t)(0, 0, 1) + H_2(r, t)(\sin \theta, -\cos \theta, 0) \\ &\quad + H_3(r, t)(\cos \theta, \sin \theta, 0) \end{aligned}$$

where $H_3 \equiv 0$ so that $\text{div } \vec{H} = 0$, and that H_1, H_2 satisfy a free boundary system similar to H_2, H_3 in the previous case of §§1–5. We finally indicate that one cannot expect to find solutions of (0.2)–(0.7) with spherically symmetric free boundary.

§1. Reduction to one-dimension. We assume that the material occupies the half-space $\{0 < x_1 < \infty\}$ and that the “normal” conductor is initially

$$\Omega_0 = \{0 < x_1 < s_0\}.$$

We assume initial conditions at $t = 0$ and boundary conditions on $x_1 = 0$ which are independent of (x_2, x_3) . We further assume that the free boundary is independent of (x_2, x_3) , i.e., it has the form $x_1 = s(t)$ where $s(t)$ is increasing (by (0.5)). If $\vec{H}$ is a solution then $\partial \vec{H} / \partial x_2$ satisfies a parabolic system with homogeneous boundary conditions. One can show that the boundary conditions

$$\text{curl } \frac{\partial \vec{H}}{\partial x_2} \times n + V_n \frac{\partial \vec{H}}{\partial x_2} = 0$$

are coercive; consequently, by uniqueness, $\partial \vec{H} / \partial x_2 \equiv 0$. Similarly $\partial \vec{H} / \partial x_3 \equiv 0$. Thus, without loss of generality we may assume that $\vec{H} = \vec{H}(x_1, t)$.

Since $n = (1, 0, 0)$, the free boundary conditions (0.2) become

$$V_n H_1 = 0, \quad \partial_{x_1} H_2 = -V_n H_2, \quad \partial_{x_1} H_3 = -V_n H_3.$$

In view of (0.5), we are interested mainly in the case

$$(1.1) \quad V_n > 0.$$

Consequently

$$H_1 = 0 \quad \text{on the free boundary}.$$

The equation $\operatorname{div} \vec{H} = 0$ in (0.4) means that $\partial H_1 / \partial x_1 = 0$. Hence

$$(1.2) \quad H_1 \equiv 0 \quad \text{in } \Omega(t).$$

Setting $x = x_1$ we get

$$\vec{H}_t = \vec{H}_{xx}, \quad 0 < x < s(t), \quad t > 0$$

with $s(0) = s_0$, and

$$\begin{aligned} \partial_x H_2 &= -s' H_2, \quad \partial_x H_3 = -s' H_3, \\ H_2^2 + H_3^2 &= H_c^2 \quad \text{on } x = s(t). \end{aligned}$$

Setting $u = H_2$, $v = H_3$ we obtain the system

$$(1.3) \quad u_t = u_{xx}, \quad 0 < x < s(t), \quad t > 0,$$

$$(1.4) \quad v_t = v_{xx}, \quad 0 < x < s(t), \quad t > 0$$

with the free boundary conditions

$$(1.5) \quad u_x = -s' u \quad \text{on } x = s(t),$$

$$(1.6) \quad v_x = -s' v \quad \text{on } x = s(t),$$

$$(1.7) \quad u^2 + v^2 = 1 \quad \text{on } x = s(t).$$

(We take $H_c = 1$, for simplicity.)

We impose initial conditions

$$(1.8) \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq s_0,$$

$$(1.9) \quad v(x, 0) = v_0(x), \quad 0 \leq x \leq s_0$$

and boundary conditions

$$(D) \quad \begin{cases} u(0, t) = g_1(t) , \\ v(0, t) = g_2(t) , \end{cases} \quad t > 0 ,$$

or

$$(N) \quad \begin{cases} u_x(0, t) = -g_1(t) , \\ v_x(0, t) = -g_2(t) , \end{cases} \quad t > 0 .$$

We shall find it useful to work with the functions

$$(1.10) \quad \tilde{u}(\xi, t) = u(x, t), \quad \tilde{v}(\xi, t) = v(x, t) \quad \text{where} \quad \xi = \frac{x}{s(t)} .$$

Clearly

$$(1.11) \quad \tilde{u}_t = \frac{1}{s^2(t)} \tilde{u}_{\xi\xi} + \xi \frac{s'(t)}{s(t)} \tilde{u}_\xi , \quad 0 < \xi < 1 , \quad t > 0 ,$$

$$(1.12) \quad \tilde{v}_t = \frac{1}{s^2(t)} \tilde{v}_{\xi\xi} + \xi \frac{s'(t)}{s(t)} \tilde{v}_\xi , \quad 0 < \xi < 1 , \quad t > 0$$

and

$$(1.13) \quad \tilde{u}_\xi = -ss' \tilde{u} \quad \text{on} \quad \xi = 1 ,$$

$$(1.14) \quad \tilde{v}_\xi = -ss' \tilde{v} \quad \text{on} \quad \xi = 1 ,$$

$$(1.15) \quad \tilde{u}^2 + \tilde{v}^2 = 1 \quad \text{on} \quad \xi = 1 ,$$

Further,

$$(1.16) \quad \tilde{u}(\xi, 0) = u_0(s_0\xi) , \quad 0 \leq \xi \leq 1 ,$$

$$(1.17) \quad \tilde{v}(\xi, 0) = v_0(s_0\xi) , \quad 0 \leq \xi \leq 1$$

and

$$(\tilde{D}) \quad \begin{cases} \tilde{u}(0, t) = g_1(t) , \\ \tilde{v}(0, t) = g_2(t) , \end{cases} \quad t > 0 ,$$

$$(\tilde{N}) \quad \begin{cases} \tilde{u}_\xi(0, t) = -s(t)g_1(t) , \\ \tilde{v}_\xi(0, t) = -s(t)g_2(t) , \end{cases} \quad t > 0 .$$

In the sequel we assume:

- (E1) there exist functions h_1, h_2 in $W^{2,p}\{0 \leq x \leq s_0, 0 \leq t \leq \delta_0\}$, for some $\delta_0 > 0$ and all $p < \infty$, which satisfy the initial and boundary conditions in (1.8), (1.9), and in either (D) or (N) (with $u = h_1, v = h_2$).

This condition is satisfied if

$$\begin{aligned} u_0, v_0 & \text{ are in } C^2[0, s_0] , \\ g_1(t), g_2(t) & \text{ are in } C^1[0, \delta_0] , \end{aligned}$$

and if the following compatibility conditions hold:

$$\begin{aligned} u_0(0) = g_1(0), v_0(0) = g_2(0) & \quad \text{for } (D) , \\ -u'_0(0) = g_1(0), -v'_0(0) = g_2(0) & \quad \text{for } (N) . \end{aligned}$$

We also assume:

- (E2) $g_1(t)$ and $g_2(t)$ belong to $C^1[0, \infty)$,

- (E3) $\min_{0 \leq x \leq s_0} u_0(x) > 0$, $\min_{0 \leq x \leq s_0} v_0(x) > 0$;

- (E4) in case (D), $g_1(t) > 0, g_2(t) > 0$ for $t \geq 0$,
in case (N), $g_1(t) \geq 0, g_2(t) \geq 0$ for $t \geq 0$;

finally,

- (E5) $\frac{\partial u_0}{\partial x}(s_0) < 0$, $\frac{\partial v_0}{\partial x}(s_0) < 0$,
 $u_0^2(s_0) + v_0^2(s_0) = 1$.

The above conditions will be sufficient for proving local existence. To prove global existence we shall require the additional conditions:

- (E6) $\frac{\partial u_0}{\partial x}(x) \leq 0$, $\frac{\partial v_0}{\partial x}(x) \leq 0$ for $0 \leq x \leq s_0$,

- (E7) in case (D) , $g'_1(t) \geq 0$, $g'_2(t) \geq 0$ for $t \geq 0$.

§2. **Existence.** We begin with local existence.

THEOREM 2.1. *Under the assumptions (E1)–(E5) the problem (1.3)–(1.9) with either (D) or (N) has a local solution (u, v, s) in $0 < t < \delta$, with s in $C^\infty(0, \delta)$ and in $C^{1+\alpha/2}[0, \delta)$ for any $0 < \alpha < 1$, and $s'(t) > 0$ if $0 \leq t < \delta$.*

Take a Banach space $C^1[0, T]$ with fixed $T > 0$, and a closed convex subset

$$B = \{s \in C^1[0, T] , \ s(0) = s_0 , \ 0 \leq s'(t) \leq K \text{ for } 0 \leq t \leq T\} ,$$

where K is a positive constant to be determined.

For any $s \in B$, we define $\tilde{u}, \tilde{v}$ to be the corresponding solution of (1.11)–(1.14), (1.16), (1.17) with either $(\tilde{D})$ or $(\tilde{N})$. Then, by the maximum principle,

$$(2.1) \quad \tilde{u}(\xi, t) > 0 , \ \tilde{v}(\xi, t) > 0 \quad \text{if } 0 \leq \xi \leq 1 , \quad 0 \leq t \leq T .$$

LEMMA 2.2. *There exists a positive constant C independent of s, K such that*

$$(2.2) \quad \tilde{u}(\xi, t) \leq C , \quad \tilde{v}(\xi, t) \leq C .$$

Proof. Consider the function

$$p(\xi, t) = C_1 t + C_2 (\xi - 1)^2 .$$

It satisfies

$$\begin{aligned} p_t - \frac{1}{s^2} p_{\xi\xi} - \xi \frac{s'(t)}{s(t)} p_{\xi} \\ = C_1 - \frac{1}{s^2} \cdot 2C_2 - \xi \frac{s'(t)}{s(t)} \cdot 2C_2 (\xi - 1) \geq C_1 - \frac{2C_2}{s_0} \geq 0 \end{aligned}$$

if $C_1 = 2C_2/s_0$. Further

$$p_{\xi} = 0 \geq \tilde{u}_{\xi} , \ p_{\xi} = 0 \geq \tilde{v}_{\xi} \quad \text{on} \quad \xi = 1$$

and

$$p(0, t) \geq C_2 , \quad -p_{\xi}(0, t) = 2C_2 .$$

Thus p is a supersolution for both $\tilde{u}$ and $\tilde{v}$, in both case (D) and (N), provided C_2 is chosen large enough. It follows that $\tilde{u} \leq p , \tilde{v} \leq p$. (In case (D), we can choose, more simply, $p = Ce^{\gamma t}$.)

□

From (2.1), (2.2) and (1.13), (1.14) we see that

$$|\tilde{u}_\xi|, |\tilde{v}_\xi| \leq C_K \quad \text{on} \quad \xi = 1,$$

where C_K depends on K (but is independent of s). By (2.1), (2.2) and a priori estimates,

$$|\tilde{u}_\xi|, |\tilde{v}_\xi| \leq C_K \quad \text{on} \quad \xi = 0$$

in case of (D). These inequalities hold of course also in case (N). Applying the maximum principle to $\tilde{u}_\xi$ and to $\tilde{v}_\xi$ we then deduce that

$$(2.3) \quad \|\tilde{u}_\xi\|_{L^\infty} \leq C_K, \quad \|\tilde{v}_\xi\|_{L^\infty} \leq C_K.$$

In case (D) we clearly have, by assumptions,

$$(2.4) \quad (\tilde{u}^2 + \tilde{v}^2)(0, t) + \int_0^T |(\tilde{u}^2 + \tilde{v}^2)_t(0, t)|^p dt \leq C \quad (C \text{ constant})$$

for any $p < \infty$; here and in the sequel all constant are independent of K unless otherwise specified. In case (N), by L^p estimates [8] and the assumption (E1), we have

$$(2.4') \quad \tilde{u}^2 + \tilde{v}^2 \in W_p^{2,1}([0, \frac{1}{2}s_0] \times [0, T]) \quad \forall p > 0.$$

We introduce an auxiliary function $w(\xi, t)$ as the solution to

$$(2.5) \quad w_t = \frac{1}{s^2(t)} w_{\xi\xi} + \xi \frac{s'(t)}{s(t)} w_\xi - \frac{2}{s(t)} (\tilde{u}_\xi^2 + \tilde{v}_\xi^2), \quad 0 \leq \xi \leq 1, \quad 0 \leq t \leq T,$$

$$(2.6) \quad w(\xi, 0) = u_0^2(s_0\xi) + v_0^2(s_0\xi), \quad 0 \leq \xi \leq 1,$$

$$(2.7) \quad w(0, t) = (\tilde{u}^2 + \tilde{v}^2)(0, t), \quad 0 \leq t \leq T,$$

$$(2.8) \quad w(1, t) = 1, \quad 0 \leq t \leq T.$$

By L^p estimates (recalling (2.4), (2.4'))

$$(2.9) \quad \|w, w_t, w_\xi, w_{\xi\xi}\|_{L^p} \leq C_{K,p} \quad \forall p < \infty,$$

and therefore, by embedding [8],

$$(2.10) \quad \|w_\xi\|_{C^{\alpha, \alpha/2}} \leq C_{K, \alpha}, \quad \text{for any } 0 < \alpha < 1.$$

We wish to determine a mapping $\bar{s} = Gs$ such that if s is a fixed point then

$$(2.11) \quad \tilde{u}^2 + \tilde{v}^2 = 1 \quad \text{on} \quad \xi = 1.$$

This leads us to studying the function $W = \tilde{u}^2 + \tilde{v}^2$. It satisfies (2.5)–(2.7) as well as

$$W_\xi = -2\tilde{u} \cdot ss' \tilde{u} - 2\tilde{v} \cdot ss' \tilde{v} = -2ss'W \quad \text{on} \quad \xi = 1.$$

The condition $W = 1$ on $\xi = 1$ which we seek to implement for the fixed point of G coincides with the relation

$$W_\xi = -2ss'$$

or

$$(s^2)' = -W_\xi \quad \text{on} \quad \xi = 1.$$

Since we further wish to have $s' \geq 0$, the last relation is replaced by

$$(s^2)' = (-W_\xi)^+(1, t).$$

In summary, $W = 1$ on $\xi = 1$ if and only if (providing $s'(t) > 0$)

$$(2.12) \quad s^2(t) = s_0^2 + \int_0^t (-W_\xi)^+(1, \tau) d\tau.$$

This suggests the following definition for G :

$$(Gs(t))^2 = s_0^2 + \int_0^t (-2\tilde{u}\tilde{u}_\xi - 2\tilde{v}\tilde{v}_\xi)^+(1, \tau) d\tau.$$

The problem which we encounter with this definition is technical: it seems difficult to estimate the integrand on the right-hand side by a constant which is independent of K .

We shall therefore proceed in a different manner. We define G by

$$(2.13) \quad (Gs(t))^2 = s_0^2 + \int_0^t (-w_\xi)^+(1, \tau) d\tau$$

where w is the auxiliary function introduced above, show that G has a fixed point, and finally establish (for the fixed point) the identity $w = W$.

The key for the success of this approach is the following estimate:

LEMMA 2.3. *There exists a constant C^* independent of K such that*

$$(2.14) \quad w_\xi(1, t) \geq -C^*$$

and, consequently,

$$(2.15) \quad 0 \leq (-w_\xi(1, t))^+ \leq C^*, \quad 0 \leq t \leq T.$$

Proof. Let $q(\xi) = 1 + C^*(1 - \xi)$. Then

$$\frac{\partial}{\partial t} q - \frac{1}{s^2} \frac{\partial^2}{\partial \xi^2} q - \xi \frac{s'(t)}{s(t)} \frac{\partial}{\partial \xi} q = -\xi \frac{s'(t)}{s(t)} (-C^*) \geq 0$$

whereas

$$w_t \leq \frac{1}{s^2} w_{\xi\xi} + \xi \frac{s'(t)}{s(t)} w_\xi.$$

Taking C^* large enough we obtain by comparison (here we use (2.1), (2.2))

$$w(\xi, t) \leq q(\xi, t).$$

Since $w(1, t) = q(1, t)$, the assertion (2.14) follows. □

From (2.13), (2.15) we see that

$$0 \leq \frac{d}{dt} (Gs) \leq \frac{(-w_\xi)^+(1, t)}{2(Gs)} \leq \frac{C^*}{2s_0}.$$

Thus G maps B into itself provided we take $K = C^*/(2s_0)$. By (2.10) we also have

$$(2.16) \quad \|Gs\|_{C^{1+\alpha/2}[0, T]} \leq C_{K, \alpha},$$

so that G is compact.

Finally we claim that G is continuous, that is, if

$$\|s_n - s\|_{C^1[0, T]} \rightarrow 0$$

then

$$\|Gs_n - Gs\|_{C^1[0, T]} \rightarrow 0.$$

Indeed, this follows from compactness and uniqueness.

Using the Schauder fixed point theorem we conclude that G has a fixed point s . From (2.16) we see that

$$(2.17) \quad s \in C^{1+\alpha/2}[0, T] \quad \text{for any } 0 < \alpha < 1.$$

We now apply the Schauder estimates [8] to deduce that

$$\tilde{u}, \tilde{v} \in C^{2+\alpha, 1+\alpha/2}([\varepsilon, 1] \times [\varepsilon, T])$$

for any $\varepsilon > 0$; in particular,

$$(2.18) \quad \tilde{u}_\xi^2 + \tilde{v}_\xi^2 \in C^{1+\alpha, \frac{1+\alpha}{2}}([\varepsilon, 1] \times [\varepsilon, T]) .$$

We can therefore apply the Schauder estimates to w (in (2.5)–(2.8)) to get

$$w \in C^{2+\alpha, 1+\alpha/2}([\varepsilon, 1] \times [\varepsilon, T]) .$$

By embedding ([8]),

$$w_\xi(1, \cdot) \in C^{\frac{1+\alpha}{2}}[\varepsilon, T]$$

and hence, by (2.13),

$$(2.19) \quad s = Gs \in C^{1+\alpha, \frac{1+\alpha}{2}}[\varepsilon, T] .$$

Applying again the Schauder estimates to w (using (2.18) and (2.19)) we get

$$(2.20) \quad w_\xi \in C^{2+\alpha, 1+\alpha/2}([\varepsilon, 1] \times [\varepsilon, T]) .$$

This implies (by (2.13) with $s = Gs$) that

$$(2.21) \quad s \in C^{2+\alpha/2} \quad \text{if} \quad s' > 0 .$$

Proof of Theorem 2.1. Let s be a fixed point of G , and define $\tilde{u}, \tilde{v}$ by (1.11)–(1.14), (1.16), (1.17) with $(\tilde{D})$ or $(\tilde{N})$, and $\tilde{w}$ by (2.5)–(2.8). Then, by (E5) $s'(0) > 0$ and therefore, by continuity,

$$(2.22) \quad -w_\xi(1, t) = 2ss'(t) > 0 \quad \text{for} \quad 0 \leq t < \delta$$

for some $\delta > 0$. The function $W = \tilde{u}^2 + \tilde{v}^2$ satisfies the same differential equation (2.5) as w , and the same initial condition at $t = 0$ and boundary condition at $\xi = 0$. Next, on $\xi = 1$,

$$W_\xi + 2ss'W = 2\tilde{u}(\tilde{u}_\xi + ss'\tilde{u}) + 2\tilde{v}(\tilde{v}_\xi + ss'\tilde{v}) = 0$$

whereas, since $w = 1$ on $\xi = 1$,

$$w_\xi + 2ss'w = w_\xi + 2ss' = 0$$

by the relation $s = Gs$, provided $-w_\xi(1, t) > 0$. It follows that w and W satisfy the same boundary conditions also at $\xi = 1$. Hence, $w \equiv W$ and, in particular,

$$\tilde{u}^2 + \tilde{v}^2 = 1 \quad \text{for} \quad \xi = 1, \quad 0 \leq t < \delta .$$

Therefore (1.15) is also satisfied, and (u, v, s) is then a solution of (1.3)–(1.9) with (D) on (N). Finally, the argument that led from (2.17) to (2.21) can be iterated to conclude that $s(t)$ is infinitely differentiable.

We now proceed to establish the existence of a global solution.

THEOREM 2.4. Under the assumptions (E1)–(E7) the problem (1.3)–(1.9) with either (D) or (N) has a solution (u, v, s) for $0 < t < \infty$, with s in $C^{1+\alpha/2}[0, \infty) \cap C^\infty(0, \infty)$ for any $0 < \alpha < 1$, and $s'(t) > 0$ for all $t \geq 0$.

Proof. In the proof of Theorem 2.1 we have obtained a fixed point s to G in B . If we can show that

$$(2.23) \quad s'(t) > 0 \quad \text{for all} \quad 0 \leq t < T$$

then (2.22) will hold for all $0 \leq t < T$, and, as before, (u, v, s) will form the asserted solution for $0 < t < T$. To get a solution for all $t > 0$, we can take a sequence $T = T_n \rightarrow \infty$ and show that the corresponding solutions converge to a solution defined for all $t > 0$; alternately we can use uniqueness (to be proved in §3) to show that the sequence of solution extend each other to a global solution.

Suppose (2.23) is not true. Then there exists a smallest value $t = t_0$ for which (2.23) fails to hold, i.e.,

$$(2.24) \quad s'(t_0) = 0.$$

In case of (D), we note by (E6), (E7) that u takes its maximum in $\{0 \leq t \leq \tau\}$ at $(0, \tau)$. Hence $u_x(0, \tau) \leq 0$. This is of course true also in case (N) (by (E4)). Since further $u_x \leq 0$ at $t = 0$ (by (E6)), and $u_x \leq 0$ on $x = s(t)$, $0 \leq t \leq t_0$, the maximum principle gives

$$u_x \leq 0 \quad \text{if} \quad 0 \leq x \leq s(t), \quad 0 \leq t \leq t_0.$$

Furthermore, since $u_x(s(t_0), t_0) = 0$ by (1.5) and (2.24), the strong maximum principle implies that

$$u_{xx}(s(t_0), t_0) > 0$$

and similarly $v_{xx}(s(t_0), t_0) > 0$, so that

$$(2.25) \quad u_t(s(t_0), t_0) > 0, \quad v_t(s(t_0), t_0) > 0.$$

On the other hand, differentiating $u^2 + v^2 = 1$ along $x = s(t)$ gives us

$$2u(u_x s' + u_t) + 2v(v_x s' + v_t) = 0$$

if $x = s(t)$, $t < t_0$; taking $t \uparrow t_0$ and using (2.24) we get $2(uu_t + vv_t) = 0$ at $(s(t_0), t_0)$, a contradiction to (2.25).

§3. Uniqueness. When we shall speak of a solution to (1.3)–(1.9) with either (D) or (N), for $0 \leq t < T$, we shall always include the following properties:

$$(3.1) \quad \begin{aligned} s &\in C^1[0, T], \quad s'(t) \geq 0, \\ u_x &\in L^\infty. \end{aligned}$$

THEOREM 3.1. *Under the assumptions (E1)–(E7), for any $T > 0$ there exists a unique solution to (1.3)–(1.9) for $0 \leq t < T$.*

Proof. Suppose (s, u, v) is a solution for (N). Let

$$(3.2) \quad U(x, t) = \int_0^x u(\xi, t) d\xi.$$

Then

$$(3.3) \quad U_t - U_{xx} = g_1(t), \quad 0 < x < s(t), \quad 0 < t < T,$$

$$(3.4) \quad U(0, t) = 0, \quad U(x, 0) = \int_0^x u_0(\xi) d\xi$$

and on the free boundary $U_{xx} + s'U_x = 0$, i.e., $U_t + s'U_x = -g_1(t)$, which integrates to

$$(3.5) \quad U(s(t), t) = - \int_0^t g_1(\tau) d\tau + \int_0^{s_0} u_0(\xi) d\xi.$$

Suppose (u_1, v_1, s_1) and (u_2, v_2, s_2) are two solutions, and denote by U_i the transform of u_i by means of (3.2). We introduce the corresponding functions $\tilde{U}_i, \tilde{V}_i$ as in (1.10) and define

$$\tilde{U}_1(\xi, t) = U_1(\xi s_1(t), t), \quad \tilde{U}_2(\xi, t) = U_2(\xi s_2(t), t).$$

Then

$$\frac{\partial}{\partial t} \tilde{U}_i = \frac{1}{r_i(t)} \frac{\partial^2}{\partial \xi^2} \tilde{U}_i + \frac{1}{2} \xi \frac{r'_i(t)}{r_i(t)} \frac{\partial \tilde{U}_i}{\partial \xi} + g_1(t), \quad 0 < \xi < 1, \quad 0 < t < T$$

where $r_i(t) = s_i^2(t)$ ($i = 1, 2$) and, consequently,

$$(3.6) \quad (\tilde{U}_1 - \tilde{U}_2)_t = \frac{1}{r_1(t)} (\tilde{U}_1 - \tilde{U}_2)_{\xi\xi} + \frac{1}{2} \xi \frac{r'_1(t)}{r_1(t)} (\tilde{U}_1 - \tilde{U}_2)_\xi + f(\xi, t)$$

where

$$(3.7) \quad f(\xi, t) = \left(\frac{1}{r_1(t)} - \frac{1}{r_2(t)} \right) (\tilde{U}_2)_{\xi\xi} + \frac{1}{2} \xi \left(\frac{r_1'(t)}{r_1(t)} - \frac{r_2'(t)}{r_2(t)} \right) (\tilde{U}_2)_\xi .$$

We take (u_2, v_2, s_2) to be the solution obtained in Theorem 2.4. Then

$$(3.8) \quad \|\tilde{U}_2, (\tilde{U}_2)_\xi, (\tilde{U}_2)_{\xi\xi}\|_{L^p} \leq A < \infty$$

so that

$$(3.9) \quad \|f\|_{L^p([0,1] \times [0,T])} \leq C_0 \|r_1 - r_2\|_{W^{1,\infty}[0,T]}$$

where C_0 depends only on A , but not on T if, say, $T \leq 1$.

Noting that $\tilde{U}_1 - \tilde{U}_2$ vanishes on $t = 0$, $\xi = 0$ and $\xi = 1$, and using L^p estimates, we get

$$\|(\tilde{U}_1 - \tilde{U}_2)_{\xi\xi}\|_{L^p([0,1] \times [0,T])} \leq C_1 \|r_1 - r_2\|_{W^{1,\infty}[0,T]}$$

and, consequently,

$$(3.10) \quad \|(\tilde{u}_1 - \tilde{u}_2)_\xi\|_{L^p([0,1] \times [0,T])} \leq C_1 \|r_1 - r_2\|_{W^{1,\infty}[0,T]} .$$

Next we consider the case for Dirichlet conditions, and make the transformation

$$U(x, t) = \int_x^{s(t)} u(\xi, t) d\xi .$$

We find that

$$U_t = U_{xx} \quad , \quad 0 < x < s(t) \quad , \quad 0 < t < T \quad ,$$

and

$$\begin{aligned} U(s(t), t) &= 0 \quad , \\ U(x, 0) &= \int_x^{s_0} u_0(\xi) d\xi \quad , \\ -U_x(0, t) &= g_1(t) \quad . \end{aligned}$$

Proceeding as before we can establish again the estimate (3.10).

Similarly,

$$(3.11) \quad \|(\tilde{v}_1 - \tilde{v}_2)_\xi\|_{L^p([0,1] \times [0,T])} \leq C_1 \|r_1 - r_2\|_{W^{1,\infty}[0,T]}$$

holds for both (N) and (D).

Next we define w_i as in (2.5)–(2.8) for $s = s_1$ and set $w = w_1 - w_2$, that is, $w = (\tilde{u}_1^2 + \tilde{v}_1^2) - (\tilde{u}_2^2 + \tilde{v}_2^2)$. Then

$$(3.12) \quad w_t = \frac{1}{r_1(t)} w_{\xi\xi} + \frac{1}{2} \xi \frac{r_1'(t)}{r_1(t)} w_{\xi} + k(\xi, t)$$

where

$$\begin{aligned} k(\xi, t) = & \left(\frac{1}{r_1} - \frac{1}{r_2} \right) (w_2)_{\xi\xi} + \frac{1}{2} \xi \left(\frac{r_1'}{r_1} - \frac{r_2'}{r_2} \right) (w_2)_{\xi} \\ & - \frac{2}{\sqrt{r_1}} [(\tilde{u}_1)_{\xi}^2 + (\tilde{v}_1)_{\xi}^2] + \frac{2}{\sqrt{r_2}} [(\tilde{u}_2)_{\xi}^2 + (\tilde{v}_2)_{\xi}^2] . \end{aligned}$$

By (3.10), (3.11) and (2.9) (which holds for w_2)

$$(3.13) \quad \|k\|_{L^p([0,1] \times [0,T])} \leq C \|r_1 - r_2\|_{W^{1,\infty}[0,T]}$$

for any $p < \infty$.

The proof of (3.10), (3.11) also gives

$$(3.14) \quad \left[\int_0^1 \int_0^T |w(\xi, t)|^p dt d\xi \right]^{1/p} \leq C \|r_1 - r_2\|_{W^{1,\infty}[0,T]} .$$

We can now use (3.13), (3.14) and $w(1, t) = 0$ to derive the interior-boundary L^p estimate

$$\|w\|_{W_p^{2,1}([\frac{1}{2}, 1] \times [0,T])} \leq C \|r_1 - r_2\|_{W^{1,\infty}[0,T]} ,$$

for any $p < \infty$. Thus, by embedding,

$$(3.15) \quad \|w_{\xi}\|_{C^{\alpha, \alpha/2}([\frac{1}{2}, 1] \times [0,T])} \leq C \|r_1 - r_2\|_{W^{1,\infty}[0,T]} ;$$

the constant C is independent of T if, say, $T < 1$.

Recalling that

$$-w_{\xi} = -(w_1 - w_2)_{\xi} = \frac{1}{2} (r_1'(t) - r_2'(t)) , \quad \xi = 1 ,$$

we obtain from (3.15)

$$(3.16) \quad \|r'\|_{C^{\alpha/2}[0,T]} \leq C \|r\|_{W^{1,\infty}[0,T]}$$

where $r = r_1 - r_2$. Since $r'(0) = 0$,

$$|r'(t)| \leq \|r'\|_{C^\alpha[0,T]} T^\alpha.$$

Also

$$|r(t)| \leq \int_0^t |r'(t)| dt \leq \|r'\|_{C^\alpha[0,T]} T^{1+\alpha}.$$

It follows from (3.16) that

$$\|r'\|_{C^{\alpha/2}[0,T]} \leq C_1 T^\alpha \|r'\|_{C^{\alpha/2}[0,T]}$$

and thus $r'(t) = 0$ and $s_1(t) = s_2(t)$ for $0 \leq t \leq T_0$ if $CT_0^\alpha = \frac{1}{2}$.

We can now proceed step-by-step to prove that $s_1(t) = s_2(t)$ for all $0 \leq t \leq T$.

§4. Asymptotic behavior in case (D). Consider the solution of (1.3)–(1.9) with the Dirichlet conditions at $x = 1$, and assume that

$$(4.1) \quad \begin{aligned} \lim_{t \rightarrow \infty} g_1(t) &= c_1 > 0, \\ \lim_{t \rightarrow \infty} g_2(t) &= c_2 > 0, \quad c_1^2 + c_2^2 > 1. \end{aligned}$$

(The inequality $c_1^2 + c_2^2 > 1$ is actually a consequence of the assumptions of Theorem 2.4.)

THEOREM 4.1. *If (4.1) holds then the free boundary $x = s(t)$ satisfies:*

$$(4.2) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} = M$$

where M is the unique solution to

$$(4.3) \quad \frac{1}{2} \frac{c_1}{\sqrt{c_1^2 + c_2^2}} M e^{M^2/4} \int_0^M e^{-\zeta^2/4} d\zeta = c_1 - \frac{c_1}{\sqrt{c_1^2 + c_2^2}}.$$

Proof. Clearly

$$\begin{aligned} \frac{d}{dt} \int_0^{s(t)} x(u^2 + v^2 - 1) dx &= \int_0^{s(t)} x(2uu_x + 2vv_x) dx \\ &= 2x(uu_x + vv_x)|_0^{s(t)} - 2 \int_0^{s(t)} (uu_x + vv_x) dx - 2 \int_0^{s(t)} x(u_x^2 + v_x^2) dx \\ &= -(s^2(t))' - 1 + g_1^2(t) + g_2^2(t) - 2 \int_0^{s(t)} x(u_x^2 + v_x^2) dx. \end{aligned}$$

Integrating in t we get

$$\begin{aligned} \frac{s^2(t)}{2} = & s_0^2 - t + \int_0^t (g_1^2 + g_2^2) d\tau - 2 \iint x(u_x^2 + v_x^2) \\ & + \int_0^{s_0} x(u_0^2 + v_0^2 - 1) dx - \int_0^{s(t)} x(u^2 + v^2) dx \end{aligned}$$

so that $s^2(t) \leq Ct$ for t large, or

$$(4.4) \quad s(t) \leq C^* \sqrt{t} \quad \text{if } t \geq t^* .$$

We introduce the auxiliary function

$$(4.5) \quad k(x, t) = \frac{1}{t^\alpha} \left(2 - e^{-\frac{C^* x}{\sqrt{t}}} \right) \quad \text{for } 0 \leq x \leq C^* \sqrt{t}, \quad t \geq t^* .$$

Then

$$k_t - k_{xx} = \frac{1}{t^{\alpha+1}} \left\{ -2\alpha + \alpha e^{-\frac{C^* x}{\sqrt{t}}} - \frac{1}{2} \frac{C^* x}{\sqrt{t}} e^{-\frac{C^* x}{\sqrt{t}}} + (C^*)^2 e^{-\frac{C^* x}{\sqrt{t}}} \right\} .$$

In the region $x \leq C^* \sqrt{t}$ the expression in braces is larger than $-2\alpha + \frac{1}{2} (C^*)^2 \exp(-C^* x / \sqrt{t})$. Hence,

$$(4.6) \quad k_t - k_{xx} \geq 0 \quad \text{if } 0 \leq x \leq C^* \sqrt{t}$$

provided $4\alpha = (C^*)^2 \exp[-(C^*)^2]$.

Clearly also

$$(4.7) \quad k(0, t) = \frac{1}{t^\alpha} > 0 \quad \text{if } t > 0 ,$$

$$(4.8) \quad k_x(s(t), t) + s'(t)k(s(t), t) > 0 \quad \text{for } t > 0 .$$

LEMMA 4.1.

$$(4.9) \quad \lim_{t \rightarrow \infty} \max_{0 \leq x \leq s(t)} |c_1 u(x, t) - c_2 v(x, t)| = 0 .$$

Proof. Let $w = c_1 u - c_2 v$. Then, by (4.1)

$$\lim_{t \rightarrow \infty} w(0, t) = 0 .$$

Also

$$w_x(s(t), t) + s'(t)w(s(t), t) = 0, \quad t > 0,$$

$$\sup_{0 \leq x \leq s(t), \quad t \geq 0} |w(x, t)| \leq A < \infty.$$

By the maximum principle we then get

$$|w(x, t)| \leq AT^\alpha k(x, t) + \sup_{\tau \geq T} |w(0, \tau)|$$

for $0 \leq x \leq s(t)$, $t \geq T$; here we used (4.4) and (4.6)–(4.8). Therefore

$$\lim_{t \rightarrow \infty} \max_{0 \leq x \leq s(t)} |w(x, t)| \leq \sup_{t \geq T} |w(0, t)|$$

and (4.9) follows.

Combining (4.9) with $u^2 + v^2 = 1$ on $x = s(t)$, we conclude that

$$(4.10) \quad \lim_{t \rightarrow \infty} u(s(t), t) = \frac{c_1}{\sqrt{c_1^2 + c_2^2}},$$

$$(4.11) \quad \lim_{t \rightarrow \infty} v(s(t), t) = \frac{c_1}{\sqrt{c_2^2 + c_2^2}}.$$

We shall now resort to the Duvaut transformation which reduces a Stefan problem to a variational inequality (cf. [3] [6]). Denote by $\sigma(x)$ the inverse of $s(t)$ and define, for $t > T$,

$$w(x, t) = \begin{cases} \int_T^t (u(x, \tau) - m(\tau)) d\tau, & 0 \leq x \leq s(T) \\ \int_{\sigma(x)}^t (u(x, \tau) - m(\tau)) d\tau, & s(T) < x < s(t) \\ 0, & x \geq s(t), \end{cases}$$

where $m(t) = u(s(t), t)$. One can compute that

$$(4.12) \quad w_t - w_{xx} = \begin{cases} -m(t) + u(x, T), & 0 \leq x < s(T) \\ -m(t), & s(T) < x < s(t) \end{cases}$$

and $w = w_x = 0$ on $x = s(t)$.

We wish to use a comparison theorem for variational inequalities ([3] [6]). We shall compare w with functions $\tilde{w}$ obtained in the following way:

Let $Z(x, t)$ be a similarity solution of the form $f\left(\frac{x}{\sqrt{t}}\right)$ to the Stefan problem with free boundary $x = N\sqrt{t}$, such that

$$\begin{aligned} Z(0, t) &= \gamma_1, \quad Z(N\sqrt{t}, t) = \gamma, \\ Z_x(N\sqrt{t}, t) &= -\frac{N}{2\sqrt{t}} \gamma. \end{aligned}$$

Then

$$(4.13) \quad f(z) = \gamma_1 - C \int_0^z e^{-\zeta^2/4} d\zeta$$

where N and C are determined by

$$\begin{aligned} 2C &= \gamma N e^{N^2/4}, \\ \gamma &= \gamma_1 - C \int_0^N e^{-\zeta^2/4} d\zeta. \end{aligned}$$

We now define $\tilde{w}$ to be the Duvaut transformation of $Z(x, t)$.

Then $\tilde{w}$ satisfies a variational inequality similar to w . By choosing $\gamma_1 < c_1$, $\gamma < c_1/\sqrt{c_1^2 + c_2^2}$ we get (by comparison) $\tilde{w}$ which is $\leq w$. On the other hand by choosing $Z(x, t) = f\left(\frac{x - s(T)}{\sqrt{t - T}}\right)$, $\gamma_1 > c_1$, $\gamma > c_1/\sqrt{c_1^2 + c_2^2}$ and comparing the corresponding $\tilde{w}$ with w in $\{x > s(T)\}$, we get (recalling that $u_x \leq 0$, $v_x \leq 0$) $\tilde{w} \geq w$. Finally, letting $\gamma \rightarrow c_1$, $\gamma_1 \rightarrow c_1/\sqrt{c_1^2 + c_2^2}$ the assertions (4.2), (4.3) follow.

The above proof does not readily extend to the case of Neumann data. The difficulty arises in trying to construct a comparison function similar to k in (4.5). In the next section we shall establish the corresponding result for the Neumann problem by entirely different method, which is of independent interest.

§5. Asymptotic behavior in case (N). For the Neumann problem we assume that

$$(5.1) \quad \begin{aligned} \lim_{t \rightarrow \infty} \sqrt{t} g_1(t) &= c_1 > 0, \\ \lim_{t \rightarrow \infty} \sqrt{t} g_2(t) &= c_2 > 0. \end{aligned}$$

THEOREM 5.1. *If (5.1) holds then the free boundary problem $x = s(t)$, in the case (N), satisfies:*

$$(5.2) \quad \lim_{t \rightarrow \infty} \frac{s(t)}{\sqrt{t}} = M.$$

Here M can be computed by taking an explicit solution $u = f_1(x/\sqrt{t})$, $v = f_2(x/\sqrt{t})$ with f_i as in (4.13), corresponding to $g_1 = c_1/\sqrt{t}$, $g_2 = c_2/\sqrt{t}$.

We first establish some estimates.

LEMMA 5.2. *The following inequalities hold:*

$$(5.3) \quad s(t) \leq B\sqrt{t} \quad , \quad t > 1 \quad ,$$

$$(5.4) \quad s(t) > b\sqrt{t} \quad , \quad t > 1 \quad (b > 0) \quad ,$$

$$(5.5) \quad s'(t) \leq \frac{C}{\sqrt{t}} \quad , \quad t > 0 \quad ,$$

$$(5.6) \quad |u_x(x, t)| + |v_x(x, t)| \leq \frac{C}{\sqrt{t}} \quad , \quad 0 < x < s(t) \quad , \quad t > 0 \quad ,$$

where B, C are positive constants.

Proof. Since

$$1 \leq u(s(t), t) + v(s(t), t) \leq \sqrt{2}$$

and $u_0(x) + v_0(x) \geq 1$ by (E6), the maximum principle gives

$$(5.7) \quad u(x, t) + v(x, t) > 1$$

and, by comparison, also

$$(5.8) \quad u(x, t) + v(x, t) < C \left(B - \frac{x}{\sqrt{t}} \right) + \sqrt{2} \quad .$$

Using the relation

$$\begin{aligned} & \int_0^t (g_1(\tau) + g_2(\tau)) d\tau + \int_0^{s_0} (u_0(y) + v_0(y)) dy \\ &= \int_0^{s(t)} (u(x, t) + v(x, t)) dx \end{aligned}$$

and (5.1), (5.7), (5.8), we easily obtain the estimates (5.3), (5.4).

The function $w = u^2 + v^2$ satisfies

$$w_t - w_{xx} \leq 0 \quad .$$

Comparing with the supersolution $1 + C(s(t) - x)/s(t)$ and using the maximum principle, we get

$$w \leq 1 + C \frac{s(t) - x}{s(t)}$$

for $t \geq 1$, which gives

$$w_x(s(t), t) \geq -\frac{C_1}{s(t)} > -\frac{C}{\sqrt{t}} \quad .$$

Since $w_x = -2s'$ on $x = s(t)$, (5.5) follows.

Next we compare $z = \sqrt{t} u_x$ with $g = C \exp(-x^2/t)$. They both satisfy the same parabolic equation

$$\zeta_t - \zeta_{xx} - \frac{1}{2t} \zeta = 0 \quad , \quad t > 1$$

But g majorizes $|z|$ on the parabolic boundary if C is large enough. Hence $\sqrt{t} |u_x| \leq C$,

and similarly $\sqrt{t} |v_x| \leq C$.

□

We now form

$$\begin{aligned} u_\alpha(x, t) &= u(\alpha x, \alpha^2 t) , \\ v_\alpha(x, t) &= v(\alpha x, \alpha^2 t) , \\ s_\alpha(t) &= \frac{1}{\alpha} s(\alpha^2 t) , \quad \text{for } \alpha \geq 1 . \end{aligned}$$

Using Lemma 5.2 we easily deduce that any sequence $\alpha \rightarrow \infty$ has a subsequence α_j for which

$$\lim_{\alpha_j \rightarrow \infty} s_{\alpha_j}(t) = s_*(t)$$

uniformly in $t \in [0, T]$, for any $T > 0$, and

$$\lim_{\alpha_j \rightarrow \infty} u_{\alpha_j}(x, t) = u_*(x, t) , \quad \lim_{\alpha_j \rightarrow \infty} v_{\alpha_j}(x, t) = v_*(x, t)$$

uniformly for $\varepsilon \leq t \leq T$ for any $\varepsilon > 0$; further,

$$(5.9) \quad \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) u_* = 0 \quad , \quad 0 < x < s_*(t), t > 0 ,$$

$$(5.10) \quad \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) v_* = 0 \quad , \quad 0 < x < s_*(t), t > 0 ,$$

$$(5.11) \quad \frac{\partial}{\partial x} u_* = -s'_*(t) u_* \quad \text{on } x = s_*(t) , \quad t > 0 ,$$

$$(5.12) \quad \frac{\partial}{\partial x} v_* = -s'_*(t) v_* \quad \text{on } x = s_*(t) , \quad t > 0 ,$$

$$(5.13) \quad u_*^2 + v_*^2 = 1 \quad \text{on } x = s_*(t) , \quad t > 0$$

and

$$(5.14) \quad s_*(0) = 0 , \quad s_*(t) \geq b\sqrt{t} , \quad s'_*(t) \geq 0 ;$$

finally, by (5.1),

$$(5.15) \quad \frac{\partial}{\partial x} u_*(0, t) = -\frac{c_1}{\sqrt{t}} \quad , \quad t > 0 ,$$

$$(5.16) \quad \frac{\partial}{\partial x} v_*(0, t) = -\frac{c_2}{\sqrt{t}} \quad , \quad t > 0 .$$

LEMMA 5.3. *The solution of (5.9)–(5.16) is unique.*

Once this lemma is proved, we may conclude that u_*, v_* are similarity solutions, and

$$s_*(t) = M\sqrt{t},$$

where M is uniquely determined by c_1, c_2 . Thus

$$\lim_{\alpha_j \rightarrow \infty} \frac{1}{\alpha_j} s(\alpha_j^2 t) = M\sqrt{t}$$

and this implies that

$$\lim_{\alpha \rightarrow \infty} \frac{1}{\alpha} s(\alpha^2 t) = M\sqrt{t}, \quad t > 0.$$

Taking $t = 1$, the assertion of Theorem 5.1 follows.

Proof of Lemma 5.3. The uniqueness proof given in §3 does not carry over the present situation where $s(0) = 0$. Hence we have to proceed differently.

Let (u_1, v_1, s_1) and (u_2, v_2, s_2) be two solutions and define

$$U_i = \int_0^x u_i(y, t) dy - 2c_1 \sqrt{t},$$

$$V_i = \int_0^x v_i(y, t) dy - 2c_2 \sqrt{t}.$$

Then

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) U_i = 0, \quad 0 < x < s_i(t), \quad t > 0,$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} \right) V_i = 0, \quad 0 < x < s_i(t), \quad t > 0,$$

$$U_i = V_i = 0 \quad \text{on} \quad x = s_i(t), \quad t > 0,$$

$$U_i = -2c_1 \sqrt{t} \quad \text{on} \quad x = 0, \quad t > 0,$$

$$V_i = -2c_2 \sqrt{t} \quad \text{on} \quad x = 0, \quad t > 0,$$

$$s_i(0) = 0, \quad s_i(t) > 0 \quad \text{for} \quad t > 0,$$

and

$$(5.17) \quad \left(\frac{\partial U_i}{\partial x} \right)^2 + \left(\frac{\partial V_i}{\partial x} \right)^2 = 1 \quad \text{on} \quad x = s_i(t), \quad t > 0.$$

For any $\tau > 0$ consider the functions

$$\begin{aligned}\tilde{U}(x, t) &= U_1(x, t + \tau) - U_2(x, t) , \\ \tilde{V}(x, t) &= V_1(x, t + \tau) - V_2(x, t) .\end{aligned}$$

Notice that

$$(5.18) \quad \tilde{U}(0, t) < 0 , \quad \tilde{V}(0, t) < 0 \quad \text{if } t \geq 0 .$$

For small t

$$(5.19) \quad s_1(t + \tau) > s_2(t)$$

since $s_2(0) = 0$ and $s_1(\tau) > 0$. We claim that (5.18) holds for all t . Indeed, otherwise there is a smallest t_0 such that $s_1(t_0 + \tau) = s_2(t_0)$. Since, by the maximum principle, $U_i < 0$ and $V_i < 0$ if $0 \leq x < s_i(t)$, $t > 0$ we get

$$(5.20) \quad \tilde{U}(s_2(t), t) < 0 , \quad \tilde{V}(s_2(t), t) < 0 \quad \text{if } t < t_0$$

and equalities hold at $t = t_0$. Noting then by the maximum principle (using (5.18) (5.20) that $\tilde{U} \leq 0$, $\tilde{V} \leq 0$ if $0 < x < s_2(t)$, $0 \leq t \leq t_0$, the strong maximum principle gives

$$\frac{\partial}{\partial x} \tilde{U}(s_2(t_0), t_0) > 0 , \quad \frac{\partial}{\partial x} \tilde{V}(s_2(t_0), t_0) > 0 ,$$

so that

$$\frac{\partial U_1}{\partial x} > \frac{\partial U_2}{\partial x} \geq 0 , \quad \frac{\partial V_1}{\partial x} > \frac{\partial V_2}{\partial x} \geq 0$$

at $x = s_2(t_0)$, $t = t_0$. This contradicts (5.17).

Having proved (5.19), we let $\tau \rightarrow 0$ and deduce that $s_1(t) \geq s_2(t)$. Similarly $s_2(t) \geq s_1(t)$. □

REMARK 5.1. The uniqueness proof extends to general boundary conditions (N) or (D) provided $s(0) = 0$ and $s(t) > 0$ if $t > 0$ ((E7) is assumed).

REMARK 5.2. The proof of Theorem 5.1 can be extended to the case of Dirichlet boundary conditions; a result similar to Lemma 5.2 can indeed be proved for this case.

§6. Cylindrical domains. In this section we study the case where the material domain is a cylinder $s_1^2 < x^2 + y^2 < \infty$ and the “normal” phase is initially $s_1^2 < x^2 + y^2 < s_0^2$. Introducing cylindrical coordinates $(r \cos \theta, r \sin \theta, z)$ we are seeking a solution to (0.2)–(0.7) for which the free boundary is given by

$$(6.1) \quad r = s(t) ;$$

the “normal” domain is $s_1 < r < s(t)$.

A priori the most general form of the solution is

$$(6.2) \quad \begin{aligned} \vec{H} = & H_1(r, \theta, t)(0, 0, 1) + H_2(r, \theta, t)(\sin \theta, -\cos \theta, 0) \\ & + H_3(r, \theta, t)(\cos \theta, \sin \theta, 0) \end{aligned}$$

(the first two vectors are normal to $\{r = 1\}$ and the last one is tangential to $\{r = 1\}$). We assume that the initial values for the H_i are functions of r only, and the boundary data on $r = s_1$ are functions of t only. As in §1, one can then show that $\vec{H}$ is independent of the variable z .

We shall work with the Laplacian in cylindrical coordinates

$$\Delta = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} .$$

The equation

$$\left(\frac{\partial}{\partial t} - \Delta \right) \vec{H} = 0$$

implies

$$(6.3) \quad \frac{\partial}{\partial t} H_1 - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) H_1 = 0 .$$

Computing also $\Delta[H_2 \cdot (\sin \theta, -\cos \theta)]$ and $\Delta[H_3 \cdot (\cos \theta, \sin \theta)]$ and equating to zero the coefficients of $(\sin \theta, -\cos \theta)$ and $(\cos \theta, \sin \theta)$ we get

$$(6.4) \quad \frac{\partial H_2}{\partial t} - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) H_2 - \frac{2}{r^2} \frac{\partial H_3}{\partial \theta} = 0 ,$$

$$(6.5) \quad \frac{\partial H_3}{\partial t} - \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) H_3 + \frac{2}{r^2} \frac{\partial H_2}{\partial \theta} = 0 .$$

Setting

$$\begin{aligned} K &= H_1(0, 0, 1) + H_2(\sin \theta, -\cos \theta, 0) , \\ L &= H_3(\cos \theta, \sin \theta, 0) , \end{aligned}$$

we proceed to compute the free boundary conditions.

Recalling that $n = (\cos \theta, \sin \theta, 0)$, we easily compute

$$(6.6) \quad \text{curl } K \times n = \begin{pmatrix} \sin \theta \partial_x (H_2 \cos \theta) + \sin \theta \partial_y (H_2 \sin \theta) \\ -\cos \theta \partial_x (H_2 \cos \theta) - \cos \theta \partial_y (H_2 \sin \theta) \\ \sin \theta \partial_y H_1 + \cos \theta \partial_x H_1 \end{pmatrix} .$$

Next, since

$$-V_n \vec{H} \cdot n = (\text{curl } \vec{H} \times n) \cdot n = 0$$

and V_n is assumed to be positive, we get $\vec{H} \cdot n = 0$, which means that

$$(6.7) \quad H_3 = 0 \quad \text{on} \quad r = s(t) .$$

Using this relation we compute

$$\text{curl } L = (0, 0, \sin \theta \partial_x H_3 - \cos \theta \partial_y H_3)$$

or $r = s(t)$, and

$$(6.8) \quad \text{curl } L \times n = \begin{pmatrix} -\sin^2 \theta \partial_x H_3 + \sin \theta \cos \theta \partial_y H_3 \\ \sin \theta \cos \theta \partial_x H_3 - \cos^2 \theta \partial_y H_3 \\ 0 \end{pmatrix} .$$

Also, by (6.7),

$$-V_n \vec{H} = -s'(t) \begin{pmatrix} \sin \theta H_2 \\ -\cos \theta H_2 \\ H_1 \end{pmatrix} .$$

Therefore the relation (0.2) gives

$$(6.9) \quad \sin \theta \partial_y H_1 + \cos \theta \partial_x H_1 = -s'(t) H_1 ,$$

$$(6.10) \quad \begin{aligned} \partial_x (H_2 \cos \theta) + \partial_y (H_2 \sin \theta) - \sin \theta \partial_x H_3 \\ + \cos \theta \partial_y H_3 = -s'(t) H_2 \end{aligned}$$

as well as (6.7). We can rewrite (6.9) in the simpler form

$$(6.11) \quad \partial_r H_1 = -s'(t) H_1$$

We next wish to simplify (6.10). Using the relations

$$\begin{aligned} \partial_x &= \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta , \\ \partial_y &= \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta , \end{aligned}$$

we can rewrite (6.10) in the form

$$\partial_r H_2 + \frac{1}{r} H_2 + \frac{1}{r} \partial_\theta H_3 = -s'(t) H_2 .$$

Recalling however that $H_3 = 0$ on $r = s(t)$, we get

$$(6.12) \quad \partial_r H_2 + \frac{1}{r} H_2 = -s'(t) H_2 .$$

Since the free boundary conditions are independent of θ , the functions

$$\frac{\partial H_1}{\partial \theta} , \quad \frac{\partial H_2}{\partial \theta} , \quad \frac{\partial H_3}{\partial \theta}$$

satisfy a parabolic system with homogeneous initial and boundary conditions. It follows, by uniqueness for parabolic systems, that these three functions must vanish identically. Consequently the H_i are functions of (r, t) only. We can now check that

$$\operatorname{div} \vec{H} = \frac{\partial}{\partial r} H_3 + \frac{H_3}{r}$$

so that, by (6.7),

$$(6.13) \quad H_3 \equiv 0$$

if and only if $\operatorname{div} \vec{H} = 0$. The final form of the system for H_1, H_2 becomes:

$$(6.14) \quad \frac{\partial}{\partial t} H_1 - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right) H_1 = 0 ,$$

$$(6.15) \quad \frac{\partial}{\partial t} H_2 - \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) H_2 = 0 ,$$

in $s_1 < r < s(t)$, with the free boundary conditions:

$$(6.16) \quad \partial_r H_1 = -s'(t) H_1 ,$$

$$(6.17) \quad \partial_r H_2 + \frac{1}{r} H_2 = -s'(t) H_2 ,$$

$$(6.18) \quad H_1^2 + H_2^2 = H_c^2 .$$

This system is similar to the system considered in §§1-5.

We summarize:

THEOREM 6.1. *Assume that the “normal” domain has the cylindrical form $s_1 < r < s(t)$ and that the initial and boundary data are independent of θ and z , and satisfy the conditions (E1)–(E7) (with x replaced by r and $x = 0$ by $r = s_1$). Then the only possible solutions of the problems (0.2)–(0.7) have the form*

$$\vec{H} = H_1(r, t)(0, 0, 1) + H_2(r, t)(\sin \theta, -\cos \theta, 0)$$

and there exists precisely one solution with $s(t) \in C^{1+\alpha/2}[0, \infty) \cap C^\infty(0, \infty)$, for any $0 < \alpha < 1$, and $s'(t) > 0$ for all $t \geq 0$. The H_i satisfy (6.14), (6.15) in $s_1 < r < s(t)$ and (6.16)–(6.18) on $r = s(t)$.

REMARK 6.1. Consider the case when the free boundary is spherically symmetric and so are the data. Trying to find a solution

$$\begin{aligned}\vec{H} = & H_1(\sin \varphi, -\cos \varphi, 0) + H_2(\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta) \\ & + H_3(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)\end{aligned}$$

we can derive a parabolic system for H_1, H_2, H_3 with free boundary conditions

$$\begin{aligned}\partial_r H_1 + \frac{1}{r} H_1 &= -s' H_1, \\ \partial_r H_2 + \frac{1}{r} H_2 &= -s' H_2, \\ H_3 &= 0, \quad H_1^2 + H_2^2 = H_c^2\end{aligned}$$

where $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. Although we can deduce that $\partial H_j / \partial \varphi = 0$, we cannot prove that $\partial H_j / \partial \theta = 0$ since the parabolic system has variable coefficients which depend on θ . (The condition $\text{div } \vec{H} = 0$ also becomes rather complicated). Thus we do not expect to find a solution with $H_j = H_j(r, t)$; this suggests that *there are no solutions of (0.2)–(0.7) with spherically symmetric free boundary*.

Acknowledgement. We would like to thank John Ockendon for introducing us to the superconductor modeling problem studied in this paper, and for valuable discussions.

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