

**SINGULAR PERTURBATION PROBLEMS WITH A  
COMPACT SUPPORT SEMILINEAR TERM**

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The meaning of the constraint condition on  $B$  in (2) is the following: if the set  $\{u = 0\}$  is negligible, then the only admissible choice for  $B$  is  $B = \{u > 0\}$ , which gives

$$F(u) = 2 \left( \int_{-\infty}^{+\infty} \beta^{1/2}(s) ds \right) P(\{u > 0\}, \Omega).$$

On the contrary, if the set  $\{u = 0\}$  is not negligible, then we have to choose a subset  $H$  of it in order to get, for  $B = \{u > 0\} \cup H$ , the least perimeter  $P(B, \Omega)$ .

We also study (see Theorem 1.1 and Proposition 1.6) the asymptotic behaviour of the minimizers of the problems

$$\int_{\Omega} \left[ \varepsilon |\nabla u|^2 + \varepsilon^{-3} \beta \left( \frac{u}{\varepsilon} \right) \right] dx + \int_{\Omega} g(x, u) dx$$

for suitable functions  $g : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$ .

## 1. Statement of the Results

Let  $\Omega$  be a bounded open subset of  $\mathbf{R}^n$  with a Lipschitz boundary. As usual we denote by  $H^1(\Omega)$  the Sobolev space of all functions  $u \in L^2(\Omega)$  with distributional derivatives in  $L^2(\Omega)$ . Let  $\beta : \mathbf{R} \rightarrow \mathbf{R}$  be a nonnegative lower semicontinuous function with compact support and let  $g : \Omega \times \mathbf{R} \rightarrow \mathbf{R}$  be a Borel function such that

- (i) for a.e.  $x \in \Omega$  the function  $g(x, \cdot)$  is continuous on  $\mathbf{R}$ ;
- (ii) there exist  $p_1, p_2 \in L^1(\Omega)$  and  $C_1, C_2 > 0$  such that for a.e.  $x \in \Omega$  and every  $s \in \mathbf{R}$

$$p_1(x) + C_1 |s|^2 \leq g(x, s) \leq p_2(x) + C_2 |s|^2.$$

Then, for every  $\varepsilon > 0$ , the well-known direct method of the Calculus of Variations (see for instance [6]) ensures the existence of a solution  $u_\varepsilon$  of the problem

$$(\mathcal{P}_\varepsilon) \quad \min \left\{ \int_{\Omega} \left[ \varepsilon |\nabla u|^2 + \varepsilon^{-3} \beta \left( \frac{u}{\varepsilon} \right) \right] dx + \int_{\Omega} g(x, u) dx : u \in H^1(\Omega) \right\}.$$

The present paper is devoted to the characterization of the asymptotic behaviour of the solutions  $u_\varepsilon$  and of the minimum values  $\min(\mathcal{P}_\varepsilon)$  as  $\varepsilon \rightarrow 0$ . In order to describe the limit problem, we introduce the functions

$$g_-(x) = \min \{g(x, t) : t \leq 0\}, \quad g_+(x) = \min \{g(x, t) : t \geq 0\},$$

we set

$$c = 2 \int_{-\infty}^{+\infty} \beta^{1/2}(s) ds,$$

and we recall some basic notions about functions of bounded variation and sets of finite perimeter.

We say that  $u \in L^1(\Omega)$  is a function with bounded variation if

$$(1.1) \quad |Du|(\Omega) = \sup \left\{ \int_{\Omega} u \operatorname{div} g \, dx : g \in C_0^1(\Omega; \mathbf{R}^n), |g| \leq 1 \right\} < +\infty.$$

The number defined in (1.1) is called total variation of  $u$  in  $\Omega$ , and is equal to the integral of  $|\nabla u|$  if  $u$  is a Lipschitz function. We denote by  $BV(\Omega)$  the vector space of all functions  $u$  such that  $|Du|(\Omega) < +\infty$ . By using Riesz's representation theorem, it can be seen that  $u \in BV(\Omega)$  if and only if there is a vector measure  $(\mu_1, \dots, \mu_n)$  with finite total variation which allows integral representation of the distributional derivative of  $u$ , i.e.

$$\int_{\Omega} u \frac{\partial g}{\partial x_i} \, dx = - \int_{\Omega} g \, d\mu_i \quad \forall g \in C_0^1(\Omega), i = 1, \dots, n.$$

If  $u = \chi_B$  is the characteristic function of a Borel set  $B$ , then the total variation of  $u$  is called perimeter of  $B$  in  $\Omega$ , and denoted by  $P(B, \Omega)$ . Sets of finite perimeter have proved to be very useful in the theory of minimal surfaces. A comprehensive treatment of this topic can be found for instance in [8] and [10].

Let  $(\mathcal{P})$  be the following problem

$$(\mathcal{P}) \quad \min \left\{ cP(B, \Omega) + \int_B g_+ \, dx + \int_{\Omega \setminus B} g_- \, dx : B \text{ Borel set} \right\}.$$

Well known compactness and lower semicontinuity properties of sets of finite perimeter (see for instance Giusti [10]) ensure that the infimum of  $(\mathcal{P})$  is attained. The following theorem is the main result of this paper.

**Theorem 1.1.** *We have*

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0^+} [\min(\mathcal{P}_\varepsilon)] = \min(\mathcal{P}).$$

*Moreover, given any sequence  $(\varepsilon_h)$  converging to 0 and any sequence  $(u_h)$  of solutions of  $(\mathcal{P}_{\varepsilon_h})$ , there exist a subsequence  $(u_{h_k})$  and a minimizer  $B$  of  $(\mathcal{P})$  such that, for almost every  $x \in \Omega$ , either  $u_{h_k}(x)$  converges to 0, or  $\operatorname{sign} u_{h_k}(x)$  converges to  $\chi_B(x) - \chi_{\Omega \setminus B}(x)$ . Finally, if*

$$(1.3) \quad g_-(x) < g(x, 0) \quad \text{and} \quad g_+(x) < g(x, 0) \quad \text{a.e. in } \Omega,$$

*then*

$$\operatorname{sign}(u_{h_k}) \rightarrow \chi_B - \chi_{\Omega \setminus B} \quad \text{a.e. in } \Omega.$$

**Remark 1.2.** By using (ii) and the projection theorem (see for instance Castaing & Valadier [7], Theorem III.22), it can be shown that  $g_-$  and  $g_+$  belong to  $L^1(\Omega)$ . Moreover, Aumann's measurable selection theorem (see [7], Theorem III.23) allows us to find two functions  $a, b \in L^2(\Omega)$  such that  $a \leq 0$ ,  $b \geq 0$  and

$$g_-(x) = g(x, a(x)) \quad \text{and} \quad g_+(x) = g(x, b(x)) \quad \text{for a.e. } x \in \Omega.$$

Notice that, if for every  $h \in \mathbf{N}$  we set  $a_h = (a \wedge h) \vee -h$  and  $(b_h \wedge h) \vee -h$ , we have

$$g_-(x) = \lim_{h \rightarrow +\infty} g(x, a_h(x)) \quad \text{and} \quad g_+(x) = \lim_{h \rightarrow +\infty} g(x, b_h(x))$$

in the strong convergence of  $L^1(\Omega)$ .

If we denote by  $\mathcal{M}(\Omega)$  the space of all real valued Borel functions defined in  $\Omega$  and endow  $\mathcal{M}(\Omega)$  with a distance inducing convergence in measure, then Theorem 1.1 can be partially rephrased in terms of  $\Gamma$ -convergence (see for instance [2] for the basic definitions) as follows.

**Theorem 1.3.** *The functionals*

$$F_\varepsilon(u) = \begin{cases} \int_{\Omega} \left[ \varepsilon |\nabla u|^2 + \varepsilon^{-3} \beta \left( \frac{u_\varepsilon}{\varepsilon} \right) \right] dx & \text{if } u \in H^1(\Omega) \\ +\infty & \text{if } u \in \mathcal{M}(\Omega) \setminus H^1(\Omega) \end{cases}$$

$\Gamma$ -converge in  $\mathcal{M}(\Omega)$ , as  $\varepsilon \rightarrow 0$ , to the functional

$$(1.4) \quad F(u) = \min \{ cP(B, \Omega) : \{u > 0\} \subset B \subset \{u \geq 0\} \}.$$

It is well-known that  $\Gamma$ -convergence and equicoercivity ensure convergence of minimizers to minimizers and of minimum values to minimum values. However, it can be easily seen that the functionals  $F_\varepsilon$  are not equicoercive in  $\mathcal{M}(\Omega)$ . For instance, if  $\Omega$  is the interval  $]0, 1[$  and  $u_\varepsilon(x) = 2 + \sin(x/\sqrt{\varepsilon})$ , for every  $\varepsilon > 0$  we have  $F_\varepsilon(u) \leq C$  for a suitable constant  $C$ , but it is impossible to extract subsequences  $(u_{\varepsilon_h})$  which converge almost everywhere. In particular, Theorem 1.1 cannot be deduced from Theorem 1.3. In order to prove (1.2) we need to consider a weaker form of convergence (see Proposition 2.1).

**Remark 1.4.** Since the perimeter  $P(B, \Omega)$  is not affected by modifications of  $B$  in negligible sets, an equivalent definition of  $F$  can be given by requiring the inclusions to hold only almost everywhere. We also remark that  $F(u) = cP(\{u > 0\}, \Omega)$  if  $u \neq 0$  almost everywhere, and  $F(u) = 0$  if either  $u \geq 0$  or  $u \leq 0$  in  $\Omega$ .

**Remark 1.5.** When we consider  $F_\varepsilon$  as functions also of the the domain of integration  $\Omega$  (and then we write  $F_\varepsilon(u, \Omega)$  instead of  $F_\varepsilon(u)$ ), it is interesting to notice that even if all

the functionals  $F_\varepsilon$  are  $\sigma$ -additive measures as functions of  $\Omega$ , the  $\Gamma$ -limit  $F$  does not share this property. In the one dimensional case, it suffices to take a function  $u$  equal to 1 in  $]0, 1/3[$ , equal to 0 in  $[1/3, 2/3]$ , and equal to  $-1$  in  $]2/3, 1[$ . Then,  $F(u, ]0, 2/3[) = 0$ ,  $F(u, ]1/3, 1[) = 0$  but  $F(u, ]0, 1[) = 1$ , so that  $A \mapsto F(u, A)$  is not an additive set function.

**Remark 1.6.** In general, there is no hope for uniqueness of the minimizing set  $B$  of the problem  $(\mathcal{P})$ . Indeed take  $\Omega = ]0, 1[$ , the same function  $u$  of Remark 1.5, and any function  $g$  such that  $g_+ = -u$  and  $g_- = +u$ . Then,  $\min(\mathcal{P}) = 1/3$  and any set  $B = ]0, t[$  with  $1/3 \leq t \leq 2/3$  is a minimizer. This phenomenon forced us to consider only the convergence of the signs of suitable subsequences of minimizers. The pointwise convergence of the minimizers is ensured under stronger assumptions on  $g$ , as the following proposition shows.

**Proposition 1.7.** *Let us assume that for almost every  $x \in \Omega$  the function  $g(x, \cdot)$  has a unique minimizer  $u_+(x) > 0$  when restricted to the half line  $[0, +\infty[$  and a unique minimizer  $u_-(x) < 0$  when restricted to the half line  $] -\infty, 0]$ . Then, given any sequence  $(\varepsilon_h)$  converging to 0 and any sequence  $(u_h)$  of minimizers of  $(\mathcal{P}_{\varepsilon_h})$ , there is a subsequence  $(u_{h_k})$  converging in measure to a function  $u \in \mathcal{M}(\Omega)$  such that  $u(x) \in \{u_-(x), u_+(x)\}$  for any  $x$  and the set  $\{u > 0\}$  minimizes  $(\mathcal{P})$ .*

## 2. Proof of the Results

The proof of Theorems 1.1 and 1.3 essentially relies on the following two propositions. We define

$$m = \inf \{s : \beta(s) \neq 0\}, \quad M = \sup \{s : \beta(s) \neq 0\}.$$

**Proposition 2.1.** *Let  $(\varepsilon_h) \subset ]0, +\infty[$  be converging to 0, and let  $(u_h) \subset H^1(\Omega)$ ,  $u \in \mathcal{M}(\Omega)$  such that*

$$\limsup_{h \rightarrow +\infty} \frac{u_h}{\varepsilon_h} \geq M$$

*almost everywhere in the set  $\{x \in \Omega : u(x) > 0\}$  and*

$$\liminf_{h \rightarrow +\infty} \frac{u_h}{\varepsilon_h} \leq m$$

*almost everywhere in the set  $\{x \in \Omega : u(x) < 0\}$ . Then,*

$$\liminf_{h \rightarrow +\infty} \int_{\Omega} \left[ \varepsilon_h |\nabla u_h|^2 + \varepsilon_h^{-3} \beta \left( \frac{u_h}{\varepsilon_h} \right) \right] dx \geq F(u),$$

*with  $F(u)$  defined in (1.4).*

**Proof.** We may assume, possibly passing to subsequences, that the liminf in the statement is a finite limit, say  $L$ . Let  $I$  be the primitive of  $\beta^{1/2}$  which vanishes for  $t \leq m$ , and let  $v_h = 2I(u_h/\varepsilon_h)$ . By the chain rule and the inequality  $2ab \leq a^2 + b^2$  we infer

$$\int_{\Omega} |\nabla v_h| dx = \frac{2}{\varepsilon_h} \int_{\Omega} \beta^{1/2} \left( \frac{u_h}{\varepsilon_h} \right) |\nabla u_h| dx \leq \int_{\Omega} \left[ \varepsilon_h |\nabla u_h|^2 + \varepsilon_h^{-3} \beta \left( \frac{u_h}{\varepsilon_h} \right) \right] dx.$$

Since  $0 \leq v_h \leq c$ , it follows that the sequence  $v_h$  is bounded in  $BV(\Omega)$ , and we can assume, by Rellich's theorem, that it converges almost everywhere to a function  $v \in BV(\Omega)$ . Moreover passing to the limit as  $h \rightarrow +\infty$  in the foregoing inequality, and recalling the lower semicontinuity of the total variation (see for instance [10], Theorem 1.9), we get  $|Dv|(\Omega) \leq L$ . We need only to show that  $F(u) \leq |Dv|(\Omega)$ . Since  $0 \leq v \leq c$ , by using the Fleming-Rishel formula (see [10], Theorem 1.23)

$$|Dv|(\Omega) = \int_0^c P(\{x \in \Omega : v(x) > t\}, \Omega) dt$$

we can find  $t \in ]0, c[$  such that  $B = \{x \in \Omega : v > t\}$  is a set of finite perimeter in  $\Omega$  and  $cP(B, \Omega) \leq |Dv|(\Omega)$ . By our assumption on  $(u_h)$ , for almost every  $x$  in the set  $\{x \in \Omega : u(x) > 0\}$  the sequence  $v_h(x)$  converges to  $c$ . In particular,  $\{x \in \Omega : u(x) > 0\} \subset B$  up to a negligible set. A similar argument shows that  $B \subset \{x \in \Omega : u(x) \geq 0\}$  up to a negligible set. Hence,

$$F(u, \Omega) \leq cP(B, \Omega) \leq |Dv|(\Omega) \leq L,$$

and the proposition is proved. ■

In the proposition below it will be very useful the so-called coarea formula (see for instance [8], 3.2.12)

$$(2.1) \quad \int_B |\nabla \varphi| dx = \int_{-\infty}^{+\infty} \mathcal{H}^{n-1}(\{x \in B : \varphi(x) = t\}) dt$$

which holds for every Borel set  $B \subset \mathbf{R}^n$  and every Lipschitz function  $\varphi$ .

**Proposition 2.2.** *For any function  $u \in \mathcal{M}(\Omega)$  it is possible to find functions  $(u_\varepsilon) \subset H^1(\Omega)$  converging to  $u$  almost everywhere, such that*

$$(2.2) \quad \lim_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left[ \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-3} \beta \left( \frac{u_\varepsilon}{\varepsilon} \right) \right] dx = F(u).$$

*Moreover, the sequence  $u_\varepsilon$  is bounded in  $L^\infty(\Omega)$  if  $u \in L^\infty(\Omega)$ .*

**Proof.** The proof is achieved in three steps. We define the functional

$$F^+(u) = \inf \left\{ \limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left[ \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-3} \beta \left( \frac{u_\varepsilon}{\varepsilon} \right) \right] dx : u_\varepsilon \rightarrow u \text{ a.e. in } \Omega \right\}.$$

A diagonal argument shows that the infimum in the definition of  $F^+$  is achieved, and  $F^+$  is lower semicontinuous with respect to the almost everywhere convergence. Then, the statement of the proposition is equivalent to the inequality  $F^+(u) \leq F(u)$  (the "liminf" inequality in (2.2) follows by Proposition 2.1).

**Step 1.** We assume that  $u$  is bounded,  $u \neq 0$  almost everywhere, and there is a bounded open set  $C \subset \mathbf{R}^n$  with a smooth boundary such that

$$(2.3) \quad C \cap \Omega = \{x \in \Omega : u(x) > 0\}.$$

We denote by  $\tau$  the distance function from  $\partial C$ , by  $C_\eta$  the set  $\{x \in C : \tau(x) < \eta\}$ , and by  $K$  a positive number such that  $[m, M] \subset [-K, K]$ . We want to define  $u_\varepsilon = a_\varepsilon \sigma_\varepsilon$ , where  $a_\varepsilon$  and  $\sigma_\varepsilon$  are functions which fulfil suitable conditions. Let  $\psi_\varepsilon \in C^\infty(\mathbf{R}^n)$  such that  $0 \leq \psi_\varepsilon \leq 1$  and

$$\psi_\varepsilon \equiv 0 \text{ in } C_{\sqrt{\varepsilon}} \quad \psi_\varepsilon \equiv 1 \text{ in } \mathbf{R}^n \setminus C_{2\sqrt{\varepsilon}} \quad |\nabla \psi_\varepsilon| \leq \frac{2}{\sqrt{\varepsilon}}.$$

We define

$$a_\varepsilon(x) = \psi_\varepsilon(x) \int_{\Omega} \frac{|u|(y)}{K \rho_\varepsilon^n} h\left(\frac{y-x}{\rho_\varepsilon}\right) dy + \varepsilon,$$

where  $h$  is any fixed convolution kernel. The functions  $a_\varepsilon$  are in  $C^\infty(\mathbf{R}^n)$ , are greater than  $\varepsilon$ , converge to  $|u|/K$  almost everywhere, and are equal to  $\varepsilon$  on  $C_{\sqrt{\varepsilon}}$ . In addition, the upper bound on  $|\nabla \psi_\varepsilon|$  yields

$$(2.4) \quad \lim_{\varepsilon \rightarrow 0^+} \varepsilon \int_{\Omega} |\nabla a_\varepsilon|^2 dx = 0$$

provided  $\rho_\varepsilon$  converges to 0 slowly enough. Now we turn to the construction of  $\sigma_\varepsilon$ . Let  $\eta_\varepsilon > 0$ ,  $\gamma_\varepsilon$  be solutions of the problem

$$\gamma' = \frac{\beta^{1/2}(\gamma(t)) + \varepsilon^{3/2}}{\varepsilon^3}, \quad \gamma(0) = -K, \quad \gamma(\eta_\varepsilon) = K.$$

By a change of variables we infer

$$(2.5) \quad \eta_\varepsilon = \int_{-K}^K \frac{\varepsilon^3}{\beta^{1/2}(t) + \varepsilon^{3/2}} dt \leq 2K\varepsilon^{3/2}.$$

The functions  $\sigma_\varepsilon$  are defined as follows:

$$\sigma_\varepsilon(x) = \begin{cases} K & \text{if } x \in C \text{ and } \tau(x) \geq \eta_\varepsilon \\ \gamma_\varepsilon(\tau(x)) & \text{if } x \in C \text{ and } \tau(x) < \eta_\varepsilon \\ -K & \text{if } x \notin C. \end{cases}$$

By (2.5), for  $\varepsilon$  small enough the functions  $a_\varepsilon$  are equal to  $\varepsilon$  in  $C_{\eta_\varepsilon}$  and, setting  $u_\varepsilon = a_\varepsilon \sigma_\varepsilon$ , we have  $\beta(u_\varepsilon/\varepsilon) = 0$  outside  $C_{\eta_\varepsilon}$ . By (2.4) we infer

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left[ \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-3} \beta\left(\frac{u_\varepsilon}{\varepsilon}\right) \right] dx = \limsup_{\varepsilon \rightarrow 0^+} \int_{B_\varepsilon} \left[ \varepsilon^3 |\nabla \sigma_\varepsilon|^2 + \varepsilon^{-3} \beta(\sigma_\varepsilon) \right] dx,$$

where we have set for simplicity  $B_\varepsilon = C_{\eta_\varepsilon}$ . Since  $|\nabla \tau| = 1$  almost everywhere (see for instance [8], 3.2.34), by using the coarea formula (2.1) and our special choice of  $\gamma_\varepsilon$  we get

$$\begin{aligned} & \limsup_{\varepsilon \rightarrow 0^+} \int_{B_\varepsilon} \left[ \varepsilon^3 |\nabla \sigma_\varepsilon|^2 + \varepsilon^{-3} \beta(\sigma_\varepsilon) \right] dx = \\ &= \limsup_{\varepsilon \rightarrow 0^+} \int_0^{\eta_\varepsilon} \left[ \varepsilon^3 |\gamma'_\varepsilon|^2 + \varepsilon^{-3} \beta(\gamma_\varepsilon) \right] \mathcal{H}^{n-1}(\{x \in C \cap \Omega : \tau(x) = t\}) dt = \\ &= \limsup_{\varepsilon \rightarrow 0^+} \int_0^{\eta_\varepsilon} \left[ 2 \gamma'_\varepsilon \beta^{1/2}(\gamma_\varepsilon) + 1 \right] \mathcal{H}^{n-1}(\{x \in C \cap \Omega : \tau(x) = t\}) dt. \end{aligned}$$

Since  $\partial C$  is smooth, for  $t$  small enough we have  $\{x \in C : \tau(x) = t\} = \{y + t\nu(y) : y \in \partial C\}$ , where  $\nu$  is the inner normal to  $C$ . In particular,

$$\begin{aligned} \mathcal{H}^{n-1}(\{x \in C \cap \Omega : \tau(x) = t\}) &\leq \mathcal{H}^{n-1}(\{y + t\nu(y) : y \in \partial C, \text{dist}(y, \Omega) < t\}) \leq \\ &\leq (1 + Lt)^{n-1} \mathcal{H}^{n-1}(\{x \in \partial C : \text{dist}(x, \Omega) < t\}). \end{aligned}$$

In the above formula,  $1 + Lt$  is greater than the Lipschitz constant of the map  $y \mapsto y + t\nu(y)$  defined in  $\partial C$ , and  $L$  is a suitable constant depending on the curvatures of  $C$ . Finally, assembling the previous inequalities we get

$$(2.6) \quad \limsup_{\varepsilon \rightarrow 0^+} \int_{\Omega} \left[ \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-3} \beta\left(\frac{u_\varepsilon}{\varepsilon}\right) \right] dx \leq 2 \left( \int_{-\infty}^{+\infty} \beta^{1/2}(s) ds \right) \mathcal{H}^{n-1}(C \cap \overline{\Omega}).$$

**Step 2.** We now remove the regularity assumption on the set  $C = \{x \in \Omega : u(x) > 0\}$ , assuming only that  $P(C, \Omega) < +\infty$ . By using Proposition 2.16 and Remark 2.13 of [10], it is possible to extend the characteristic function of  $C$  to a function  $v \in BV(\mathbf{R}^n)$  with compact support such that  $0 \leq v \leq 1$  and  $|Dv|(\partial\Omega) = 0$ . Let  $(\rho_h)$  be a sequence of mollifiers, and let  $v_h = v * \rho_h$ ; by Sard's theorem, almost every level set of  $v_h$  is smooth. Moreover, given any  $\eta \in ]0, 1/2[$ , by the coarea formula (2.1) for every  $h \in \mathbf{N}$  we can find  $t_h \in ]\eta, 1 - \eta[$  such that the set

$$C_h = \{x \in \mathbf{R}^n : v_h(x) > t_h\}$$

is smooth and

$$(2.7) \quad \mathcal{H}^{n-1}(C_h \cap \overline{\Omega}) \leq \frac{1}{1 - 2\eta} \int_{\Omega} |\nabla v_h| dx = \frac{1}{1 - 2\eta} |Dv_h|(\Omega).$$



Now we set

$$u_h(x) = \begin{cases} |u|(x) & \text{if } x \in C_h \\ -|u|(x) & \text{if } x \in \Omega \setminus C_h. \end{cases}$$

Since  $v_h$  converges to  $\chi_C$  almost everywhere in  $\Omega$ , it can be easily seen that  $u_h$  converges to  $u$  almost everywhere. Since  $u_h$  fulfil condition (2.3), by (2.6) and (2.7) we infer

$$F^+(u_h) \leq \frac{c}{1-2\eta} |Dv_h|(\Omega).$$

Since  $|Dv|(\partial\Omega) = 0$ , the sequence  $|Dv_h|(\Omega)$  converges to  $|Dv|(\Omega) = P(C, \Omega)$  (see for instance [10], Proposition 1.15). By letting  $h \rightarrow +\infty$  we get

$$F^+(u) \leq \frac{1}{1-2\eta} F(u),$$

and the inequality follows by letting  $\eta \rightarrow 0$ .

**Step 3.** Let  $B$  be a minimizing set in the definition of  $F(u)$ . Let  $u_h$  be the functions defined by

$$u_h(x) = \begin{cases} (h \wedge u(x)) \vee -h & \text{if } u(x) \neq 0 \\ 1/h & \text{if } u(x) = 0 \text{ and } x \in B \\ -1/h & \text{if } u(x) = 0 \text{ and } x \notin B. \end{cases}$$

Since the functions  $u_h$  are bounded and are nowhere equal to zero, the first two steps yield

$$F^+(u_h) \leq F(u_h) = F(u).$$

By letting  $h \rightarrow +\infty$  and using the lower semicontinuity of  $F^+$  we obtain the desired inequality. ■

**Proof of Theorem 1.1.** Let  $B$  be a minimizer of  $(\mathcal{P})$ , and let  $a_h, b_h$  be the functions in Remark 1.2. We define

$$u_h(x) = \begin{cases} b_h(x) & \text{if } x \in B \\ a_h(x) & \text{if } x \notin B. \end{cases}$$

By Proposition 2.2, for every integer  $h$  we can find a sequence  $(u_\varepsilon)$  converging to  $u_h$  almost everywhere, bounded in  $L^\infty(\Omega)$ , and such that

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0^+} \left[ \int_{\Omega} \left[ \varepsilon |\nabla u_\varepsilon|^2 + \varepsilon^{-3} \beta \left( \frac{u_\varepsilon}{\varepsilon} \right) \right] dx + \int_{\Omega} g(x, u_\varepsilon) dx \right] = \\ & = F(u_h) + \int_{\Omega} g(x, u_h) \leq cP(B, \Omega) + \int_{\Omega} g(x, u_h) dx. \end{aligned}$$

In particular,

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} [\min(\mathcal{P}_\varepsilon)] &\leq \lim_{h \rightarrow +\infty} \left[ cP(B, \Omega) + \int_{\Omega} g(x, u_h) dx \right] = \\ &= cP(B, \Omega) + \int_B g_+ dx + \int_{\Omega \setminus B} g_- dx = \min(\mathcal{P}). \end{aligned}$$

In order to show the inequality

$$(2.8) \quad \liminf_{\varepsilon \rightarrow 0^+} [\min(\mathcal{P}_\varepsilon)] \geq \min(\mathcal{P}),$$

we choose a sequence  $(\varepsilon_h)$  converging to 0 such that

$$\liminf_{\varepsilon \rightarrow 0^+} [\min(\mathcal{P}_\varepsilon)] = \lim_{h \rightarrow +\infty} [\min(\mathcal{P}_{\varepsilon_h})] = L$$

and we assume  $L < +\infty$  (the inequality being trivial if  $L = +\infty$ ). Let  $u_h \in H^1(\Omega)$  be a minimizer of  $(\mathcal{P}_{\varepsilon_h})$ , and let  $v_h = 2I(u_h/\varepsilon_h)$  be as in the proof of Proposition 2.1. The same argument of Proposition 2.1 shows that the sequence  $(v_h)$  is bounded in  $BV(\Omega)$ . Hence, it is not restrictive to assume that  $v_h$  converges almost everywhere to a function  $v \in BV(\Omega)$ . We define

$$\begin{aligned} B_1 &= \left\{ x \in \Omega : \limsup_{h \rightarrow +\infty} \frac{u_h(x)}{\varepsilon_h} \geq M \right\} \\ B_2 &= \left\{ x \in \Omega : \liminf_{h \rightarrow +\infty} \frac{u_h(x)}{\varepsilon_h} \leq m \right\} \\ u(x) &= \begin{cases} 1 & \text{if } x \in B_1 \\ -1 & \text{if } x \in B_2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By Proposition 2.1 we get

$$(2.9) \quad \liminf_{h \rightarrow +\infty} \int_{\Omega} \left[ \varepsilon_h |\nabla u_h|^2 + \varepsilon_h^{-3} \beta \left( \frac{u_h}{\varepsilon_h} \right) \right] dx \geq F(u).$$

If  $x \in B_1$  and  $v_h(x)$  converges, then necessarily the limit of  $v_h$  is equal to  $c$ , and

$$\liminf_{h \rightarrow +\infty} \frac{u_h(x)}{\varepsilon_h} \geq M,$$

because  $2I(t) < c$  for all  $t < M$ . In particular,

$$\liminf_{h \rightarrow +\infty} u_h(x) \geq 0$$

for almost every  $x \in B_1$ , hence

$$\liminf_{h \rightarrow +\infty} \int_{B_1} g(x, u_h) dx \geq \int_{B_1} \liminf_{h \rightarrow +\infty} g(x, u_h) dx \geq \int_{B_1} g_+ dx.$$

A similar argument gives

$$\liminf_{h \rightarrow +\infty} \int_{B_2} g(x, u_h) dx \geq \int_{B_2} g_- dx.$$

Moreover, for every  $x \in \Omega \setminus (B_1 \cup B_2)$  the sequence  $(u_h)$  converges to 0, so that

$$\liminf_{h \rightarrow +\infty} \int_{\Omega \setminus (B_1 \cup B_2)} g(x, u_h) dx \geq \int_{\Omega \setminus (B_1 \cup B_2)} g(x, 0) dx.$$

The last three inequalities yield

$$(2.10) \quad \liminf_{h \rightarrow +\infty} \int_{\Omega} g(x, u_h) dx \geq \int_B g_+ dx + \int_{\Omega \setminus B} g_- dx$$

for any Borel set  $B$  containing  $B_1$  and contained in  $\Omega \setminus B_2$ . By taking as  $B$  the minimizing set in the definition of  $F(u)$ , the inequality (2.8) follows by (2.9) and (2.10).

This proves (1.2). The last statements of the theorem can be shown by repeating the same argument leading to (2.9) with an arbitrary sequence  $(\varepsilon_h)$ . Finally, if (1.3) holds and if the sequence  $(u_{h_k})$  were converging to 0 in a set of positive measure, then the strict inequality in (2.10) would imply a strict inequality in (2.8), that is a contradiction. ■

**Proof of Theorem 1.3.** By Proposition 2.1 we infer

$$\liminf_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) \geq F(u)$$

whenever  $u_\varepsilon \rightarrow u$  in measure. In fact, sequences converging in measure admit subsequences converging almost everywhere. By Proposition 2.2 we get a sequence  $(u_\varepsilon)$  converging to  $u$  in measure such that

$$\lim_{\varepsilon \rightarrow 0^+} F_\varepsilon(u_\varepsilon) = F(u).$$

This proves (see for instance [2]) the  $\Gamma$ -convergence of  $F_\varepsilon$  to  $F$ . ■

**Proof of Proposition 1.7.** By Theorem 1.1, we may assume, with no loss of generality, that  $\text{sign}(u_h)$  converges almost everywhere to  $\chi_B - \chi_{\Omega \setminus B}$  for a suitable minimizer  $B$  of  $\mathcal{P}$ . We shall prove that  $u_h$  converges in measure to the function defined by

$$u(x) = \begin{cases} u_+(x) & \text{if } x \in B \\ u_-(x) & \text{if } x \in \Omega \setminus B. \end{cases}$$

Let  $\delta > 0$  be given; by the assumptions on  $g$  there exists  $\gamma(x) > 0$  such that

$$\begin{aligned} g(x, t) &\geq \gamma(x) + g_+(x) && \text{if } t \geq 0 \text{ and } |t - u_+(x)| \geq \delta \\ g(x, t) &\geq \gamma(x) + g_-(x) && \text{if } t \leq 0 \text{ and } |t - u_-(x)| \geq \delta. \end{aligned}$$

Therefore, setting

$$\Omega_h = \{x \in \Omega : \text{sign } u_h(x) = \text{sign } u(x), |u_h(x) - u(x)| \geq \delta\}$$

it is easy to obtain

$$(2.11) \quad \int_{\Omega} g(x, u_h) dx \geq \int_{\Omega_h} \gamma dx + \int_{\{u_h \geq 0\} \cap B} g_+ dx + \int_{\{u_h < 0\} \setminus B} g_- dx.$$

By Theorem 1.1 and Proposition 2.1 we get

$$(2.12) \quad \limsup_{h \rightarrow +\infty} \int_{\Omega} g(x, u_h) dx \leq \int_B g_+ dx + \int_{\Omega \setminus B} g_- dx$$

so that, by (2.11) and (2.12),

$$\lim_{h \rightarrow +\infty} \int_{\Omega_h} \gamma dx = 0.$$

Since  $\gamma > 0$  and  $\text{sign}(u_h) \rightarrow \text{sign}(u)$  a.e. in  $\Omega$ , this implies

$$\lim_{h \rightarrow +\infty} \text{meas}(\{x \in \Omega : |u_h(x) - u(x)| \geq \delta\}) = 0$$

and, since  $\delta > 0$  is arbitrary, we obtain the convergence in measure of  $u_h$  to  $u$ . ■

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