INFORMATION FLOWS INTRINSIC TO THE
STABILITY OF ECONOMIC EQUILIBRIUM

BY

J.S. JORDAN

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UNIVERSITY OF MINNESOTA

514 Vincent Hall

206 Church Street S.E.

Minneapolis, Minnesota 55455

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ABSTRACT

Most economic theories specify equilibrium relationships among economic variables, and, at least implicitly, the equilibrium information flows among economic decision-makers. However, the adjustment process which moves the system to an equilibrium position is usually not specified. This paper develops a method for analyzing a general class of such adjustment processes. The principal goal of this analysis is to determine the minimal communication and coordination among economic decision-makers needed to achieve equilibrium. This can be compared with the information structure of the equilibrium model. If the latter is not adequate for the former, equilibrium cannot generally be achieved, and the equilibrium relationships cannot reliably approximate actual economic behavior. This information analysis is proposed as a new source of discipline in the evaluation of economic models.

1. Introduction

An economic equilibrium concept describes a relationship between the endogenous variables and the exogenous parameters of the economic environment. Competitive pure trade equilibrium, for example, relates prices and net trade profiles to profiles of trader preferences and endowments. Most research on economic equilibrium concepts has been directed toward proving existence and establishing the optimality, incentive-compatibility or other characteristic properties of the equilibria. However, since economic models are intended to elucidate real phenomena, it is also important to determine whether, and how, an equilibrium could be brought about.

To most theorists writing before the dawn of "general competitive analysis", this was never in question. They explicitly interpreted market equilibrium as the inevitable outcome of natural economic forces. For example, John Stuart Mill (1871, p. 448) wrote:

...If the demand increases, the value rises; if the demand diminishes, the value falls: again, if the supply falls off, the value rises; and falls if the supply is increased. The rise or the fall continues until the demand and supply are again equal to one another: and the value which a commodity will bring in any market is no other than the value which, in that market, gives a demand just sufficient to carry off the existing or expected supply....

However, Samuelson (1947) convincingly demonstrated that, when extended to multiple markets, this informal dynamics is completely inadequate. When later theorists attempted to make the dynamics precise by writing the "tatonnement" price adjustment process as a system of differential equations, they discovered that the stability of equilibrium could be ensured only under very special conditions. Scarf (1960) provided an example of a pure trade economy with a unique equilibrium which was not even locally stable. The subsequent Sonnenschein-Mantel-Debreu characterization of market excess demand functions

showed that the dynamically worst cases are fully consistent with general competitive equilibrium theory. Attempts to establish the stability of competitive equilibrium continue, using alternatives to the tatonnement process, but no satisfactory alternative has yet been found.

With respect to other, more recent equilibrium concepts, the situation is even worse. The past decade has seen the introduction of numerous new equilibrium concepts for economies and organizations. In particular, there have been many adaptations of noncooperative game theoretic solution concepts to model self-interested behavior in organizations with differential information. However, theorists have typically ignored the question of how an equilibrium could be brought about, or how it could be reestablished in response to a change in the exogenous parameters of the model.

Over the past decade, the methodology of intertemporal equilibrium has become the prevalent approach to economic dynamics. An intertemporal equilibrium involves the simultaneous determination of the endogenous variables at all points in time, or, in models with uncertainty, the simultaneous determination of the joint probability distribution of the endogenous variables over time. In competitive models, this requires prices for presently traded commodities to be sensitive to the planned behavior of traders on markets in the distant future, or, in the case of overlapping generations models, to the behavior of traders yet unborn. The required coordination and interdependence of the activity of disparate economic agents at distant points in time begs even more strongly the question of how such an equilibrium could be reached. The methodology of intertemporal equilibrium has succeeded in harnessing the power of optimization theory to the study of economic behavior over time, but it offers little hope of explaining how economic variables adjust in response to disequilibrium.

It would seem that the only solution is to model disequilibrium behavior directly. Indeed, one could argue that a model of disequilibrium adjustment should precede a concept of equilibrium, since an equilibrium is simply a stationary state of an adjustment process. However, the attempt to model disequilibrium behavior leads quickly to a methodological paradox. Since self-interested behavior is the basis of most economic theory, models of disequilibrium, like all economic models, are sharply criticized if the actions of individuals are not chosen optimally. As one expert in this field has written concerning models of unemployment, "My belief is that non-Walrasian models of unemployment will stand or fall on the issue of rationality of behavior" (Drazen (1980), p. 299). Unfortunately, except for those few models which admit dominant strategies, the optimality of one individual's actions depends on the actions of others, and the simultaneous optimization by all agents requires considerable coordination. Thus we are led to concepts of "equilibrium disequilibrium", such as the fixed-price equilibrium concepts of Dreze (1975) and Benassy (1975), for example. Of course these and other authors have made a valuable contribution by modeling the idea that inefficient allocations can be stationary states, which is a fundamental theme of Keynesian macroeconomics. But we are left with the question of how much coordination is needed to reach "equilibrium disequilibrium", and the troubling possibility that a similar degree of coordination could achieve an efficient equilibrium.

On the other hand, if we attempt to model disequilibrium behavior in the absence of individual optimization, we lose the guidance and discipline of economic theory. Indeed, it is not likely that any particular scenario of unfulfilled expectations, incompatible plans, and self-defeating decisions would prove more appealing than others. Yet if we accept the fact that disequilibrium dynamics lies in a class of models among which we cannot choose,

the obvious solution is not to choose but to study the class. This is the motivation for a new methodology, which I will describe in this paper.

An equilibrium concept establishes a relation between a set of economic environments, E , and a "state space" of endogenous variables S . For the moment, it is easiest to think of this relation as a function $\mu: E \rightarrow S$, although we will weaken this requirement to local uniqueness below. We are concerned with the class of continuous-time dynamic adjustment processes which make each equilibrium $\mu(e)$ locally stable. Of course, there are many such processes, just as there is a nondenumerable infinity of paths from an initial state s^0 to the equilibrium $\hat{s} = \mu(e)$. This class of processes might seem to have little in common; we certainly cannot draw any qualitative conclusions about the time derivative $\dot{s}$. However, suppose that the environment changes slightly to e' , with a new equilibrium $\mu(e') = s' \neq \hat{s}$. Then somewhere on the path from s^0 to $\hat{s}$ the process must recognize the change and move the state to s' instead. If the change from e to e' consists of a change in parameters which are the private information of agent i , it is agent i who must somehow signal the change. Exactly what action agent i takes to signal the change, or how this action is motivated, we will not be able to determine. But it must be the case that some communication is made, and we can reasonably hope to establish the minimal information which must be communicated.

Section 2 below describes the class of locally stable dynamic adjustment processes associated with any given equilibrium concept, and the techniques we use to analyze this class. We call this class of processes the "intrinsic dynamics" of the equilibrium concept, to emphasize that since no behavioral restrictions are imposed on the adjustment processes, any properties they possess as a class are intrinsic to the equilibrium concept itself. In particular, the minimal communication required for stability is an intrinsic property. If

the required communication exceeds the capacity of the institutions the equilibrium concept attempts to model, or if, in a strategic equilibrium model, the additional communication violates the incentive-compatibility of the equilibria, the equilibrium concept needs to be reconsidered. Thus we hope to obtain from stability an additional source of discipline for evaluating equilibrium concepts.

Section 3 below discusses the intrinsic dynamics of Walrasian equilibria for exchange environments. Stability of Walrasian equilibrium requires each trader to communicate, at each price, information about the derivative of his demand function as well as his vector of excess demands. Less communication is required by Pareto-improving adjustment processes which converge to an arbitrary Pareto optimum. Hence, when stability is considered, the Walrasian equilibrium concept loses its informational efficiency.

Section 4 is devoted to the intrinsic dynamics of Nash equilibria in an abstract noncooperative game. It is shown that the information structure of the game does not generally enable the players to adjust their strategies so as to stabilize Nash equilibria. Section 5 contains some concluding remarks. To avoid interrupting the exposition, some results are proved in the three appendices to this paper.

The analytical techniques developed in Section 2 involve a certain amount of homotopy theory which, though not advanced, is not commonly used in economic theory¹. For this reason we will relax somewhat the usual standard of rigor to emphasize the intuition behind our techniques and their applications.

¹ However, a recent series of papers on social choice theory by Graciela Chichilnisky, some in collaboration with Geoffrey Heal, makes extensive use of homotopy methods.

2. Intrinsic Dynamics

This section describes a general model of economic adjustment processes. The space of possible economic environments is a product space $E = \prod_{i=1}^N E^i$. Each E^i is the set of characteristics for agent i . Put somewhat differently, E^i is the space of exogenous private information directly observed by agent i . The endogenous variables which are determined in equilibrium lie in a space S of "state" variables.

The equilibrium relationships are represented implicitly as the zero set of a function $\psi: E \times S \rightarrow \mathbb{R}^n$. That is, the "equilibrium correspondence" $\mu: E \rightarrow S$ is defined by $\mu(e) = \{s \in S: \psi(e, s) = 0\}$, so $\mu(e)$ is the set of states which constitute equilibria for the environment $e = (e^1, \dots, e^N)$. We assume by way of regularity conditions that S is an open subset of $\mathbb{R}^n$, that ψ is continuous, that ψ is differentiable in s , and that the map $(e, s) \mapsto D_s \psi(e, s)$ is continuous. We also assume that $n \geq 2$, since most of the problems discussed below do not arise, or are trivial, in the one-dimensional case.

For example, if E is a space of pure exchange economies, so E^i is a space of preferences and endowments for trader i , let S be the price simplex Δ . Let $z^i: E^i \times S \rightarrow \mathbb{R}^n$ be the excess demand function for trader i , and define $\psi: E \times S \rightarrow \mathbb{R}^n$ by $\psi(e, p) = \sum_{i=1}^N z^i(e^i, p)$. Then $\mu(e)$ is the set of Walrasian equilibrium prices for e . Of course trade N -tuples could also be included as state variables. In this case, let $S = \Delta \times \mathbb{R}^{Nn}$, and define ψ by $\psi(e; p; y^1, \dots, y^N) = (z^1(e^1, p) - y^1, \dots, z^N(e^N, p) - y^N, \sum_{i=1}^N y^i)$. Then $\mu(e)$ is the set of Walrasian prices and trade profiles for e .

As a second example, consider the Cournot oligopoly model. Let $c^i: \mathbb{R}_+ \rightarrow \mathbb{R}$ denote the i^{th} firm's cost function, and suppose that market demand is summarized by an inverse demand function $d: \mathbb{R}_+ \rightarrow \mathbb{R}$ which is known to all

firms. Let S denote the space of quantity N -tuples, and define $\psi: E \times S \rightarrow \mathbb{R}^N$ by

$$\psi(e; q^1, \dots, q^N) = \left(\frac{\partial}{\partial q^i} \{d(\sum_{j=1}^N q^j)q^i - e^i(q^i)\} \right)_{i=1}^N.$$

Then assuming that the function $q^i \rightarrow d(\sum_{j \neq i} q^j + q^i)q^i - e^i(q^i)$ is concave for each i , and neglecting boundary cases, $\mu(e)$ is the set of Cournot equilibria for e .

The model of equilibrium is summarized by the triple (E, S, ψ) . There is of course a large class of functions ψ with the same zero set, and thus the same equilibrium correspondence μ . Nonetheless, we take the function ψ as primitive, for two reasons. The first is a matter of convenience. Our machinery for analyzing local stability works best at a locally unique equilibrium, and local uniqueness is easy to ensure by assuming that $D_S \psi(e, s)$ is non-singular. The second involves a substantive informational issue. We would like to interpret the value $\psi(e, s)$ as information which is naturally generated in the equilibrium model. This interpretation is justified in both of the above examples, although in the Cournot oligopoly example each coordinate of ψ is observed only by the respective firm. With this interpretation, we can pose the stability question as follows: Can the adjustment $\dot{s}$ be chosen depending on s and $\psi(e, s)$ alone, or is more information about e required, such as information about $D_S \psi(e, s)$? The answer to this form of the stability question is generally not invariant to the choice of ψ .

With this general model of equilibrium, a dynamic adjustment process can be described as follows. Each agent i observes the current state s and his private information e^i , and communicates a "control" message c^i . An adjustment $\dot{s}$ in the state variables is then made depending on the current state s and the N -tuple of control messages $(c^i)_{i=1}^N$. More precisely, let C^i denote the i^{th} agent's space of control variables, and let $f^i: E^i \times S \rightarrow C^i$

denote the rule by which agent i selects a control message c^i in response to e^i and s . We will call f^i the i^{th} agent's response function. The adjustment $\dot{s}$ is determined by an adjustment function $a: ((c^i)_i, s) \rightarrow \dot{s}$. Combining a and the response functions yields an ordinary differential equation $\dot{s} = F(e, s)$, where $F(e, s) = a((f^i(e^i, s))_i, s)$. Given an initial state s^0 , the evolution of the state is described by the system

$$(*) \quad \begin{aligned} \dot{s} &= F(e, s) \\ s(0) &= s^0. \end{aligned}$$

Of course, we also require that $F(e, s) = 0$ if and only if $\psi(e, s) = 0$ so that the stationary states of the dynamic system $(*)$ correspond to equilibria as originally defined. By way of regularity conditions, we assume that for each i , C^i is an open subset of a finite-dimensional euclidean space, that f^i is continuous, that $f^i(e^i, \cdot)$ is C^1 for each $e^i \in E^i$, and that a is C^1 . Then F is continuous and $F(e, \cdot)$ is Lipschitzian, so that solutions to $(*)$ exist locally and are unique.

Taken literally, the response functions $(f^i)_i$, together with the adjustment function a constitute rules for communication and decision. However, such a literal interpretation is deceptive since the dynamic model is merely intended to formalize the idea that economic variables adjust in response to the actions of economic agents.

The analytical force of the model derives from requiring the state adjustment to depend on the environment only through the current state and the N -tuple $(c^i)_i$, and requiring agent i 's selection of c^i to depend on other agents' private information only through the state s . This forces all of the interaction among agents which is implicitly needed to identify and stabilize equilibria to be represented explicitly by the variables $(c^i)_i$ and s .

Further restrictions which may be appropriate in specific contexts, such as requiring f^i to depend only on certain coordinates of s , are also possible. In some contexts, such as the Cournot oligopoly example above, the state s is a vector of individual state variables s^i , so agent i is most naturally modeled as choosing $\dot{s}^i$ directly. That is, $f^i: (e^i, s) \mapsto \dot{s}^i$ and $a: ((\dot{s}^i)_i, s) \mapsto (\dot{s}^i)_i$. In the Cournot example, the gradient process:

$$\dot{q}^i = \frac{\partial}{\partial q^i} (d(\sum_j q_j) q^i - e^i(q^i))$$

is known to be globally stable in the traditional linear demand, nonincreasing returns to scale case (Arrow and Hurwicz (1960)).

An indirect relationship between $\dot{c}^i$ and $\dot{s}$ is needed in other models where agents do not have direct control over state variables. In the tatonnement price adjustment model, where e^i represents trader i 's preferences and endowment, and the state variable is the price vector p , $f^i(e^i, p)$ is trader i 's excess demand, and $\dot{p} = a((c^i)_i, p) = \sum_i c^i$. To construct a nontatonnement adjustment process, let the state variables consist of the price vector p , together with a profile $(y^i)_{i=1}^N$, of trade vectors. Let $f^i(e^i; p, (y^j)_j)$ be the preference maximizing trade at p , given the endowment $w^i + y^i$, where w^i is the original endowment. The adjustment function a can be defined by

$$\dot{p} = \sum_i c^i$$

as before, and by using a rationing scheme to determine $(y^i)_i$ as a function of $(c^i)_i$, so that $\sum_i y^i = 0$. It is also possible to construct Pareto-improving (and thus stable) nontatonnement processes (e.g., Section 4 of Jordan (1982)).

Given an equilibrium model (E, S, ψ) , we are interested in the class of adjustment processes which make the equilibria locally stable. For each

$(e, s^0) \in E \times S$, and each t , let $s^*(e, s^0, t)$ denote the solution of

$$(*) \quad \begin{aligned} \dot{s} &= F(e, s) \\ s(0) &= s^0 \end{aligned}$$

at time t , if the solution at time t exists. Let $D = \{(e, s): s^*(e, s^0, t) \text{ exists for all } t \in [0, \infty)\}$. An adjustment process $((f^i)_i, a)$ is locally stable if, for each $\bar{e} \in E$, each $\bar{s} \in S$ with $\psi(\bar{e}, \bar{s}) = 0$ and $D_s \psi(\bar{e}, \bar{s})$ nonsingular, and each neighborhood $\bar{S}$ of $\bar{s}$, there is a neighborhood S' of $\bar{s}$ and a neighborhood E' of $\bar{e}$, such that for each $(e, s^0) \in E' \times S'$,

- a) $s^*(e, s^0, t) \in \bar{S}$ for all t ; and
- b) $\lim_{t \rightarrow \infty} s^*(e, s^0, t)$ exists and is in $\mu(e)$.

Requirement (a) above is often termed "Lyapunov stability" and (b) "asymptotic stability". With e regarded as fixed (e.g., take E to be a one point set), these are the conventional requirements of local stability (e.g., Arrow and Hahn (1971), p. 279). However, we have also required stability under perturbations of the environment. If $\bar{s} \in \mu(\bar{e})$, and the environment is perturbed to some $e \in E''$, the adjustment process must move the state to some $\hat{s} \in \mu(e)$. Since ψ is C^1 and $D_s \psi(\bar{e}, \bar{s})$ is nonsingular, μ is locally single-valued and continuous, so $\bar{s}$ is close to $\mu(e)$. Thus a perturbation of the environment from $\bar{e}$ to e induces the same adjustment equation as a perturbation from the equilibrium $\hat{s}$ to $\bar{s}$ in the state space when the environment is e . Hence, stability in response to slight changes in the environment would follow from stability in response to state perturbations alone, provided that the radius of admissible state perturbations is locally bounded away from zero.

From an economic point of view, exogenous changes in the environment seem a more likely source of disequilibrium than exogenous disturbances in the endogenous state variables. Quirk and Saposnik (1968, pp. 149-150), motivate their discussion of stability with the statement "The parameters of the competitive model, tastes and resource holdings of consumers and production possibilities available to producers, are volatile, and the optimality properties of the competitive mechanism would hold little interest if, when equilibrium is disturbed by some change in these parameters, the system had no tendency to converge to equilibrium". Thus the local stability property is best interpreted as the ability of an adjustment process to seek a new equilibrium in response to a sufficiently small exogenous change in the environment. This interpretation also guides our analytical approach, which is described below.

Given an equilibrium model (E, S, ψ) , let $\mathcal{D}(E, S, \psi)$ denote the set of all locally stable adjustment processes $((f^i)_i, \psi)$ having the given equilibria as stationary states. We will call $\mathcal{D}(E, S, \psi)$ the intrinsic dynamics of (E, S, ψ) . The terminology emphasizes that we are not interested in particular locally stable adjustment processes, but only those properties, especially informational properties, common to all. Such properties are, of course, dictated by the equilibrium model itself.

The analytical apparatus I propose to apply has two stages. The first stage investigates the general system

$$(*) \quad \begin{aligned} \dot{s} &= F(e, s) \\ s(0) &= s^0 \end{aligned}$$

to determine the information about e that F must depend on in order to achieve local stability as defined above. The second stage determines the information about e^i that must be communicated by the response function f^i

in order to generate the requisite information about e . I expect that in most applications, the second stage will be fairly straightforward, at least for anyone reasonably familiar with informational decentralization theory. The first stage is the more challenging, and represents the principal analytical innovation in this paper.

To analyze the system $(*)$, suppose that $((f^i)_i, a)$ is locally stable. Let $\bar{s} \in S$ and let $E^* = \{e \in E: \psi(e, \bar{s}) \neq 0 \text{ but } e \in E'(\bar{e}) \text{ for some } \bar{e} \in E \text{ with } \psi(\bar{e}, \bar{s}) = 0 \text{ and } D_s \psi(\bar{e}, \bar{s}) \text{ nonsingular}\}$, where $E'(\bar{e})$ is the neighborhood of $\bar{e}$ given in the definition of local stability above. The set E^* consists of small perturbations of environments for which $\bar{s}$ is a regular equilibrium. In the definition of E^* , we take the neighborhoods $E'(\bar{e})$ smaller, if necessary, so that there is a continuous function $\mu^*: E^* \times [0,1] \rightarrow S$ with

- i) $\psi(e, \mu^*(e, \lambda)) - \lambda \psi(e, \bar{s}) = 0$ for all (e, λ) ;
- ii) $D_s \psi(e, \mu^*(e, \lambda))$ is nonsingular for all (e, λ) ;
- iii) $\mu^*(e, 1) = \bar{s}$ for all $e \in E^*$; and
- iv) $\lim_{t \rightarrow \infty} s^*(e, \bar{s}, t) = \mu^*(e, 0)$.

The existence of a function μ^* satisfying (i-iii) is ensured by the Implicit Function Theorem. Since $D_s \psi(\bar{e}, \bar{s})$ is nonsingular and e is near $\bar{e}$, e has a unique equilibrium near $\bar{s}$. This equilibrium is $\mu^*(e, 0)$, which, by the Lyapunov stability condition (a), must be the equilibrium achieved by the adjustment process. Since equilibria in $\mu(e)$ other than $\mu^*(e, 0)$ are irrelevant, we will set $\mu = \mu^*(\cdot, 0)$ to economize on notation. For each $e \in E^*$, since $\psi(e, \bar{s}) \neq 0$, $F(e, \bar{s}) \neq 0$ and $\mu(e) - \bar{s} \neq 0$. Recall that $\dim S = n$, so

- a) $F(\cdot, \bar{s}): E^* \rightarrow \mathbb{R}^n - 0$, and
- b) $\mu(\cdot) - \bar{s}: E^* \rightarrow \mathbb{R}^n - 0$,

where $R^n - 0 = \{x \in R^n: x \neq 0\}$. The first function indicates the initial direction of movement in response to an equilibrium-changing perturbation of the environment, and the second function describes the actual direction of the new equilibrium. The latter function is completely determined by the equilibrium model. Hence, the best way to determine the dependence of F on e is to relate $F(\cdot, \bar{s})$ to $\mu(\cdot) - \bar{s}$. The strongest relationship which appears to hold in general is that of "homotopy equivalence", which can be established simply by following the solution paths of (*). More precisely, we can add the point " ∞ " to the time interval (as the "one-point compactification" of $[0, \infty]$ (Kelley (1955), p. 150)) and define the function $H: E^* \times [0, \infty] \rightarrow R^n - 0$ by

$$H(e, t) = \begin{cases} F(e, \bar{s}) & \text{if } t = 0; \\ (s^*(e, \bar{s}, t) - \bar{s})/t & \text{if } 0 < t \leq 1; \\ s^*(e, \bar{s}, t) - \bar{s} & \text{if } 1 < t < \infty; \text{ and} \\ \mu(e) - \bar{s} & \text{if } t = \infty. \end{cases}$$

Then H is a homotopy, or continuous deformation of $F(\cdot, \bar{s})$ into $\mu(\cdot) - \bar{s}$, so the equilibrium model determines $F(\cdot, \bar{s})$ up to a homotopy equivalence class. In order to prove that H is continuous, it is necessary to prove that $H(\cdot, t)$ converges continuously to $\mu(\cdot) - \bar{s}$ as $t \rightarrow \infty$. This is a straightforward application of the definition of local stability, so we omit the details. The continuity of H at $t < \infty$ follows from a standard result on the continuity of the solutions of ordinary differential equations.

In general, continuous functions f and g on a space X to a space Y are homotopic, written $f \approx g$, if there is a continuous function $H: X \times [0, 1] \rightarrow Y$ with $H(\cdot, 0) = f$ and $H(\cdot, 1) = g$. The function H is called a homotopy. The relation $\approx$ is an equivalence relation on the set of continuous functions on X to Y . In our application above, the compactified half-

line $[0, \infty]$ is homeomorphic to $[0, 1]$, and thus can be substituted for the latter interval in defining a homotopy. A space Y is contractible if there is a homotopy $H: Y \times [0, 1] \rightarrow Y$ such that $H(\cdot, 0)$ is the identity function on Y , and $H(\cdot, 1)$ is a constant function. The path $H(y, \cdot)$ takes each $y \in Y$ to the same point $y^\circ = H(y, 1)$. For example, R^n is contractible, by the homotopy $H(y, \lambda) = (1 - \lambda)y$. Similarly, any convex set is contractible. If Y is contractible by a homotopy H , and f and g are any two continuous functions on X to Y , then $f \approx g$ by the homotopy H' defined by

$$H'(x, \lambda) = \begin{cases} H(f(x), 2\lambda) & \text{if } 0 \leq \lambda \leq \frac{1}{2}; \text{ and} \\ H(g(x), 2-2\lambda) & \text{if } \frac{1}{2} \leq \lambda \leq 1. \end{cases}$$

Thus, in our application above, it is important that H takes values in $R^n - 0$, since $F(\cdot, \bar{s})$ and $\mu(\cdot) - \bar{s}$ are trivially homotopic as functions on E^* to R^n . The space $R^n - 0$ is not contractible, since any attempt to deform the identity function to a constant must pull points through the excluded origin.

Samuelson (1947, p. 258) used the term "correspondence principle" to denote the relation between the stability of certain dynamical systems and the comparative statics of the equilibrium model. Since the local comparative statics are given by the function $\mu(\cdot) - \bar{s}$, we will state the result as a "homotopy correspondence principle":

$$(HCP) \quad F(\cdot, \bar{s}) \approx \mu(\cdot) - \bar{s}: E^* \rightarrow R^n - 0.$$

The HCP depends only on μ , and is otherwise invariant to the choice of ψ . It should also be emphasized that the HCP is necessary but not in general sufficient for local stability.

The informational implications of the HCP are easiest to pursue in terms of

the function ψ . If the adjustment process reveals, for each (e,s) , the value $\psi(e,s)$ and the derivative $D_s\psi(e,s)$, no additional information about e is needed. For example, the "Newton method"

$$F(e,s) = -D_s\psi(e,s)^{-1}\psi(e,s)$$

is locally stable, since the derivative of the right hand side at an equilibrium ($\psi(e,s) = 0$) is $-I$, which obviously satisfies the usual characteristic root condition. The amount of information about e^i that the control message c^i must contain so that $(c^1, \dots, c^N; s)$ reveals $\psi(e,s)$ and $D_s\psi(e,s)$ will of course depend on the particular function ψ . We will explore this issue in the Walrasian equilibrium example in Section 3 below. However, if we interpret the value $\psi(e,s)$ as information naturally generated in the equilibrium model, we can ask whether information about $D_s\psi(e,s)$ is also necessary for stability. The HCP implies that the answer to this question is affirmative if there is no continuous deformation of $\mu(\cdot) - \bar{s}$ which ignores $D_s\psi(e, \bar{s})$.

The existence or nonexistence of such a deformation is easiest to analyze by treating $D_s\psi(e, \bar{s})$ and $\psi(e, \bar{s})$ as parameters of the environment e . For each $e \in E^*$, let $A_e = D_s\psi(e, \bar{s})$ and $b_e = \psi(e, \bar{s})$. Then A_e is nonsingular and $b_e \neq 0$. Let $GL(n)$ denote the space of $n \times n$ nonsingular matrices (the "general linear group") topologized as a subspace of R^{n^2} , and let $P = \{(A,b) \in GL(n) \times R^n - 0 : (A,b) = (A_e, b_e) \text{ for some } e \in E^*\}$. Suppose there is a map $\pi : P \rightarrow E^*$ satisfying

- v) for each $(A,b) \in P$, if $e = \pi(A,b)$ then $(A,b) = (A_e, b_e)$;
- vi) the functions $(A,b;s) \mapsto \psi(\pi(A,b), s)$ and $(A,b;s) \mapsto D_s\psi(\pi(A,b), s)$ are continuous; and
- vii) the function $(A,b;s) \mapsto F(\pi(A,b), s)$ is continuous.

Conditions (vi) and (vii) ensure that the parameterization $\pi: P \rightarrow E^*$ is sufficiently continuous for our purposes. Condition (vi) implies that $\mu \circ \pi$ is continuous, and together with the HCP, (vi) and (vii) imply that $F(\pi(\cdot), \bar{s}) = \mu \circ \pi(\cdot) - \bar{s}: P \rightarrow R^n - 0$. Of course, (vi) and (vii) will be satisfied if π is continuous, but if E is a topologically complicated space, such as a space of preference relations, it may be easier to verify (vi) and (vii) directly.

The homotopy class of $\mu \circ \pi(\cdot) - \bar{s}$ has a simple representative. Define $H: P \times [0,1] \rightarrow R^n - 0$ by $H(A,b;\lambda) = [\mu^*(\pi(A,b),\lambda) - \bar{s}]/(1-\lambda)$. Then $H(\cdot;0) = \mu^*(\pi(\cdot),0) - \bar{s} = \mu \circ \pi(\cdot) - \bar{s}$ and by the implicit differentiation of (i) above, $H(A,b;1) = -A^{-1}b$. Thus $\mu \circ \pi(\cdot) - \bar{s}$ is homotopic to the function $(A,b) \mapsto -A^{-1}b$. Let $L: P \rightarrow R^n - 0$ denote the latter function. The question whether the adjustment process can ignore $D_s\psi(e,s)$ now becomes the question whether L can be continuously deformed to a function which ignores A . More precisely, let $P_b = \{b \in R^n - 0: b = b_e \text{ for some } (A_e, b_e) \in P\}$. The question is:

(Q) Does there exist a continuous function $g: P_b \rightarrow R^n - 0$ such that L is homotopic to the map $(A,b) \mapsto g(b)$?

If the answer to (Q) is negative, then the environmental information $D_s\psi(e,s)$ cannot be completely ignored.

The function L depends on ψ only through the definition of the domain, P . Hence the answer to (Q) is determined by the nature of P . Although the HCP depends on ψ only through μ , the parameter space P is clearly sensitive to ψ as well as μ , so we cannot make any general assertions concerning the nature of P . However, we will discuss some typical cases, which will also appear in the applications in Sections 3 and 4 below.

Since the environments in E^* , which are parameterized by P , consist of

small perturbations of environments $\bar{e}$ with $\psi(\bar{e}, \bar{s}) = 0$, the vectors b_e will typically lie near 0. Hence, let $\Sigma^{n-1} = \{b \in \mathbb{R}^n - 0 : \|b\| = \epsilon\}$, for some small $\epsilon > 0$, and suppose that $P_1 \times \Sigma^{n-1} \subset P$ for a subset $P_1 \subset GL(n)$. Thus, we are supposing that E^* is rich enough to permit the value of ψ to be perturbed a small distance from zero in all directions, and to permit independent variations of $D_S \psi(e, \bar{s})$ within the set P_1 . We can now restrict the domain of L to $P_1 \times \Sigma^{n-1}$, and think of the function g mentioned in (Q) as a function on Σ^{n-1} to $\mathbb{R}^n - 0$. Let $A_0 \in P_1$, and consider the linear function $b \mapsto -A_0^{-1}b$. If the answer to (Q) is positive, then, in particular, the function g must be homotopic to the function $b \mapsto -A_0^{-1}b$. Since $\mathbb{R}^n - 0$ is homotopy equivalent to Σ^{n-1} , the theorem of Hopf (e.g., Milnor (1965), p. 51) implies that this is possible if and only if the "degree" of g is the same as the degree of this linear function, which is ± 1 as $\det(-A_0^{-1}) \gtrless 0$. This implication is easiest to visualize when $n = 2$. Then if b moves around the circle Σ^1 , $-A_0^{-1}b$ moves around the origin in the same direction if $\det(-A_0^{-1}) > 0$, and in the opposite direction if $\det(-A_0^{-1}) < 0$. The Hopf Theorem states that the degree is the unique homotopy invariant of maps on Σ^{n-1} to $\mathbb{R}^n - 0$, so we can rephrase (Q) by replacing g with any map having the same degree as the function $b \mapsto -A_0^{-1}b$. In particular, we can assume without loss of generality that $\det(-A_0) > 0$, and restate (Q) as:

(Q') Is the function $L: P_1 \times \Sigma^{n-1} \rightarrow \mathbb{R}^n - 0$ homotopic to the function $(A, b) \mapsto b$?

The Hopf Theorem also yields the following immediate result:

- 1) The answer to Q' is negative if P_1 also contains some A' with $\text{sign } \det A' = -\text{sign } \det A_0$.

The above result is perhaps the simplest implication of the HCP, but it is not generally satisfactory for two reasons. First, it suggests that $\text{sign det } D_s \psi(e, s)$ may be all the additional information needed for stability, which we will demonstrate below is far from the case. Second, and more seriously, in the presence of boundary conditions commonly used to ensure the existence of equilibrium, together with the uniqueness of equilibrium, the hypothesis of (1) is not satisfied. Thus far we have taken the existence of equilibrium for granted. This is somewhat deceptive, since most equilibrium models restrict the class of environments to ensure that equilibria exist, and we must take such restrictions into account when stating richness conditions for E . The existence of equilibrium is often ensured by a boundary condition, which states that as s diverges to the boundary of S , or to infinity if S is unbounded, the vector $\psi(e, s)$ points back into S , or at least does not point in the direction of divergence. More formally, given $e \in E$, suppose $\psi(e, \cdot)$ satisfies

- B) There exists $s^0 \in S$ and a compact cube $K \subset S$ with $s^0 \in \text{int } K$ such that for each $s \in \partial K$, $\psi(e, s) \neq 0$ and $\|s - s^0\|^{-1}(s - s^0) \neq \|\psi(e, s)\|^{-1}\psi(e, s)$,

where ∂K denotes the boundary of K .¹ In the Walrasian equilibrium model, where $\psi(e, \cdot)$ is an aggregate excess demand function, condition (B) follows from the monotonicity of preferences (see Section 3 below). Appendix I below shows that (B) implies the existence of $\bar{s} \in K$ with $\psi(e, \bar{s}) = 0$.

Since equilibria are commonly far from the boundary, one would not expect a boundary condition to place any restrictions on the behavior of $\psi(e, \cdot)$ near an equilibrium. Surprisingly, boundary conditions are quite restrictive when the

¹This boundary condition is adapted from Varian (1977).

equilibrium is unique. Appendix II below shows that if $\bar{s}$ is a unique equilibrium and $D_s\psi(e, \bar{s})$ is nonsingular, then $\det(-D_s\psi(e, \bar{s})) > 0$. The Walrasian model with two commodities offers an intuitive example of this implication. Although, with respect to the result (1) above, an environment $e \in E^*$ does not have $\bar{s}$ as an equilibrium, e has an equilibrium s sufficiently near $\bar{s}$ that $\text{sign } \det D_s\psi(e, s) = \text{sign } \det D_s\psi(e, \bar{s})$. Thus, if we assume that a boundary condition like (B) is satisfied and that the equilibrium is globally unique rather than merely locally unique, the parameter set P_1 must be confined to matrices A with $\det(-A) > 0$.

Since we are concerned merely with local stability, the global uniqueness of equilibrium might seem an irrelevant issue. However, our definition of local stability requires stability at all equilibria. If there are multiple equilibria, it might seem more natural to expect that some equilibria will be stable and others unstable, as in the two commodity Walrasian model. We could change the definition to require stability only for at least one equilibrium, but this would entail explaining how the selection of the stable equilibrium behaves in response to changes in the environment. If the equilibrium is unique, these issues are avoided. Thus, wherever possible, conceptual clarity is best served by restricting attention to environments with a unique equilibrium.

When P_1 is confined to matrices with determinants of the same sign, more subtle aspects of the space $GL(n)$ come into play. The sign of the determinant splits $GL(n)$ into two disjoint homeomorphic subsets. We will use the symbol $GL(n)^+$ to denote the set of matrices A with $\det(-A) > 0$. Since the set of singular $n \times n$ matrices forms an $(n^2 - 1)$ -dimensional submanifold of R^{n^2} , one might expect $GL(n)^+$ to behave as an open half-space of R^{n^2} . This analogy holds to the extent that any two matrices in $GL(n)^+$ can be connected by a continuous path in $GL(n)^+$ (e.g., Warner (1971), p. 131), but the analogy fails

in that $GL(n)^+$ is not contractible. In fact, given any vector b in the sphere Σ^{n-1} , the map $A \mapsto Ab$ on $GL(n)^+$ to $R^n - 0$ is not homotopic to a constant map (e.g., Steenrod (1951), p. 54). Similarly, the map $A \mapsto -A^{-1}b$ is not homotopic to a constant map.

Another fact which is quite useful for our purposes is that $GL(n)^+$ has a nonzero "fundamental group". Although this is not the place for a lecture on the fundamental group, we can still give a simple explanation of the geometry behind this result. Let $O(n)^+$ denote the subspace of $GL(n)^+$ consisting of orthonormal matrices, and let $\gamma: GL(n)^+ \rightarrow O(n)^+$ denote the Gram-Schmidt projection. That is, the first column of $\gamma(A)$ is the first column of A , normalized to euclidean norm one, and the second column of $\gamma(A)$ is the second column of A , projected on the subspace orthogonal to the first column and then normalized, etc. Then γ is a continuous deformation of $GL(n)^+$ onto $O(n)^+$. If $i: O(n)^+ \rightarrow GL(n)^+$ is the inclusion map, then $i \circ \gamma$ is homotopic to the identity map on $GL(n)^+$, and $\gamma \circ i$ is (and is thus homotopic to) the identity map on $O(n)^+$, so that $O(n)^+$ and $GL(n)^+$ are "homotopy equivalent". Thus the homotopy properties of $GL(n)^+$ and $O(n)^+$ are identical. The case $n = 2$ is transparent. Every matrix in $O(2)^+$ is of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

for some $(x,y) \in S^1$, where S^1 is the unit circle in R^2 . Thus $O(2)^+$ is homeomorphic to S^1 , so it is clear that $O(2)^+$ is not contractible. If $n > 2$, $O(n)^+$ is more complicated. In particular, $O(3)^+$ is homeomorphic to real projective 3-space (Husemoller (1966), p.93). However, the map $\sigma: S^1 \rightarrow O(n)^+$ defined by

$$\sigma(x,y) = \left(\begin{array}{cc|c} x & y & 0 \\ -y & x & \\ \hline 0 & & -I_{n-2} \end{array} \right),$$

where I_{n-2} denotes the $(n-2) \times (n-2)$ identity matrix, is still not homotopic to a constant map. In fact, the map σ generates the fundamental group of $O(n)^+$. If $n \geq 3$, the fundamental group is isomorphic to the integers mod 2, so every map on S^1 to $GL(n)^+$ is homotopic to either σ or a constant map (Whitehead (1942b)). Of more direct relevance to the HCP is the fact that the map on $S^1 \times \Sigma^{n-1}$ to $R^n - 0$ defined by $(x,y;b) \mapsto \sigma(x,y)b$ is not homotopic to the map $(x,y;b) \mapsto b$ (Whitehead (1942a), Theorem 4, p. 638). Similarly, the map $(x,y;b) \mapsto -\sigma(x,y)^{-1}b$ is not homotopic to the map $(x,y;b) \mapsto b$.

The facts mentioned above can be directly applied to Q^- . If the answer to Q^- is positive, then for any $b^0 \in \Sigma^{n-1}$, the map $A \mapsto -A^{-1}b^0$ on P_1 to $R^n - 0$ is homotopic to the constant map $A \mapsto b^0$. As noted above, this is impossible if P_1 is equal to (or homotopy equivalent to) $GL(n)^+$. Hence:

- 2) The answer to Q^- is negative if P_1 is homotopy equivalent to $GL(n)^+$.

The map σ described above can be used to obtain a more general result. Since the map $(x,y;b) \mapsto -\sigma(x,y)^{-1}b$ on $S^1 \times (R^n - 0)$ is not homotopic to the map $(x,y;b) \mapsto b$,

- 3) If there is a homotopy $H: S^1 \times [0,1] \rightarrow GL(n)^+$ with $H(\cdot,0) = \sigma$ and $H(S^1,1) \subset P_1$, then the answer to Q^- is negative.

Of course, if P_1 is homotopy equivalent to $GL(n)^+$, the hypothesis of (3) is satisfied, so (3) implies (2). The hypothesis of (3) is existential, and there-

fore will usually have to be verified by the explicit construction of H , but the simplicity of σ eases this requirement considerably, as will be seen in Section 4 below and later on in this Section.

The results (1-3) are all that is needed in the applications in Sections 3 and 4 below. The reader may wish to turn to those sections now, before reading the remainder of the present Section which is devoted to a discussion of alternative techniques and other comments.

It may be useful at this point to compare the HCP with a more straightforward and potentially more transparent technique developed by Saari and Simon (1978) in the context of the Walrasian model, where $\psi(e, \cdot)$ is the aggregate excess demand function. They do not consider the role of individual agents, so the environment is the aggregate excess demand function itself, which we will denote by the function $\psi: S \rightarrow R^n$, suppressing the argument e . Translated into our notation, the approach used by Saari and Simon can be described as follows. Suppose the adjustment process takes the form

$$(**) \quad \dot{s} = G(D_s \psi(s), \psi(s)),$$

where $G: GL(n) \times R^n \rightarrow R^n$ is a C^1 function, and $D_b G(A, 0)$ is nonsingular for all $A \in GL(n)$, where b represents the second argument of G . The focus of this approach is the function $A \mapsto D_b G(A, 0)$. Differentiating the right hand side of (**) at equilibrium, that is, where $\psi(s) = 0$, yields the matrix product $D_b G(A, 0)A$, since $G(\cdot, 0) \equiv 0$. If $D_b G(A, 0)$ is not sufficiently sensitive to A , then manipulating A can cause the product $D_b G(A, 0)A$ to have a characteristic root with positive real part, thus preventing local stability.

To measure the sensitivity of $D_b G(\cdot, 0)$, Saari and Simon introduced the notion of "ignorable coordinate". The ij th entry of A is an ignorable coor-

dinate if, when b is restricted to a neighborhood of 0 in R^n , G is independent of a_{ij} for all (A,b) . Saari and Simon showed that the local stability of the differential equation (**) implies some surprising restrictions on the number and location of ignorable coordinates. First, if local stability at every equilibrium is required, then the matrix A varies over the entire space $GL(n)$. Therefore, for any ij , A can be chosen so that a sufficiently large change in a_{ij} changes $\text{sign det } A$. If the entry ij is ignored, then $\det D_b G(A,0)$ is unaffected by the change in a_{ij} , so the sign of $\det (D_b G(A,0)A)$ is changed, changing the sign of the real part of at least one characteristic root. Hence, if (**) is locally stable at every equilibrium, there are no ignorable coordinates. Second, if local stability is required only at a unique equilibrium, then as discussed above, the matrices A are confined to $GL(n)^+$. In this case Saari and Simon obtained restrictions on the number and location of ignorable coordinates, depending on n , for $n = 2, 3$ and 4 . We will say that entries ij and $k\ell$ are rc-distinct if $i \neq k$ and $j \neq \ell$. The restrictions obtained by Saari and Simon are all subsumed in the statement that no two rc-distinct coordinates are ignorable. Thus, $D_b G(\cdot, 0)$ can ignore a row or column of A , but cannot ignore any two entries in distinct rows and columns. Saari and Simon gave an example of a function G which achieves the stability requirement and ignores a column of A , and the example is easily modified to ignore a row of A , so the above restriction is the sharpest possible result.

We now show that the prohibition of rc-distinct ignorable coordinates, for any n , follows easily from the HCP. Consider the function G restricted to $P_1 \times \Sigma^{n-1}$, and suppose by way of contradiction that G ignores a_{ij} and $a_{k\ell}$, with $i \neq k$ and $j \neq \ell$. By applying a homeomorphism to P_1 which interchanges rows and columns of A as necessary, we can assume that $ij = 11$ and $k\ell = 22$. Recall the function $\sigma: S^1 \rightarrow GL(n)^+$ defined by

$$\sigma(x,y) = \left(\begin{array}{cc|c} x & y & 0 \\ -y & x & \\ \hline & & -I_{n-2} \end{array} \right),$$

and define the map $\beta : S^1 \rightarrow GL(n)^+$ by

$$\beta(x,y) = \left(\begin{array}{cc|c} 2 & y & 0 \\ -y & 2 & \\ \hline & & -I_{n-2} \end{array} \right).$$

Since G ignores a_{11} and a_{22} , $G(\sigma(\cdot), \cdot) = G(\beta(\cdot), \cdot)$. Define

$H : S^1 \times [0,1] \rightarrow GL(n)^+$ by

$$H(x,y;\lambda) = \left(\begin{array}{cc|c} -2 + \lambda & y(1-\lambda) & 0 \\ -y(1-\lambda) & -2 + \lambda & \\ \hline & & -I_{n-2} \end{array} \right).$$

Define $H' : S^1 \times \Sigma^{n-1} \times [0,1] \rightarrow \mathbb{R}^n - 0$ by $H'(x,y;b;\lambda) = G(H(x,y;\lambda), b)$. Then $H'(x,y;b;0) = G(\beta(x,y), b) = G(\sigma(x,y), b)$, and $H'(\cdot, 1)$ is the function $(x,y;b) \mapsto G(-I, b)$, which ignores (x,y) . We have already used, in (3) above, the fact that a map which ignores (x,y) is not homotopic to the map $(x,y;b) \mapsto -\sigma(x,y)^{-1}b$. But the HCP requires $G(\sigma(\cdot), \cdot)$ to be homotopic to the latter map, and this contradiction proves that G cannot ignore a_{ij} and a_{kl} .

The method developed by Saari and Simon is technically much simpler than the HCP, since it involves merely the manipulation of matrices to change the sign of the real parts of at least one characteristic root. However, it requires stronger regularity conditions on the adjustment process and, more importantly, the necessary matrix manipulations may be sufficiently prolix to

be intractable unless n is fairly small. Nonetheless, it is certainly to be expected that their method would prove easier than the HCP in some applications. Also, I would like to emphasize that the results of Saari and Simon provided invaluable guidance in the development of the techniques used in the present paper.

If we strengthen the regularity conditions in our model of adjustment processes, and replace our definition of local stability with the conventional sufficient condition involving characteristic roots, we can obtain a version of the HCP which is somewhat more transparent, and more similar to Saari's and Simon's analysis of the function $D_b G(\cdot, 0)$. We will say that an adjustment process is linearly stable if for each $e \in E$ and each $s \in S$ with $\psi(e, s) = 0$ and $D_s \psi(e, s)$ nonsingular, all characteristic roots of $D_s F(e, s)$ have a negative real part. It is easy to verify that linear stability implies local stability. The term "linear" is used because this condition allows stability to be verified by considering only the linear part of the Taylor expansion of $F(e, \cdot)$.¹ Now add back to E^* the environments $\bar{e}$ for which $\bar{s}$ is a regular equilibrium, to define $E_0^* = E^* \cup \{e \in E: \psi(e, \bar{s}) = 0 \text{ and } D_s \psi(e, \bar{s}) \text{ is nonsingular}\}$. Similarly, we need to extend the parameterization of E^* to include parameter pairs (A_e, b_e) with $b_e = 0$, so let U be an open neighborhood of 0 in R^n with $\Sigma^{n-1} \subset U$ and $\pi(P_1 \times U) \subset E_0^*$. Define $\psi^*: P_1 \times U \times S \rightarrow R^n$ and $F^*: P_1 \times U \times S \rightarrow R^n$ in the obvious way, namely $\psi^* = \psi(\pi(\cdot), \cdot)$ and $F^* = F(\pi(\cdot), \cdot)$. If ψ, F and our parameterization of E_0^* are sufficiently well-behaved, we can differentiate the equations

$$\psi^*(A, b, s) = 0$$

$$F^*(A, B, s) = 0$$

¹ This terminology was suggested to me by Stanley Reiter as an alternative to the mathematically conventional term "hyperbolic".

with respect to b and s at $(A, 0, \bar{s})$ to obtain

$$D_b F^*(A, 0, \bar{s}) = D_s F^*(A, 0, \bar{s}) A^{-1}.$$

In particular, $D_b F^*(A, 0, \bar{s})$ is nonsingular, so $D_b F^*(\cdot, 0, \bar{s}): P_1 \rightarrow GL(n)$. Now consider the linear differential equation on $GL(n)$ given by

$$\begin{aligned} \dot{B} &= D_s F^*(A, 0, \bar{s}) B + D_b F^*(A, 0, \bar{s}) \\ B(0) &= 0. \end{aligned}$$

This equation is just the "first variation" of the differential equation (*) which we have analyzed up to now. By the characteristic root condition on $D_s F$, this system is stable, so the solution $B(t)$ converges to the equilibrium $B = -A^{-1}$. Since $\dot{B}(0) = D_b F^*(A, 0, \bar{s})$, the same argument used to prove the HCP shows that the maps $D_b F^*(\cdot, 0, \bar{s})$ and $A \mapsto -A^{-1}$ on P_1 to $GL(n)$ are homotopic. The latter map is of course a homeomorphism of P_1 onto a subset of $GL(n)$, so this is a strong result. We state this as the "linear homotopy correspondence principle":

(LHCP) There is a homotopy $H: P_1 \times [0, 1] \rightarrow GL(n)$ with $H(\cdot, 0) = D_b F^*(\cdot, 0, \bar{s})$ and $H(A, 1) = -A^{-1}$ for all $A \in P_1$.

We can use the LHCP to obtain a weaker richness criterion for P_1 . We say that P_1 is "contractible in $GL(n)$ " if there is a homotopy $H': P_1 \times [0, 1] \rightarrow GL(n)$ such that $H'(\cdot, 1)$ is the inclusion map and $H'(\cdot, 0)$ is a constant function. If F ignores $D_s \psi(e, \bar{s})$, then $F^*(\cdot, \bar{s})$ is a function of b alone, so $D_b F^*(\cdot, 0, \bar{s})$ is a constant function. Define $H': P_1 \times [0, 1] \rightarrow GL(n)$ by $H'(A, \lambda) = -H(A, \lambda)^{-1}$. Then $H'(A, 1) = A$ for all $A \in P_1$, and $H'(\cdot, 0) = -D_b F^*(\cdot, 0, \bar{s})^{-1}$, a constant function. Hence

- 4) If the adjustment process is linearly stable, then $F^*(\cdot, \bar{s})$ is not independent of A if P_1 is not contractible in $GL(n)$.

The richness criteria in (1-3) above each imply that P_1 is not contractible in $GL(n)$. If the condition in (3) is satisfied, then P_1 supports the "fundamental group" of $GL(n)^+$, and is therefore not contractible in $GL(n)$. If P_1 contains matrices A_0 and A' with $\text{sign det } A_0 = -\text{sign det } A'$, which is the condition in (1), then there is no continuous path from A_0 to A' in $GL(n)$. If P_1 were contractible in $GL(n)$, then $H(A_0, \lambda)$ from $\lambda = 1$ to $\lambda = 0$, followed by $H(A', \lambda)$ from $\lambda = 0$ to $\lambda = 1$ would be such a path.

As an illustration of (4), suppose that ψ^* takes the form $\psi^*(A, b, s) = A(s - \bar{s}) + b$, and that $F^*(A, b, s) = \psi^*(A, b, s)$, so $F^*(A, b, \bar{s}) = b$, and thus $F^*(\cdot, \bar{s})$ ignores A . The equation $\dot{s} = A(s - \bar{s}) + b$ is linearly stable if and only if all of the characteristic roots of A have negative real parts. Then if we set P_1 equal to the set of "stable" matrices, (4) implies that the set of stable matrices is contractible in $GL(n)$.

The results (1-4) above provide homotopy theoretic conditions on the equilibrium model (E, S, ψ) . Each condition implies that the stable adjustment of the state variables s requires more information about the environment e than the value $\psi(e, s)$. If $\psi(e, s)$ represents the information generated in the equilibrium model, then the information structure of the equilibrium model is not sufficient for stability. These results, together with the HCP and LHCP which were used to derive them, may be criticized because they do not provide sufficient conditions for stability, and their application involves technicalities not familiar to many theorists. We hope to mitigate the second criticism by describing the application to particular equilibrium models in Sections 3 and 4, below. Regarding the first criticism, although the homotopy equivalence expressed in the HCP and LHCP are not sufficient for stability, they may provide useful insights to guide the construction of stable adjustment processes.

For example, we have noted above that the "Newton Method", $\dot{s} = -D_s \psi(e,s)^{-1} \psi(e,s)$, is locally stable. In fact it is linearly stable, since differentiating the right hand side at equilibrium yields the matrix $-I$. We have already noted that $GL(n)$ is homotopy equivalent to $O(n)$, via the Gram-Schmidt orthogonalization process. Since our analytical approach relies on homotopy equivalence, we cannot distinguish between the knowledge of $D_s \psi(e,s)$, which is an n^2 -dimensional variable, and the knowledge of its orthogonalization $\gamma(D_s \psi(e,s))$, which is an $n(n-1)/2$ -dimensional variable. However, the equation $\dot{s} = -\gamma(D_s \psi(e,s))' \psi(e,s)$, where the prime denotes transpose, is also linearly stable since differentiation of the right hand side at equilibrium yields $\gamma(D_s \psi(e,s))' D_s \psi(e,s)$, an upper triangular matrix with negative diagonal elements. In practice, this "orthogonal Newton method" may be easier to use, since computing $\gamma(D_s \psi(e,s))'$ may be easier than computing $D_s \psi(e,s)^{-1}$, and the normalization of the columns of $\gamma(D_s \psi(e,s))$ is of course not necessary for linear stability.

3. Walrasian Equilibria

This section applies the analysis developed above to a simplified version of the Walrasian equilibrium model. Let $n \geq 2$ and suppose that there are $n + 1$ commodities, with commodity $n + 1$ as numeraire. There are $N \geq n$ traders, and each trader is represented by a linear excess demand function $p \mapsto A^i(p - \bar{p}) + b^i$, where $p \in R_{++}^n$. The pair (A^i, b^i) should be interpreted as a local linear approximation to trader i 's excess demand function for commodities $1, \dots, n$, with the excess demand for commodity $n + 1$ being determined by the budget equality. We will suppose that A^i is nonsingular, so that $E^i = GL(n) \times R^n$, with generic element $e^i = (A^i, b^i)$. This is of course inconsistent with consumer theory since not all nonsingular $n \times n$ matrices can be represented as the sum of a substitution matrix and a matrix of income effects. Also, the pairs (A^i, b^i) merely serve to parameterize the underlying characteristics, namely preferences and endowments. These problems can be remedied by suitably restricting the set of matrices A^i , and using locally quadratic preferences. This is done in Section 3 of Jordan (1982), but these issues will be ignored here.

The state space S is the price space R_{++}^n , and the function ψ is defined by

$$\psi((A^i, b^i)_{i=1}^N, p) = (\sum_{i=1}^N A^i)(p - \bar{p}) + \sum_{i=1}^N b^i$$

for each $e \in E$, let $A_e = \sum_{i=1}^N A^i$ and $b_e = \sum_{i=1}^N b^i$. Then $\mu(e) - \bar{p} = -A_e^{-1} b_e$. We seek to determine how much about (A^i, b^i) trader i must communicate through control messages c^i in order to ensure stability. Note that trader i communicates c^i according to some response function $c^i = f^i(A^i, b^i; p)$, so c^i is chosen without knowledge of (A^j, b^j) for $j \neq i$. Since b^i is trader i 's excess demand at $\bar{p}$, the price $\bar{p}$ is an equilibrium if and only if

$\sum_i b^i = 0$. Hence trader i must communicate b^i to enable the system to determine whether or not $\bar{p}$ is an equilibrium. Note also that for this purpose, b^i is all that need be communicated.

We will use the HCP to assess the additional communication required for stability. Let A^1 and A'^1 be any two distinct matrices in $GL(n)$, and fix A^i for each $i \geq 2$ so that $\text{sign det } (A^1 + \sum_{i \geq 2} A^i) = -\text{sign det } (A'^1 + \sum_{i \geq 2} A^i)$. Let $b^i = 0$ for all $i \geq 2$, and let b^1 vary on the small sphere Σ^{n-1} in $R^n - 0$. Let $A_0 = A^1 + \sum_{i \geq 2} A^i$, $A' = A'^1 + \sum_{i \geq 2} A^i$, and let $P_1 = \{A_0, A'\}$, with the parameterization $\pi: P_1 \times \Sigma^{n-1} \rightarrow E$ defined by $\pi(A, b) = (A - \sum_{i \geq 2} A^i, b; A^2, 0, \dots, A^N, 0)$. Then by the result (1) in Section 2 above, $F(\cdot, \bar{p})$ cannot ignore the distinction between A_0 and A' , and thus the distinction between A^1 and A'^1 . Hence trader 1 must communicate this distinction. Since A^1 and A'^1 were chosen arbitrarily, trader 1 must communicate enough information to reveal A^1 exactly. The model is symmetric in traders, so stability requires each trader to communicate the derivative of his excess demand function, as well as the value of his excess demand. This result, stated more carefully to take account of the restrictions on matrices which are derivatives of individual excess demand functions, is Theorem 3.6 of Jordan (1982).

The requirement that traders communicate the derivatives of their excess demand functions, as well as their excess demands, dramatically illustrates the additional communication requirements of stability. This result depends on the class of environments E being rich enough to permit changing the sign of the determinant of the derivative of the aggregate excess demand at an equilibrium. However, if the equilibrium is unique, and the aggregate excess demand function $\psi(e, \cdot)$ satisfies the boundary condition:

B) There exists $p^0 \in R_{++}^n$ and a compact cube $K \subset R_{++}^n$ with $p^0 \in \text{int } K$ such that for each $p \in \partial K$, $\psi(e, p) \neq 0$ and $\|p - p^0\|^{-1}(p - p^0) \neq \|\psi(e, p)\|^{-1}\psi(e, p)$,

then $\text{sign det } (D_{\bar{p}} \psi(e, \bar{p})) = (-1)^n$. We will first relate (B) to the conventional boundary behavior of excess demand functions. Recall that $\psi(e, \cdot)$ is the excess demand for commodities $1 \leq j \leq n$, so including the numeraire, the full excess demand function is $p \mapsto (\psi(e, p), -p\psi(e, p))$. If $\psi(e, \cdot)$ is derived from a well-behaved exchange economy, it satisfies

- i) $\psi(e, \cdot)$ is bounded from below;
- ii) the function $p \rightarrow -p\psi(e, p)$ is bounded from below; and
- iii) for any price sequence $\{p^k\}_{k=1}^{\infty}$ in R_{++}^n which has no cluster points in R_{++}^n , $\|\psi(e, p^k), -p^k\psi(e, p^k)\| \rightarrow \infty$.

Conditions (i) and (ii) reflect the fact that aggregate excess supply cannot exceed the aggregate endowment. Condition (iii) follows if there is at least one trader with strictly increasing preferences and a positive endowment of every commodity. It is straightforward to derive (B) from (i-iii) by choosing K to be a large cube in R_{++}^n , parallel to the axes with its lower corner near 0, and choosing p^0 at the center of K .

Hence, to study local stability for well-behaved exchange environments with unique equilibria, we need to assume $P_1 \subset GL(n)^+$. In fact, Mas-Colell (1977, Corollary 2, p. 119) has shown that if A is any $n \times n$ matrix with $\text{sign det } (A) = (-1)^n$, and $\bar{p} \in R_{++}^n$, then there is an exchange economy with $\bar{p}$ as its unique equilibrium price, and A as the derivative of the aggregate excess demand function at $\bar{p}$. Thus we can take $P_1 = GL(n)^+$, and conclude from (2) of Section 2 that $F(\cdot, \bar{p})$ cannot ignore the derivative of aggregate excess

demand. Thus each trader i must communicate some information about A^i , although it is not clear what the information must consist of.

If the object of an adjustment process is merely to achieve a Pareto optimal allocation, less communication is required. Let $S = \{(y^i)_i \in R^{nN} : \sum_i y^i = 0\}$, the set of feasible trade profiles. Instead of parameterizing individual characteristics, let E^i be the set of pairs (u^i, w^i) , where u^i is a well-behaved utility function and w^i is an endowment for trader i . Let $f^i(u^i, w^i; (y^j)_j) = \|Du^i(w^i + y^i)\|^{-1} Du^i(w^i + y^i) = c^i$, the normalized utility gradient at $w^i + y^i$. Then the adjustment function $a: ((c^i)_i, (y^i)_i) \mapsto (\dot{y}^i)_i$ can be defined so that $(\dot{y}^i)_i = 0$ if all c^i 's are equal, and $c^i y^i > 0$ for all i , otherwise. This adjustment process is Pareto-improving, so for each environment $(u^i, w^i)_i$, the set of interior Pareto optima, $\{(y^i)_i \in S : w^i + y^i \in R_{++}^n \text{ for all } i \text{ and } \|Du^i(w^i + y^i)\|^{-1} Du^i(w^i + y^i) = \|Du^j(w^j + y^j)\|^{-1} Du^j(w^j + y^j) \text{ for all } i, j\}$, is a set of locally stable equilibria. The defining equilibrium condition is the equality of all normalized utility gradients, so this process does not require traders to communicate any more information than is needed to verify an equilibrium.

4. Nash Equilibrium

This section applies intrinsic dynamics to the model of Nash equilibrium in an abstract noncooperative game. There are $N \geq 2$ players, and each player's strategy space is the real line, $S^i = \mathbb{R}$. The state space is the joint-strategy space $S = \prod_i S^i = \mathbb{R}^N$. We will consider only pure strategies. Each player is characterized by a "payoff function" on S to $\mathbb{R}$, specifying his payoff as a function of his own strategy and the strategies of the other players. Given the other players' strategies $(s^j)_{j \neq i}$, player i seeks the value of s^i which maximizes his payoff. Assuming this choice is single-valued, the resulting function on $\prod_{j \neq i} S^j$ to S^i is usually termed player i 's "reaction function". A Nash equilibrium is a strategy N -tuple which simultaneously satisfies all players' reaction functions. Since a player's payoff function affects the equilibria only through the derived reaction function, we will model each player's characteristic as his reaction function. For the moment, we will confine attention to linear reaction functions, so let $E^i = \mathbb{R}^N$, with generic element $e^i = ((a_j^i)_{j \neq i}, b^i)$, which parameterizes the reaction function $s^i = \sum_{j \neq i} a_j^i s_j + b^i$. Define $\psi: E \times S \rightarrow \mathbb{R}^N$ by $\psi(e, s) = (-s^i + \sum_{j \neq i} a_j^i s_j + b^i)_{i=1}^N$. Then s is a Nash equilibrium for e if and only if $\psi(e, s) = 0$. Given an environment $e = ((a_j^i)_{j \neq i}, b^i)_i$, let A_e denote the $N \times N$ matrix $(a_j^i)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq N}}$, where $a_i^i = -1$ for each i ; and let b_e denote the N -vector $(b^i)_i$. Then each environment e is completely described by the parameters (A_e, b_e) , and ψ can be written $\psi(e, s) = A_e s + b_e$. If A_e is nonsingular, $-A_e^{-1} b_e$ is the unique Nash equilibrium for e .

The reaction functions themselves specify a discrete-time adjustment process:

$$s_t^i = \sum_{j \neq i} a_j^i s_{t-1}^j + b^i,$$

for which the equilibrium is stable if the characteristic roots of $(A_e + I)$ all have modulus less than unity. The continuous-time analogue:

$$\dot{s}^i = -s^i + \sum_{j \neq i} a_j^i s^j + b^i$$

is stable if the characteristic roots of A_e all have negative real parts. This raises the question whether there is any adjustment process which is stable for all environments with A_e nonsingular. Of course, to retain the information structure of the game, player i 's strategy adjustment must depend only on the current joint-strategy s and his own characteristic e^i . Thus, each player i has a strategy-adjustment function $f^i: E^i \times S \rightarrow R$, and the functions f^i specify the system of differential equations

$$\dot{s}^i = f^i(e^i, s), \quad 1 \leq i \leq N.$$

In order to ensure that $(f^i(e^i, s))_i = 0$ if and only if $\psi(e, s) = 0$, it must be the case that for each i , $f^i(e^i, s) = 0$ if and only if $\psi^i(e^i, s) = 0$, where $\psi^i(e^i, s) = -s^i + \sum_{j \neq i} a_j^i s^j + b^i$. We will say that f^i is

sign preserving if $\text{sign } f^i(e^i, s) = \text{sign } \psi^i(e^i, s)$ for all (e^i, s) ;

sign reversing if $\text{sign } f^i(e^i, s) = -\text{sign } \psi^i(e^i, s)$ for all (e^i, s) ;

and

degenerate if $\text{sign } f^i(e^i, s)$ is constant on $\{(e^i, s): \psi^i(e^i, s) \neq 0\}$.

Since f^i is continuous and the sets $\{(e^i, s): \psi^i(e^i, s) > 0\}$ and $\{(e^i, s): \psi^i(e^i, s) < 0\}$ are each path-connected, these are the only possible cases. The HCP can be used to prove that if the strategy adjustment process is locally stable, then no f^i is degenerate. However, this result is also obvious from

a phase diagram, so we will not give the proof. One would most naturally expect each f^i to be sign preserving, but we will not exclude the sign-reversing case.

We will use the HCP to show that local stability cannot be achieved for all environments e with A_e nonsingular. Let $\bar{s} = 0$, so that $b_e = \psi(e, \bar{s})$ and $A_e = D_s \psi(e, \bar{s})$. Let A_0 and A' be nonsingular $N \times N$ matrices with -1 's on the diagonal and with $\text{sign det } A_0 = -\text{sign det } A'$. Let $P_1 = \{A_0, A'\}$, and let Σ^{N-1} be a small sphere about 0 in $R^N - 0$. Then each $(A, b) \in P_1 \times \Sigma^{N-1}$ identifies an environment in E . The result (1) in Section 2 above implies that $(f^i(\cdot, \bar{s}))_i$ cannot be homotopic to any function on $P_1 \times \Sigma^{N-1}$ to $R^N - 0$ which ignores the argument A . However, for each i , define

$$d^i = \begin{cases} +1 & \text{if } f^i \text{ is sign preserving; and} \\ -1 & \text{if } f^i \text{ is sign reversing.} \end{cases}$$

Then the function $H: P_1 \times \Sigma^{N-1} \times [0, 1] \rightarrow R^N - 0$ defined by $H(A, b; \lambda) = (\lambda d^i b^i + (1-\lambda)f^i((a_j^i)_{j \neq i}, b^i; \bar{s}))_i$ is a homotopy with $H(\cdot, 0) = (f^i(\cdot, \bar{s}))_i$, and $H(\cdot, 1)$ ignores A . Hence, the adjustment process is not locally stable.

The class of environments E is somewhat unsatisfactory because it includes environments which have no equilibria. The existence of equilibria can be ensured by requiring each player's reaction function to be continuous and bounded. The bounds on reaction functions can be used to replace S with a compact cube in R^N , so that the existence of equilibrium can be proved using the Brouwer fixed-point theorem. Although boundedness is inconsistent with the global linearity of reaction functions, we can still assume that the parameters $((a_j^i)_{j \neq i}, b^i)$ describe the reaction function on a neighborhood of equilibrium. If, in addition, the response function $f^i(\cdot, s)$ depends on the

reaction function only on a neighborhood of s , then the above analysis remains valid for environments with bounded reaction functions. That is, the above analysis remains valid as it pertains to the local stability of the equilibrium $-A_e^{-1}b_e$, which lies in the "linear region" of each player's bounded reaction function. However, the boundedness of the players' reaction functions directly implies the boundary condition (B) stated in Section 2 above. Therefore, if $-A_e^{-1}b_e$ is the unique Nash equilibrium, $\det(-A_e) > 0$. It is natural to restrict attention to environments which possess bounded reaction functions (to ensure the existence of equilibrium) and no more than one equilibrium, so that demonstrating the instability of one equilibrium is conclusive. This leads to further restrictions on the parameter space.

In particular, we must have $\det(-A_e) > 0$, and since the parameters (A_e, b_e) describe reaction functions, all diagonal entries of A_e are -1 . If $N = 2$, we thus have $\text{trace } A_e < 0$ and $\det A_e > 0$, so the system

$$\dot{s} = A_e s + b_e$$

is stable. However, we will show that if $N = 3$, there is no adjustment process which is locally stable for all environments with bounded reaction functions and unique equilibria. Our proof extends directly to the case $N > 3$.

Let Q denote the boundary of the square in R_+^2 centered at 0 , with sides of length 2 . That is $Q = \{(x,y) \in R^2: x \in \{-1,1\} \text{ and } -1 \leq y \leq 1, \text{ or } y \in \{-1,1\} \text{ and } -1 \leq x \leq 1\}$. For each $(x,y) \in Q$ define the matrix

$$A(x,y) = \begin{pmatrix} -1 & x-1 & y-1 \\ y-1 & -1 & x-1 \\ x-1 & y-1 & -1 \end{pmatrix}.$$

It is straightforward to verify that $\det(-A(x,y)) > 0$. Let $P_1 = \{A(x,y): (x,y) \in Q\}$, and let Σ^2 denote a small sphere in R^3 , centered at 0. It is demonstrated in an appendix to this paper that for each $(A,b) \in P_1 \times \Sigma^2$, the reaction functions described locally by the parameters (A,b) can be extended to bounded continuous functions in such a way that $-A^{-1}b$ is the unique Nash equilibrium.

As before, let $\bar{s} = 0$, and consider the adjustment process $(f^i(\cdot, \bar{s}))_i = F(\cdot, \bar{s})$ as a function on $P_1 \times \Sigma^2$ to $R^3 - 0$. We have already shown that $F(\cdot, \bar{s})$ is homotopic to a function which ignores A , so instability can be proved by showing that P_1 is sufficiently rich that the equilibrium function $\mu = L: (A,b) \mapsto -A^{-1}b$ cannot be homotopic to a function which ignores A . We will do this by showing that the map $A(\cdot): Q \rightarrow GL(3)$ is homotopic to the map

$$(x,y) \mapsto \begin{pmatrix} x & y & 0 \\ -y & x & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Deforming Q to the unit circle S^1 by radial projection describes a homotopy between the above map and its analogue defined on S^1 , which was denoted σ in Section 2 above. Thus P_1 supports the fundamental group of $\{A \in GL(3): \det A < 0\}$, and by result (3) of Section 2 above, L cannot ignore A .

To study the map $A(\cdot): Q \rightarrow GL(3)$, note that for every $(x,y) \in Q$, the vector $(1,1,1)$ is an eigenvector of $A(x,y)$, with eigenvalue $x + y - 3$. This suggests changing coordinates to the basis $\{(1,1,1), (1,-1,0), (1,1,-2)\}$. Let

$$R = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix} \quad \text{and} \quad R^{-1} = (1/6) \begin{pmatrix} 3 & -3 & 0 \\ 1 & 1 & -2 \\ 2 & 2 & 2 \end{pmatrix}.$$

Define the map $B: Q \rightarrow GL(3)$ by

$$B(x,y) = R^{-1}A(x,y)R = \begin{pmatrix} -3(x+y) & 9(x-y) & 0 \\ -3(x-y) & -3(x+y) & 0 \\ 0 & 0 & 2(x+y-3) \end{pmatrix}.$$

Since $\det R > 0$, there is a path $p: [0,1] \rightarrow GL(3)$ with $p(0) = I$ and $p(1) = R$. Then the function $H: Q \times [0,1] \rightarrow GL(3)$ defined by $H(x,y;\lambda) = p(\lambda)^{-1}A(x,y)p(\lambda)$ is a homotopy between $A(\cdot)$ and $B(\cdot)$. At this point it should be apparent that $B(\cdot)$ is essentially the same as σ , but we will continue the argument to show how homotopies in $GL(n)$ can be used to manipulate and simplify matrices. Since $x + y - 3 < 0$ for all $(x,y) \in Q$, inserting the path $\lambda \mapsto \lambda(-1) + (1-\lambda)2(x+y-3)$ into the lower right corner of $B(x,y)$ determines a homotopy between $B(\cdot)$ and the function $C(\cdot): Q \rightarrow GL(3)$ defined by

$$C(x,y) = \begin{pmatrix} -3(x+y) & 9(x-y) & 0 \\ -3(x-y) & -3(x+y) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Similarly, the path $\lambda \mapsto \lambda 3(x-y) + (1-\lambda)9(x-y)$, inserted in the second entry in the first row, gives a homotopy between $C(\cdot)$ and the map $D(\cdot)$ defined by

$$D(x,y) = \begin{pmatrix} -3(x+y) & 3(x-y) & 0 \\ -3(x-y) & -3(x+y) & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, define

$$R' = (1/6) \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 6 \end{pmatrix},$$

so the desired function is $R'D(\cdot)$. Since $\det R' > 0$, there is a path $p': [0,1] \rightarrow GL(3)$ with $p'(0) = I$ and $p'(1) = R'$, so the function $(x,y,\lambda) \mapsto p(\lambda)D(x,y)$ is a homotopy between $D(\cdot)$ and $R'D(\cdot)$. This proves that if $N = 3$, there is no adjustment process which is locally stable for every environment with a unique, regular ($\det A \neq 0$) Nash equilibrium. The proof extends directly to the case $N > 3$, by redefining $A(\cdot)$ as

$$A(x,y) = \left(\begin{array}{ccc|c} -1 & x-1 & y-1 & \\ y-1 & -1 & x-1 & 0 \\ x-1 & y-1 & -1 & \\ \hline & 0 & & -I_{N-3} \end{array} \right)$$

where I_{N-3} is the $(N-3) \times (N-3)$ identity matrix, and, of course, replacing Σ^2 with Σ^{N-1} .

We have shown that, if $N \geq 3$, even if reaction functions are constrained to be bounded and possess a unique, regular Nash equilibrium, there are no adjustment rules for the players' strategies which generally make the equilibrium locally stable. As with the previous results in this paper, stability requires the adjustment $\dot{s}$ to depend on $D_s \psi(e,s)$ as well as the value $\psi(e,s)$. Moreover, the i^{th} player's adjustment $\dot{s}^i$ must depend on more information about A than player i can deduce from the current joint strategy s and his own characteristic alone. Thus the information structure of the game is not sufficient to permit stable adjustment processes.

The concept of Nash equilibrium has been criticized on the grounds that,

in the absence of dominant strategies, a player cannot generally compute his Nash equilibrium strategy unilaterally unless he knows the characteristics of the other players. The fact that in most situations of economic interest, individual economic agents cannot independently achieve equilibrium is the major motivation of the present paper. However, instead of modeling the achievement of Nash equilibrium through the interaction of the players, a popular alternative approach has been to use the concept of Bayesian equilibrium to obtain a unilateral equilibrium for each player even in games which do not admit dominant strategies.

The Bayesian equilibrium model, first formulated by Harsanyi (1967-68) gives each player a subjective probability of the other players' characteristics. In order to avoid giving each player an infinite sequence of subjective probabilities of all players' subjective probabilities, etc., most applications impose "rational expectations" by assuming that the players' characteristics are generated by a joint probability distribution which is known to all players. Thus the environment e is a random variable, generated by a publicly known probability distribution π . Player i observes e^i , and chooses s^i , without observing $(s^j)_{j \neq i}$, to maximize his expected payoff. An equilibrium for player i is an expected payoff maximizing function $\sigma^i(\cdot, \pi): e^i \mapsto s^i$. The probability distribution of $(s^j)_{j \neq i}$ facing player i when he chooses s^i depends on π and the functions $(\sigma^j)_{j \neq i}$. However, since each player knows π , each player, like the theorist, can derive the equilibrium N -tuple $(\sigma^i)_i$. Thus player i can compute his equilibrium σ^i independently, at least assuming the equilibrium is unique.

Aside from the possibility of multiple equilibria, the principal drawback of the Bayesian equilibrium concept is the assumption that all players know π . Suppose that the game is in a Bayesian equilibrium, but π changes slightly to a

new distribution π' , which leaves the marginal distribution of some e^i unaffected. If the game were repeated, with the environment e generated independently at each repetition, player i might eventually observe that his distribution of payoffs is inconsistent with his expectations, but this would not generally be sufficient to infer the new distribution π' . Even if the strategy N-tuple s became public knowledge after each repetition, the distribution of the environment could not generally be identified. Of course, π' could be learned if the environment e became public knowledge after each play of the game. However, this possibility is usually excluded because it would permit monitoring, and the Bayesian equilibrium model is usually applied to situations where the characteristics of the players cannot be monitored to enforce decision-rules other than the self-enforcing equilibrium $(\sigma^i)_i$. Thus, in most applications, the Bayesian equilibrium model requires players to possess knowledge that the information structure of the game would not permit them to acquire.

This conflict could be avoided by assuming that in equilibrium, player i knows the joint distribution of e^i and $(s^j)_{j \neq i}$, which could be learned from repeated ex-post observations of s . However, this distribution is endogenous, so the unilateral equilibrium property is lost. Instead, players must adjust their decision rules in response to their observations. If the environment distribution has finite support, so that the decision rules and all of the relevant probability distributions are finite-dimensional, then the method of intrinsic dynamics can be applied to this adjustment problem, provided that we suppose that as the decision rules $(\sigma^i)_i$ are adjusted the resulting joint distributions of e^i and $(s^j)_{j \neq i}$ are learned instantaneously. The idealization that "learning-time" proceeds infinitely faster than "adjustment-time" has been applied to rational expectations equilibrium models by Bray (1982) and Radner (1982). A more general approach would combine the learning and strategy

adjustment processes. The general method of formulating a class of adjustment processes and determining the communication required for convergence to equilibrium seems applicable to this problem, but the techniques formulated in Section 2 above do not seem directly applicable to adjustment processes which involve statistical learning.

5. Conclusion

Intrinsic Dynamics offers a new perspective on economic equilibrium concepts. It provides techniques for testing whether the information structure of an equilibrium model is sufficient to permit the local stability of equilibria. Since Intrinsic Dynamics imposes no behavioral assumptions on the model in disequilibrium states, the results it generates are intrinsic to the equilibrium concept itself.

The major potential contribution of Intrinsic Dynamics probably lies in future applications to specific equilibrium concepts. Nonetheless, there are several issues which it would be interesting to pursue at the general level. In Section 2 above, we simply assumed that the equilibrium model generated sufficient information to identify the values of the function ψ . However, it is not difficult to construct abstract examples in which the dimension of each E^i is much larger than n , but in order to generate enough information to determine $\psi(e,s)$, each agent i must communicate his entire characteristic e^i . In such examples, the communication needed to identify $\psi(e,s)$ obviously identifies $D_s\psi(e,s)$ as well. Thus, although the treatment in Section 2 will probably suffice for most applications, more work is needed on a general criterion to determine whether stability requires more communication than is needed to verify the equilibrium conditions. Recent research by Hurwicz, Reiter and Saari (1980), and Saari (1983) on the construction of minimal message processes in a static context should be useful in determining the communication needed to identify $\psi(e,s)$, or, more generally, the communication needed merely to verify the condition $\psi(e,s) = 0$. It would also be extremely useful to develop a general method to construct, for any given equilibrium function $\mu: E \rightarrow S$, a locally stable adjustment process which uses the minimal necessary communication.

Appendix I: The Existence of Equilibrium

Let $e \in E$, and suppose that S is an open subset of R^n , and that $\psi(e, \cdot): S \rightarrow R^n$ is continuous. Suppose also that $\psi(e, \cdot)$ satisfies the following boundary condition:

- (B) There exists $s^\circ \in S$, and a compact cube $K \subset S$ with $s^\circ \in \text{int } K$, such that for each $s \in \partial K$, $\psi(e, s) \neq 0$ and $\|s - s^\circ\|^{-1}(s - s^\circ) \neq \|\psi(e, s)\|^{-1}\psi(e, s)$,

where ∂K denotes the boundary of K . Then there is some $\bar{s} \in K$ with $\psi(e, \bar{s}) = 0$.

To prove this result, suppose by way of contradiction that $\psi(e, s) \neq 0$ for all $s \in K$. Then define the function $g: K \rightarrow R$ by $g(s) = \max \{\lambda: s^\circ + \lambda\psi(e, s) \in K\}$, and define $h: K \rightarrow K$ by $h(s) = s^\circ + g(s)\psi(e, s)$. It is straightforward to verify that g , and thus h , is continuous, so h has a fixed-point. Since $h(s) \in \partial K$ for each $s \in K$, $h(s^*) = s^*$ for some $s^* \in \partial K$. Thus $s^\circ + g(s^*)\psi(e, s^*) = s^*$, but $g(s^*) > 0$, which contradicts (B). Hence there is some $\bar{s} \in K$ with $\psi(e, \bar{s}) = 0$.

Appendix II: The Boundary Condition and Uniqueness of Equilibrium

Let $e \in E$, and suppose that S is an open subset of R^n and that $\psi(e, \cdot): S \rightarrow R^n$ is C^1 . Assume also that $\psi(e, \cdot)$ satisfies the boundary condition (B) in Appendix I above. Let $\bar{s} \in K$ with $\psi(e, \bar{s}) = 0$, and suppose $\bar{s}$ is the unique equilibrium. Then if $D_S \psi(e, \bar{s})$ is nonsingular, $\det (-D_S \psi(e, \bar{s})) > 0$.

To keep the geometry of the argument uncluttered, we will also assume that $\bar{s} = s^0 = 0$, and that K is centered at 0, but these assumptions are easily dropped. First, define $H: [0, 1] \times \partial K \rightarrow R^n - 0$ by $H(\lambda, s) = \lambda^{-1} \psi(e, \lambda s)$. Then $H(1, \cdot) = \psi(e, \cdot)$ and $H(0, \cdot)$ is the linear map $s \mapsto D_S \psi(e, 0)s$. The fact that H takes values in $R^n - 0$ follows from the uniqueness of the equilibrium $\bar{s} = 0$ and the nonsingularity of $D_S \psi(e, 0)$. Now define $H': [0, 1] \times \partial K \rightarrow R^n - 0$ by $H'(\lambda, s) = \lambda(-s) + (1-\lambda)\psi(e, s)$. Then $H'(0, \cdot) = \psi(e, \cdot)$ and $H'(1, \cdot)$ is the linear map $s \mapsto -s$. The fact that H' takes values in $R^n - 0$ follows from the boundary condition. Thus the linear maps $s \mapsto D_S \psi(e, 0)s$ and $s \mapsto -s$ are homotopic. Since ∂K and the unit sphere in $R^n - 0$ are homeomorphic, the Hopf degree theorem (Milnor (1965)) implies that $\text{sign det } D_S \psi(e, 0) = \text{sign det } (-I)$. Hence $\det (-D_S \psi(e, 0)) > 0$.

This type of argument also underlies a more general result which pertains also to multiple equilibria. If each equilibrium $\bar{s}_j$ is regular, and we define $i(\bar{s}_j) = \pm 1$ as $\det (D_S \psi(e, \bar{s}_j)) \gtrless 0$, then $\sum_j i(\bar{s}_j) = (-1)^n$. This result was obtained by Dierker (1972) for the Walrasian model.

Appendix III: Uniqueness of Equilibrium

With Bounded Reaction Functions

This appendix demonstrates that each (A,b) in a certain parameter set, which was used in Section 4 above, can describe the behavior of bounded continuous reaction functions on a neighborhood of the unique equilibrium $-A^{-1}b$. Let $Q = \{(x,y) \in \mathbb{R}^2: x \in \{-1,1\} \text{ and } -1 \leq y \leq 1, \text{ or } y \in \{-1,1\} \text{ and } -1 \leq x \leq 1\}$. For each $(x,y) \in Q$, define

$$A(x,y) = \begin{pmatrix} -1 & x-1 & y-1 \\ y-1 & -1 & x-1 \\ x-1 & y-1 & -1 \end{pmatrix}.$$

Let Σ^2 denote a small sphere centered at 0 in $\mathbb{R}^3$. As a preliminary step, let A denote the set of matrices of the form

$$\begin{pmatrix} -1 & a & b' \\ c & -1 & \beta \\ \gamma & a & -1 \end{pmatrix}$$

where $-2 \leq a, b', c, \beta, \gamma \leq 0$, and either

- i) $a = \beta = \gamma = -2$; or
- ii) $a = \beta = -2$ and $a = c = 0$, or
 $\beta = a = -2$ and $a = b' = 0$, or
 $\gamma = a = -2$ and $b' = c = 0$; or
- iii) $a = b' = c = 0$.

It is easily verified that all matrices in A are nonsingular, and A is compact, so let $d > 0$ such that $\| -A^{-1}b \| < d$ for all $A \in A$ and $b \in \Sigma^2$. Now choose (x,y) arbitrarily in Q and b arbitrarily in Σ^2 . We first

consider the case in which $x < 1$ and $y < 1$. Then either $x = -1$ or $y = -1$, so without loss of generality suppose that $x = -1$. Let $A = A(x,y)$. Then for each $1 \leq i \leq 3$, the parameters (A,b) describe the linear reaction function $s^i = \sum_{j \neq i} a_j^i s^j + b^i$. Define the function $g_y: R \rightarrow P$ by

$$g_y(r) = \begin{cases} (y-1) & \text{if } |r| \leq d \\ (d+1 - |r|)(y-1) & \text{if } d < |r| \leq d+1 \\ 0 & \text{if } |r| > d. \end{cases}$$

Define the function $g_x: R \rightarrow R$ by

$$g_x(r) = \begin{cases} (x-1) & \text{if } |r| \leq 4(d+1) \\ (4d+5 - |r|)(x-1) & \text{if } 4(d+1) \leq |r| \leq 4d+5 \\ 0 & \text{if } |r| > 4d+5, \end{cases}$$

where we assume, for the moment, that $x = -1$. Given b , define the reaction functions

$$\begin{aligned} s^1 &= g_x(s^2)s^2 + g_y(s^3)s^3 + b^1 \\ s^2 &= g_y(s^1)s^1 + g_x(s^3)s^3 + b^2, \text{ and} \\ s^3 &= g_x(s^1)s^1 + g_y(s^2)s^2 + b^3. \end{aligned}$$

These reaction functions are continuous, and since $|g_y(r)r| < 2(d+1)$ and $|g_x(r)r| < 10(d+1)$ for all $r \in R$, they are bounded as well. For each s with $|s^i| \leq d$ for all i , these reaction functions agree with those specified by (A,b) . By the choice of d , this region contains the point $s^\circ = -A^{-1}b$ in its interior, so s° is an equilibrium and the parameters (A,b) describe the reaction functions on a neighborhood of s° . We need to show that s° is the unique equilibrium. That is, we must show that s° is the only solution to the

equation

$$(*) \quad \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 & g_x(s^2) & g_y(s^3) \\ g_y(s^1) & -1 & g_x(s^3) \\ g_x(s^1) & g_y(s^2) & -1 \end{pmatrix} \begin{pmatrix} s^1 \\ s^2 \\ s^3 \end{pmatrix} + \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix}.$$

Let $A(s)$ denote the matrix in (*), and let $S_d = \{s \in R^3: |s^i| \leq d \text{ for all } i\}$. Then for each $s \in S_d$, $A(s) = A$, so s^0 is the only solution to (*) in S_d . Let $S_{4(d+1)} = \{s \in R^3: |s^i| \leq 4(d+1) \text{ for all } i\}$. Then for each $s \in S_{4(d+1)}$, $g_x(s^1) = g_x(s^2) = g_x(s^3) = -2$, so $A(s)$ satisfies case (i) in the definition of A above. Hence for each $s \in S_{4(d+1)}$, $-A(s)^{-1}b \in S_d$, so s^0 is the unique solution in $S_{4(d+1)}$. Now suppose $s \notin S_{4(d+1)}$. Without loss of generality, suppose $|s^1| > 4(d+1)$. Then $g_y(s^1) = 0$. Also, since $|g_y(s^3)s^3| < 2(d+1)$ for all s^3 , and $|g_x(s^2)| \leq 2$ for all s^2 , $|s^1| > 4(d+1)$ implies that $|s^2| > d+1$, which implies that $g_y(s^2) = 0$. Thus we have $g_y(s^1) = g_y(s^2) = 0$. If $g_y(s^3) < 0$, the same reasoning in reverse implies that $g_x(s^3) = -2$ and $g_x(s^2) = -2$, so $A(s)$ satisfies case (ii) in the definition of A . If $g_y(s^3) = 0$, then $A(s)$ satisfies case (iii), so in either case $A(s) \in A$ and $-A(s)^{-1}b \in S_d$. Hence s^0 is the unique solution to (*). This proves the desired result if $x = -1$ and $-1 \leq y \leq 1$. If $y = 1$ and $-1 \leq x < 1$, the same proof applies, but the proof can be shortened since $g_y(\cdot) \equiv 0$, so $A(s)$ always satisfies case (iii) in the definition of A . If $x = 1$ and $y = 1$, the reaction functions described by (A,b) are constant, and thus require no modification. This completes the proof for $(x,y) \in Q$ with $x \leq y$, and the case $x > y$ is symmetric.

If $N > 3$, then as $A(x,y)$ is defined in Section 4, the reaction function described by $(A(x,y),b)$ for each player $i > 3$ is constant, and thus requires no modification, so the same proof applies to $N > 3$. Also, it is

clear that the reaction functions constructed above could be smoothed without affecting the proof. Hence the global smoothness of ψ can also be retained.

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