**THE DAM PROBLEM WITH LEAKY
BOUNDARY CONDITIONS**

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0. Introduction

Let Ω be a bounded locally Lipschitz domain in $\mathbf{R}^2$. Ω represents the section of a porous medium, the points in $\mathbf{R}^2$ will be denoted by (x, y) .

The boundary Γ of Ω is divided into three parts: an impervious part S_1 , a part in contact with air S_2 and finally a part covered by fluid S_3 (see the figure 1).

Figure 1

For convenience, we will assume that S_1, S_3 are relatively open in Γ and we will denote by $S_{3,i}, i = 1, \dots, N$ the different connected components of S_3 .

Assuming that the flow in Ω has reached a steady state we are concerned with finding the pressure p of the fluid and the part of the porous medium where some flow occurs – i.e. the wet subset A of Ω .

Note that this two dimensional model could describe for instance the steady flow in the cross section Ω of a longitudinal porous medium.

The paper is divided as follows. In section 1 we give a strong formulation of the problem explaining the model we would like to consider here and which differs from the classical one where Dirichlet boundary conditions are imposed on S_3 : indeed, on this part

of the boundary the flux of fluid will be controlled by some nonlinear law. In section 2, using the analysis of Brezis-Kinderlehrer-Stampacchia [B.K.S.] we transform the problem into a weak form. In section 3 we develop a theory of existence. In section 4 we give some preliminary results regarding the solution of the weak problem and we show that uniqueness could only hold modulo "pools" -i.e. particular functions. We show also that p is not necessarily positive below S_3 so that unsaturated regions may occur. In section 5 we study different examples, one of which reduces to a variational inequality in the spirit of [Ba.₁]. We show in particular that the function χ defined in (P) (see below) is not necessarily a characteristic function of a set. This is an important difference with the classical dam problem with Dirichlet boundary conditions (compare to [C.C.]).

1. Strong formulation

The boundary of A , that we will denote by ∂A , is divided into four parts: an impervious part Γ_1 , a free boundary Γ_2 , a part covered by the fluid Γ_3 and finally a seepage front Γ_4 where the fluid is flowing outside Ω but does not remain there in a significant amount to modify the pressure (see figure 1).

The velocity v of the fluid in A is given, according to Darcy's law, by

$$v = -k\nabla(p + y) \quad (1.0)$$

where p is the pressure, k is the coefficient of permeability of the medium.

We assumed the medium homogeneous and the fluid to be for instance water with a specific weight equal to 1. In fact, from now on, we will assume that everything has been scaled in such a way that $k = 1$.

Then, if the fluid is incompressible we have

$$\operatorname{div} v = 0 \quad \text{in } A$$

or

$$\Delta p = 0 \quad \text{in } A. \quad (1.1)$$

Next, on $\Gamma_1 \cup \Gamma_2$ there is no flux of fluid through this part of boundary. So, if ν denotes the outward unit normal to ∂A one has

$$v \cdot \nu = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2$$

or by (1.0)

$$\frac{\partial}{\partial \nu}(p + y) = 0 \quad \text{on} \quad \Gamma_1 \cup \Gamma_2. \quad (1.2)$$

On Γ_4 the fluid is free to exit the porous medium and thus one has

$$v \cdot \nu \geq 0 \quad \text{on} \quad \Gamma_4$$

which reads again by (1.0)

$$\frac{\partial}{\partial \nu}(p + y) \leq 0 \quad \text{on} \quad \Gamma_4. \quad (1.3)$$

We will denote by φ the outside pressure on $S_2 \cup S_3$. For instance, if we assume that we are in the case of the figure 1, and if the atmospheric pressure has been scaled to 0, then φ is given by

$$\begin{aligned} \varphi(x, y) &= 0 \quad \text{if} \quad (x, y) \in S_2 \\ &= h_i - y \quad \text{if} \quad (x, y) \in S_{3,i}, \quad i = 1, \dots, N. \end{aligned}$$

h_i denotes the level of the reservoir covering $S_{3,i}$. (We assume that the fluid is water and the units chosen in such a way that the hydrostatic pressure at (x, y) of a reservoir of level h is given by $h - y$). Clearly φ is a Lipschitz continuous function. In what follows we could consider the case of a general Lipschitz function φ . However we will restrict our analysis to φ given by the above expression – i.e. to the case where the external pressure is due to reservoirs hanging around.

Besides (1.2) there is a second condition on Γ_2 – i.e. the pressure p coincides with the atmospheric pressure – in such a way that

$$p = 0 \quad \text{on} \quad \Gamma_2. \quad (1.4)$$

In [C.C] or [C] we considered the case of Dirichlet boundary condition – i.e. the case where p is prescribed equal to φ on S_3 . Here we would like to prescribe the flux on this

part of boundary. More precisely we would like to consider the case where the flux is governed by a function of the jump of pressure across S_3 . This kind of conditions is called a “leaky boundary condition” and we refer the reader to [Be] for physical justifications of such a model.

So, we will assume that

$$\frac{\partial}{\partial \nu}(p + y) = \beta(x, \varphi - p) \quad \text{on } S_3. \quad (1.5)$$

Here x denotes a point in $\mathbf{R}^2$. We use the same notation for the first entry of such a point but we do not think that this will create any confusion. On β we can assume for instance

$$\beta(x, 0) \in L^2(S_3) \quad (1.6)$$

where $L^2(S_3)$ denotes the usual space of functions of square integrable on S_3 for the superficial measure σ (see for instance [N.])

$$x \rightarrow \beta(x, u) \quad \text{is measurable for every } u \in \mathbf{R}, \quad (1.7)$$

and also $\exists C > 0$ such that

$$|\beta(x, u_1) - \beta(x, u_2)| \leq C|u_1 - u_2| \quad \sigma\text{-a.e. } x \in S_3, \quad \forall u_1, u_2 \in \mathbf{R}, \quad (1.8)$$

$$u \rightarrow \beta(x, u) \quad \text{is nondecreasing for } \sigma\text{-a.e. } x \in S_3, \quad (1.9)$$

$$\beta(x, u) \geq 0 \quad \sigma\text{-a.e. } x \in S_3, \quad \forall u \geq 0. \quad (1.10)$$

A particular case is when

$$\beta(x, u) = \beta_i(u) \quad \text{for } x \in S_{3,i}.$$

One could also think to extend our results to the case where β_i is a maximal monotone graph. For instance if β_i is for every $i = 1, \dots, N$ the multivalued graph

$$\beta_i = (0, \mathbf{R})$$

then one recovers the Dirichlet conditions of [C.C.] (see [Br] for this theory and also [R] for results in this direction). However, since our concern is mainly to stress out the differences between this model and the classical one we will not consider such a generalization here.

So, assuming that everything is smooth the problem is to find a pair (p, A) such that (1.1)-(1.5) holds. This is what will be referred as the strong formulation.

2. Weak formulation

First remark that find the pair (p, A) is equivalent to find the pair (p, χ_A) where χ_A denotes the characteristic function of the set A . Then, following [B.K.S.] (see also [A.]), for any smooth function ξ one has

$$\begin{aligned} \int_A \nabla p \cdot \nabla \xi + \xi_y dx &= \int_A \nabla(p+y) \cdot \nabla \xi dx \\ &= \int_A -\Delta(p+y) \cdot \xi dx + \int_{\partial A} \frac{\partial p+y}{\partial \nu} \cdot \xi d\sigma(x). \end{aligned}$$

So, if we assume that p is a smooth function satisfying (1.1)-(1.5) one gets by (1.1)

$$\int_A \nabla p \cdot \nabla \xi + \xi_y dx = \int_{\partial A} \frac{\partial p+y}{\partial \nu} \cdot \xi d\sigma(x).$$

Using (1.2) and (1.5) this implies:

$$\int_A \nabla p \cdot \nabla \xi + \xi_y dx = \int_{S_3} \beta(x, \varphi - p) \cdot \xi d\sigma(x) + \int_{\Gamma_4} \frac{\partial p+y}{\partial \nu} \cdot \xi d\sigma(x)$$

and if we assume that

$$\xi \geq 0 \quad \text{on } \Gamma_4 \tag{2.1}$$

we get by (1.3)

$$\int_A \nabla p \cdot \nabla \xi + \xi_y dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi d\sigma(x) \leq 0. \tag{2.2}$$

Now, see (1.4), assume that we extend p by 0 outside of A and that we still denote by p this extension. Then, clearly, if p is smooth up to Γ_2 one deduces from (2.2) that

$$\int_{\Omega} \nabla p \cdot \nabla \xi + \chi_A \xi_y dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi d\sigma(x) \leq 0 \quad \forall \xi \geq 0 \quad \text{on } \Gamma_2. \tag{2.3}$$

The part Γ_4 is an unknown of the problem so we have assumed that $\xi \geq 0$ on S_2 and this implies in particular $\xi \geq 0$ on Γ_4 .

So, we are led to look for a pair $(p, \chi) = (p, \chi_A)$ satisfying (2.3). Recasting this within reasonable spaces the problem becomes:

Find $(p, \chi) \in H^1(\Omega) \times L^\infty(\Omega)$ such that

$$\begin{aligned} \text{(i)} \quad & p \geq 0, \quad 0 \leq \chi \leq 1 \quad \text{a.e. in } \Omega, \quad \chi = 1 \quad \text{on } [p > 0] = \{(x, y) \in \Omega \mid p(x, y) > 0\}, \\ \text{(ii)} \quad & p = 0 \quad \text{on } S_2, \\ \text{(iii)} \quad & \int_{\Omega} \nabla p \cdot \nabla \xi + \chi \xi_y dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi d\sigma(x) \leq 0 \quad \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \quad \text{on } S_2. \end{aligned} \tag{P}$$

We will refer to (P) as the weak formulation of our initial problem. Clearly if (1.1)-(1.5) has a solution (p, A) and if p denotes also the extension of p by 0 outside A then we have shown above that (p, χ_A) is a solution to (P) and thus any strong solutions to our initial problem will be found among those of (P).

We now study the question of the existence of a solution to (P).

3. Existence of solution

We argue as in [B.K.S.] and we first introduce the following approximated problem.

Find $p_\epsilon \in H^1(\Omega)$ such that

$$p_\epsilon = 0 \quad \text{on } S_2 \quad \text{and}$$

$$\int_{\Omega} \nabla p_\epsilon \nabla \xi + H_\epsilon(p_\epsilon) \cdot \xi_y dx - \int_{S_3} \beta(x, \varphi - p_\epsilon) \cdot \xi d\sigma(x) = 0 \quad \forall \xi \in H^1(\Omega), \quad \xi = 0 \quad \text{on } S_2 \quad (P_\epsilon)$$

H_ϵ is the approximation of the Heaviside graph defined by

$$H_\epsilon(p) = 0 \vee \frac{1}{\epsilon} p \wedge 1 \tag{3.1}$$

where $\vee$ denotes the maximum of two functions and $\wedge$ the minimum, ϵ is positive.

Then we have:

THEOREM 1: Assume that $\beta(x, u)$ is a function satisfying (1.6)-(1.10). Then, under the above assumptions, for any $\epsilon > 0$ there exists a unique solution p_ϵ to (P_ϵ) . Moreover, one has

$$0 \leq p_\epsilon \quad \text{a.e. in } \Omega. \quad (3.2)$$

Proof: Set

$$V = \{v \in H^1(\Omega) \mid v = 0 \text{ on } S_2\}.$$

For $p \in V$ let us consider the map from V into $\mathbf{R}$:

$$\xi \rightarrow \int_{\Omega} \nabla p \cdot \nabla \xi \, dx - \int_{S_3} \beta(x, \gamma_0(\varphi - p)) \cdot \xi \, d\sigma(x) \quad (3.3)$$

where γ_0 denotes the usual trace operator (see [N] or [R.T.]). We will sometimes drop for simplicity the notation γ_0 in the second integral of (3.3).

Since β is Lipschitz continuous and satisfies (1.6), (1.8) one has

$$\left| \int_{S_3} \beta(x, \gamma_0(\varphi - p)) \cdot \xi \, d\sigma(x) \right| \leq C |\gamma_0(\varphi - p)|_{2, S_3} |\xi|_{2, S_3} + |\beta(x, 0)|_{2, S_3} |\xi|_{2, S_3} \leq K |\xi|_{1, 2} \quad (3.4)$$

$|\cdot|_{2, S_3}$ denotes the usual L^2 -norm on $L^2(S_3)$ and $|\cdot|_{1, 2}$ denotes the $H^1(\Omega)$ -norm. We assume V equipped with this last norm. We deduce from (3.4) that (3.3) defines a continuous linear form on V that we denote by $A(p)$. Thus if $\langle \cdot, \cdot \rangle$ denotes the pairing between V' and V , we have defined an operator A from V into V' through the formula

$$\langle A(p), \xi \rangle = \int_{\Omega} \nabla p \cdot \nabla \xi \, dx - \int_{S_3} \beta(x, \gamma_0(\varphi - p)) \cdot \xi \, d\sigma(x). \quad (3.5)$$

It is easy to check that A is monotone on V . More precisely one has

$$\langle A(p) - A(p'), p - p' \rangle \geq \int_{\Omega} |\nabla(p - p')|^2 \, dx. \quad (3.6)$$

and A is coercive on V (see [L.]). Moreover, A restricted to finite dimensional spaces is clearly continuous. So, see [L.], A is a one-to-one operator from V into V' . Now, if $v \in L^2(\Omega)$ then

$$\xi \rightarrow - \int_{\Omega} H_{\varepsilon}(v) \xi_y dx$$

defines a continuous linear form on V . Thus, for any $v \in L^2(\Omega)$ there exists a unique $u_{\varepsilon} = \tau_{\varepsilon}(v)$ such that

$$u_{\varepsilon} \in V, \quad \langle A(u_{\varepsilon}), \xi \rangle = - \int_{\Omega} H_{\varepsilon}(v) \xi_y dx \quad \forall \xi \in V. \quad (3.7)$$

So, to prove that there exists a solution to (P_{ε}) we now only need to show that the map τ_{ε} has a fixed point. For that, if φ denotes a Lipschitz continuous extension of φ to Ω , we remark that $u_{\varepsilon} - \varphi \in V$, thus from (3.7) we obtain

$$\langle A(u_{\varepsilon}), u_{\varepsilon} - \varphi \rangle = - \int_{\Omega} H_{\varepsilon}(v) (u_{\varepsilon} - \varphi)_y dx$$

hence for some constant C independent of ε

$$\begin{aligned} \langle A(u_{\varepsilon}) - A(\varphi), u_{\varepsilon} - \varphi \rangle &= - \langle A(\varphi), u_{\varepsilon} - \varphi \rangle - \int_{\Omega} H_{\varepsilon}(v) \cdot (u_{\varepsilon} - \varphi)_y dx \\ &= - \int_{\Omega} \nabla \varphi \cdot \nabla (u_{\varepsilon} - \varphi) dx + \int_{S_3} \beta(x, 0) (u_{\varepsilon} - \varphi) d\sigma(x) - \int_{\Omega} H_{\varepsilon}(v) (u_{\varepsilon} - \varphi)_y dx \\ &\leq (\|\nabla \varphi\|_2 + |\Omega|^{1/2} + C|\beta(x, 0)|_{2, S_3}) \|\nabla (u_{\varepsilon} - \varphi)\|_2 \end{aligned} \quad (3.8)$$

by Cauchy-Schwarz inequality. ($\|\cdot\|$ denote the Euclidean norm in $\mathbf{R}^n$ or the Lebesgue measure, $\|\cdot\|_2$ is the L^2 norm on $L^2(\Omega)$, we used the continuity of the trace operator, see [N.]). Recalling (3.6) we obtain

$$\int_{\Omega} |\nabla (u_{\varepsilon} - \varphi)|^2 \leq (\|\nabla \varphi\|_2 + |\Omega|^{1/2} + C|\beta(x, 0)|_{2, S_3})^2 \quad (3.9)$$

and thus

$$\|u_{\varepsilon}\|_{1,2} \leq C \quad (3.10)$$

where C is some constant independent of ε . The existence of a fixed point for τ_{ε} follows then easily from the Schauder fixed point theorem. Hence the existence of p_{ε} .

To prove uniqueness one argues like in [B.K.S.] -see also [C.M.] for more general results. More precisely, set for $\delta > 0$

$$f_\delta(x) = \begin{cases} (1 - \frac{\delta}{x})^+ & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (3.11)$$

(()⁺ denotes the positive part of functions). Then f_δ is a Lipschitz continuous function and if p_ϵ, p'_ϵ are solutions to (P_ϵ) then

$$f_\delta(p_\epsilon - p'_\epsilon) \in V.$$

Thus one deduces from (P_ϵ) written for p_ϵ and p'_ϵ

$$\begin{aligned} \int_{\Omega} \nabla(p_\epsilon - p'_\epsilon) \cdot \nabla f_\delta(p_\epsilon - p'_\epsilon) dx &= - \int_{\Omega} (H_\epsilon(p_\epsilon) - H_\epsilon(p'_\epsilon)) f_\delta(p_\epsilon - p'_\epsilon)_y dx \\ &+ \int_{S_3} \beta(x, \varphi - p_\epsilon) - \beta(x, \varphi - p'_\epsilon) \cdot f_\delta(p_\epsilon - p'_\epsilon) d\sigma(x). \end{aligned} \quad (3.12)$$

Hence, by the monotonicity of β

$$\int_{\Omega} \nabla(p_\epsilon - p'_\epsilon) \cdot \nabla f_\delta(p_\epsilon - p'_\epsilon) dx \leq - \int_{\Omega} (H_\epsilon(p_\epsilon) - H_\epsilon(p'_\epsilon)) f_\delta(p_\epsilon - p'_\epsilon)_y dx$$

So, if we set $q_\epsilon = p_\epsilon - p'_\epsilon$, $[q_\epsilon > \delta] = \{x \in \Omega : q_\epsilon(x) > \delta\}$ we obtain easily using (3.11) and the Lipschitz continuity of H_ϵ

$$\int_{[q_\epsilon > \delta]} \frac{|\nabla q_\epsilon|^2}{q_\epsilon^2} dx \leq \frac{1}{\epsilon} \int_{[q_\epsilon > \delta]} \frac{|\nabla q_\epsilon|}{q_\epsilon} dx.$$

Hence, by Cauchy-Schwarz Inequality

$$\int_{\Omega} \left| \nabla \ln \left(1 + \frac{(q_\epsilon - \delta)^+}{\delta} \right) \right|^2 dx = \int_{[q_\epsilon > \delta]} \frac{|\nabla q_\epsilon|^2}{q_\epsilon^2} dx \leq \frac{|\Omega|}{\epsilon^2}.$$

By Poincaré's Inequality we obtain

$$\int_{\Omega} \left| \ln \left(1 + \frac{(q_\epsilon - \delta)^+}{\delta} \right) \right|^2 dx \leq C$$

where C is independent of δ . Letting $\delta \rightarrow 0$ we deduce

$$q_\epsilon \leq 0 \quad \text{a.e. in } \Omega$$

and the uniqueness of p_ϵ follows by exchanging the roles of p_ϵ and p'_ϵ .

To prove (3.2) if we set $\xi = (p_\epsilon)^-$ in (P_ϵ) we obtain:

$$-\int_{\Omega} |\nabla p_\epsilon^-|^2 dx - \int_{S_3} \beta(x, \varphi - p_\epsilon) \cdot p_\epsilon^- d\sigma(x) = 0$$

hence by (1.10)

$$\int_{\Omega} |\nabla p_\epsilon^-|^2 dx \leq 0$$

and (3.2) follows.

Remark 1: We have in fact that if φ, φ' are such that

$$\varphi \leq \varphi' \quad \text{on } S_2 \cup S_3$$

then the corresponding solutions p_ϵ and p'_ϵ are such that

$$p_\epsilon \leq p'_\epsilon \quad \text{a.e. in } \Omega.$$

Indeed $f_\delta(p_\epsilon - p'_\epsilon) \in V$ and the last integral in (3.12) written with φ and φ' is non positive.

In fact one can also prove existence of a non negative solution to (P_ϵ) when $u \rightarrow \beta(x, u)$ is not assumed to be nondecreasing. Indeed one has

THEOREM 1': *Assume that $\beta(x, u)$ is a Carathéodory function - i.e. measurable in x for every u and continuous in u for σ -a.e. x and such that for some constant a, b one has*

$$|\beta(x, u)| \leq a|u| + b \quad \sigma\text{-a.e. } x \in S_3, \quad \forall u \in \mathbf{R}. \quad (3.13)$$

Then if a is small enough there exists a solution to (P_ϵ) . Moreover if (1.10) holds then one has

$$p_\epsilon \geq 0 \quad \text{a.e. in } \Omega.$$

Proof: For $v \in L^2(S_3)$ there exists a unique $u_\epsilon \in V$ such that

$$\int_{\Omega} \nabla u_\epsilon \cdot \nabla \xi \, dx = - \int_{\Omega} H_\epsilon(u_\epsilon) \xi_y \, dx + \int_{S_3} \beta(x, \varphi - v) \cdot \xi \, d\sigma(x). \quad (3.14)$$

This is an easy consequence of the Schauder fixed point theorem (see the proof above).

Set

$$\tau_\epsilon(v) = \gamma_0(u_\epsilon)$$

where γ_0 denotes the trace on S_3 . Clearly, τ_ϵ is a continuous map from $L^2(S_3)$ into itself.

Set

$$K = \{v \in L^2(S_3) \mid |v - \varphi|_2 \leq R\}.$$

Then K is a closed convex of $L^2(S_3)$. Taking $\xi = u_\epsilon - \varphi$ in (3.14) we get :

$$\begin{aligned} \int_{\Omega} |\nabla(u_\epsilon - \varphi)|^2 \, dx &= - \int_{\Omega} \nabla \varphi \cdot \nabla(u_\epsilon - \varphi) \, dx \\ &\quad - \int_{\Omega} H_\epsilon(u_\epsilon) \cdot (u_\epsilon - \varphi)_y \, dx + \int_{S_3} \beta(x, \varphi - v) \cdot (u_\epsilon - \varphi) \, d\sigma(x) \end{aligned}$$

(we assume φ extended to Ω into a Lipschitz function -see [E. T.]). Applying Cauchy-Schwarz Inequality and (3.13) we get easily

$$\|\nabla(u_\epsilon - \varphi)\|_2^2 \leq (\|\nabla \varphi\|_2 + |\Omega|^{1/2}) \|\nabla(u_\epsilon - \varphi)\|_2 + (a|v - \varphi|_{2,S_3} + b|S_3|^{1/2}) |u_\epsilon - \varphi|_{2,S_3}$$

with obvious notation. Now clearly for some constant C one has

$$|u_\epsilon - \varphi|_{2,S_3} \leq C \|\nabla(u_\epsilon - \varphi)\|_2$$

(see for instance [R.T.]) and one deduces

$$\|\nabla(u_\epsilon - \varphi)\|_2 \leq \|\nabla \varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2} + aC|v - \varphi|_{2,S_3} \quad (3.15)$$

and thus

$$|u_\epsilon - \varphi|_{2,S_3} \leq C \|\nabla(u_\epsilon - \varphi)\|_2 \leq C \|\nabla \varphi\|_2 + C|\Omega|^{1/2} + bC^2|S_3|^{1/2} + aC^2|v - \varphi|_{2,S_3}$$

If we choose $v \in K$ then $u_\epsilon = \tau_\epsilon(v) \in K$ provided

$$C\|\nabla\varphi\|_2 + C|\Omega|^{1/2} + bC^2|S_3|^{1/2} + aC^2R \leq R$$

i.e. if $aC^2 \leq 1$ provided that

$$R \geq (1 - aC^2)^{-1} \cdot C\{\|\nabla\varphi\|_2 + |\Omega|^{1/2} + bC|S_3|^{1/2}\}. \quad (3.16)$$

Thus if we assume that $aC^2 < 1$ and (3.16) holds then τ_ϵ is an operator from K into K . Moreover, by (3.15) u_ϵ is uniformly bounded in $H^1(\Omega)$ and τ_ϵ is compact. The existence of a fixed point for τ_ϵ is then a consequence of the Schauder fixed point theorem. This proves the existence of p_ϵ . To prove that $p_\epsilon \geq 0$ one argues as in the end of the proof of Theorem 1.

Remark 2: Theorem 1' applies for instance when for some constants α, γ

$$|\beta(x, u)| \leq \alpha|u|^{1-\epsilon} + \gamma \quad \sigma\text{-a.e. } x \in S_3, \quad \forall u \in \mathbf{R}. \quad (3.17)$$

Indeed, in this case, it is easy to show that (3.13) holds for a as small as we wish.

We are now able to show

THEOREM 2: Assume that β is a function satisfying (1.10) and (1.6)-(1.9) or (3.13), then there exists a solution (p, χ) to the problem (P) .

Proof: Let p_ϵ be the solution to (P_ϵ) . From (3.10), (3.15) one deduces

$$|p_\epsilon|_{1,2} \leq C$$

where C is some constant independent of ϵ . So, using classical compactness arguments (see [N.]) one can extract a subsequence of ϵ , still denoted by ϵ , such that for some $p \in V$

$$p_\epsilon \rightharpoonup p \quad \text{in } H^1(\Omega) \quad , \quad p_\epsilon \rightarrow p \quad \text{in } L^2(\Omega) \quad \text{and a.e. on } \Omega \quad (3.18)$$

$$\gamma_0(p_\epsilon) \rightarrow \gamma_0(p) \quad \text{in } L^2(S_3) \quad (3.19)$$

Since $\{v \in V \mid v(x) \geq 0 \text{ a.e. on } \Omega\}$ is closed and convex it is weakly closed and thus p is in this set so that

$$p \geq 0 \text{ a.e. in } \Omega. \quad (3.20)$$

Since $H_\epsilon(p_\epsilon)$ is uniformly bounded there exists a function χ such that

$$H_\epsilon(p_\epsilon) \rightharpoonup \chi \text{ in } L^2(\Omega). \quad (3.21)$$

The set

$$\{f \in L^\infty(\Omega) \mid 0 \leq f \leq 1 \text{ a.e. in } \Omega\}$$

being closed and convex is weakly closed and by (3.21) one has

$$0 \leq \chi \leq 1 \text{ a.e. in } \Omega. \quad (3.22)$$

By (3.18) on $[p > 0]$ one has

$$H_\epsilon(p_\epsilon) \rightarrow 1 \text{ a.e.}$$

and thus by the Lebesgue theorem

$$H_\epsilon(p_\epsilon) \rightarrow 1 \text{ in } L^2([p > 0]).$$

Since by (3.21) one has also

$$H_\epsilon(p_\epsilon) \rightharpoonup \chi \text{ in } L^2([p > 0])$$

one deduces

$$\chi = 1 \text{ on } [p > 0] \quad (3.23)$$

and thus (P) (i), (iii) follows.

Next, for $\xi \in H^1(\Omega)$, $\xi \geq 0$ on S_2 one has for any $\delta > 0$

$$\xi \wedge \frac{p_\epsilon}{\delta} \in V.$$

Thus inserting this function in (P_ε) we obtain

$$\int \nabla p_\varepsilon \cdot \nabla \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right) dx + H_\varepsilon(p_\varepsilon) \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right)_y dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \xi \wedge \frac{p_\varepsilon}{\delta} d\sigma(x) = 0.$$

Thus, we have also

$$\begin{aligned} \int_{[\xi \leq \frac{p_\varepsilon}{\delta}]} \nabla p_\varepsilon \cdot \nabla \xi dx + \int_{\Omega} H_\varepsilon(p_\varepsilon) \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right)_y dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right) d\sigma(x) \\ = - \int_{[\frac{p_\varepsilon}{\delta} < \xi]} |\nabla p_\varepsilon|^2 / \delta dx \leq 0. \end{aligned} \quad (3.24)$$

(If f, g are two functions we denote by $[f < g]$, $[f \leq g]$ the sets defined by $[f < g] = \{(x, y) \in \Omega \mid f(x, y) < g(x, y)\}$, $[f \leq g] = \{(x, y) \in \Omega \mid f(x, y) \leq g(x, y)\}$. We will use this notation subsequently without further notice). Now, using the divergence theorem, remark that

$$\int_{\Omega} H_\varepsilon(p_\varepsilon) \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right)_y dx = - \int_{\Omega} (H_\varepsilon(p_\varepsilon))_y \cdot \xi \wedge \frac{p_\varepsilon}{\delta} dx + \int_{\partial\Omega} H_\varepsilon(p_\varepsilon) \cdot \nu_y \cdot \xi \wedge \frac{p_\varepsilon}{\delta} d\sigma(x) \quad (3.25)$$

where ν_y denote the second entry of the outward normal to Ω . Note that in this formula as well as in (3.24) we use the fact that

$$\gamma_0 \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right) = \gamma_0(\xi) \wedge \gamma_0 \left(\frac{p_\varepsilon}{\delta} \right).$$

Letting $\delta \rightarrow 0$ in (3.25) and since

$$\xi \wedge \frac{p_\varepsilon}{\delta} \rightarrow \xi \quad \text{a.e. on } p_\varepsilon > 0$$

we obtain by the Lebesgue theorem

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int_{\Omega} H_\varepsilon(p_\varepsilon) \left(\xi \wedge \frac{p_\varepsilon}{\delta} \right)_y dx &= - \int_{\Omega} H_\varepsilon(p_\varepsilon)_y \cdot \xi dx + \int_{\partial\Omega} H_\varepsilon(p_\varepsilon) \cdot \nu_y \cdot \xi d\sigma(x) \\ &= \int_{\Omega} H_\varepsilon(p_\varepsilon) \cdot \xi_y dx. \end{aligned}$$

Next, passing in the limit in (3.24) we get easily for any $\xi \in H^1(\Omega)$, $\xi \geq 0$ on S_2

$$\int_{\Omega} \nabla p_\varepsilon \cdot \nabla \xi dx + \int_{\Omega} H_\varepsilon(p_\varepsilon) \xi_y dx - \int_{S_3} \beta(x, \varphi - p_\varepsilon) \cdot \xi d\sigma(x) \leq 0. \quad (3.26)$$

The only difficulty is maybe to pass to the limit in the last integral of the left hand side of (3.24). For that note that

$$\begin{aligned}
& - \int_{S_3} \beta(x, \varphi - p_\epsilon) \cdot (\xi \wedge \frac{p_\epsilon}{\delta}) d\sigma(x) = \\
& - \int_{S_3 \cap [p_\epsilon > 0]} \beta(x, \varphi - p_\epsilon) \cdot (\xi \wedge \frac{p_\epsilon}{\delta}) d\sigma(x) - \int_{S_3 \cap [p_\epsilon = 0]} \beta(x, \varphi) \cdot (\xi \wedge 0) d\sigma(x) \\
& \geq - \int_{S_3 \cap [p_\epsilon > 0]} \beta(x, \varphi - p_\epsilon) \cdot (\xi \wedge \frac{p_\epsilon}{\delta}) d\sigma(x) - \int_{S_3 \cap [p_\epsilon = 0]} \beta(x, \varphi) \cdot \xi d\sigma(x)
\end{aligned}$$

since $\beta(x, \varphi) \geq 0$. Letting $\delta \rightarrow 0$, by the Lebesgue theorem, we deduce

$$\lim_{\delta \rightarrow 0} - \int_{S_3} \beta(x, \varphi - p_\epsilon) \cdot (\xi \wedge \frac{p_\epsilon}{\delta}) d\sigma(x) \geq - \int_{S_3} \beta(x, \varphi - p_\epsilon) \cdot \xi d\sigma(x)$$

and (3.26) follows.

Then letting $\epsilon \rightarrow 0$ in (3.26) and using (3.18), (3.19), (3.21) we obtain for any $\xi \in H^1(\Omega), \xi \geq 0$ on S_2

$$\int_{\Omega} \nabla p \cdot \xi dx + \int_{\Omega} \chi \xi_y dx - \int_{S_3} \beta(x, \varphi - p) \cdot \xi d\sigma(x) \leq 0$$

and (P) (iii) follows.

4. Some properties of the solutions

In this section we describe some useful properties of any solution (p, χ) to (P). Some of these results are similar to the ones in [C.C.] but proved with new methods.

First we have

PROPOSITION 1: *Let (p, χ) be a pair solution to (P). Then one has in the distributional sense*

$$\Delta p + \chi_y = 0 \quad \text{in } \Omega \tag{4.1}$$

$$\Delta p \geq 0 \quad , \quad \chi_y \leq 0 \quad \text{in } \Omega. \tag{4.2}$$

Proof: If $\xi \in \mathcal{D}(\Omega)$, $\mathcal{D}(\Omega)$ is the space of C^∞ - functions with compact support in Ω , then $\pm\xi$ is a test function for (P) (iii) and one gets since ξ vanishes on $\partial\Omega$

$$\int_{\Omega} \nabla p \cdot \nabla \xi + \chi \xi_y dx = 0 \quad \forall \xi \in \mathcal{D}(\Omega)$$

which is (4.1).

Next, if $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ then $\pm H_\varepsilon(p)\xi$ is a test function for (P) (iii). (H_ε is defined by (3.1)).

From (P) (iii) we then deduce

$$\int_{\Omega} \nabla p \cdot \nabla (H_\varepsilon(p)\xi) + \chi \cdot (H_\varepsilon(p)\xi)_y dx = 0.$$

Hence, since on $[p = 0]$, $H_\varepsilon(p) = 0$ and on $[p > 0]$, $\chi = 1$:

$$\int_{\Omega} H_\varepsilon(p) \nabla p \cdot \nabla \xi + (H_\varepsilon(p)\xi)_y dx = - \int_{\Omega} H'_\varepsilon(p) \xi \cdot |\nabla p|^2 dx \leq 0.$$

Since $H_\varepsilon(p)\xi = 0$ on $\partial\Omega$, we get applying the divergence theorem for the second integral

$$\int_{\Omega} H_\varepsilon(p) \nabla p \cdot \nabla \xi dx \leq 0 \quad \forall \xi \in \mathcal{D}(\Omega), \xi \geq 0.$$

Letting $\varepsilon \rightarrow 0$, by the Legesgue convergence theorem we obtain

$$\int_{\Omega} \nabla p \cdot \nabla \xi dx \leq 0 \quad \forall \xi \in \mathcal{D}(\Omega), \xi \geq 0. \quad (4.3)$$

The first inequality of (4.2) follows, the second results from (4.1).

As a consequence we have:

COROLLARY 1: *Let (p, χ) a pair of solution to (P) then for any $s > 1$*

$$p \in W_{loc}^{1,s}(\Omega).$$

Proof: This is a consequence of (4.1) since $\chi_y \in W^{-1,s}(\Omega)$ for any s and from the usual regularity theory -see for instance [B.L.].

Remark 3: As a consequence of our corollary

$$[p > 0] = \{(x, y) \in \Omega \mid p(x, y) > 0\}$$

is open in Ω since p is continuous in Ω . (see [G.T.], [K.S.]). In fact p is continuous at any point of $\Omega \cup S_2$.

We have also

PROPOSITION 2: *Let (p, χ) be a solution to (P). Let $(x_0, y_0) \in \Omega$. If $p(x_0, y_0) > 0$ then there exists $\varepsilon > 0$ such that the cylinder*

$$C_\varepsilon = \{(x, y) \in \Omega \mid |x - x_0| < \varepsilon, y < y_0 + \varepsilon\}$$

lies in the set $[p > 0]$.

If $p(x_0, y_0) = 0$ then $p(x_0, y) = 0 \quad \forall (x_0, y) \in \Omega, y > y_0$.

Proof: The proof follows [C.C.] but we reproduce it for the reader's convenience. If $p(x_0, y_0) > 0$ then since the set $[p > 0]$ is open there exists some $\varepsilon > 0$ such that the square

$$Q_\varepsilon = \{(x, y) \in \Omega \mid |x - x_0| < \varepsilon, |y - y_0| < \varepsilon\}$$

is included in $[p > 0]$. On Q_ε by (P)(i) one has $\chi = 1$, hence since by (4.2) χ is non decreasing in y one has $\chi = 1$ on C_ε . By (4.1) we deduce that p is harmonic on C_ε . If p should vanish on C_ε we would then get a contradiction with the maximum principle. This proves the first part of the proposition. The second part follows directly from the first.

In the case of Dirichlet boundary conditions -and in a case like for instance on figure 1- the pressure remains positive below S_3 (see [C.C.]). We would like to show that this is no more the case in the present situation.

First let us show

PROPOSITION 3: *Assume that $\beta \equiv 0$. Then every solution to (P) is given by*

$$(p, \chi) = (h - y, 1) \tag{4.4}$$

on every connected component of $[p > 0]$,

$$(p, \chi) = (0, 0) \quad (4.5)$$

elsewhere.

Proof: If $\beta \equiv 0$ one has

$$\int_{\Omega} \nabla p \cdot \nabla \xi + \chi \xi_y dx \leq 0, \quad \forall \xi \geq 0 \quad \text{on } S_2.$$

Taking $\xi = \pm p$ we deduce

$$\int_{\Omega} |\nabla p|^2 + \chi p_y dx = 0. \quad (4.6)$$

Taking $\xi = k + y$ for k large enough in such a way that $k + y \geq 0$ on S_2 we get

$$\int_{\Omega} p_y + \chi dx \leq 0.$$

Since $\chi = 1$ on $[p > 0]$ and $0 \leq \chi \leq 1$ this last inequality implies

$$\int_{\Omega} \chi p_y + \chi^2 dx \leq 0. \quad (4.7)$$

Adding (4.6) and (4.7) we get

$$\int_{\Omega} |\nabla p|^2 + 2\chi p_y + \chi^2 dx \leq 0$$

hence

$$\int_{\Omega} p_x^2 + (p_y + \chi)^2 dx \leq 0. \quad (4.8)$$

Thus we obtain

$$\nabla p = (0, -\chi) \quad \text{a.e. in } \Omega.$$

In particular on any connected component of $[p > 0]$

$$\nabla p = (0, -1)$$

and thus $p = (h - y)$. Outside

$$(0, 0) = \nabla p = (0, -\chi)$$

and the proposition follows.

Remark 4: It is clear that every pair given by (4.4), (4.5) defines a solution to (P) for $\beta = 0$. For instance in the case of figure 1 the pair

$$(p, \chi) = ((k - y)^+, \chi_{[y < k]})$$

is a solution to (P) for $\beta = 0$ provided that k is small enough (see figure 1). Clearly different values of k are suitable here so that (P) does not have a unique solution. We are encountering “pool” solutions as in the case of the dam problem with Dirichlet boundary condition — see [C.C]. We will see that again in Theorem 3.

We are now able to prove:

PROPOSITION 4: *The region below S_3 is not in general saturated — i.e. one does not have in general $p > 0$ below S_3 .*

Proof: Consider β satisfying the assumptions of theorem 2. Then for any $0 < \eta < 1$ there exists (p_η, χ_η) solution of (P) corresponding to $\eta\beta$. In particular (P) (iii) reads

$$\int_{\Omega} \nabla p_\eta \nabla \xi + \chi_\eta \xi_y dx - \int_{S_3} \eta \beta(x, \varphi - p_\eta) \cdot \xi d\sigma(x) \leq 0 \quad (4.9)$$

for any $\xi \in H^1(\Omega)$, $\xi \geq 0$ on S_2 . Taking $\xi = p_\eta - \varphi$ in this inequality one deduces very easily (compare to (3.10), (3.15)) that

$$|p_\eta|_{1,2} \leq C$$

where C is independent of η . Hence, up to a subsequence,

$$\begin{aligned} p_\eta \rightharpoonup p \quad \text{in } H^1(\Omega) \quad , \quad p_\eta \rightarrow p \quad \text{in } L^2(\Omega) \quad , \quad p_\eta \rightarrow p \quad \text{in } L^2(S_3) \\ p_\eta \rightarrow p \quad \text{a.e. in } \Omega \quad , \quad \chi_\eta \rightarrow \chi \quad \text{in } L^2(\Omega). \end{aligned}$$

when $\eta \rightarrow 0$. Passing to the limit in (4.9) we obtain

$$\int_{\Omega} \nabla p \cdot \nabla \xi + \chi \xi_y \leq 0 \quad \forall \xi \in H^1(\Omega), \quad \forall \xi \in H^1(\Omega), \xi \geq 0 \quad \text{on } S_2. \quad (4.10)$$

Moreover, see the proof of theorem 2, one has clearly

$$0 \leq p, \quad p \in V, \quad 0 \leq \chi \leq 1, \quad \chi = 1 \quad \text{on } [p > 0]$$

and thus (p, χ) is a solution of (P) for $\beta \equiv 0$. If for any η one had $p > 0$ below S_3 one would have $\chi_\eta \equiv 1$ below S_3 and thus at the limit $\chi \equiv 1$ below S_3 . This is clearly not the case by proposition 3.

Thus, roughly speaking, provided β is small enough an unsaturated region could develop. We will see in section 5 examples where $p = 0$ in a neighbourhood below S_3 and make further comments about this question.

Since the properties of (p, χ) in the model that we are considering differs significantly from the one of (p, χ) in the case of Dirichlet boundary conditions, it is worthwhile to check that problem (P) is a free boundary problem.

For that we can prove.

THEOREM 3: *Let us assume that β is chosen as in Theorem 2. Moreover let us assume that*

$$\beta(x, u) \cdot u \geq 0 \quad \forall u \in \mathbf{R}, \quad \sigma\text{-a.e. } x \in S_3. \quad (4.11)$$

Let (p, χ) be a solution to (P) and let us denote by h_1 the level of the highest reservoir. Then one has

$$(p, \chi) = (k - y, 1)$$

on any connected component of $[p > 0]$ that intersects $[y > h_1]$. Moreover, outside these connected components one has

$$p \leq (h_1 - y)^+. \quad (4.12)$$

Proof: Let (p, χ) be a solution of (P) . Taking

$$\xi = (p - (h_1 - y)^+)^+$$

in (P) (iii) we obtain

$$\int_{\Omega} \nabla p \cdot \nabla (p - (h_1 - y)^+)^+ + \chi (p - (h_1 - y)^+)_y^+ dx - \int_{S_3} \beta(x, \varphi - p) (p - (h_1 - y)^+)^+ d\sigma(x) \leq 0.$$

One integrates only on $p > (h_1 - y)^+ > \varphi$, so on this set, by (4.11), one has

$$\beta(x, \varphi - p) \leq 0$$

and the above inequality becomes

$$\int_{\Omega} \nabla p \cdot \nabla (p - (h_1 - y)^+)^+ + \chi (p - (h_1 - y)^+)_y^+ dx \leq 0.$$

Now, on the set $p > (h_1 - y)^+$ we have $p > 0$ and thus $\chi = 1$ and we obtain

$$\int_{\Omega} \nabla p \cdot \nabla (p - (h_1 - y)^+)^+ + (p - (h_1 - y)^+)_y^+ dx \leq 0$$

which can be written

$$\int_{[y \leq h_1]} |\nabla (p - (h_1 - y))^+|^2 dx + \int_{[y > h_1]} |\nabla p|^2 + p_y dx \leq 0. \quad (4.13)$$

Taking now $\xi = (y - h_1)^+$ in (P) (iii) we obtain since this function vanishes on S_3

$$\int_{[y > h_1]} p_y + \chi dx \leq 0$$

Noting that $\chi^2 \leq \chi$ and $\chi p_y = p_y$ a.e. we obtain

$$\int_{[y > h_1]} \chi p_y + \chi^2 dx \leq 0. \quad (4.14)$$

Adding (4.13) and (4.14) we get

$$\int_{[y \leq h_1]} |\nabla (p - h_1 - y))^+|^2 dx + \int_{[y > h_1]} p_x^2 + (p_y + \chi)^2 dx \leq 0 \quad (4.15)$$

from which we deduce

$$\nabla p = (0, -\chi) \quad \text{on} \quad [y > h_1]. \quad (4.16)$$

Thus, on any connected component C of $[p > 0]$ that intersects $[y > h_1]$ one has

$$p = k - y.$$

Indeed $p = k - y$ on the part of C intersecting $[y > h_1]$. By analytic continuation since, by (4.1), $\Delta p = 0$ in C and thus p is analytic in C , one has $p = k - y$ on C . Outside of these connected components one has $p = 0$ -and thus (4.12)- or by(4.15)

$$\nabla(p - (h_1 - y)^+)^+ = 0$$

when $y \leq h_1$. Thus, on any connected component of the set $[y < h_1]$ one has

$$(p - (h_1 - y)^+)^+ = Cst.$$

But any such a component touches somewhere the line $y = h_1$ where the constant is 0 since $p = 0$ (if p was not equal to 0 we would be on a component C of $[p > 0]$ intersecting $[y > h_1]$). If the constant vanishes then clearly (4.12) holds. This completes the proof.

Remark 5: Considerer for instance the following situation:

Figure 2

Then for $y > k$ one has $p = 0$. If not, one would have $p = h - y$ for $h > k$ and this will lead to a contradiction to $p = 0$ on S_2 . For the same reason one has $(p, \chi) = (0, 0)$ in O (see figure 2 and (4.16)).

Taking $\xi = \pm p \chi_C$ (χ_C is the characteristic function of C) in (P) (iii) we get

$$\int_C |\nabla p|^2 + \chi p_y dx = 0.$$

Taking $\xi = (y - k) \chi_C$ we get

$$\int_C p_y + \chi dx \leq 0$$

and proceeding as above we obtain

$$\nabla p = (0, -\chi) \quad \text{in } C$$

and thus

$$p = (h - y)^+ \quad \text{in } C \tag{4.17}$$

for some $h \leq k$. Conversely any function $((h - y)^+, \chi_{[y < h]})$ on C extended by (p, χ) outside of C is a solution to (P) . Hence, as in the case of Dirichlet boundary conditions uniqueness results for this problem will only be up to “pools” -i.e. functions $((h - y)^+, \chi_{[y < h]})$ - see [C.C.].

We would like now to stress out some other differences of this model compared to the classical one - i.e. the one with Dirichlet boundary conditions. We will do that through some examples.

5. Some particular examples

The first case we will consider is described in figure 3, where we assume that $S_1 = \emptyset$

and β is independent of x .

Figure 3

Then we would like to show that in this case when

$$0 \leq \beta(\varphi)/\nu_y \leq 1 \quad (5.1)$$

the only solution to (P) is given by

$$(p, \chi) = (0, (\beta(\varphi)/\nu_y)\chi_I)$$

where χ_I denotes the characteristic function of the region below S_3 and denoted by I on the figure 3, ν_y is the y entry of the unit outward normal to Γ on S_3 (we have denoted by $\beta(\varphi)/\nu_y$ the function independent of y equal to $\beta(\varphi)/\nu_y$ on S_3). Thus this is a particular case where uniqueness holds. However, we see that we cannot expect in general χ to be a characteristic function of a set (χ is not if $\beta(\varphi)/\nu_y < 1$). Moreover, the porous medium is completely unsaturated. So, the situation is quite different of the one in the classical dam problem (see [C.C.]).

So, let us prove:

PROPOSITION 5: Assume that (5.1) holds and that β is nondecreasing with $\beta(0) = 0$. Then the problem (P) corresponding to the figure 3 has a unique solution given by

$$(p, \chi) = (0, (\beta(\varphi)/\nu_y)\chi_I). \quad (5.2)$$

Proof: First let us check that (p, χ) given by (5.2) satisfies (P). We only have to check (P) (iii). For that note that since $\beta(\varphi)/\nu_y$ is a function of x only

$$\begin{aligned} \int_{\Omega} \nabla p \cdot \nabla \xi + \chi \xi_y dx - \int_{S_3} \beta(\varphi - p) \cdot \xi d\sigma(x) &= \int_I \frac{\beta(\varphi)}{\nu_y} \xi_y dx - \int_{S_3} \beta(\varphi) \cdot \xi d\sigma(x) \\ &= \int_I \left(\frac{\beta(\varphi)}{\nu_y} \cdot \xi \right)_y dx - \int_{S_3} \beta(\varphi) \cdot \xi d\sigma(x) \\ &= \int_{\partial I \setminus S_3} \frac{\beta(\varphi)}{\nu_y} \cdot n_y \cdot \xi d\sigma(x) \leq 0 \end{aligned} \quad (5.3)$$

for any $\xi \geq 0$ on S_2 , n_y denotes the y entry of the outward unit normal to $\partial I \setminus S_3$ and thus $n_y \leq 0$ (see figure 3). ∂I denotes the boundary of I . So, (p, χ) given by (5.2) is a solution to (P). Note that from (5.3) one deduces easily that

$$\int_{\Omega} \chi \cdot \xi_y dx - \int_{S_3} \beta(\varphi) \xi d\sigma(x) = 0 \quad \forall \xi \in H^1(\Omega), \xi = 0 \text{ on } \partial I \cap S_2. \quad (5.4)$$

Let us now denote by (p', χ') an other solution to (P). Thus, one has

$$\int_{\Omega} \nabla p' \cdot \nabla \xi + \chi' \xi_y dx - \int_{S_3} \beta(\varphi - p') \cdot \xi d\sigma(x) \leq 0 \quad \forall \xi \in H^1(\Omega), \xi \geq 0 \text{ on } S_2. \quad (5.5)$$

Taking $\xi = p'$ in (5.5) and $\xi = -p'$ in (5.4) and adding one gets

$$\int_{\Omega} |\nabla p'|^2 + (\chi' - \chi) \cdot p'_y dx - \int_{S_3} \beta(\varphi - p') - \beta(\varphi) \cdot p' d\sigma(x) \leq 0. \quad (5.6)$$

But

$$\begin{aligned} \int_{\Omega} (\chi' - \chi) p'_y dx &= \int_{\Omega} (1 - \chi) p'_y dx \\ &= \int_I \left(1 - \frac{\beta(\varphi)}{\nu_y} \right) p'_y dx + \int_{II} p'_y dx \\ &= \int_I \left(\left(1 - \frac{\beta(\varphi)}{\nu_y} \right) p' \right)_y dx \\ &= \int_{S_3} \left(1 - \frac{\beta(\varphi)}{\nu_y} \right) \cdot \nu_y \cdot p' d\sigma(x) \geq 0. \end{aligned} \quad (5.7)$$

Moreover, from the fact that β is nondecreasing one has

$$-\int_{S_3} \beta(\varphi - p') - \beta(\varphi) \cdot p' d\sigma(x) \geq 0. \quad (5.8)$$

Combining (5.6), (5.7), (5.8) one deduces

$$\int_{\Omega} |\nabla p'|^2 dx \leq 0$$

and thus $p' = p = 0$. But now (5.5) reads

$$\int_{\Omega} \chi' \xi_y dx - \int_{S_3} \beta(\varphi) \cdot \xi d\sigma(x) \leq 0 \quad \forall \xi \in H^1(\Omega), \xi \geq 0 \text{ on } S_2. \quad (5.9)$$

Taking $\xi \in \mathcal{D}(\Omega)$ - or using (4.1) one deduces

$$\chi'_y = 0 \quad \text{or} \quad \chi' = \chi'(x).$$

Then if we denote by T and B respectively the upper and the bottom part of Γ one deduces from (5.9) for $\xi \in H^1(\Omega)$, $\xi \geq 0$ on S_2

$$\begin{aligned} 0 &\geq \int_{\Omega} (\chi' \xi)_y dx - \int_{S_3} \beta(\varphi) \cdot \xi d\sigma(x) \\ &= \int_T \chi' \cdot \xi \cdot \nu_y d\sigma(x) + \int_B \chi' \cdot \xi \cdot \nu_y d\sigma(x) - \int_{S_3} \beta(\varphi) \cdot \xi d\sigma(x). \end{aligned} \quad (5.10)$$

Taking in (5.10) any ξ that vanishes on S_2 one gets

$$\int_{S_3} (\chi' \nu_y - \beta(\varphi)) \cdot \xi d\sigma(x) = 0$$

for such a ξ . Hence $\chi' = \beta(\varphi)/\nu_y$ on S_3 . Then (5.10) becomes

$$0 \geq \int_{\Gamma \setminus S_3} \chi' \cdot \xi \cdot \nu_y d\sigma(x) \quad \forall \xi \in H^1(\Omega), \xi \geq 0 \text{ on } S_2.$$

Taking $\xi \geq 0$ on T , $\xi = 0$ on B one deduces

$$0 \geq \int_{T \setminus S_3} \chi' \cdot \xi \cdot \nu_y d\sigma(x)$$

hence $\chi' = 0$ on $T \setminus S_3$ and the result follows. (We have assumed $\nu_y > 0$ on T). This completes the proof of the theorem.

Remark 6: We don't know in general if uniqueness of a solution to (P) holds modulo "pools". However, the above example and the one below seem to indicate that this is the case.

Remark 7: If one assumes that some part of B is impervious then $p = 0$ is no more solution. Indeed, if it was the case then we would have (5.9) and thus (5.2), see the above proof. But then, clearly, one would not have (P) (iii). Thus in this case the solution is saturated i.e. the set $[p > 0]$ has a positive measure. The same happens when $\beta(\varphi)/\nu_y > 1$ on a set of positive measure.

Remark 8: If $p = 0$ on some "rectangle" D below S_3 as in the figure 4.

Figure 4.

then one has necessarily

$$\chi = \beta(\varphi)/\nu_y \quad \text{on } D. \quad (5.11)$$

In particular, due to (P) (i), this situation is impossible (see also (1.5)) when

$$\beta(\varphi)/\nu_y > 1 \quad \text{on } \partial D \cap S_3.$$

To prove (5.11) note that if ξ is a function vanishing on $\partial D \setminus S_3$ and if we extend this function by 0 outside D we obtain

$$\int_D \chi \cdot \xi_y - \int_{S_3 \cap \partial D} \beta(\varphi) \cdot \xi \, d\sigma(x) = 0. \quad (5.12)$$

Thus

$$\chi_y = 0 \text{ on } D \text{ or } \chi = \chi(x) \text{ on } D.$$

Then (5.12) reads

$$\int_{S_3 \cap \partial D} (\chi \nu_y - \beta(\varphi)) \cdot \xi \, d\sigma(x) = 0$$

for any ξ vanishing on $\partial D \setminus S_3$ and the result follows.

We would like to show now that the situation described in Remark 8 -i.e. $p = 0$ below S_3 - could happen even if the bottom part of Γ is impervious (see Remark 7), in other words saturation depends strongly on β . In this example we will also show that the solution to (P) is unique and is related via the Baiocchi transform (see [Ba₁]) to the solution of some variational inequality. We consider the porous medium Ω described in the figure 5 -i.e. a rectangle

figure 5

whose bottom is assumed to be impervious, the top covered with water and the lateral sides in contact with the atmosphere. We assume also that β is independent of x and

nondecreasing. h is the level of water, D the thickness of the porous medium, L its horizontal size.

Remark that if $p = 0$ on S_3 then $\beta(\varphi - p) = \beta(h)$ on S_3 . So, consider K the closed convex set of $H^1(\Omega)$ defined by

$$K = \{v \in H^1(\Omega) \mid v \geq 0, v = 0 \text{ on } S_2 \cup S_3, v = \frac{\beta(h)}{2} \cdot x(L - x) \text{ on } S_1\} \quad (5.13)$$

and u the solution of the variational inequality

$$u \in K, \quad \int_{\Omega} \nabla u \cdot \nabla (v - u) \, dx \geq \int_{\Omega} (\beta(h) - 1) \cdot (v - u) \, dx \quad \forall v \in K. \quad (5.14)$$

It is clear that this variational inequality has a unique solution (see [K.S.]). Moreover, the boundary data of u admits a $C^\infty(\overline{\Omega})$ extension. So, combining the well known techniques of regularity for variational inequalities (see [B.S.]) and for elliptic problems in domains with corners (see [G.]) one can show that for any $p \geq 1$ and any $\alpha \in (0, 1)$ one has

$$u \in W^{2,p}(\Omega) \cap C^{1,\alpha}(\overline{\Omega}). \quad (5.15)$$

Moreover, one has

LEMMA 1: *Let u be the solution of the variational inequality (5.13), (5.14). Assume that*

$$\beta(h) = \beta < \frac{4D^2}{4D^2 + L^2} \quad (5.16)$$

then

$$0 \leq u \leq \frac{1 - \beta}{2} \left\{ \left(\frac{L}{2} \sqrt{\frac{\beta}{1 - \beta}} - y \right)^+ \right\}^2 \quad \text{a.e. in } \Omega. \quad (5.17)$$

In particular $u(x, y)$ vanishes when $y \in [\frac{L}{2} \sqrt{\frac{\beta}{1 - \beta}}, D]$.

Proof: First note that (5.16) is equivalent to

$$\frac{L}{2} \sqrt{\frac{\beta}{1 - \beta}} < D.$$

For $k \in (\frac{L}{2}\sqrt{\frac{\beta}{1-\beta}}, D)$ consider the function

$$z = \frac{1-\beta}{2} \cdot (k-y)^2.$$

One has

$$-\Delta z = \beta - 1$$

and thus z is the solution of the variational inequality

$$\begin{aligned} z \in K' &= \{v \in H^1(\Omega) \mid v \geq 0, \quad v = \frac{1-\beta}{2} \cdot (k-y)^2 \text{ on } \Gamma\}, \\ \int_{\Omega} \nabla z \cdot \nabla (v-z) \, dx &\geq \int_{\Omega} (\beta(h)-1) \cdot (v-z) \, dx \quad \forall v \in K'. \end{aligned}$$

Moreover on S_1 one has

$$z = \frac{1-\beta}{2} \cdot k^2 \geq \frac{\beta}{2} \cdot \frac{L^2}{4} \geq \frac{\beta}{2} \cdot x(x-L) = u.$$

Since such variational inequality has its solution that varies monotonically with respect to the data (see [Br.], [C.M.]) one has

$$0 \leq u \leq z = \frac{1-\beta}{2} \cdot (k-y)^2 \quad \text{in } \Omega$$

and the result follows since the inequality holds for any $k \in [\frac{L}{2}\sqrt{\frac{\beta}{1-\beta}}, D]$.

Then we can prove

THEOREM 4 : Assume that we are in the case of the figure 5 and that (5.16) holds.

Let (p, χ) be a solution to (P). If one sets

$$u(x, y) = \int_y^D p(x, t) \, dt \tag{5.18}$$

then u is the solution of the variational inequality (5.13), (5.14).

Proof: Let us denote provisionally by u' the integral in (5.18) and by u the solution of the variational inequality (5.13), (5.14). First we have clearly

$$u' = 0 \quad \text{on } S_2 \cup S_3. \tag{5.19}$$

Next taking $\xi = \psi(x)$ where $\psi \in \mathcal{D}(0, L)$ in (P) (iii) one deduces

$$\int_{\Omega} p_x \psi_x dx - \int_{S_3} \beta(h - p(x, D)) \psi d\sigma(x) = 0$$

which can be written also as

$$\int_0^L \left\{ \left(\int_0^D p(x, t) dt \right)_x \cdot \psi_x - \beta(h - p(x, D)) \right\} \cdot \psi dx = 0 \quad \forall \psi \in \mathcal{D}(0, L).$$

Thus in the distributional sense one has

$$-u'(x, 0)_{xx} = \beta(h - p(x, D)) \leq \beta(h) = -u(x, 0)_{xx} \quad (5.20)$$

and thus by the maximum principle

$$u'(x, 0) \leq u(x, 0) \quad \text{on } S_1. \quad (5.21)$$

(Recall that both $u'(x, 0)$ and $u(x, 0)$ vanish at the end points of $(0, L)$).

It follows from (5.19), (5.21) that

$$u' \leq u \quad \text{on } \partial\Omega. \quad (5.22)$$

Next, when ζ is a smooth function vanishing on S_2 ,

$$\xi = \int_0^y \zeta(x, t) dt$$

is a suitable test function for (P) (iii). So, noting that

$$p = -u'_y$$

we deduce

$$\int_{\Omega} -\nabla u'_y \cdot \nabla \int_0^y \zeta(x, t) dt + \chi \zeta dx - \int_{S_3} \beta(h - p(x, D)) \cdot \int_0^D \zeta(x, t) dt d\sigma(x) = 0.$$

This reads also

$$\int_{\Omega} -\nabla u'_y \cdot \nabla \int_0^y \zeta(x, t) dt + (\chi - \beta(h - p(x, D))) \cdot \zeta dx = 0. \quad (5.23)$$

(dx is the Lebesgue measure on $\mathbf{R}^2$). Remark that

$$-\nabla u'_y \cdot \nabla \int_0^y \zeta(x, t) dt = (-\nabla u' \cdot \nabla \int_0^y \zeta(x, t) dt)_y + \nabla u' \cdot \nabla \zeta. \quad (5.24)$$

Then, (5.22) becomes

$$\begin{aligned} \int_{\Omega} \nabla u' \cdot \nabla \zeta + (\chi - \beta(h - p(x, D))) \cdot \zeta dx &= \int_{\Omega} (\nabla u' \cdot \nabla \int_0^y \zeta(x, t) dt)_y dx \\ &= - \int_{\Omega} (p \cdot \zeta)_y dx + \int_{\Omega} (u'_x \cdot \int_0^y \zeta_x(x, t) dt)_y dx. \end{aligned}$$

Remark that the function

$$(u'_x \cdot \int_0^y \zeta_x(x, t) dt)_y \in L^2(\Omega)$$

and that $u'_x = 0$ on S_3 since $u' = 0$ there. Moreover, $\int_0^y \zeta_x(x, t) dt = 0$ on S_1 . So, by the divergence theorem we deduce

$$\int_{\Omega} \nabla u' \cdot \nabla \zeta + (\chi - \beta(h - p(x, D))) \cdot \zeta dx = \int_{S_1} p \zeta d\sigma(x) - \int_{S_3} p \zeta d\sigma(x) \quad (5.25)$$

for every smooth ζ vanishing on S_2 . By an easy density argument, (5.25) holds for every $\zeta \in H^1(\Omega)$, $\zeta = 0$ on S_2 .

Let us set

$$K' = \{v \in H^1(\Omega) \mid v \geq 0, v = u' \text{ on } \partial\Omega\}.$$

Then, clearly taking $\zeta = v - u'$ in (5.25) one deduces

$$\int_{\Omega} \nabla u' \cdot \nabla v - u' dx = \int_{\Omega} (\beta(h - p(x, D)) - \chi) \cdot v - u' dx \quad \forall v \in K'. \quad (5.26)$$

Now, one has

$$[u' > 0] = [p > 0]. \quad (5.27)$$

Indeed, if $u'(x_0, y_0) > 0$ then $p(x, y) > 0$ for some $(x, y) \in \Omega$ such that $y > y_0$ and thus, by Proposition 2, $p(x_0, y_0) > 0$. Conversely if $p(x_0, y_0) > 0$ then $p(x, y) > 0$ in a

neighbourhood of (x_0, y_0) and by (5.18) $u'(x_0, y_0) > 0$. This proves (5.27). Then, from (5.27) one deduces that $\chi u' = u'$ and thus

$$\chi(v - u) = \chi v - u' \leq v - u'.$$

By (5.26) this implies that u' satisfies

$$u' \in K' \quad , \quad \int_{\Omega} \nabla u' \cdot \nabla v - u' \, dx \geq \int_{\Omega} (\beta(h - p(x, D)) - 1) \cdot v - u' \, dx \quad \forall v \in K' \quad (5.28).$$

Since

$$\beta(h - p(x, D)) \leq \beta(h) = \beta \quad , \quad u' \leq u \quad \text{on } \partial\Omega$$

one deduces from the monotonicity of the solution of variational inequalities with respect to the data (see [C.M.]) that

$$u' \leq u \quad \text{on } \Omega.$$

Then, by Lemma 1, $u' = 0$ on a neighbourhood of S_3 and so does p . From (5.20) we have now

$$-u'_{xx} = \beta = -u_{xx}$$

hence $u' = u$ on $\partial\Omega$ and $K' = K$. Moreover, (5.28) becomes equivalent to (5.13), (5.14) and $u' = u$. The result follows.

We have also

LEMMA 2: *There exists a smooth function $\varphi : (0, L) \rightarrow \mathbf{R}^+$ such that*

$$[u > 0] = \{(x, y) \in \Omega \mid 0 < y < \varphi(x)\}.$$

Proof: Since u is continuous we know that u is positive in a neighbourhood of any point of S_1 . Moreover, by (5.18) u is decreasing in y so that if $u(x_0, y_0) > 0$ then, $u(x_0, y) > 0$ for any $y < y_0$. The result follows then by setting

$$\varphi(x) = \sup\{y \mid u(x, y) > 0\}.$$

The smoothness of φ follows from well known results on variational inequalities (see [K.N.S.]).

Then we can show

COROLLARY 2: *Assume that we are in the case of the figure 5 and that (5.16) holds. Then the solution (p, χ) to (P) is unique and given by*

$$p = -u_y \quad (5.29)$$

$$\chi = \begin{cases} \beta & \text{on } [u = 0] \\ 1 & \text{on } [u > 0]. \end{cases} \quad (5.30)$$

where u is the solution of the variational inequality (5.13), (5.14).

Proof: If (p, χ) is a solution to (P), and we know that such a solution exists, then by Theorem 4 one has

$$p = -u_y.$$

Next by Lemma 2 and remark 8 one deduces that (5.30) holds.

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