# NUMERICAL ANALYSIS OF OSCILLATIONS IN NONCONVEX PROBLEMS

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# NUMERICAL ANALYSIS OF OSCILLATIONS IN NONCONVEX PROBLEMS

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## 1. Introduction

Let $\Omega$ be a bounded domain of $\mathbf{R}^n$. For the simplicity of the numerical analysis we will assume that $\Omega$ is a polygonal domain. Let $w_i \in \mathbf{R}^n$, $i = 1, \ldots, k$, $k \geq 2$, and consider a function $\varphi : \mathbf{R}^n \to \mathbf{R}$ such that

$$\varphi(w_i) = 0 \quad \forall\, i = 1, \ldots, k, \tag{1.1}$$

$$\varphi(w) > 0 \quad \forall\, w \neq w_i, \quad i = 1, \ldots, k. \tag{1.2}$$

In physical terms, $\varphi$ is some energy that vanishes at wells $w_i$. Let us denote by $\partial\Omega$ the boundary of $\Omega$ and by $\Gamma_0$ some portion of $\partial\Omega$ of positive measure.

Let $a$ be in the convex hull of the $w_i$'s, such that

$$a \neq w_i \quad \forall i = 1, \ldots, k. \tag{1.3}$$

If $V_a$ denotes some space of functions $v$ such that

$$v(x) = a \cdot x \quad \text{on} \quad \Gamma_0 \tag{1.4}$$

($a \cdot x$ denotes the scalar product of $a$ and $x$) we would like to consider the problem

$$\operatorname*{Inf}_{v \in V_a} \int_\Omega \varphi(\nabla v(x))\, dx. \tag{1.5}$$

As we will see, this problem in general does not admit a minimizer, i.e., the infimum of (1.5) is not attained for some $u \in V_a$. In other words minimizing sequences of (1.5) do not converge in a strong enough topology to allow their limit points to be a minimizer. In such an event the problem seems to lack interest. However, it turns out that it arises in numerous physical settings. For instance (see [B.J.$_1$], [B.J.$_2$], [Er.], [F.$_1$], [F.$_2$], [J.K.], [K.], [Ko.], [S.]), some crystals have their deformations governed by energies $\varphi$ experiencing some potential wells $w_i$. For such a crystal, even if no unique equilibrium is available, there is some organized way to reach its lowest energy level. In terms of problem (1.5) the minimizing sequences have some pattern that we would like to investigate here. Note that we are considering here the so called scalar case –i.e. the $w_i$'s are vectors and not matrices. So, in particular hyperelasticity for ordered material seems to be beyond our scope. We will comment on that at the end of the paper. We refer also the reader to [C.C.K.] for results in this direction.

Some similar problems have been first considered by C. Collins, D. Kinderlehrer and M. Luskin in one dimension (see [C.K.L.], [C.L.$_1$] - [C.L.$_4$]).

One can add also to the energy functional defined in (1.5) some kind of weak coerciveness by considering instead as in [B.M.]

$$\int_\Omega \varphi(\nabla v(x)) + (v - a \cdot x)^2 \, dx. \tag{1.6}$$

In this event, the second term under the integral suffies to force the pattern of the minimizing sequences. We refer the reader to [C.C.] for a complete analysis of this problem. On the other hand, when no such a term is introduced, the oscillations of the minimizing sequences are forced, as we will see, only by the boundary condition that has to be matched.

The paper is divided as follows. In section 2 we estimate in terms of the mesh size the infimum (1.5) using Lagrange elements of type $P_1$. We give also an existence result for the approximated problem. In section 3 we show that (1.5) does not admit in general a minimizer. We also analyse the Young measure associated to the problem proving in particular its uniqueness. In section 4 we investigate the pattern of the minimizing sequences. In section 5 we comment on some numerical computations for these types of problems.

## 2. Energy estimate on a finite element space

In what follows we will always assume

$$a = 0. \tag{2.1}$$

Indeed, there is no loss of generality in doing so. To see this, remark that if we set $v = u - a \cdot x$ then

$$\int_\Omega \varphi(\nabla u(x))\,dx = \int_\Omega \varphi(\nabla v(x) + a)\,dx \tag{2.2}$$

thus minimize the left hand side of (2.2) on $V_a$ reduces to minimize the right hand side of (2.2) on $V_0$. We did not yet define $V_a$ but, as we will see, for the problem that we have in mind, we just need $V_a$ to be a space of functions equal to $a \cdot x$ on $\Gamma_0$ and containing the piecewise affine functions (by a piecewise affine function we mean a continuous function affine on simplices covering $\Omega$). But then we are led to a problem identical to the one we had with now

$$\tilde{\varphi}(w) = \varphi(w + a)$$

i.e. with a function $\tilde{\varphi}$ having $w_i - a$ as wells, with 0 in the convex hull of these wells (since clearly $a$ in the convex hull of the $w_i$'s is equivalent to 0 in the convex hull of the $w_i - a$'s).

Let $\tau$ a triangulation of $\Omega$ with simplices of diameters less then $h$. If $K$ is some simplex of $\tau$ denote by $P_1(K)$ the space of polynomials of degree 1 on $K$. Set

$$V_h^0 = \{v : \Omega \to \mathbf{R},\ \text{continuous} \ \mid v_{|K} \in P_1(K) \quad \forall\, K \in \tau,\ v = 0 \text{ on } \Gamma_0\}.$$

($v_{|K}$ denotes the restriction of $v$ to $K$).

Then one has:

**Theorem 1:** *Assume that $\varphi$ is bounded on bounded subsets of $\mathbf{R}^n$. Then there exists a constant $C$ such that*

$$E_h = \operatorname*{Inf}_{v \in V_h^0} \int_\Omega \varphi(\nabla v(x))\,dx \le C \cdot h^{1/2}. \tag{2.3}$$

**Remark 1:** Note that $\nabla v(x)$ and thus $\varphi(\nabla v(x))$ are constant on every simplex of $\tau$ so that no further assumption on $\varphi$ is needed for the integral of (2.3) to make sense.

**Proof of Theorem 1:** Clearly since $a = 0$ belongs to the convex hull of the $w_i$'s one can find $w_i$'s, that we will denote by $w_1, \ldots, w_p$, $p \geq 2$, such that

$$w_i - w_1, \quad i = 2, \ldots, p \quad \text{are linearly independent} \tag{2.4}$$

and such that for some unique $\alpha_i \in (0,1)$

$$\sum_{i=1}^{p} \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^{p} \alpha_i = 1. \tag{2.5}$$

Set

$$w_h(x) = \bigwedge_{i=1}^{p} w_i \cdot x + h^\alpha \tag{2.6}$$

where $\wedge$ denotes the minimum of two functions and $\alpha$ is some number in (0,1). Since $\alpha \in (0,1)$ it should be noticed that

$$h^\alpha >> h. \tag{2.7}$$

First, remark that

$$w_h(x) \leq h^\alpha \quad \forall\, x. \tag{2.8}$$

Indeed if not we would have for some $x$

$$w_i \cdot x > 0 \quad \forall\, i = 1, \ldots, p$$

and by (2.5)

$$\sum_{i=1}^{p} \alpha_i w_i \cdot x = 0$$

hence a contradiction.

Moreover, this "hat" function $w_h$ has exactly $p$ different slopes, i.e. for every $i$ the set of $x$ such that $w_h(x) = w_i \cdot x + h^\alpha$ has a non empty interior. Indeed, otherwise for some $i_0$ we would have

$$w_{i_0} \cdot x + h^\alpha \geq w_i \cdot x + h^\alpha \quad \forall\, i, \quad \forall\, x,$$

thus

$$\sum_{i=1}^{p} \alpha_i w_{i_0} \cdot x \geq \sum_{i=1}^{p} \alpha_i w_i \cdot x = 0 \quad \forall\, x.$$

It follows that

$$w_{i_0} \cdot x \geq 0 \quad \forall x$$

which is impossible unless $w_{i_0} = 0 = a$ which has been excluded.

Consider the set

$$S_h = \{x \in \mathbf{R}^n | w_h(x) \geq 0\}$$

$$= \{x \in \mathbf{R}^n | w_i \cdot x + h^\alpha \geq 0 \quad \forall i = 1, \ldots, p\}.$$

Clearly $S_h$ is the intersection of $p$ half spaces and thus a convex domain having $p$ edges. For instance when $n = 2$, if $p = 2$ then $S_h$ is a strip, if $p = 3$ $S_h$ is a triangle. When $n = 3$, if $p = 2$, $S_h$ is a strip, if $p = 3$, $S_h$ is a cylinder with a triangular basis, if $p = 4$, $S_h$ is a tetrahedron...

Let us denote by $W$ the subspace of $\mathbf{R}^n$ spanned by the $w_i$'s. Then, $S_h \cap W$ is a $(p-1)$-simplex with vertices $v_0, v_1, \ldots, v_{p-1}$. Note that the functions $w_i \cdot x$ are constant on any subspace orthogonal to $W$. For any $z = (z_1, \ldots, z_{p-1}) \in \mathbf{Z}^{p-1}$ set

$$w_{h,z}(x) = w_h(x - \sum_{i=1}^{p-1} z_i(v_i - v_0)) \tag{2.9}$$

Clearly, $w_{h,z}$ is a piecewise affine function, non negative only on each of the sets

$$S_{h,z} = S_h + \sum_{i=1}^{p-1} z_i(v_i - v_0). \tag{2.10}$$

Note that these subsets are disjoint or have in common a subspace parallel to $W^\perp$, the orthogonal of $W$. Set

$$u_h(x) = \bigvee_{z \in \mathbf{Z}^{p-1}} w_{h,z}(x) \tag{2.11}$$

where $\vee$ denotes the supremum of functions. Then $u_h$ is a piecewise affine function equal to $w_{h,z}$ on $S_{h,z}$. Note that for a given $x$ the supremum in (2.11) is taken only on a finite number of $z$ such that $S_{h,z}$ neighbours $x$. Now clearly one has

$$|u_h(x)| \le Ch^\alpha \tag{2.12}$$

for some constant $C$. Moreover, at every point $x$ of in $\mathbf{R}^n$ one has

$$\nabla u_h(x) = w_i \tag{2.13}$$

except on the edges of the function $u_h$. To obtain a function $u_h$ which is affine on each simplex of $\tau$ we modify $u_h$ in the following way. On each simplex of $\tau$ where $u_h$ has some edge we replace it by the affine function that coincides with $u_h$ on its vertices. Clearly, (2.12) is preserved. Now the volume of the simplices where this modification occurred is no more than

$$h \cdot (n \text{ - 1-area of the edge of } u_h) \tag{2.14}$$

(the edge of $u_h$ is cutting one simplex at a time, and any dimension of this simplex in bounded by $h$).

Denote by $N(h^\alpha)$ the number of $S_{h,z}$ covering $\Omega$. By a scaling argument one has

$$\begin{aligned} n - 1\text{-area of the edge of } u_h &= C.N(h^\alpha)(h^\alpha)^{p-2} \\ &\le C'N(1)(h^\alpha)^{p-2}/(h^\alpha)^{p-1} \\ &\le Ch^{-\alpha}. \end{aligned}$$

Hence, by (2.14) the volume where the modification has occurred is less then

$$C.h^{1-\alpha} \tag{2.15}$$

i.e.

$$\nabla u_h = w_i \tag{2.16}$$

except on a part of $n$-dimensional volume $C \cdot h^{1-\alpha}$.

To match the boundary condition we modify $u_h$ once more by replacing $u_h$ by the affine function which coincides with

$$\text{dist}\,(.,\partial\Omega)_\wedge u_{h\vee} - \text{dist}\,(.,\partial\Omega) \tag{2.17}$$

at the node of $\tau$ (dist $(x, \partial\Omega)$ denotes the distance from $x$ to $\partial\Omega$). Let us still denote this last function by $u_h$. Clearly the gradient of $u_h$ coincide with the previous gradient of $u_h$ except (see (2.12)) on a neighbourhood of $\Gamma$ of volume

$$C|\partial\Omega|h^\alpha \tag{2.18}$$

Thus, see (2.13) – (2.17), we have obtained a function $u_h \in V_h^0$ such that (2.13) holds except on a set $S$ of volume

$$Ch^{1-\alpha} + C|\partial\Omega|h^\alpha$$

where, of course, $u_h$ has a bounded gradient. Thus we have

$$\begin{aligned}\int_\Omega \varphi(\nabla u_h(x))\,dx &= \int_{\Omega\setminus S} \varphi(\nabla u_h(x))\,dx + \int_S \varphi(\nabla u_h(x))\,dx \\ &\leq C(h^{1-\alpha} + h^\alpha)\end{aligned}$$

where $C$ is some constant independent of $h$.

We thus obtain, since $\alpha \in (0,1)$, and $h$ could be assumed to be less then 1

$$E_h \leq \int_\Omega \varphi(\nabla u_h(x))\,dx \leq 2Ch^{\min(1-\alpha,\alpha)}$$

Now, the function $\min(1-\alpha,\alpha)$ is maximum for $\alpha = \frac{1}{2}$ which gives the result.

**Remark 2:** Of course, for some triangulation, the rate of $h^{1/2}$ is not sharp, but this rate holds for any $\varphi$ and any triangulation. In particular we did not assume the triangulation to be quasi-uniform.

**Remark 3:** In the case of a quasi-uniform mesh and a slow growth of $\varphi$ at infinity one can improve our estimate. Indeed assume that for some $r < 1$ one has

$$|\varphi(\xi)| \leq C|\xi|^r \quad \forall\, \xi \in \mathbf{R}^n. \tag{2.19}$$

Then, instead of using (2.17) to match the boundary condition we can set $u_h$ to 0 on the nodes of $\partial\Omega$ and leave it unchanged elsewhere.

Due to (2.12) the gradient of $u_h$ is bounded by $Ch^\alpha/h$ on a neighbourhood of $\partial\Omega$ of thickness $h$. So, by (2.19), (2.15) we obtain

$$\int_\Omega \varphi(\nabla u_h(x))\,dx \leq Chh^{(\alpha-1)r} + Ch^{1-\alpha}$$
$$\leq Ch^{\min(1-\alpha, 1-r(1-\alpha))}.$$

The function in the exponent of $h$ is maximum when

$$1-\alpha = 1 - r(1-\alpha) \quad \Leftrightarrow \quad \alpha = 1 - \frac{1}{1+r}$$

which leads to

$$E_h \leq \int_\Omega \varphi(\nabla u_h(x))\,dx \leq Ch^{\frac{1}{1+r}} \tag{2.20}$$

which is slightly better than (2.3)

**Remark 4:** When $r \to 0$ the rate $h^{\frac{1}{1+r}}$ is almost sharp in the sense that, in general, one cannot expect a rate better than $h$.

Indeed, consider for instance

$$n = 2, \quad \Omega = (-1,+1)^2, \quad \Gamma_0 = \partial\Omega,$$
$$w_1 = (-1,0), \quad w_2 = (1,0),$$
$$\varphi(\xi) = \operatorname*{Min}_i |\xi - w_i|^2 \text{ for } |\xi - w_i| \leq 1, \quad i = 1,2,$$
$$\varphi(\xi) \geq 1 \quad \text{for} \quad |\xi - w_1| \geq 1 \quad \text{or} \quad |\xi - w_2| \geq 1.$$

Assume that our mesh is quasi-uniform. If $u_h = 0$ on $(-1,+1) \times \{1\}$ then necessarily $\frac{\partial u_h}{\partial x} = 0$ on the triangles touching this part of the boundary and one obtains

$$\int_\Omega \varphi(\nabla u_h(x))\,dx \geq \quad \text{measure of these triangles}$$
$$= Ch.$$

and thus the rate of convergence cannot be better than $h$ in general.

It could be useful to know that the approximated problem

$$\operatorname*{Inf}_{v\in V_h^0}\int_\Omega \varphi(\nabla v(x))\,dx \tag{2.21}$$

admits a solution. In this spirit we can prove:

**Theorem 2:** *Assume that $\varphi$ is continuous and that*

$$\varphi(\xi)\to+\infty \text{ when } |\xi|\to+\infty \tag{2.22}$$

*then there exists $u_h\in V_h^0$ such that*

$$\int_\Omega \varphi(\nabla u_h(x))\,dx = \operatorname*{Inf}_{v\in V_h^0}\int_\Omega \varphi(\nabla v(x))\,dx. \tag{2.23}$$

**Proof:** A function $v_h\in V_h^0$ is completely determined by its value at the nodes of the triangulation. Let $X$ denote the vector of the values of $v_h$ at the nodes not on $\Gamma_0$. Then

$$\int_\Omega \varphi(\nabla v_h(x))\,dx$$

becomes a continuous function $F$ of $X$ that we have to minimize over $\mathbf{R}^N$ where $N$ is the number of nodes ouside $\Gamma_0$. If we can show that

$$F(X)=\int_\Omega \varphi(\nabla v_h(x))\,dx\to+\infty \tag{2.24}$$

when $|X|\to+\infty$ we are led to minimize $F$, which is continuous, over a compact set of $\mathbf{R}^N$ and we are done. If $|X|\to+\infty$, then one of the value of $v_h$ at some node $N_i$ must go to $+\infty$. Then due to the 0 boundary condition one must have on some simplex

$$|\nabla v_h|\geq\frac{v_h(N_i)}{\text{dist }(N_i,\Gamma_0)}.$$

(Since we do not assume that $\Omega$ is convex, dist $(N_i, \Gamma_0)$ is here the length of the shortest path linking $N_i$ to $\Gamma_0$ and passing through the different nodes).

Hence if we denote by $K$ such a simplex

$$\int_\Omega \varphi(\nabla v_h(x))\, dx \geq |K| \varphi(\nabla v_{h|_K})$$

where $|K|$ denotes the measure of $K$ and (2.24) follows by (2.22).

## 3. Oscillation phenomenon

In order to obtain a unique pattern for the minimizing sequences of (1.5) we have to limit the number of wells and assume that $\varphi$ has at most $n$ wells i.e.:

$$2 \leq k \leq n. \tag{3.1}$$

Recall that $a = 0$ belongs to the convex hull of the $w_i$'s. It will be useful to assume:

$$w_i - w_1, \quad i = 2, \ldots, k \text{ are linearly independent} \tag{3.2}$$

and for some $\alpha_i \in (0, 1)$ one has

$$\sum_{i=1}^{k} \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^{k} \alpha_i = 1. \tag{3.3}$$

It results from (3.1) and the fact that $a$ belongs to the convex hull of the $w_i$'s that the $w_i$'s span a subspace $W$ which is of dimension strictly less than $n$. This fact, as we will see later, is crucial. Let us denote by $W^\perp$ the orthogonal complement of $W$. We will assume that

$$\forall\, x \in \Omega, \exists\, x_0 \in \Gamma_0 \quad \text{such that } (x, x_0) \subset \Omega \text{ and } \; x - x_0 \in W^\perp \tag{3.4}$$

($(x, x_0)$ denotes the segment with end points $x$ and $x_0$. Note that this assumption is automatically satisfied when $\Gamma_0 = \partial\Omega$).

Under the above assumptions we can prove

**Theorem 3:** *Let us assume that $\varphi$ is bounded on bounded subsets of $\mathbf{R}^n$. Let $V_0 \subset W^{1,1}(\Omega)$ be a space of functions vanishing on $\Gamma_0$, containing the continuous piecewise affine functions. If (3.1), (3.4) hold then the problem*

$$\mathop{Inf}_{v \in V_0} \int_\Omega \varphi(\nabla v(x))\, dx \tag{3.5}$$

*cannot have a minimizer.*

**Remark 5:** $W^{1,1}(\Omega)$ is the usual Sobolev space of functions in $L^1(\Omega)$ with first derivatives in $L^1(\Omega)$–(see [G.T]). In the integral (3.5) we make the convention that

$$\int_\Omega \varphi(\nabla v(x))\, dx = +\infty \tag{3.6}$$

if $\varphi(\nabla v(x)) \notin L^1(\Omega)$. Recall that $\varphi$ is a nonnegative function and this convention avoids imposing some growth conditions on $\varphi$.

**Proof of Theorem 3:** Let $\nu_1, \ldots, \nu_\ell$ an orthonormal basis of $W^\perp$. Due to (3.4) we have a Poincaré inequality of the type

$$\int_\Omega |v(x)|\, dx \le C \int_\Omega |\nabla_\nu v(x)|\, dx \quad \forall v \in V_0 \tag{3.7}$$

($\nabla_\nu$ denotes the vector of components $\frac{\partial}{\partial \nu_i}$, $|\quad|$ the Euclidean norm).

By Theorem 1 one has

$$\mathop{\mathrm{Inf}}_{v \in V_0} \int_\Omega \varphi(\nabla v(x))\, dx = 0.$$

If for some $u \in V_0$

$$\int_\Omega \varphi(\nabla u(x))\, dx = 0$$

then by (1.1), (1.2) one has:

$$\nabla u(x) = w_i \ne 0 \quad \text{a.e in } \Omega \tag{3.8}$$

hence since $\frac{\partial u}{\partial \nu_j} = \nabla u \cdot \nu_j$

$$\nabla_\nu u(x) = 0 \quad \text{a.e in } \Omega.$$

Then from (3.7) applied to $u$ one deduces

$$u(x) = 0 \quad \text{a.e in } \Omega. \tag{3.9}$$

But then $\nabla u = 0$ and a contradiction with (3.8).

So, in the absence of minimizers we turn to the study of minimizing sequences. As we will see, they have a common pattern which is explained by the uniqueness of the Young measure associated to them. More precisely, recall that if $u_h \in W^{1,\infty}(\Omega)$,

$$|u_h|_\infty, \quad ||\nabla u_h||_\infty \leq C \tag{3.10}$$

where $C$ is some constant independent on $h$, then $u_h$ defines a Young measure in the sense that there exists a subsequence of $u_h$, that for simplicity we will still denote by $u_h$, a parametrized probability measure $\nu_x$ on $\mathbf{R}^n$, such that for any bounded Carathéodory function $F(x,\xi)$ one has

$$\lim_{h\to 0} \int_\Omega F(x, \nabla u_h(x))\, dx = \int_\Omega \int_{\mathbf{R}^n} F(x,\lambda)\, d\nu_x(\lambda) dx. \tag{3.11}$$

(We refer for instance to [B.], [C.K.], [K.], [K.P.$_1$], [K.P.$_2$], [D.], [Ev.], [T.], [V.] for this question, recall that $| \quad |_\infty$ denotes the usual $L^\infty$ norm-see [G.T.] for more information on Sobolev spaces).

Then we can prove:

**Theorem 4:** *Assume that (3.1)-(3.4) hold, $\varphi$ continuous. Let $u_h$ be a minimizing sequence of (3.5). If $u_h$ defines a Young measure $\nu_x$ -i.e. if (3.10) holds- then one has necessarily*

$$\nu_x = \sum_{i=1}^{k} \alpha_i \delta_{w_i} \quad a.e\ x \in \Omega \tag{3.12}$$

*where the $\alpha_i$'s are the constants appearing in (3.3) and $\delta_{w_i}$ denotes the Dirac mass at $w_i$.*

**Proof:** Up to a subsequence one can assume that one has for some $u \in W^{1,\infty}(\Omega)$

$$u_h \to u \quad \text{uniformly in } \Omega \tag{3.13}$$

$$\nabla u_h \rightharpoonup \nabla u \quad \text{in } L^\infty(\Omega) \ \text{ weak} - *. \tag{3.14}$$

((3.14) means that each component of the gradient is converging in $L^\infty(\Omega)$ weak $- *$). Let $\nu_x$ be a Young measure defined by $u_h$.

From (3.11) applied with $F(x,\xi) = \varphi(\xi)$ we have

$$0 = \lim_{h\to 0} \int_\Omega \varphi(\nabla u_h(x))\, dx = \int_\Omega \int_{\mathbf{R}^n} \varphi(\lambda)\, d\nu_x(\lambda) dx.$$

It results that

$$\int_{\mathbf{R}^n} \varphi(\lambda)\, d\nu_x(\lambda) = 0 \quad \text{a.e in } \Omega.$$

Hence, by (1.1), (1.2) the support of $\nu_x$ has to be contained in the $w_i$'s for a.e $x$, i.e.

$$\nu_x = \sum_{i=1}^k \beta_i(x) \delta_{w_i}. \tag{3.15}$$

Applying (3.11) again we have from (3.15)

$$\int_\Omega |\nabla_\nu u_h(x)|\, dx = \int_\Omega [\sum_{j=1}^{\ell} (\nabla u_h(x) \cdot \nu_j)^2]^{1/2}\, dx \to \int_\Omega \int_{\mathbf{R}^n} [\sum_{j=1}^{\ell} (\lambda \cdot \nu_j)^2]^{1/2}\, d\nu_x(\lambda) dx$$

$$= \int_\Omega \sum_{i=1}^k \beta^i(x) [\sum_{j=1}^{\ell} (w_i \cdot \nu_j)^2]^{1/2}\, dx = 0$$

since the $\nu_j$'s belong to $W^\perp$.

Recalling (3.7) and (3.13) we obtain

$$\int_\Omega |u(x)|\, dx = \lim_{h\to 0} \int_\Omega |u_h(x)|\, dx = 0.$$

Hence $u = 0$.

Let $B$ be a ball included in $\Omega$.

Since the limit of any subsequence of $u_h$ is 0, (see (3.14), we have when $h \to 0$

$$\nabla u_h \rightharpoonup 0 \text{ in } L^\infty(\Omega) \ \text{ weak} - *.$$

From (3.11) applied with $F(x,\xi) = X_B(x)\xi$ (this function is not bounded in $\xi$, but the gradient of $u_h$ is!) we obtain

$$0 = \lim_{h \to 0} \int_B \nabla u_h(x)\, dx = \int_B \int_{\mathbf{R}^n} \lambda\, d\nu_x(\lambda) dx \tag{3.16}$$

By (3.15) this reads

$$0 = \sum_{i=1}^k \int_B \beta_i(x)\, dx \cdot w_i.$$

Hence, by (3.3) and the uniqueness of barycentric coordinates

$$\frac{1}{|B|} \int_B \beta_i(x)\, dx = \alpha_i \quad \forall\, i = 1, \ldots, k.$$

(Recall that since $\nu_x$ is a probability measure $\sum_{i=1}^k \beta_i(x) = 1$). Since this is true for any $B$ it results from the Lebesgue differentiation theorem that

$$\beta_i(x) = \alpha_i \quad \text{a.e in } \Omega, \quad \forall\, i = 1, \ldots, k$$

and thus (3.15) reads

$$\nu_x = \sum_{i=1}^k \alpha_i \delta_{w_i} \quad \text{a.e in } \Omega.$$

This shows that $\nu_x$ is necessarily the homogeneous Young measure (3.12) (see [K.P.$_1$]) and concludes the proof.

**Remark 6:** The construction of $u_h$ given in section 2 provides a minimizing sequence such that (3.10) holds.

To end this section we would like to discuss the necessity of our assumptions. For instance in theorem 3, the assumption (3.1) cannot be relaxed. Indeed consider the one dimensional case and assume for instance that $\varphi\colon \mathbf{R} \to \mathbf{R}$ satisfies

$$\varphi(-1) = \varphi(1) = 0 \quad , \quad \varphi(\xi) > 0 \quad \forall\, \xi \neq \pm 1. \tag{3.17}$$

Take $\Omega = (0,1)$, $\Gamma_0 = \partial\Omega$, $a = 0$. So, we are in the case $n = 1$, $k = 2$. Then clearly any piecewise affine function such that

$$u'(x) = \pm 1 \quad , \quad u(0) = u(1) = 0$$

satisfies

$$0 = \int_\Omega \varphi(u'(x))\, dx = \operatorname*{Inf}_{v \in V_0} \int_\Omega \varphi(v'(x))\, dx$$

where $V_0$ is any space containing the piecewise affine functions vanishing on $\partial\Omega$. So, the problem (3.5) admits infinitely many minimizers. The situation is exactly the same in higher dimensions. Consider for instance the case $n = 2$, $k = 3$. Assume *that* $\Omega$ is an equilateral triangle centered at 0 and of sides of length 1. Consider the "hat" function $u$ defined on this triangle i.e. such that

$$u(0) = 1, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

and interpolates linearly between these points. Let $w_1$, $w_2$, $w_3$ the different gradients of this function and $\varphi$ a function vanishing on these $w_i$'s and positive elsewhere and satisfying also the assumptions of theorems 3 and 4. Clearly $u$ is a minimizer of

$$\int_\Omega \varphi(\nabla v(x))\, dx$$

on $W_0^{1,1}(\Omega)$. So the problem (3.5) admits in this case a minimizer. Now, if we split $\Omega$ into equilateral triangles (not necessarily of the same size) and if on each of these triangles we define $u$ to be equal to 0 on the boundary and to the length side at the center, $u$ interpolating linearly between these points then $u$ is an other minimizer and we get that way infinitely many of them.

To see now the necessity of (3.4) we can argue for instance in dimension 2 with $\Omega = (-1,1)^2$, $\Gamma_0 = \{-1\} \times (-1,1)$, $w_1 = (1,0)$, $w_2 = -w_1$. Then any function depending on $x$ only, vanishing on $\Gamma_0$ and climbing or going down with a slope $\pm 1$ is a minimizer.

A similar example can provide an understanding of the necessity of (3.2), (3.3) in theorem 4. Indeed assume

$$\Omega = (-1,1)^2 \quad , \quad \Gamma_0 = \partial\Omega \quad , \quad w_1 = (1,0) \quad , \quad w_2 = -w_1 \quad , \quad w_3 = -2w_1$$

then

$$0 = \frac{1}{2}w_1 + \frac{1}{2}w_2 \quad \text{or} \quad 0 = \frac{2}{3}w_1 + \frac{1}{3}w_3$$

and

$$\nu_x = \frac{1}{2}\delta_{w_1} + \frac{1}{2}\delta_{w_2} \quad \text{or} \quad \nu_x = \frac{2}{3}\delta_{w_2} + \frac{1}{3}\delta_{w_3}$$

are two admissible Young measures for the problem (3.5) and indeed (see the proof of Theorem 1 for some insight on this question) one can construct minimizing sequences corresponding to both of them.

## 4. Probabilistic analysis of oscillations

In this section we would like to analyse the behaviour of the minimizing sequences of (1.5). As we will see, under the assumptions of theorem 4, they have a common pattern. In particular in order to minimize the energy they are choosing their gradients around each of the wells with a probability which tends to be constant. We will give an estimate of this probability in terms of $h$ when the minimizing sequence considered is in $V_h^0$. First let us make precise our assumptions.

In all this section we will assume that (3.1)–(3.4) hold. Moreover, we denote by $\pi$ the projection on the $w_i$'s (see [C.K.L.]) i.e. $\pi$ denotes the function $\pi$ defined by

$$\pi(\xi) = w_i$$

$$\text{where } w_i \text{ is the well of smallest index i such that} \tag{4.1}$$

$$|\xi - w_i| = \operatorname*{Min}_j |\xi - w_j|$$

($|\quad|$ denotes the Euclidean norm of $\mathbf{R}^n$). The function $\pi$ takes only a finite number of values -i.e. the $w_k$'s. Moreover, it is clearly a Borel function so that if $\xi(x)$ is any measurable function, $\pi(\xi(x))$ will be measurable.

Then, we will assume that there exists $\wedge > 0$, $p \geq 1$ such that

$$\begin{aligned} \varphi(\xi) &\geq \wedge \cdot |\xi - \pi(\xi)|^p \\ &= \wedge \cdot \operatorname*{Min}_i |\xi - w_i|^p \quad \forall\, \xi \in \mathbf{R}^n. \end{aligned} \tag{4.2}$$

Now, let us denote by $R$ some positive number such that

$$R < \frac{1}{2}|w_i - w_j| \quad \forall\, i, j = 1, \ldots, k,\ i \neq j. \tag{4.3}$$

If $v \in W^{1,\infty}(\Omega)$ and if $B$ is some subset of $\Omega$ we will denote by $B_i^R$ or $B_i^R(v)$ (we will drop sometimes for convenience the dependence in $v$) the set

$$B_i^R = B_i^R(v) = \{x \in B \mid \nabla v(x) \in B(w_i, R)\} \tag{4.4}$$

where $B(w_i, R)$ is the ball of center $w_i$ and radius $R$ in $\mathbf{R}^n$ –i.e.

$$B(w_i, R) = \{w \in \mathbf{R}^n \mid |w - w_i| < R\}.$$

If $|\quad|$ denotes the Lebesgue measure in $\mathbf{R}^n$, $|B_i^R|/|B|$ represents the probability for $v$ to have its gradient on $B$ in $B(w_i, R)$. For convenience we introduce also the notation

$$B_{ex}^R = B \setminus \bigcup_{i=1}^{k} B_i^R. \tag{4.5}$$

We are first going to assume that $B$ is a Lipschitz domain for which the inequality

$$\int_{\partial B} |v(x)|\, d\sigma(x) \le C \int_B |v| + |\nabla_\nu v(x)|\, dx \quad \forall\, v \in W^{1,1}(\Omega) \tag{4.6}$$

holds (see (3.7) for the meaning of $\nabla_\nu$). It is easy to see that (4.6) holds for instance for a Lipschitz domain of $\Omega$ such that the outward normal $n(x)$ satisfies: there exists $i \in 1, \dots, \ell$ and a constant $c$ such that

$$|n(x) \cdot \nu_i| > c \quad \sigma - \text{a.e} \quad x \in \partial B. \tag{4.7}$$

Recall that $\nu_1, \dots, \nu_\ell$ denotes an orthonormal basis of $W^\perp$.

We will set

$$E(v) = \int_\Omega \varphi(\nabla v(x))\, dx. \tag{4.8}$$

Then we can prove:

**Lemma 1:** *Assume that (3.1)-(3.4) hold. Moreover assume that $\varphi$ satisfies (4.2). If $B$ is a Lipschitz domain such that (4.6) holds, then one has for some constant $C$*

$$\left| \sum_{i=1}^{k} |B_i^R(v)| w_i \right| \le C \cdot E(v)^{\frac{1}{p}} \tag{4.9}$$

*for any $v \in W^{1,p}(\Omega)$, $v$ vanishing on $\Gamma_0$.*

*(Recall that $|\quad|$ denotes the Lebesgue measure or the usual Euclidean norm in $\mathbf{R}^n$)*

**Proof:** Note first that

$$\begin{aligned} \sum_{i=1}^{k} |B_i^R| w_i &= \int_{\underset{i}{\cup} B_i^R} \pi(\nabla v(x))\, dx \\ &= \int_B \pi(\nabla v(x))\, dx - \int_{B_{ex}^R} \pi(\nabla v(x))\, dx \\ &= \int_B \pi(\nabla v(x)) - \nabla v(x)\, dx + \int_B \nabla v(x)\, dx - \int_{B_{ex}^R} \pi(\nabla v(x))\, dx \\ &\overset{\text{def}}{=} I_1 + I_2 + I_3. \end{aligned} \tag{4.10}$$

To estimate $I_1$ note that by Hölder's inequality and (4.2)

$$\begin{aligned} |I_1| &\le \int_B |\pi(\nabla v(x)) - \nabla v(x)|\, dx \\ &\le \left(\int_B |\pi(\nabla v(x)) - \nabla v(x)|^p\, dx\right)^{\frac{1}{p}} |B|^{1-\frac{1}{p}} \\ &\le \Lambda^{-\frac{1}{p}} |B|^{1-\frac{1}{p}} E(v)^{\frac{1}{p}}. \end{aligned} \tag{4.11}$$

To estimate $I_2$ one applies the divergence theorem to get

$$I_2 = \int_B \nabla v(x)\, dx = \int_{\partial B} v(x) n(x)\, d\sigma(x)$$

where $n(x)$ denotes the outward normal to $\partial B$. Hence, by (4.6), (3.7):

$$\begin{aligned} |I_2| &\le \int_{\partial B} |v(x)|\, d\sigma(x) \\ &\le \int_B |v(x)| + |\nabla_\nu v(x)|\, dx \\ &\le \int_\Omega |v(x)| + |\nabla_\nu v(x)|\, dx \\ &\le C \int_\Omega |\nabla_\nu v(x)|\, dx \end{aligned} \tag{4.12}$$

On the other hand from (4.2) we deduce

$$\varphi(\xi) \ge \Lambda \cdot \operatorname*{Min}_i |\xi - w_i|^p \ge \Lambda \cdot |\xi - P_W(\xi)|^p$$

where $P_W$ denotes the orthogonal projection on $W$. Due to the equality

$$\xi = P_W(\xi) + P_{W^\perp}(\xi)$$

one deduces

$$\varphi(\xi) \ge \Lambda \cdot |P_{W^\perp}(\xi)|^p \quad \forall \xi \in \mathbf{R}^n. \tag{4.13}$$

Since

$$P_{W^\perp}(\nabla v(x)) = \sum_{i=1}^{\ell} (\nabla v(x) \cdot \nu_i) \cdot \nu_i$$

we obtain for almost every $x$

$$\varphi(\nabla v(x)) \geq \wedge \cdot |\nabla_\nu v(x)|^p$$

and integrating (see (3.6)) we obtain

$$\int_\Omega \varphi(\nabla v(x))\, dx \geq \wedge \cdot \int_\Omega |\nabla_\nu v(x)|^p\, dx \quad \forall\, v \in V_0. \tag{4.14}$$

Hence by the Hölder Inequality and (4.12)

$$\begin{aligned} |I_2| &\leq C|\Omega|^{1-\frac{1}{p}} \left( \int_\Omega |\nabla_\nu v(x)|^p\, dx \right)^{\frac{1}{p}} \\ &\leq C \wedge^{-\frac{1}{p}} |\Omega|^{1-\frac{1}{p}} E(v)^{\frac{1}{p}}. \end{aligned} \tag{4.15}$$

To estimate $I_3$, note that

$$\begin{aligned} |I_3| &\leq \int_{B^R_{ex}} |\pi(\nabla v(x))|\, dx \\ &\leq \operatorname*{Max}_i |w_i|\ |B^R_{ex}|. \end{aligned}$$

Now, clearly

$$\begin{aligned} R|B^R_{ex}| &\leq \int_{B^R_{ex}} |\pi(\nabla v(x)) - \nabla v(x)|\, dx \\ &\leq \int_B |\pi(\nabla v(x)) - \nabla v(x)|\, dx. \end{aligned}$$

Hence, using Hölder's inequality

$$|B^R_{ex}| \leq \frac{1}{R} |B|^{1-\frac{1}{p}} \wedge^{-\frac{1}{p}} E(v)^{\frac{1}{p}} \tag{4.16}$$

and thus

$$|I_3| \leq \operatorname*{Max}_i |w_i| \frac{1}{R} |B|^{1-\frac{1}{p}} \wedge^{-\frac{1}{p}} E(v)^{\frac{1}{p}} \tag{4.17}$$

Combining (4.10), (4.11), (4.15), (4.17) we obtain clearly

$$\left| \sum_{i=1}^k |B^R_i(v)| w_i \right| \leq C E(v)^{\frac{1}{p}}$$

which is (4.9).

As a consequence we have:

**Theorem 5:** *Assume that (3.1)-(3.4) hold. Moreover assume that $\varphi$ satisfies (4.2). If $B$ is a Lipschitz domain such that (4.6) holds then for some constant $C$*

$$\left|\alpha_i - \frac{|B_i^R(v)|}{|B|}\right| \leq C \cdot E(v)^{\frac{1}{p}} \quad \forall\, i = 1, \ldots, k \tag{4.18}$$

*for any $v \in W^{1,p}(\Omega)$, $v$ vanishing on $\Gamma_0$.*

**Proof:** Let $M$ denote the $n+1 \times k$ matrix

$$M = \begin{bmatrix} w_1 & \ldots & w_k \\ 1 & \ldots & 1 \end{bmatrix}. \tag{4.19}$$

Since the vectors $w_i - w_1$ are linearly independent this matrix has rank $k$. In particular, the system

$$My = b \tag{4.20}$$

has at most one solution which, when it exists, is given by

$$y = (M^T M)^{-1} M^T b$$

($M^T$ denotes the transpose of $M$) and one has

$$|y| \leq \|(M^T M)^{-1} M^T\|\, |b| \tag{4.21}$$

where $\|\quad\|$ denotes the matrix norm corresponding to the Euclidean norm.

Since

$$\sum_{i=1}^{k} \alpha_i w_i = 0 \quad , \quad \sum_{i=1}^{k} \alpha_i = 1$$

one has

$$\begin{aligned} \sum_{i=1}^{k} (\alpha_i |B| - |B_i^R|) w_i &= -\sum_{i=1}^{k} |B_i^R| w_i \\ \sum_{i=1}^{k} (\alpha_i |B| - |B_i^R|) &= |B_{ex}^R|. \end{aligned} \tag{4.22}$$

Hence, the vector with entries $\alpha_i|B| - |B_i^R|$ satisfies (4.20) for $b$ given by the right hand side of (4.22). The result follows then by combining (4.21), (4.9) and (4.16).

As a consequence of this result we see that for any minimizing sequence $v_h$ of (3.5) one has

$$\frac{|B_i^R(v_h)|}{|B|} \to \alpha_i \quad \forall\, i = 1, \ldots, k$$

at a rate proportional to $E(v_h)^{1/p}$. In particular if one sets

$$E_h = \operatorname*{Inf}_{v \in V_h^0} \varphi(\nabla v(x))\, dx \tag{4.23}$$

one has:

**COROLLARY 1:** *Under the assumption of Theorem 5, if $B$ is a Lipschitz domain such that (4.6) holds and if $v_h \in V_h^0$ is such that*

$$E(v_h) \leq 2E_h$$

*then there exists a constant $C$ such that*

$$\left| \alpha_i - \frac{|B_i^R(v_h)|}{|B|} \right| \leq C h^{\frac{1}{2p}} \quad \forall\, 1, \ldots, k. \tag{4.24}$$

**Proof:** This results from Theorem 5 and (2.3).

**Remark 7:** Note that this estimate holds in particular for any minimizer $u_h \in V_h^0$ of (2.23). In the case where some low growth condition on $\varphi$ is assumed then the rate $h^{\frac{1}{2p}}$ can be improved (see Remark 3).

We now assume that $B$ is any Lipschitz domain of $\Omega$ and we use the same notation than before in particular for $B_i^R$. Then we have:

**Lemma 2:** *Assume that (3.1)-(3.4) hold. Moreover assume that $\varphi$ satisfies (4.2) with $p > 1$. If $B$ is a Lipschitz domain included in $\Omega$ then there exists a constant $C$ such that:*

$$|\sum_{i=1}^{k} |B_i^R(v)|\; w_i| \leq C \cdot (E(v)^{\frac{1}{p}} + E(v)^{\frac{1}{pq}} \cdot \|v\|_{1,p,B}^{\frac{1}{p}}) \tag{4.25}$$

*for any* $v \in W^{1,p}(\Omega)$, $v$ *vanishing on* $\Gamma_0$.

*(q denotes the conjugate exponent of* $p$, $\|v\|_{1,p,B} = (\int_B |v(x)|^p + |\nabla v(x)|^p\,dx)^{\frac{1}{p}}$).

**Proof:** One proceeds as in Lemma 1 –i.e. one writes (4.10). There is no change in the estimate of $I_1$, $I_3$, and (4.11) and (4.17) hold. To estimate $I_2$ one notices as before that

$$|I_2| = \left| \int_{\partial B} v(x) n(x)\, d\sigma(x) \right| \leq \int_{\partial B} |v(x)|\, d\sigma(x). \tag{4.26}$$

Now, for any function in $W^{1,1}(B)$ one has for some constant $C$

$$\int_{\partial B} |v(x)|\, d\sigma(x) \leq C \int_B |v(x)| + |\nabla v(x)|\, dx.$$

Applying this inequality to $v = v^p$ we get

$$\int_{\partial B} |v(x)|^p\, d\sigma(x) \leq C \int_B |v|^p + (p-1)|v|^{p-1}|\nabla v|\, dx. \tag{4.27}$$

Now, by Hölder's inequality

$$\int_B |v|^{p-1}|\nabla v|\, dx \leq \left( \int_B |v|^p\, dx \right)^{\frac{1}{q}} \left( \int_B |\nabla v|^p\, dx \right)^{\frac{1}{p}}$$

where $q$ is the conjugate exponent of $p$ –i.e. $\frac{1}{p} + \frac{1}{q} = 1$. From (4.27) we deduce

$$\int_{\partial B} |v(x)|^p\, d\sigma(x) \leq C \left( \int_B |v|^p\, dx \right)^{\frac{1}{q}} \cdot \left( \int_B |v|^p + |\nabla v|^p\, dx \right)^{\frac{1}{p}}$$

for some constant $C$.

Recalling (4.26) and using Hölder's ineqality we obtain

$$|I_2| \leq \int_{\partial B} |v(x)|\, d\sigma(x) \leq |\partial B|^{1-\frac{1}{p}} \left( \int_{\partial B} |v(x)|^p\, d\sigma(x) \right)^{\frac{1}{p}}$$

($|\partial B|$ denotes the $n-1$ dimensional Hausdorff measure of $\partial B$). Hence for some positive constant $C$

$$|I_2| \leq C \left( \int_B |v|^p\, dx \right)^{\frac{1}{pq}} \|v\|_{1,p,B}^{\frac{1}{p}}. \tag{4.28}$$

Now, due to (3.4) we have a Poincaré Inequality of the type

$$\int_\Omega |v|^p \, dx \le C \int_\Omega |\nabla_\nu v(x)|^p \, dx$$

for any $v \in W^{1,p}(\Omega)$, $v$ vanishing on $\Gamma_0$.

Hence by (4.14)

$$\int_B |v|^p \, dx \le \int_\Omega |v|^p \, dx \le C \cdot \int_\Omega \varphi(\nabla v(x)) \, dx$$

and (4.23) becomes

$$|I_2| \le C \cdot E(v)^{\frac{1}{pq}} \|v\|_{1,p,B}^{\frac{1}{p}}.$$

One concludes then as in lemma 1.

As a conssequence we can prove

**THEOREM 6:** *Assume that (3.1)-(3.4) hold. Moreover assume that $\varphi$ satisfies (4.2) with $p > 1$. Then if $B$ is a Lipschitz domain included in $\Omega$ one has for some constant $C$*

$$\left| \alpha_i - \frac{|B_i^R(v)|}{|B|} \right| \le C \cdot (E(v)^{\frac{1}{pq}} + E(v)^{\frac{1}{p}}) \quad \forall\, i = 1, \dots, k \tag{4.29}$$

*for any $v \in W^{1,p}(\Omega)$, $v$ vanishing on $\Gamma_0$.*

**Proof:** Denote by $| \;\; |_{1,p}$ the usual norm on $W^{1,p}(\Omega)$. One argues exactly as in Theorem 5 but one replaces now (4.9) by (4.25). Noting first that

$$|v|_{1,p,B} \le |v|_{1,p}$$

one obtains

$$|\sum_{i=1}^k |B_i^R(v)| \; w_i| \le C(E(v)^{\frac{1}{p}} + |v|_{1,p}^{\frac{1}{p}} E(v)^{\frac{1}{pq}}) \tag{4.30}$$

Now, for any $v \in W^{1,p}$, vanishing on $\Gamma_0$ one has

$$\int_\Omega |v|^p \, dx \le C \int_\Omega |\nabla v|^p \, dx$$

and thus

$$
\begin{aligned}
|v|_{1,p} &\le C\left(\int_\Omega |\nabla v(x)|^p\,dx\right)^{\frac{1}{p}} \\
&\le C\left(\int_\Omega |\nabla v - \pi(\nabla v)|^p\,dx\right)^{\frac{1}{p}} + C\left(\int_\Omega |\pi(\nabla v)|^p\,dx\right)^{\frac{1}{p}} \\
&\le C_1 E(v)^{\frac{1}{p}} + C_2
\end{aligned}
$$

where $C_1$, and $C_2$ are constants independent of $v$. (Recall that $|\pi(\nabla v)| \le \mathrm{Max}_i\,|w_i|$).

Thus

$$|v|_{1,p}^{\frac{1}{p}} \le C_1 E(v)^{\frac{1}{p^2}} + C_2$$

for some other constants $C_1$, $C_2$ and (4.30) becomes

$$|\sum_{i=1}^{k} |B_i^R(v)|\; w_i| \le C(E(v)^{\frac{1}{pq}} + E(v)^{\frac{1}{p}} + E(v)^{\frac{1}{p^2}+\frac{1}{pq}})$$

Hence (4.29) follows, since $\frac{1}{p} + \frac{1}{q} = 1$.

As a consequence we see that any minimizing sequence $v_h$ of (3.5) is such that

$$\frac{|B_i^R(v_h)|}{|B|} \to \alpha_i \quad \forall\, i, \ldots, k$$

at a rate proportional to $E(v_h)^{\frac{1}{pq}}$ ($\frac{1}{pq} < \frac{1}{p}$).

In particular one can show

**COROLLARY 2:** *Under the assumptions of Theorem 6, if $B$ is a Lipschitz domain included in $\Omega$ and if $v_h \in V_h^0$ is such that*

$$E(v_h) \le 2E_h$$

*then there exists a constant $C$ such that*

$$\left|\alpha_i - \frac{|B_i^R(v_h)|}{|B|}\right| \le Ch^{\frac{1}{2pq}} \quad \forall\, i = 1, \ldots, k. \tag{4.31}$$

**Proof:** This is an immediate consequence of (2.3) and Theorem 6. Of course we assume here that $h$ is small – i.e. $h < 1$.

**Remark 8:** This estimate holds in particular for any minimizer $u_h \in V_h^0$ of (2.23). See also remark 7.

**Remark 9:** To see how minimizing sequences have their gradients that approach the Young measure $\nu_x$ defined by (3.12) one could evaluate both against some function $F(x,.)$ and estimate the difference. We refer the reader to [C.C.] for this kind of results.

## 5. Numerical results and concluding remarks

Let us assume in this section that $\Omega$ is the square $(-1,1)^2$, $\Gamma_0 = \partial\Omega$, and consider the case of two wells

$$w_1 = (-1,1) \quad , \quad w_2 = (1,-1).$$

Our results apply since $n = 2$. Moreover set

$$\varphi(\xi) = |\xi - w_1|^2 \; |\xi - w_2|^2.$$

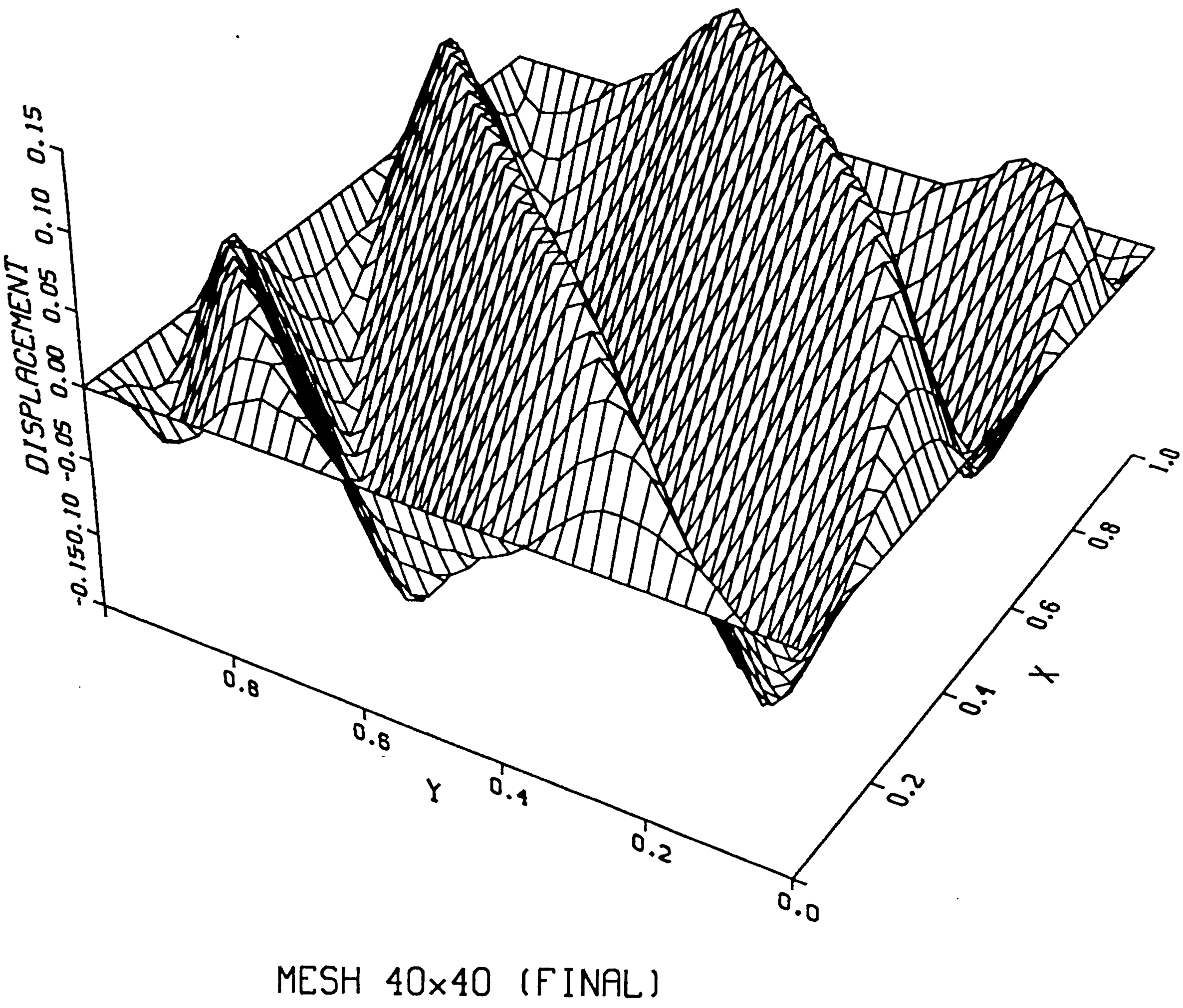
It is not difficult to check that these data fit our previous analysis. Since

$$0 = \frac{1}{2}w_1 + \frac{1}{2}w_2$$

the minimizing sequences of (1.5) are choosing their gradient alternatively around $w_1$ and $w_2$ with a probability which tends to $\frac{1}{2}$ (see Theorem 6 and Corollary 2). This can be observed on the figure below.

In fact, in order to minimize the energy, the computer uses almost exactly the technique developed in Theorem 1. Indeed, with two wells the function $u_h$ built in Theorem 1 looks as the one produced by the computer. Of course, the oscillations of the minimizing sequences are stopped at the level of the grid.

Our numerical codes use standard iteration algorithms. We start with a random distribution of initial values of the function close to 0 and we decrease the energy using a conjugate gradient method. An example of the final result is ploted on the following figure.




Computational Result for $h = 1/40$.

As we mentioned earlier the most interesting examples of applications of our results occur in the study of deformations of ordered materials such as crystals. In this case the potential wells are matrices. Moreover, due to the frame indifference, these wells are infinite in number. Also deformations have to satisfy the compatiblity condition which makes the problem more involved. For some results in this direction and complements we refer the reader to [C.C.K.] and [Co.].

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