# GEOMETRIC EVOLUTION OF PHASE-BOUNDARIES

By

Yoshikazu Giga

and

Shun'ichi Goto

# GEOMETRIC EVOLUTION OF PHASE-BOUNDARIES

*In the memory of Professor Kôsaku Yosida*

YOSHIKAZU GIGA* AND SHUN'ICHI GOTO**

**1. Introduction.** This paper continues our study [CGG], [GG] of a motion of phase-boundaries whose speed locally depends on the normal vector field and curvature tensors.

Material science provides a lot of examples of such a motion. Let $D_t$ denote a bounded open set in $\mathbb{R}^n$ where one phase, say, solid of material occupies at time $t$. Another phase, say, liquid occupies outside $D_t$ and two phases are bounded by an interface $\Gamma_t$ called a phase boundary. We are interested in evolution of the phase boundary $\Gamma_t$. To write down the equation for $\Gamma_t$ we temporary assume that $\Gamma_t$ is smooth hypersurface and $\Gamma_t$ equals $\partial D_t$, the boundary of $D_t$. Let $\mathbf{n}$ denote the unit exterior normal vector field of $\Gamma_t = \partial D_t$. Let $V = V(t, x)$ denote the speed of $\Gamma_t$ at $x \in \Gamma_t$ in the exterior normal direction. The equation for $\Gamma_t$ we consider here is of form

$$(1.1) \qquad V = f(t, \mathbf{n}(x), \nabla \mathbf{n}(x)) \text{ on } \Gamma_t,$$

where $f$ is a given function and $-\nabla \mathbf{n}$ is essentially the curvature tensor. For later convenience we extend $\mathbf{n}$ to a vector field (still denoted by $\mathbf{n}$) on a tubular neighborhood of $\Gamma_t$ such that $\mathbf{n}$ is constant in the normal direction of $\Gamma_t$ and $\nabla$ stands for spatial derivatives in $\mathbb{R}^n$. An interesting example is

$$(1.2) \qquad V = -\frac{1}{\beta(\mathbf{n})}\left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\frac{\partial H}{\partial p_i}(\mathbf{n}) + c(t)\right)$$

which is referred as evolution of an isothermal interface [Gu1], [Gu2], [AG]. Physically speaking $H \geq 0$ represents the interfacial energy and defined on the unit sphere $S^{n-1}$; we extend $H$ to $\mathbb{R}^n$ such that $H(\lambda p) = \lambda H(p)$, $\lambda > 0$. The function $c(t)$ represents the energy-difference between bulk phases and $\beta(\mathbf{n}) > 0$ is a kinetic constant which measures the drag opposing interfacial motion. When $H(p) = |p|, \beta(\mathbf{n}) = 1$, $c(t) = 0$ the equation (1.2) becomes

$$(1.3) \qquad V = -\operatorname{div} \mathbf{n}$$

*Department of Mathematics, Hokkaido University, Sapporo 060, JAPAN and IMA, University of Minnesota.
**Department of Applied Science, Faculty of Engineering 36, Kyushu University, Fukuoka 812, JAPAN.

which is often referred as the mean curvature flow equation. The equation (1.2) is interpreted as an anisotropic version of (1.3) (with driving force $c$).

A fundamental analytic question is to construct a solution of (1.1) for arbitrary initial data. Since solution $\Gamma_t$ may develop singularities in a finite time even for (1.3) with smooth initial data [Gr], we should introduce a notion of weak solutions to track the whole evolution of $\Gamma_t$. The first attempt is done by Brakke [B] where he constructed a global varifold solution. However his solution may not be unique. Recently alternative weak formulation of solutions are introduced. The main idea is to interpret $\Gamma_t$ as a level set of a function $u$. This idea goes back to [Se] for $n = 2$ and extended by Osher and Sethian [OS] for numerical study of

$$(1.4) \qquad\qquad V = -\mathrm{div}\,\mathbf{n} + c \quad (c: \text{ constant }).$$

In [CGG] Y.-G. Chen and the authors introduced a weak notion $\Gamma_t$ of (1.1) through viscosity solutions $u$ in $(0, \infty) \times \mathbb{R}^n$ of the equation induced by (1.1). We constructed a *unique* global weak solution $\{\Gamma_t\}_{t \geq 0}$ with arbitrary initial data for a certain class of (1.1) including (1.2) - (1.4) as a special examples (where $H$ is $C^2$ outside origin and convex, and $\beta$ is continuous). In [GG] we clarify the class of (1.1) which our theory apply. Roughly speaking, our theory applies to (1.1) provided that the equation is degenerate parabolic and that $f$ grows linearly in $\nabla \mathbf{n}$ (see [GG]). Note that this includes the case when $H$ is convex not necessarily strictly convex for (1.2). The case when $f$ depends on $x$ will be discussed in [GGI] and [G] using comparison results in [GGIS]. Almost the same time as [CGG], Evans and Spruck [ES1] constructed the same unique solution but only for (1.3). Their method of construction is approximation while ours are Perron's method which applies to more general equation than (1.3). Very recently based on results in [CGG] Soner [S] recasted the definition of solutions and obtained the asymptotic behavior of weak solutions to (1.2) when $c < 0$ where $c$ is independent of $t$. After we completed this work, we learned that analysis in [CGG] and [ES1] for (1.3) is extended by Ilmanen [I] on manifolds.

In this paper we discuss relation between classical solutions and our weak solutions when the former exists. In [ES1] Evans and Spruck proved that their solutions agree with the classical solution up to the time the latter exists. In this paper we extend their results to general equations (1.1) even if the equation is degenerate parabolic (see §3). In the meanwhile we construct a local classical solution to (1.1) provided that the equation is uniformly parabolic by a level surface approach in [ES2] where they discussed (1.3) only. Our generalization clarify the key ingredient of this method (see §2). In §4 we prove a result which suggests that the cone like singularity is impossible to appear.

The bibliography of [CGG], [GG] includes many references related to the mean curvature flow equation (1.3). We take this opportunity to note some other, related articles not cited there and not mentioned elsewhere in this paper. The equation (1.3) is derived formally as a singular limit of the Allen-Cahn equation. If the motion is smooth, the convergence is proved by Bronsard and Kohn [BK] and de Mottoni and Schatzman [MS]

and the proof is simplified by [Ch] very recently. After we completed this work, we learned that Evans, Soner and Songanidis [ESS] proved the convergence even if $\Gamma_t$ has singularities (assuming that $\Gamma_t$ develops no interior). We also learned that Evans and Spruck [ES3] proved that $\partial\Gamma_t$ is countably $n-1$ rectifiable for almost all time $t$; however, little is known for the regularity of weak solution $\Gamma_t$ even for (1.3). For anisotropic version (1.2) less is known. S. Angenent discussed this topic when $n = 2$ but $H$ is not necessarily convex (see his article in this proceeding). We note that (1.2) is also derived as a formal limit of some Ginzburg-Landau equations [Ca]. Very recently Taylor [T] analyzed (1.2) by completely different method when $H$ is a crystalline energy (so is not $C^2$). This topic is discussed also in [AG].

**2. Local smooth solutions.** There seems to be a couple of ways to construct a unique local-in-time smooth solution $\{\Gamma_t\}$ of

(2.1)         (a)         $V = f(t, \mathbf{n}, \nabla\mathbf{n})$ on $\Gamma_t$,

              (b)         $\Gamma_{t|t=0} = \Gamma_0$,

where $\Gamma_0$ is a closed smooth hypersurface in $\mathbb{R}^n$ provided that the equation is uniformly parabolic. Usually one introduces coordinates to represent $\Gamma_t$ to get equations of mappings parametrizing $\Gamma_t$. However, there are a lot of freedom to choose coordinates so the equations become highly degenerate. Hence we are forced to use the Nash-Moser implicit function theorem to get a local solution (cf. [Ha]). However, if we fix the parametrization of $\Gamma_t$ one can apply usual and more familiar implicit function theorem to get local solutions. Indeed, if we express $\Gamma_t$ by "height" from $\Gamma_0$ we get a single uniform parabolic equations on $\Gamma_0$ for the height function. This approach is carried out by [B] (and [C]) for the mean curvature flow (plus a driving force) equation; see also [Hu] where the proof is implicit. Since the equation is considered on the manifold $\Gamma_0$, there needs a lot of notations from differential geometry to write down the proof, although this approach is theoretically simpler. The third alternative method is introduced by Evans and Spruck [ES2] to solve the mean curvature flow equation locally. Their idea is to construct a signed distance function of $\Gamma_t$ instead of $\Gamma_t$ itself. Our goal in this section is to extend their method to more general equations (2.1). Our generalization will clarify the core of the argument.

For (2.1) we only assume smoothness of $f$ and a uniform parabolicity to prove the local existence of smooth solutions which we should state below. Let $S^{n-1}$ denote an $n-1$ dimensional unit sphere in $\mathbb{R}^n$. Let $\mathsf{M}_n$ denote the space of $n \times n$ real matrices and $\mathsf{S}_n$ denote the space of $n \times n$ real symmetric matrices equipped with the usual ordering. We often use the following projection:

$$Q_{\overline{p}}(X) = R_{\overline{p}} X R_{\overline{p}}, \quad R_{\overline{p}} = I - \overline{p} \otimes \overline{p} \in \mathsf{S}_n, \quad \overline{p} \in S^{n-1}, \quad X \in \mathsf{M}_n,$$

where $I$ denotes the identity matrix (see [GG]). We now list our assumptions on $f$.

(2.2)                    $f : [0, T) \times S^{n-1} \times \mathsf{M}_n \to \mathbb{R}$ is smooth .

3

(2.3)          There is a constant $\theta, 0 < \theta < 1$ such that

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \{ f(t, -\overline{p}, -Q_{\overline{p}}(X + \varepsilon Y)) - f(t, -\overline{p}, -Q_{\overline{p}}(X)) \}$$

$$\geq \theta \text{ trace } Q_{\overline{p}}(Y) \quad \text{if } Y \geq 0$$

$$\text{for all } (t, \overline{p}, X) \in [0, T) \times S^{n-1} \times \mathbb{S}_n, Y \in \mathbb{S}_n.$$

The second assumption can be interpreted as a uniform ellipticity of $-f$.

THEOREM 2.1. *Assume (2.2) and (2.3). Let $\Gamma_0$ be a smooth hypersurface which is a boundary of a bounded domain $D_0$ in $\mathbb{R}^n$. There are a positive number $T_* < T$ and a unique smooth family $\{\Gamma_t\}_{0 \leq t < T_*}$ of smooth closed hypersurfaces which solves (2.1).*

We derive an equation for the signed distance function of $\Gamma_t$ satisfying (2.1a). From now on we assume that $f$ is independent of $t$ for simplicity since the proof can be trivially modified for $f$ depending on $t$. Suppose that the desired solution $\Gamma_t$ exists. Let $D_t$ be the bounded domain surrounded by $\Gamma_t$. If a function $u(t, x) > 0$ in $D_t$ and $u = 0$ on $\Gamma_t$, we see (2.1a) is equivalent to

(2.4)          $u_t + F(\nabla u, \nabla^2 u) = 0$ on $\Gamma_t$    $(u_t = \partial u / \partial t)$

with

(2.5)          $F(p, X) = -|p| f(-\overline{p}, -Q_{\overline{p}}(X)/|p|), \quad \overline{p} = p/|p|$

provided that the gradient $\nabla u$ does not vanish on $\Gamma_t$ (see [GG]). Here $\nabla^2 u$ denotes the Hessian matrix of $u$. If $u$ is a signed distance function of $\Gamma_t$ defined by

$$u(t, x) = \begin{cases} d(x, \Gamma_t), & x \in D_t \\ -d(x, \Gamma_t), & x \notin D_t, \end{cases}$$

then (2.4) is equivalent to

(2.6)          $u_t + \widetilde{F}(\nabla u, \nabla^2 u) = 0$    on $\Gamma_t$

with

(2.7)          $\widetilde{F}(p, X) = F(p, X) - \text{ trace } (X \overline{p} \otimes \overline{p}).$

Indeed, since $|\nabla u| = 1$ in a tubular neighborhood of $\Gamma_t$, differentiation yields

$$\nabla^2 u (\nabla u \otimes \nabla u) = 0$$

4

which shows that the last term in (2.7) is not affected for the signed distance function. There is an advantage of (2.6) over (2.4) since (2.6) is no longer degenerate in $\nabla u$ direction (see Proposition 2.6). A geometric consideration shows that

$$\nabla^2 d(x + r\mathbf{n}) = \nabla^2 d(x)(I + r\nabla^2 d(x))^{-1}, \quad x \in \Gamma_t, r : \text{ small },$$

where $d(x) = d(x, \Gamma_t)$. Therefore, for the signed distance function $u$ solving (2.6) on $\Gamma_t$ we have

$$u_t = G(u, \nabla u, \nabla^2 u)$$

in a *tubular neighborhood* of $\Gamma_t$ (not only on $\Gamma_t$) with

(2.8) $$G(r, p, X) = -\widetilde{F}(p, X(I - rX)^{-1}).$$

The above observation shows that Theorem 2.1 follows from the following Proposition 2.2 and Lemma 2.3.

PROPOSITION 2.2. *Suppose that $f$ and $\Gamma_0$ satisfy hypotheses of Theorem 2.1. Let $v_0$ be the signed distance function of $\Gamma_0$. For $\delta_0 > 0$ we set*

$$\Omega = \{x \in \mathbb{R}^n; |v_0(x)| < \delta_0\}, \quad Q_T = (0, T) \times \Omega.$$

*Then for sufficiently small $\delta_0$ there are $T_* > 0$ and a unique function $v \in C^\infty([0, T_*) \times \Omega) \times C^2(\overline{Q}_{T_*})$ which solves*

(2.9) $$v_t = G(v, \nabla v, \nabla^2 v) \text{ in } Q_{T_*}$$

(2.10) $$|\nabla v| = 1 \text{ on } (0, T_*) \times \partial\Omega$$

(2.11) $$v|_{t=0} = v_0 \text{ in } \Omega$$

*in the classical sense, where $G$ is defined by (2.8).*

LEMMA 2.3. *If $v$ solves (2.9) - (2.11) with $|\nabla v_0| = 1$ in $\Omega$, then $|\nabla v| = 1$ in $Q_{T_*}$.*

Proposition 2.2 is based on a standard theory [LUS] of parabolic equations and an iteration argument, and it is rather straight forward extension of [ES2]. We just sketch the proof in the last part of this section. Lemma 2.3 reflects an interesting structure of (2.9) so we give its proof. The key ingredient is a generalization of a calculus identify

$$\frac{\partial}{\partial r}\left(\frac{x}{1 - rx}\right) = \frac{x^2}{(1 - rx)^2} = x^2\frac{\partial}{\partial x}\left(\frac{x}{1 - rx}\right);$$

see the following two lemmas.

5

LEMMA 2.4. *For $r \in \mathbb{R}$ let $\Phi_r(X)$ be the Yosida approximation of $X \in \mathsf{M}_n$ i.e.*

$$\Phi_r(X) = X(I - rX)^{-1}.$$

*Then*

$$d\Phi_r(X)[Z] = (I - rX)^{-1}Z(I - rX)^{-1}, Z \in \mathsf{M}_n,$$

*where $d\Phi_r(X)$ denotes the Fréchet derivative of $\Phi_r : \mathsf{M}_n \to \mathsf{M}_n$ at $X$.*

*Proof.* The left hand side equals the directional derivative of $\Phi_r$ at $X$ in the direction of $Z$. This implies

$$d\Phi_r(X)[Z] = \lim_{h \to 0} \left\{ (X + hZ)(I - r(X + hZ))^{-1} - (I - rX)^{-1}X \right\} h^{-1}$$

$$= \lim_{h \to 0} (I - rX)^{-1} \left\{ (I - rX)(X + hZ) - X(I - r(X + hZ)) \right\} (I - r(X + hZ))^{-1} h^{-1}$$

$$= \lim_{h \to 0} (I - rX)^{-1} Z(I - r(X + hZ))^{-1} = (I - rX)^{-1}Z(I - rX)^{-1}. \quad \square$$

LEMMA 2.5. *(i) $d\Phi_r(X)[X^2] = (I - rX)^{-2}X^2 = \frac{\partial}{\partial r}\Phi_r(X).$*
*(ii) For $g : \mathsf{M}_n \to \mathbb{R}$ let $g_r(X)$ be*

$$g_r(X) = g \circ \Phi_r(X) = g(\Phi_r(X)).$$

*Then*

$$\frac{\partial}{\partial r}g_r(X) = dg_r(X)[X^2] = \sum_{1 \le i,j,k \le n} \frac{\partial g_r(X)}{\partial X_{ij}} X_{ik}X_{ij}, \qquad X = (X_{ij}) \in \mathsf{M}_n.$$

*Proof.* Lemma 2.4 and

$$\frac{\partial}{\partial r}\Phi_r(X) = (I - rX)^{-1}X^2$$

yield (i). To show (ii) we use the chain rule to get

$$(2.12) \qquad dg_r(X)[Z] = dg(X_r) \circ d\Phi_r(X)[Z], \quad X_r = \Phi_r(X)$$

$$= dg(X_r)[(I - rX)^{-1}Z(I - rX)^{-1}] \quad \text{(by Lemma 2.4)}.$$

Applying (i) yields

$$dg_r(X)[X^2] = dg(X_r) \left[ \frac{\partial}{\partial r}\Phi_r(X) \right] = \frac{\partial}{\partial r}g_r(X).$$

This proves (ii) since we have

$$dg_r(X)[X^2] = \sum_{i,j,k} \frac{\partial g_r(X)}{\partial X_{ij}} X_{ik}X_{kj}$$

by the definition of differentials. $\square$

We next show a uniformly ellipticity of $G = G(r, p, X)$ in (2.8) near $|p| = 1$ and $r = 0$. For $\delta > 0$, $K > 0$ we get

$$(2.13) \qquad U = U_{\delta K} = \left\{ (r, p, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathsf{S}_n; |r| < \delta, \frac{1}{2} < |p| < 2, |X| < K \right\},$$

where $|X|$ denotes the operator norm of the self-adjoint operator $X$.

6

PROPOSITION 2.6. *Assume (2.2) and (2.3) for $f$. Assume that $\delta K$ is sufficiently small, say $\delta K \leq 3/8$. Then there is a constant $c, 0 < c < 1$ such that*

$$c|\xi|^2 \leq \sum_{1 \leq i,j \leq n} \frac{\partial G}{\partial X_{ij}}(r,p,X)\xi_i\xi_j \leq c^{-1}|\xi|^2$$

*for all $(r,p,X) \in \overline{U}_{\delta K}$, $\xi = (\xi_1, \cdots \xi_n) \in \mathbb{R}^n$.*

Proof. The upper bound is trivial since $\partial G/\partial X_{ij}$ is continuous in $\overline{U}$. To get the lower bound we first observe that (2.3) implies

$$(2.14) \qquad \sum_{i,j} \frac{\partial \widetilde{F}}{\partial X_{ij}} Y_{ij} \leq -\theta \text{ trace } Y \quad \text{for } Y \geq 0,$$

where $\widetilde{F}$ is defined by (2.7). Indeed from (2.3) and (2.5) it follows that

$$\widetilde{F}(p, X + \varepsilon Y) - \widetilde{F}(p,X) \leq -\varepsilon(\theta \text{ trace } (R_{\overline{p}} Y R_{\overline{p}}) + \text{ trace } (Y\overline{p} \otimes \overline{p}))$$
$$= -\varepsilon(\theta \text{ trace } Y + (1 - \theta) \text{ trace } (Y\overline{p} \otimes \overline{p}))$$
$$\leq -\varepsilon\theta \text{ trace } Y \quad \text{for } Y \geq 0, \varepsilon > 0$$

since trace $(Y\overline{p} \otimes \overline{p}) \geq 0$. Here we have used $R_{\overline{p}}^2 = R_{\overline{p}}$ so that

$$\text{trace } (R_{\overline{p}} Y R_{\overline{p}}) = \text{ trace } (Y R_{\overline{p}}).$$

We thus obtain (2.14).

We apply the chain rule (2.12) to get

$$\sum_{i,j} \frac{\partial G}{\partial X_{ij}}(r,p,X)\xi_i\xi_j = dg_r(X)[\xi \otimes \xi], \qquad g_r(X) = -\widetilde{F}(p, X_r)$$

$$= d(-\widetilde{F})(X_r) \circ d\Phi_r(X)[\xi \otimes \xi],$$

where $X_r = \Phi_r(X)$ is the Yosida approximation of $X$. Lemma 2.4 now yields

$$(2.15) \qquad \sum_{i,j} \frac{\partial G}{\partial X_{ij}}\xi_i\xi_j = \sum_{i,j} \frac{\partial(-\widetilde{F})}{\partial X_{ij}} Y_{ij} \text{ with } Y = (I - rX)^{-1}\xi \otimes \xi(I - rX)^{-1}.$$

Since $|rX| \leq 3/8$ implies $(I - rX)^{-1} \geq 2I/5$, we have

$$Y \geq \left(\frac{2}{5}\right)^2 \xi \otimes \xi \geq 0.$$

7

Applying (2.14) to (2.15) yields

$$\sum_{i,j} \frac{\partial G}{\partial X_{ij}} \xi_i \xi_j \geq \frac{4\theta}{25} |\xi|^2. \quad \square$$

*Proof of Lemma 2.3.* Since $v$ solves (2.9), a calculation shows that the function

$$w = |\nabla v|^2 - 1$$

solves

$$(2.16) \qquad w_t = \sum_{i,j} \frac{\partial G}{\partial X_{ij}} w_{ij} + \sum_{\ell} \frac{\partial G}{\partial p_\ell} w_\ell + 2\frac{\partial G}{\partial r} w$$

$$+ 2 \left[ \frac{\partial G}{\partial r} - \sum_{i,j,k} \frac{\partial G}{\partial X_{ij}} v_{ik} v_{jk} \right],$$

where all partial derivatives of $G$ is evaluated at $(v, \nabla v, \nabla^2 v)$ and $w_\ell = \partial w / \partial x_\ell$, $w_{ij} = \partial^2 w / \partial x_i \partial x_j$ etc. Since $v_{kj} = v_{jk}$, from Lemma 2.5 (ii) it follows that

$$\frac{\partial G}{\partial r} = \sum_{i,j,k} \frac{\partial G}{\partial X_{ij}} v_{ik} v_{jk}.$$

The equation (2.16) now becomes

$$(2.17) \qquad w_t = \sum_{i,j} \frac{\partial G}{\partial X_{ij}} w_{ij} + \sum_{\ell} \frac{\partial G}{\partial p_\ell} w_\ell + 2\frac{\partial G}{\partial r} w \text{ in } Q_{T_*}.$$

Since (2.17) is parabolic by Proposition 2.6, applying the maximum principle with (2.10) and $|\nabla v_0| = 1$ yields $w = 0$ in $Q_{T_*}$. This is the same as $|\nabla v| = 1$ in $Q_{T_*}$. $\square$

*Remark 2.7.* If $f$ in (2.1) depends on $x$ in addition, (2.16) should be altered. Apparently, straight forward extension of Theorem 2.1 for $f$ depending on $x$ is not easy. The proof of Proposition 2.2 is easily extended while there is a difficulty in the proof of Lemma 2.3. We do not get the equation like (2.17) yielding $w = 0$, when $f$ depends on $x$.

We conclude this section to give a brief sketch of the proof of Proposition 2.2; see [ES2] for the detail. We set

$$h = G(v_0, \nabla v_0, \nabla^2 v_0)$$

and write the equation for $\tilde{v} = v - v_0 - th$ with the boundary conditions. The function $\tilde{v}$ can be interpreted as a small disturbance for $v_0 + th$. Since the linearized equation around $v_0 + th$ is uniformly parabolic in region we are interested in, we apply Schauder's estimates

[LUS] to carry out the standard iteration procedure which yields a local classical solution $\tilde{v}$. A standard regularity theory [LUS] guarantees the higher regularity of $v$ up to $t = 0$.

*Remark 2.8.* In the method just sketched there is a curious point on minimal regularity assumptions on the initial data $v_0$ in (2.11) to construct the local classical solution. We are forced to assume $v_0 \in C^{4+\alpha}(0 < \alpha < 1)$ guarantee that $v_0 + th$ is $C^{2+\alpha,1+\alpha/2}$ which is the least regularity assumption to get solution $\tilde{v} \in C^{2+\alpha,1+\alpha/2}$ (see [LUS] for the definition). We wonder whether or not $v_0 \in C^{2+\alpha}$ is enough to get $C^{2+\alpha,1+\alpha/2}$ solution $v$ of (2.9) - (2.11).

**3. Consistency with weak solutions.** The smooth solution constructed in §2 may collapse in a finite time. In [CGG] weak solutions for (2.1) is introduced so that one can track whole evolution of $\Gamma_t$; see also [GG]. In this section we show that the smooth solution agrees with the weak solution in the time interval where the former exists. This result is proved by [ES1] for the mean curvature flow equation (1.3). Although the basic idea is similar, our proof given below is simpler and more general than theirs because we do not approximate solutions.

We first recall the definition of weak solutions in [CGG, GG]. Let $\{(\Gamma_t, D_t)\}_{t \geq 0}$ be a family of compact sets and bounded open sets in $\mathbb{R}^n$. Suppose that for some $\alpha < 0$ there is a viscosity solution $u \in C_\alpha([0, T] \times \mathbb{R}^n)$ for

$$(3.1) \qquad u_t + F(t, \nabla u, \nabla^2 u) = 0 \text{ in } (0, \infty) \times \mathbb{R}^n$$

such that zero level sets of $u(t, \cdot)$ at $t \geq 0$ equals $\Gamma_t$ and that the set where $u(t, \cdot) > 0$ equals $D_t$. Here $F$ is determined from $f$ through (2.5) where $t$-dependence is suppressed. If $(\Gamma_t, D_t)|_{t=0} = (\Gamma_0, D_0)$ we say $\{(\Gamma_t, D_t)\}_{t \geq 0}$ is a *weak solution* of (2.1) (with initial data $(\Gamma_0, D_0)$). Here $T > 0$ is arbitrary and $v \in C_\alpha(A)$ means $v - \alpha$ is continuous with compact support in $A$.

It is now well-known that there is a unique (global) weak solution of (2.1) if $D_0$ is bounded open and $\Gamma_0 (\subset \mathbb{R}^n \setminus D_0)$ contains $\partial D_0$ provided that (2.1) is degenerate parabolic and $f$ grows linearly in $\nabla \mathbf{n}$ ([CGG, Theorem 7.3], [GG, Theorem 3.8]). For example if $f = f(t, p, Z)$ satisfies

$(2.2')$ \qquad $f : [0, \infty) \times S^{n-1} \times \mathsf{M}_n \to \mathbb{R}$ is continuous ,

$(2.3')$ \qquad $f(t, -\overline{p}, -Q_{\overline{p}}(X)) \geq f(t, -\overline{p}, -Q_{\overline{p}}(Y))$ for $X \geq Y$, \quad $\overline{p} \in S^{n-1}$ and $t \geq 0$

(which are weaker than (2.2) and (2.3)) and

$$(3.2) \qquad C_{RT} = \sup \left\{ \frac{\partial f}{\partial Z_{ij}}; |Z| \leq R, \overline{p} \in S^{n-1}, 0 < t < T, 1 \leq i, j \leq n \right\} < \infty,$$

then there is a unique global weak solution with given initial data. The condition $(2.3')$ means a degenerate ellipticity of $-f$. Our theory applies to (1.2) provided that $H$ is convex and $C^2$ outside the origin and that $\beta : S^{n-1} \to \mathbb{R}$ and $c : [0, \infty) \to \mathbb{R}$ is continuous.

THEOREM 3.1. *Assume (2.2′), (2.3′) and (3.2) for $f$ in (2.1). Let $\Gamma_0$ be a smooth hypersurface which is a boundary of a bounded domain $D_0$ in $\mathbb{R}^n$. Let $\{\Gamma_t^s\}_{0 \leq t < T_0}$ be the local smooth solution of (2.1) with initial data $\Gamma_0$. Let $\{(\Gamma_t, D_t)\}_{t \geq 0}$ be the weak solution of (2.1) with initial data $(\Gamma_0, D_0)$. Then $\Gamma_t = \Gamma_t^s$ for $0 \leq t < T_0$.*

If we assume (2.2), (2.3), we proved in §2 that there is a unique smooth local solution. However under (2.2′), (2.3′) we do not know the existence of unique classical solution of (2.1).

We again assume that $f$ is independent of $t$ for simplicity. Let $v$ denote the signed distance function of $\Gamma_t^s$, i.e.

$$v(t, x) = \begin{cases} d(x, \Gamma_t^s), & x \in D_t^s \\ -d(x, \Gamma_t^s), & x \notin D_t^s, \end{cases}$$

where $D_t^s$ is the bounded domain bounded by $\Gamma_t^s$. For the proof we construct a viscosity solution $u$ of (3.1) in $(0, T_0) \times \mathbb{R}^n$ such that the signature of $u$ agrees with $v$. Since $\Gamma_t^s$ is a smooth solution of (2.1) we see as in §2 for each $T_1, 0 < T_1 < T_0$ there is $\delta > 0$ such that $v$ solves

$$(3.3) \qquad v_t = G(v, \nabla v, \nabla^2 v) \text{ in } R_{T_1}^\delta = \bigcup_{0 < t < T_1} \{t\} \times \Omega_\delta(t)$$

with $\Omega_\delta(t) = \{x \in \mathbb{R}^n; |v(t, x)| < \delta\}$, where $G$ is defined by (2.8).

LEMMA 3.2. *For sufficiently small $\delta$ the function $v$ satisfies*

$$(3.4) \qquad v_t + F(\nabla v, \nabla^2 v) \leq 0 \text{ in } R_{T_1}^\delta \cap \{v \leq 0\}$$

*in the classical sense, where $F$ is as in (2.5).*

*Proof.* We take $\delta$ small so that $|v\nabla^2 v| \leq 3/8$ in $R_{T_1}^\delta$. Since $|rX| \leq 3/8$ implies

$$X(I - rX)^{-1} - X = rX^2(I - rX)^{-1} \leq 0 \text{ for } r \leq 0,$$

we see, by (3.3),

$$(3.5) \qquad v_t + \widetilde{F}(\nabla v, \nabla^2 v) \leq v_t - G(v, \nabla v, \nabla^2 v) = 0 \text{ in } R_{T_1}^\delta \cap \{v \leq 0\},$$

where $\widetilde{F}$ is defined by (2.7). Here we use the fact that (2.3′) implies the degenerate ellipticity

$$\widetilde{F}(p, X) \leq \widetilde{F}(p, Y) \quad \text{if } X \geq Y.$$

Since $|\nabla v| = 1$ implies

$$\nabla^2 v (\nabla v \otimes \nabla v) = 0,$$

(3.4) now follows from (3.5). □

LEMMA 3.3. *For sufficiently small $\delta$ there is $\sigma_0$ such that*

$$w(t, x) = e^{-\sigma t} v(t, x), \quad \sigma > \sigma_0$$

*satisfies*

(3.6) $$w_t + F(\nabla w, \nabla^2 w) \leq 0 \text{ in } R_{T_1}^{\delta} \cap \{v \geq 0\}.$$

*Proof.* Since $v$ solves (3.3), using

$$w_t = -\sigma e^{-\sigma t} v + e^{-\sigma t} v_t,$$
$$\widetilde{F}(\lambda p, \lambda X) = \lambda \widetilde{F}(p, X),$$

we see

$$w_t + \sigma w + \widetilde{F}(\nabla w, \nabla^2 w (I - v \nabla^2 v)^{-1}) = 0$$

or

(3.7) $$w_t + F(\nabla w, \nabla^2 w) = F(\nabla w, \nabla^2 w) - F(\nabla w, \nabla^2 w (I - v \nabla^2 v)^{-1}) - \sigma w \text{ in } R_{T_1}^{\delta}.$$

By taking $\delta$ small, we may always assume that

$$|v \nabla^2 v| \leq 3/8, \quad |\nabla^2 w| \leq |\nabla^2 v| \leq M \text{ in } R_{T_1}^{\delta}$$

with some $M$ independent of $t$ and $x$. A calculation now shows

$$|F(p, X) - F(p, Z)| \leq K|X - Z|, \quad K = C_{3M, T_1}$$

for $X = \nabla^2 w, Y = \nabla^2 v, Z = X(I - rY)^{-1}, r = v$, where $C_{RT}$ is as in (3.2). From (3.7) it follows that

$$w_t + F(\nabla w, \nabla^2 w) \leq K|X - Z| - \sigma w.$$

Since

$$X(I - rY)^{-1} - X = rXY(I - rY)^{-1}$$
$$= v(\nabla^2 w)(\nabla^2 v)(I - v\nabla^2 v)^{-1}$$

for $\sigma > \sigma_0$ with

$$\sigma_0 = K \sup_{R_{T_1}^{\delta}} |w(\nabla^2 v)^2 (I - v\nabla^2 v)^{-1}|$$

we have

$$w_t + F(\nabla w, \nabla^2 w) \leq (\sigma_0 - \sigma)w$$
$$\leq 0 \qquad \qquad \text{for } w \geq 0$$

This is the same as (3.6). □

11

PROPOSITION 3.4. *For each $T_1$, $0 < T_1 < T_0$ there are (viscosity) sub- and supersolutions $w_-$ and $w_+$ of*

(3.8)
$$u_t + F(\nabla u, \nabla^2 u) = 0 \text{ in } (0, T_1) \times \mathbb{R}^n$$

*such that $w_- \leq w_+$ and $w_\pm$ has the same signature as $v$ and satisfies*

$$w_\pm(0, x) = v(0, x) \text{ if } |v| < \delta_0$$
$$w_\pm(t, x) = -\delta_0 \quad \text{ if } v < -\delta_0$$
$$w_\pm(t, x) = \delta_0 \quad \text{ if } v > \delta_0$$

*for some $\delta_0 > 0$.*

*Proof.* Let $\delta$ be as in Lemmas 3.2 and 3.3. We set

$$w_- = \begin{cases} \min(w, \delta_0) & \text{if } v \geq 0 \\ \max(v, -\delta_0) & \text{if } v \leq 0 \end{cases}$$

with $\delta_0 = \delta/2$ where $t \in (0, T_1)$. Clearly $w_-(0, x) = v(0, x)$ for $|v| < \delta_0$ and $w_-$ has the same signature as $v$. Since $w$ is a subsolution of (3.8) in $\{v \geq 0\}$ and $v$ is a subsolution in $\{v \leq 0\}$ and since $F$ is geometric

$$\min(w, \delta_0), \quad \max(v, -\delta_0)$$

are subsolutions of (3.8) in $\{v \geq 0\}$, $\{v \leq 0\}$, respectively (see [CGG, Theorem 5.2]).

We shall prove that $w_-$ is a subsolution of (3.8) in $(0, T_1) \times \mathbb{R}^n$. To show this it remains to show that $w_-$ is a subsolution on the set of $v = 0$. Since $\Gamma_t^s$ is smooth, it is not difficult to prove that

$$\mathcal{P}^{2,+} w_-(t, x) \subset \mathcal{P}^{2,+} w \cup \mathcal{P}^{2,+} v(t, x),$$

where $\mathcal{P}^{2,+}$ denotes the parabolic super 2-jets (see e.g. [GGIS] for the definition). On $\Gamma_t^s$ we have

$$w_t + F(\nabla w, \nabla^2 w) \leq 0$$
$$v_t + F(\nabla v, \nabla^2 v) \leq 0$$

in the classical sense so for $(\tau, p, X) \in \mathcal{P}^{2,+} w_-(t, x)$ such that $v(t, x) = 0$ we see

$$\tau + F(p, X) \leq 0$$

which implies $w_-$ is a subsolution on the set $\{v = 0\}$.

12

In the same way we see $v$ and $w$ are supersolutions in $\{v \geq 0\}$ and $\{v \leq 0\}$, respectively. We thus conclude

$$w_+ = \left\{ \begin{array}{ll} \min(v, \delta_0) & \text{if } v \geq 0 \\ \max(w, -\delta_0) & \text{if } v \leq 0 \quad (w = e^{-\sigma t}v) \end{array} \right.$$

is the desired supersolution; $w_- \leq w_+$ follows from the definition. $\square$

*Proof of Theorem 3.1.* Using $w_\pm$ in Proposition 3.4, Perron's method [CGG, Theorem 4.5] provides a viscosity solution $u$ of (3.8) in $(0, T_1) \times \mathbb{R}^n$ such that $w_- \leq u \leq w_+$ on $[0, T_1) \times \mathbb{R}^n$. Since $u$ is forced to have the same signature as $v$, we find $\Gamma_t = \Gamma_t^s$, $D_t = D_t^s$ for $0 \leq t < T_1$. Since $T_1$ is an arbitrary positive number less than $T_0$, the proof is now complete. $\square$

*Remark 3.5.* After this work is completed, Evans, Soner and Songanidis [ESS] proved that the signed distance function $d$ of weak solution $\Gamma_t$ satisfies

$$d_t - \Delta d \leq 0 \text{ in } \{d < 0\}$$

*globally* in space in the viscosity sense provided that $\Gamma_t$ solves the mean curvature flow equation (1.3). This can be interpreted as an improvement of Lemma 3.2 for (1.3). It would be interesting to study whether

$$d_t + \widetilde{F}(\nabla d, \nabla^2 d) \leq 0 \text{ in } \{d < 0\}$$

holds for (2.1) with general $f$.

**4. Singularities.** Smooth local solutions may collapse in a finite time even for the mean curvature flow equation (1.3). It is shown in [Gr] that a barbell with a long, thin handle actually becomes singular (see also [A]). A natural question is what kind of singularities actually appears. In [Hu] Huisken considered a two-dimensional rotational symmetric hypersurface with positive mean curvature and proved that the singularity behaves like cylinders asymptotically. Note that this does not exclude cusp singularities. The place of singularities related to this problem is discussed in [K].

Our result in this section suggests that asymptotically cone like singularity is impossible to occur in the motion by mean curvature. We show in particular that if the singularity is nondegenerate in some sense, then it looks like sphere asymptotically.

THEOREM 4.1. *Let* $\{(\Gamma_t, D_t)\}_{t>0}$ *be a weak solution of (1.3). Let $u$ be a defining function of $(\Gamma_t, D_t)$ i.e.*

$$\Gamma_t = \{x \in \mathbb{R}^n; u(t, x) = 0\},$$
$$D_t = \{x \in \mathbb{R}^n; u(t, x) > 0\}$$

13

and $u$ solves

(4.1)
$$u_t - |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

in the viscosity sense. Let $t_0 > 0$ and $x_0 \in \Gamma_{t_0}$. Suppose that

(4.2)
$$u(t, x) = c\tau + \frac{1}{2}\nabla^2 u(t_0, x_0)y \cdot y + 0(|y|^2 + |\tau|) \text{ as } |y|^2 + |\tau| \to 0,$$

where $\tau = t - t_0, y = x - x_0, t \leq t_0$. Then $u$ is of the form

(4.3)
$$u(t, x) = c\tau + \lambda \sum_{i=1}^{k} y_i^2 + o(|y|^2 + |\tau|)$$

with $c = 2(k-1)\lambda, \lambda \neq 0$ for some $k \leq n$.

*Proof.* We rescale $u$ around $(t_0, x_0)$:

$$u_\sigma(\tau, y) = \frac{1}{\sigma^2}u(\sigma^2\tau + t_0, \sigma y + x_0), \quad \sigma > 0.$$

By (4.2) $u_\sigma$ converges to a polynomial $v$ of form

$$v(\tau, y) = c\tau + Ay \cdot y, \quad A \in \mathbb{S}_n$$

uniformly in every compact sets in $(-\infty, 0] \times \mathbb{R}^n$. By the stability [CGG, Proposition 2.4] we observe that $v$ is a viscosity solution of (4.1) in $(-\infty, 0) \times \mathbb{R}^n$. We may assume $A$ is diagonal by a rotation. Plugging

$$v(\tau, y) = c\tau + \sum_{j=1}^{k}\lambda_j y_j^2 \quad (\lambda_j \neq 0 \text{ for all } 1 \leq j \leq k)$$

in (4.1) yields $\lambda_1 = \lambda_2 = \cdots = \lambda_k$ and

$$v(\tau, y) = c\tau + \lambda \sum_{j=1}^{k} y_j^2, c = 2(k-1)\lambda$$

for some $k$. This is the same as (4.3). $\square$

*Remark 4.2.* If $\det \nabla^2 u(t_0, x_0) \neq 0$, then $k = n$. This means singularities look like sphere, which is very related to results in [Hu]. The assumption (4.2) is fulfilled if we assume $u$ is $C^2$ and $\nabla u(t_0, x_0) = 0$. However, we still do not know whether or not one can take such a defining function $u$ near singularities. After we completed this work, we learned that Ilmanen [I] showed that $u$ is not necessarily $C^2$ even if $u$ is initially $C^2$.

# REFERENCES

[A]    S. ANGENENT, *Shrinking doughnuts*, Proc. of the conference on elliptic and parabolic equations held at Gregymog, Wales, August 1989.

[AG]   S. ANGENENT AND M. GURTIN, *Multiphase thermomechanics with interfacial structure. 2. Evolution of an isothermal interface*, Arch. Rational Mech. Anal. 108 (1989), pp. 323–391.

[B]    K.A. BRAKKE, *The motion of a surface by its mean curvature*, Princeton University Press, 1978.

[BK]   L. BRONSARD AND R. KOHN, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, preprint.

[Ca]   G. CAGINALP, *The role of microscopic anisotropy in the macroscopic behavior of a phase boundary*, Annals of Physics, 172 (1986), pp. 136–155.

[Ch]   X. CHEN, *Generation and propagation of the interface for reaction-diffusion equation*, preprint.

[C]    X.-Y. CHEN, *Dynamics of interfaces in reaction diffusion systems*, Hiroshima Math. J. 21, to appear (1991).

[CGG]  Y.-G. CHEN, Y. GIGA AND S. GOTO, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geometry 33, to appear (1991); (Announcement: Proc. Japan Acad. Ser. A, 65 (1989), pp. 207–210 ).

[ESS]  L.C. EVANS, H.M. SONER AND P.E. SONGANIDIS, *The Allen-Cahn equation and generalized motion by mean curvature*, manuscript.

[ES1]  L.C. EVANS AND J. SPRUCK, *Motion of level sets by mean curvature I*, J. Differential Geometry, to appear.

[ES2]  L.C. EVANS AND J. SPRUCK, *Motion of level sets by mean curvature II*, J. Differential Geometry, to appear..

[ES3]  L.C. EVANS AND J. SPRUCK, *Motion of level sets by mean curvature III*, preprint 1990.

[GG]   Y. GIGA AND S. GOTO, *Motion of hypersurfaces and geometric equations*, J. Math. Soc. Japan, to appear.

[GGIS] Y. GIGA, S. GOTO, H. ISHII AND M.-H. SATO, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J., to appear.

[GGI]  Y. GIGA, S. GOTO AND H. ISHII, *Global existence of weak solutions for interface equations coupled with diffusion equations*, preprint 1990.

[G]    S. GOTO, *A level surface approach to interface dynamics*, in preparation.

[Gr]   M. GRAYSON, *A short note on the evolution of a surface by its mean curvature*, Duke Math. J. 58 (1989), pp. 555–558.

[Gu1]  M. GURTIN, *Towards a nonequilibrium thermodynamics of two-phase materials*, Arch. Rational Mech. Anal. 100 (1988), pp. 275–312.

[Gu2]  M. GURTIN, *Multiphase thermomechanics with interfacial structure. 1, Heat conduction and the capillary balance law*, Arch. Rational Mech. Anal. 104 (1988), pp. 195–221.

[Ha]   R.S. HAMILTON, *Three manifolds with positive Ricci curvature*, J. Differential Geometry 17 (1982), pp. 255–306.

[Hu]   G. HUISKEN, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Differential Geometry 31 (1990), pp. 285–299.

[I]    T. ILMANEN, *Generalized flow of sets by mean curvature on a manifold*, preliminary report.

[K]    B. KAWOHL, *Remarks on quenching, blow up and dead core*, Proc. of the conference on elliptic and parabolic equations held at Gregymog, Wales, August 1989.

[LUS]  O.A. LADYZHENSKAYA, V. SOLONNIKOV AND N. URAL'CERA, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, vol. 23, AMS, 1968.

[MS]   P. DeMOTTONI AND M. SCHATZMAN, *Geometrical evolution of developed interfaces*, preprint; (Announcement: Evolution géometric d'interfaces, C.R. Acad. Sci. Paris, 309 (1989), pp. 453–458 ).

[OS]   S. OSHER AND J.A. SETHIAN, *Front propagating with curvature dependent speed: Algorithms based on Hamilton-Jacobi formulations*, J. Comput. Phys. 79 (1988), pp. 12–49.

[Se]   J.A. SETHIAN, *Curvature and evolution of fronts*, Comm. Math. Physics 101 (1985), pp. 487–499.

[S]    H.M. SONER, *Motion of a set by the curvature of its boundary*, preprint 1990.

[T]    J. TAYLOR, *Motion of curves by crystalline curvature*, preliminary report.

15

| # | Author/s | Title |
|---|---|---|
| 662 | **Geneviève Raugel and George R. Sell**, Navier-Stokes equations in thin 3d domains: Global regularity of solutions I |
| 663 | **Emanuel Parzen**, Time series, statistics, and information |
| 664 | **Andrew Majda and Kevin Lamb**, Simplified equations for low Mach number combustion with strong heat release |
| 665 | **Ju. S. Il'yashenko**, Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation |
| 666 | **James F. Reineck**, Continuation to gradient flows |
| 667 | **Mohamed Sami Elbialy**, Simultaneous binary collisions in the collinear N–body problem |
| 668 | **John A. Jacquez and Carl P. Simon**, Aids: The epidemiological significance of two different mean rates of partner-change |
| 669 | **Carl P. Simon and John A. Jacquez**, Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations |
| 670 | **Matthew Stafford**, Markov partitions for expanding maps of the circle |
| 671 | **Ciprian Foias and Edriss S. Titi**, Determining nodes, finite difference schemes and inertial manifolds |
| 672 | **M.W. Smiley**, Global attractors and approximate inertial manifolds for abstract dissipative equations |
| 673 | **M.W. Smiley**, On the existence of smooth breathers for nonlinear wave equations |
| 674 | **Hitay Özbay and Janos Turi**, Robust stabilization of systems governed by singular integro-differential equations |
| 675 | **Mary Silber and Edgar Knobloch**, Hopf bifurcation on a square lattice |
| 676 | **Christophe Golé**, Ghost circles for twist maps |
| 677 | **Christophe Golé**, Ghost tori for monotone maps |
| 678 | **Christophe Golé**, Monotone maps of $T^n \times R^n$ and their periodic orbits |
| 679 | **E.G. Kalnins and W. Miller, Jr.**, Hypergeometric expansions of Heun polynomials |
| 680 | **Victor A. Pliss and George R. Sell**, Perturbations of attractors of differential equations |
| 681 | **Avner Friedman and Peter Knabner**, A transport model with micro- and macro-structure |
| 682 | **E.G. Kalnins and W. Miller, Jr.**, A note on group contractions and radar ambiguity functions |
| 683 | **George R. Sell**, References on dynamical systems |
| 684 | **Shui-Nee Chow, Kening Lu and George R. Sell**, Smoothness of inertial manifolds |
| 685 | **Shui-Nee Chow, Xiao-Biao Lin and Kening Lu**, Smooth invariant foliations in infinite dimensional spaces |
| 686 | **Kening Lu**, A Hartman–Grobman theorem for scalar reaction-diffusion equations |
| 687 | **Christophe Golé and Glen R. Hall**, Poincaré's proof of Poincaré's last geometric theorem |
| 688 | **Mario Taboada**, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations |
| 689 | **Peter Rejto and Mario Taboada**, Weighted resolvent estimates for Volterra operators on unbounded intervals |
| 690 | **Joel D. Avrin**, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion |
| 691 | **Susan Friedlander and Misha M. Vishik**, Lax pair formulation for the Euler equation |
| 692 | **H. Scott Dumas**, Ergodization rates for linear flow on the torus |
| 693 | **A. Eden, A.J. Milani and B. Nicolaenko**, Finite dimensional exponential attractors for semilinear wave equations with damping |
| 694 | **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Inertial sets for dissipative evolution equations |
| 695 | **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Hölder continuity for the inverse of Mañé's projection |
| 696 | **Michel Chipot and Charles Collins**, Numerical approximations in variational problems with potential wells |
| 697 | **Huanan Yang**, Nonlinear wave analysis and convergence of MUSCL schemes |
| 698 | **László Gerencsér and Zsuzsanna Vágó**, A strong approximation theorem for estimator processes in continuous time |
| 699 | **László Gerencsér**, Multiple integrals with respect to $L$-mixing processes |
| 700 | **David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation |
| 701 | **Bo Deng**, Symbolic dynamics for chaotic systems |
| 702 | **P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses |
| 703 | **Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem |
| 704 | **Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j-radii of polytopes in finite-dimensional normed spaces |
| 705 | **A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation |