**A NUMERICAL STUDY OF
A ROTATIONALLY DEGENERATE HYPERBOLIC SYSTEM
PART I: THE RIEMANN PROBLEM**

By

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A NON-STRICTLY HYPERBOLIC SYSTEM

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ABSTRACT. We study numerically the Riemann problem for a 2×2 system of conservation laws with a cubic flux function, a particular case of the class of models introduced by Keyfitz and Kranzer. The system is not strictly hyperbolic, and the classical Lax theory for hyperbolic systems is not directly applicable. Correspondingly, some numerical schemes which are accurate for strictly hyperbolic systems are not well behaved for this example. When they do work, different schemes yield markedly different results for some data. We explain this effect by observing that the vanishing viscosity method does not converge to a continuous solution operator. This fact does not contradict the well-posedness of the hyperbolic model; it does imply that precise control of the viscosity introduced into a computational method is crucial for generating the correct numerical solutions. We examine all of these issues, and comment on their implications for similar systems which arise in continuum mechanics.

1 INTRODUCTION

Consider the system of hyperbolic conservation laws given by

$$U_t + (|U|^2 U)_x = 0 \quad (1.1)$$

where $U = (u, v) \in \mathbb{R}^2$. This system is not strictly hyperbolic, since the two eigenvalues of the Jacobian $D(|U|^2 U)$, viz. $\lambda^r(U) = 3|U|^2$ and $\lambda^a(U) = |U|^2$, are equal at the origin $U = 0$. Away from $U = 0$, the corresponding eigenvectors are given by (u, v) and $(v, -u)$, respectively, so the fast radial mode (corresponding to λ^r) is genuinely non-linear in the sense of Lax [26], while the slow angular mode (corresponding to λ^a) is linearly degenerate.

The present article is the first in a series of papers in which we study system (1.1) numerically. In this Part I we restrict attention to the Riemann problem. The Riemann problem consists of finding a (weak) solution of (1.1) which assumes initial values

$$U(x, 0) = U^0(x) = \begin{cases} U_l & x < 0 \\ U_r & x > 0. \end{cases} \quad (1.2)$$

For purposes of our discussion, it is often helpful to consider polar coordinates in phase space, i.e. $r : \mathbb{R}^2 \rightarrow [0, \infty)$, $\theta : \mathbb{R}^2 \rightarrow \mathbb{R} \bmod 2\pi$ defined by $U = r(U)(\cos(\theta(U)), \sin(\theta(U)))$. An intuitively plausible way to assign a centered, piecewise continuous solution to (1.1)-(1.2) is to combine (i) a contact discontinuity in the slow family, which rotates the state variables from U_l to an intermediate state U_m , where $r(U_l) = r(U_m)$, with (ii) a shock or rarefaction wave from U_m to U_r , along which $\theta = \text{const}$ (see [11]).

For comparison, consider also the reduced system obtained by setting $v \equiv 0$ in (1.1):

$$u_t + (u^3)_x = 0. \quad (1.3)$$

The Riemann problem for this scalar equation may be solved by following the construction of Liu [28] and Wendroff [40] for non-convex flux functions. In addition to the “classical” solutions to (1.2) consisting of either a single shock or rarefaction wave, the non-convexity of (1.2) introduces composite wave solutions, consisting of a shock juxtaposed with a rarefaction. Of course, completing such a scalar solution with a v -component which is identically equal to zero gives solutions to the full problem for special, co-linear data.

These two constructions suggest a non-uniqueness of solutions and/or a non-continuous dependence of the solutions on initial data. In order to illustrate this difficulty, consider the Riemann problem (1.1)-(1.2) with data

$$U_l = (1, 0) \quad U_r = (\cos \delta, \sin \delta) \quad (1.4)$$

for values of δ near π . If $\delta = \pi$ exactly, the two constructions yield two different solutions. One solution consists of a contact discontinuity joining U_l to U_r ; the other solution is composed of a shock joining U_l to a middle state $U_c = (-1/2, 0)$ and an adjacent rarefaction wave joining U_c to U_r . In both solutions, $v \equiv 0$. The composite wave solution, being the solution of a restricted problem, might seem more natural. On the other hand, the contact wave solution may readily be extended to data (1.4) with $\delta \neq \pi$. The continuity of the contact wave solution operator, as data moves off of the line $v = 0$, seems a desirable property. Indeed, as long as δ is different from π , there is no centered, piecewise smooth solution of (1.1)-(1.4) which would, in the limit $\delta \rightarrow \pi$, approach the composite wave solution. Conversely, any contact discontinuity is the limit of smooth, quasistationary solutions of (1.1). However one resolves this apparent non-uniqueness/non-continuous dependence theoretically, it seems to us an issue of obvious importance how numerical methods respond to the discrepancy.

A major motivation to study system (1.1) is its rôle as a prototype of larger systems which arise in continuum mechanics ([11]). At the level of the Riemann problem, there is a precise isomorphism between the wave pattern of (1.1) and (a specific) part of the wave pattern of any generic rotationally degenerate system of hyperbolic conservation laws (see [11]). Systems which describe plane wave propagation of isotropic, multi-dimensional systems are rotationally degenerate. Indeed, by asymptotic expansion techniques Brio and Hunter [4] have derived (1.1) from such isotropic systems.

Two fields from continuum mechanics to which these comments apply are magnetohydrodynamics and isotropic, neo-Hookian elasticity. With respect to the analytical treatment of the Riemann problem for these systems, we mention the work of Gogosov [19] for MHD, and Keyfitz and Kranzer [24], Shearer [37] for the elastic string problem, as well as the paper [10] which addresses MHD and the elastic body. Regarding computational results, we refer to Brio and Wu [5], Wu [41] for MHD, and to Fehribach and Shearer [9], Gilquin and Serre [15] and Tait [38] for the elastic string. We also mention

that very recently some computations on model (1.1) were presented in [23]. The systematic numerical study of this model, the first part of which we present here, continues the analytic work begun in [10, 11, 12].

We conclude this introduction by outlining the format for the rest of this paper. Section 2 describes the computational methods employed in this study, and Section 3 presents numerical results. Section 4 interprets these results, and returns to our discussion of how (1.1) relates to more realistic continuum mechanics models.

2 COMPUTATIONAL METHODS

In this section, we discuss issues pertinent to the numerical computation of solutions to (1.1). After reviewing some fundamental ideas and results in the theory of (strictly) hyperbolic conservation laws in sub-section 2.1, in sub-sections 2.2 thru 2.4 we give a brief introduction to numerical methods for such systems. We address the issue of non-strict hyperbolicity in sub-section 2.5. Finally, in 2.6 we comment on the numerical schemes we use in our study.

Sub-sections 2.1 thru 2.4 are not intended to give a complete review of the theory of numerical schemes for hyperbolic systems of conservation laws; for this purpose, we refer the reader to the paper of Harten, Lax, and van Leer [20]. Here we restrict ourselves to those basic ideas which we need in order to explain our selection of computational schemes, and to have sufficient background for discussing the numerical results. On the other hand, the scope of this section is broader than one would need for the Riemann problem alone; it is meant to establish a basis for the whole of our study, including [13, 14]. Thus the relationship between this section and the rest of this paper is that a number of general issues are collected here, and are addressed in the other sections within the relatively simple framework of the Riemann problem for (1.1).

2.1. Introductory Theory. In this sub-section we introduce some of the basic facts on hyperbolic systems of conservation laws. Such systems are of the general form

$$w_t + (f(w))_x = 0 \quad w(x, 0) = w^0(x) \quad (2.1)$$

where $w : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}^m$ and $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$. (2.1) is called hyperbolic if the Jacobian Df of the flux function is diagonalizable over $\mathbb{R}$; of particular importance is the case of strict hyperbolicity, when all the eigenvalues of Df are distinct.

It is known that smooth solutions of (2.1) typically break down after finite time. Therefore it is necessary to broaden our concept of solution, and admit weak solutions of the system, i.e. functions w which satisfy (2.1) in the sense of distributions:

$$\int_0^\infty \int_{-\infty}^\infty (\varphi_t(x, t)w(x, t) + \varphi_x(x, t)f(w(x, t)))dxdt + \int_{-\infty}^\infty \varphi(x, 0)w^0(x)dx = 0 \quad (2.2)$$

for all $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ with compact support. Across any curve of discontinuity, (2.2) enforces the Rankine-Hugoniot conditions:

$$\sigma(w^+ - w^-) = f(w^+) - f(w^-). \quad (2.3)$$

Here, σ is the propagation speed of the discontinuity and w^- , w^+ indicate the states just to the left and right of the discontinuity. (These notions are due to Lax [26].) For strictly hyperbolic systems, Glimm [16] (see also [30]) proved the existence of weak solutions for initial data of small total variation. (Remark: The analytic construction of [16] has been incorporated into the so-called random choice numerical scheme by Glimm, Chorin, and others; see [7] for references.)

Unfortunately, weak solutions of (2.1) are not uniquely determined by the initial data, and a selection principle is required to choose the physically relevant solution. For strictly hyperbolic systems, this selection problem may be considered as resolved in a satisfactory manner. In the case of scalar conservation laws ($m = 1$), the vanishing viscosity approach has been carried out completely, see Lax [26], Oleinik [33], and Kruskov [25]. For arbitrary measurable initial data a unique solution may be obtained as the limit, when $\epsilon \rightarrow 0$, of the viscous regularization

$$w_t + (f(w))_x = \epsilon w_{xx}; \quad (2.4)$$

this viscosity solution depends $\mathcal{L}_{loc}^1$ -continuously on the data. When this solution is piecewise smooth (which is a generic property, see [35]), it may be characterized by an explicit admissibility criterion which must be satisfied

pointwise at discontinuities, namely the entropy condition of Lax [26] or that of Oleinik [33].

The entropy condition carries over to strictly hyperbolic systems, see Lax [26] and Liu [28]. While convergence of the viscosity method for general strictly hyperbolic systems is still an open question, Lax's and Liu's conditions have been shown to be equivalent, for small amplitude quasistationary step discontinuities, to the existence of viscous profiles, i.e. traveling wave solutions of (2.4) which approximate these quasistationary discontinuities as $\epsilon \downarrow 0$ [32]. These ideas have been widely applied in the development of numerical schemes for (2.1). However, they do not seem to carry over readily to non-strictly hyperbolic systems (see Section 4.1 and [12]).

2.2. Conservative differencing. In order to describe numerical schemes for (2.1), discretize (a finite interval of) space into N small intervals of equal length Δx , and time into discrete steps Δt . Assume we know w at space points $i\Delta x$, $i \in 1, \dots, N+1$, at time $n\Delta t$. We write

$$w_i^n = w(i\Delta x, n\Delta t) \quad f_i^n = f(w_i^n),$$

and ask how to advance w^n to w^{n+1} .

Equation (2.1) describes the conservation of a field variable w whose flux is $f(w)$; we mimic this conservation principle by considering conservative numerical schemes. A conservative scheme in which w_i^{n+1} depends only on the three values $w_{i-1}^n, w_i^n, w_{i+1}^n$ may be written as a numerical flux difference

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (g(w_{i+1}^n, w_i^n) - g(w_i^n, w_{i-1}^n)). \quad (2.5)$$

Here, the time step Δt must be restricted by the Courant-Friedrichs-Lewy condition, which states $\frac{a\Delta t}{\Delta x} \leq 1$, where a is the maximum of the (absolute value of the) eigenvalues of A . The function $g(\cdot, \cdot)$ is called the numerical flux, and must be consistent with the physical flux $f(\cdot)$ in the sense that

$$g(w, w) = f(w).$$

The simplest scheme one might think of trying would be to approximate (2.1) by difference quotients in time and space, with the spatial differentiation following the characteristics of the system, i.e., upwind differencing. For (1.1) where characteristic speeds are always non-negative, upwinding gives

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (f_i^n - f_{i-1}^n). \quad (2.6)$$

Note that upwind differencing is conservative, with a numerical flux

$$g(w_{i+1}, w_i) = f(w_i).$$

The difference scheme (2.6) is of first order accuracy (for smooth solutions). The behavior of this and other schemes near discontinuities of (2.1) will be addressed in sub-section 2.4.

2.3. Godunov and Random Choice schemes. Many modern schemes for solving hyperbolic systems (2.1) are based on explicitly resolving the local wave dynamics defined by (2.1), accounting for the nonlinearity of f and possible discontinuities in the solution. These schemes have their origins in the method introduced by Godunov [18]. To describe Godunov's method, consider an approximation $\bar{w}_i^0$ of the initial data $w^0(x)$, constant over each cell $(i - 1/2)\Delta x < x < (i + 1/2)\Delta x$. At each cell edge, there is a (possible) discontinuity in $\bar{w}^0$, and one must solve a Riemann problem locally. Restrict the timestep $\Delta t < \frac{\Delta x}{2a}$, so that waves from adjacent Riemann problems do not interact. For $0 \leq t \leq \Delta t$ consider the exact solution to (2.1), composed of the solutions to all of these local Riemann problems. Define w^1 as the restriction of this solution to the line $t = \Delta t$. Now find a piecewise constant approximation $\bar{w}^1$ to w^1 , which will in turn serve as data for advancing the solution from $t = \Delta t$ to $t = 2\Delta t$. Continue in this manner to find w^{n+1} and $\bar{w}^{n+1}$ with w^{n+1} the restriction of an exact solution (defined on the strip $n\Delta t \leq t \leq (n+1)\Delta t$) by data $\bar{w}^n$ at $t = n\Delta t$ to $t = (n+1)\Delta t$, and $\bar{w}^{n+1}$ its piecewise constant approximation.

Different numerical schemes may be obtained by altering the ansatz for solving the local Riemann problems, and for obtaining $\bar{w}^n$ from w^n . Godunov obtained his approximation by solving each Riemann problem exactly, and then averaging w^{n+1} over each cell $((i - 1/2)\Delta x, (i + 1/2)\Delta x)$. We remark that Godunov's method may be put in conservative form [20].

In his existence proof [16], Glimm proposed an alternative procedure for obtaining $\bar{w}_i^{n+1}$. Instead of averaging over the computational cells, Glimm samples the Riemann solution at one point in each cell. Providing the sampling points are uniformly distributed within every interval, then with probability one all waves will be correctly described. Glimm's original proposal was to randomly sample the interval; Liu [29] showed that equidistributed sampling is sufficient for the existence proof. Colella [7] suggests using the equidistributed van der Corput sequence for computational purposes.

To implement a Godunov or random choice scheme, one must specify a procedure for solving a Riemann problem at each cell edge, for each time step. For strictly hyperbolic systems, this procedure is well defined (when the entropy condition is invoked) when the jump $w_r - w_l$ in the Riemann data is small, and yields a unique pattern of self-similar waves and intermediate constant states [26, 28]. In essence, the Riemann solver constructs a path in phase space joining w_l with w_r via intermediate states. Carrying out this construction in a numerical scheme is, in general, an expensive process. Moreover, only a limited amount of information from the exact solution is usually ever used (e.g., the average over subintervals in Godunov’s scheme). Therefore, it is often advantageous to replace the exact Riemann solver with an approximate solver which preserves salient features of the solution. [20] describes the construction of such approximate Riemann solvers.

The Godunov and random choice schemes, as we have described them above, are first order accurate in space and time. Higher order modifications of the Godunov scheme have been developed, beginning with the work of van Leer [27]. These schemes attain high order accuracy in space by choosing the approximants $\bar{w}_i^n$ as a piecewise high order polynomial (often linear) instead of piecewise constant. High accuracy in time is usually attained by predictor-corrector methodology. That is, a predictor value $w_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ is computed, approximating the solution of (2.1) at time $t = (n + 1/2)\Delta t$, using data $\bar{w}^n$. This predictor step usually involves solving Riemann problems at the cell edges. The solution at time $(n + 1)\Delta t$ is then found by a conservative differencing $f(w_{i+\frac{1}{2}}^{n+\frac{1}{2}}) - f(w_{i-\frac{1}{2}}^{n+\frac{1}{2}})$. These high order schemes must include certain “slope limiting” or “flux limiting” procedures to prevent the introduction of spurious extrema caused by the discrete representation of the solution (see [27] and sub-section 2.6).

2.4. Artificial viscosity. As remarked in sub-section 2.1, an admissibility condition must be imposed in order to select a unique and physically meaningful weak solution of (2.1). In numerical schemes for strictly hyperbolic systems, a small viscosity is sufficient for this purpose. This viscosity is not usually added explicitly to (2.1), but rather is introduced through the design of the scheme.

Consider, for example, the special case of upwinding for a linear, scalar conservation law $w_t + aw_x = 0$, with $a > 0$. It is easy to see that in this case

(2.6) may be rewritten

$$w_i^{n+1} = w_i^n - \frac{a\Delta t}{2\Delta x}(w_{i+1}^n - w_{i-1}^n) + \frac{a\Delta t}{2\Delta x}(w_{i+1}^n - 2w_i^n + w_{i-1}^n). \quad (2.7)$$

The last term is a difference approximation to w_{xx} . Thus, scheme (2.6) may be viewed as solving, with second order accuracy, the parabolic equation (2.4) with a viscosity coefficient

$$\epsilon = \frac{a\Delta x}{2}\left(1 - \frac{a\Delta t}{\Delta x}\right). \quad (2.8)$$

The high order Godunov schemes sketched above incorporate a selective amount of artificial viscosity. In particular, near a large gradient the “limiter” reduces the spatial interpolation to zeroth order (i.e. constant), and the whole scheme (essentially) reverts to upwind differencing (see sub-section 2.6). We emphasize that this extra dose of viscosity is included only near large jumps in the data, such as found at a shock; away from sharp transitions, the limiters are turned off and the artificial viscosity is reduced. Indeed, it is precisely this well regulated inclusion of viscosity which gives the Godunov methods their high degree of resolution. For strictly hyperbolic systems, employing artificial viscosity as a mechanism for shock smoothing connects up neatly with the theoretical success of the vanishing viscosity approach.

By employing probabilistic sampling, the random choice method introduces no artificial viscosity. This allows for perfect shock resolution; that is, the width of the shock transition is zero. The question of admissibility is addressed in the analytic construction of the Riemann solver.

2.5. Non-strictly hyperbolic systems. When hyperbolicity is no longer strict, results are available only for special systems or classes of systems. In particular, for systems whose flux function is of the form $f(w) = \varphi(w)w$, existence results have been obtained via Glimm’s construction by Liu and Wang [31] and Temple [39]. Such fluxes were introduced by Keyfitz and Kranzer [24], and include (1.1). Other results concerning the Riemann problem for special cases of non-strictly hyperbolic systems were derived by Isaacson [21], Shearer [37], Schaeffer, Shearer, Marchesin, and Paes-Leme [36], Isaacson, Marchesin, Plohr, and Temple [22], and Freistühler [10, 11].

A few attempts and suggestions have been made to apply some of the computational methods outlined above to non-strictly hyperbolic systems.

For schemes which do not explicitly refer to the wave geometry of the Riemann solution in phase space, such extensions are direct. This has been done for MHD in [5]. For schemes which do refer to the wave curve geometry, Bell, Colella, and Trangenstein [3] have suggested modifications of the high order Godunov methodology which account for the special character of this geometry near points in phase space where strict hyperbolicity is lost.

Speaking generally, when studying non-strictly hyperbolic systems numerically, the typical experience is to find that “the nature of the computationally resolved wave patterns depends sensitively on the choice of the numerical method” [17]. The lack of a general solution theory is no explanation for such inconsistency. Rather, the lack of theory and the computational sensitivity may have a common source. It is our contention that the vanishing viscosity approach is not, in general, well-behaved, and the indiscriminate inclusion of artificial viscosity in numerical schemes leads to errors. One must make a clear distinction between solving the hyperbolic system (2.1) and the related parabolic regularization (2.4). (Of course, the hyperbolic setting makes sense only when based on a theory which is independent of the ill-defined vanishing viscosity approach.)

2.6. Schemes used in this study. In our study, we solved (1.1) using the random choice scheme, a simple upwind scheme, and two second order Godunov-type schemes, that of Bell, Colella and Trangenstein (BCT), [3] and that of Roe [34]. We also used a variant of the random choice method. This variant integrates the viscous problem

$$U_t + (|U|^2 U)_x = \epsilon U_{xx} \quad \epsilon > 0 \quad (2.9)$$

by fractional steps, first solving $U_t + (|U|^2 U)_x = 0$ by random choice, then using Crank-Nicholson to solve $U_t = \epsilon U_{xx}$. (In fact, we used a Strang splitting, calling the random choice solver for time $\frac{\Delta t}{2}$, Crank-Nicholson for Δt , and finishing with another call to random choice for a half-step.) We view (2.9) as introducing a controlled amount of diffusion into the calculation, and have compared the results obtained from this scheme for different values of ϵ with the results produced by the other methods.

The random choice method was coded using the “intuitive” centered wave construction introduced in Section 1 (see Section 4.1 for analytic results on this solution). Following [7], we used the van der Corput sequence for the sampling in the projection step.

Upwind differencing is completely defined by equation (2.6).

The two second order Godunov-type schemes are predictor-corrector methods, and consist of four similar steps: (i) slope limiting; (ii) time-centered predictor; (iii) (approximate) Riemann solver; (iv) conservative flux differencing.

In each cell, a piecewise linear interpolant $\bar{w}^n$ is defined, where the slopes are “limited” to prevent unphysical wiggles from appearing near rapid transitions (as happens in Lax-Wendroff schemes). In cell i the slope is chosen to be the minimum of the centered, forward, and backward difference quotients (more correctly, one limits the components of Mw where M is the matrix of left eigenvectors of Df). Next cell edge predictor states $w_{i+1/2}^{n+1/2,L}$ and $w_{i-1/2}^{n+1/2,R}$ are calculated by tracing characteristics from $x = i\Delta x$ to, respectively, $(i + \frac{1}{2})\Delta x$ and $(i - \frac{1}{2})\Delta x$. In this manner, a mid-time left and right state are defined at each cell edge. One now solves a Riemann problem at the cell edges, say $i + \frac{1}{2}$, and determines that state $w_{i+1/2}^*$ in the Riemann solution which propagates with zero speed. Finally the updated solution is computed by a conservative differencing

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x} (f(w_{i+1/2}^*) - f(w_{i-1/2}^*)).$$

This procedure is formally second order accurate in both space and time.

The exact Riemann solver in the third step above may be expensive to employ and an approximate Riemann solver will often suffice [20]. For (1.1), calculating the stationary state is easy since both waves have non-negative speed. More generally, BCT [3] suggest projecting the jump $w_{i-1/2}^{n+1/2,R} - w_{i+1/2}^{n+1/2,L}$ along the right eigenvectors of Df .

Roe’s approximate Riemann solver is defined by a linear wave decomposition of the jump $w_{i-1/2}^{n+1/2,R} - w_{i+1/2}^{n+1/2,L}$. This approximate solver is the only essential difference with the BCT version. Details are in [34, 20].

To facilitate comparison of results, we always use the same space step Δx , and the time step $\Delta t = 0.9 \Delta x / \max(3|U^0|^2)$, i.e. 90 % of the maximal value allowed by the CFL-condition. Specifics of the issues we shall illustrate in the next section would be changed if we refined the grid, but all of the issues have counterparts on the refined grids. (Remark: The reader should recall that the size of artificial viscosity introduced in a numerical scheme, as in (2.8), depends on space and time steps.)

3 NUMERICAL RESULTS

Here we present the results from a series of numerical experiments, carried out for different initial data (1.2) and with different schemes. In Section 4.2, we draw together these numerical findings by calling upon analytic results about (1.1).

For our numerical experiments, the Riemann data were of the form

$$U_l = (1, 0) \quad U_r = (\rho \cos \delta, \rho \sin \delta) \quad (3.1)$$

with ρ and δ chosen from the values $\rho = 0.25, 0.75, 1.0, 1.1$, and $\delta = 0.0, 0.25\pi, 0.75\pi, 0.9\pi, 0.95\pi, \pi$. Using each of the schemes described in Section 2.6, we computed the solutions up to time $t = 0.75$ on the interval $-0.15 \leq x \leq 3.0$, thus allowing sufficient travel distance for all waves. With a space step $\Delta x = 0.01$, this means there were a total of 316 grid points in space. Computations were performed on a Sun SparcStation 1.

We present a large selection of numerical results in order to address the following three topics: (a) a comparison between explicit resolution and implicit approximation of the wave pattern; (b) the strong dependence of solutions to certain data on the size of artificial viscosity; (c) the subtle balance between the ideal dynamics and viscous effects for such data.

The results obtained by the BCT and the Roe schemes were always very similar, both qualitatively and quantitatively. Also the results from the first order upwind scheme differed only slightly from these second order methods. Thus we restrict our graphic display here to output from the random choice scheme, the Bell-Colella-Trangenstein second order Godunov scheme and the combination of random choice with Crank-Nicholson.

To illustrate (a), the pictures in Figure 1 show solutions obtained by random choice and the Godunov scheme, for the four values of ρ (Fig. 1.1-1.4)., and $\delta = 0.25\pi, 0.75\pi, \pi$.

Aspect (b) is illustrated by the pictures in Figure 2, which show solutions obtained by the various schemes. Figure 2.1 and 2.2 show results of solving (2.9) as described with different values of ϵ . In Figure 2.1, $\epsilon = 0.0075$ (i.e., $\epsilon = \frac{3}{4}\Delta x$); in Figure 2.2, $\epsilon = 0.005$ (i.e., $\frac{1}{2}\Delta x$). Figure 2.3 presents the results using the Godunov scheme; For comparison, Figure 2.4 shows results using the (inviscid) random choice scheme. Data are given by (3.1) with $\rho = 0.75$, $\delta = 0.9\pi$.

Finally, topic (c) is made visible in the graphics of Figure 3, which presents a sequence of data that approach the interesting case $\delta = \pi$. Shown are results using: (Figure 3.1) equation (2.9) with $\epsilon = 0.0075$; (Figure 3.2) equation (2.9) with $\epsilon = 0.005$; (Figure 3.3) second order Godunov method; (Figure 3.4) the (inviscid) random choice method. The data are given by $\delta = 0.75\pi, 0.95\pi, \pi$, with constant $\rho = 0.75$.

4 INTERPRETATION

In this section, we will try to elucidate the origin and meaning of the results from the numerical experiments presented in Section 3. Specifically, we will collect some analytical results on (1.1) (in Sub-section 4.1), discuss details of the individual experiments (in Sub-section 4.2), and give more details on the relationship of (1.1) to physical models (in Sub-section 4.3).

4.1. Some analytical results. We begin by examining more closely the “intuitive” way of solving the Riemann problem for (1.1) that was introduced at the very beginning of this paper. To any Riemann data (1.2), this approach assigns a solution $U(x, t) = \hat{U}(x/t)$ via:
if $|U_l| < |U_r|$:

$$\hat{U}(\sigma) = \begin{cases} U_l & \sigma < |U_l|^2 \\ U_m & |U_l|^2 < \sigma < 3|U_l|^2 \\ (\sigma/3)^{\frac{1}{2}} U_r / |U_r| & 3|U_l|^2 < \sigma < 3|U_r|^2 \\ U_r & \sigma > 3|U_r|^2, \end{cases} \quad (4.1.1)$$

if $|U_l| \geq |U_r|$:

$$\hat{U}(\sigma) = \begin{cases} U_l & \sigma < |U_l|^2 \\ U_m & |U_l|^2 < \sigma < |U_l|^2 + |U_l||U_r| + |U_r|^2 \\ U_r & \sigma > |U_l|^2 + |U_l||U_r| + |U_r|^2, \end{cases} \quad (4.1.2)$$

with $U_m = (|U_l|/|U_r|)U_r$ in both cases. Denoting by $S(U^0)$ the solution thus assigned to data U^0 we have the following property, see [12]:

Lemma 1. *(Dynamical stability.) S is continuous with respect to the $\mathcal{L}_{loc}^1$ -topology. Discontinuities are stable against small perturbations.*

For the general Cauchy problem, we have:

Lemma 2. *Let U^0 be any initial data for (1.1) such that the total variation of $r \circ U^0$ and $\theta \circ U^0$ is bounded. Then, for any pair $(\Delta x, \Delta t)$ satisfying the Courant–Friedrichs–Lewy condition: $3 \sup |U^0|^2 \Delta t \leq \Delta x$, using the solution operator S for the Riemann problem of (1.1) in Glimm’s scheme defines a step function $U = U(x, t)$ for all $x \in \mathbb{R}, t \geq 0$. If ${}^k(\Delta x, \Delta t)$ is an appropriate sequence of such pairs converging to zero, the corresponding sequence of step functions kU contains a subsequence which converges to a weak solution of (1.1).*

In [31], Liu and Wang have proved analogous statements for a class of rotationally invariant systems under certain assumptions; only slight modifications are required to extend their proof to imply Lemma 2.

Next, we examine the solution operator for the Riemann problem of (1.3) given by [28, 40]. To any data

$$u^0(x) = \begin{cases} u_l & x < 0 \\ u_r & x > 0 \end{cases} \quad (4.2)$$

this solution operator assigns a function $u(x, t) = \hat{u}(x/t)$ as follows:

if $u_l \in [0, 1]u_r$:

$$\hat{u}(\sigma) = \begin{cases} u_l & \sigma < 3u_l^2 \\ (\sigma/3)^{\frac{1}{2}}u_r/|u_r| & 3u_l^2 < \sigma < 3u_r^2 \\ u_r & \sigma > 3u_r^2, \end{cases} \quad (4.3.1)$$

if $u_r \in [-\frac{1}{2}, 1]u_l$:

$$\hat{u}(\sigma) = \begin{cases} u_l & \sigma < u_l^2 + u_l u_r + u_r^2 \\ u_r & \sigma > u_l^2 + u_l u_r + u_r^2, \end{cases} \quad (4.3.2)$$

if $u_r \in (-\infty, -\frac{1}{2})u_l$:

$$\hat{u}(\sigma) = \begin{cases} u_l & \sigma < \frac{3}{4}u_l^2 \\ (\sigma/3)^{\frac{1}{2}}u_r/|u_r| & \frac{3}{4}u_l^2 < \sigma < 3u_r^2 \\ u_r & \sigma > 3u_r^2. \end{cases} \quad (4.3.3)$$

Denoting by $s(u^0)$ the solution u thus assigned to given data u^0 , we characterize this procedure as follows:

Lemma 3. (i) *(Dynamical stability.) s is continuous with respect to the $\mathcal{L}_{loc}^1$ -topology. Discontinuities are stable against small perturbations.* (ii)

(*Vanishing viscosity.*) For given data u^0 , $s(u^0)$ is the unique limit obtained by the vanishing viscosity method.

These statements follow from Kruskov's theory, see [25]. By rotational invariance, (1.3) represents the restriction of (1.1) to any line (in phase space) through the origin. Thus s may be lifted in an obvious manner to yield a solution operator for the Riemann problem of (1.1) with co-linear data. The difficulty with the vanishing viscosity approach for (1.1) is now apparent; solutions obtained from s are different from those obtained from S for data such that $0 \in (U_l, U_r)$. The following discrepancy has been proved in [12]:

Lemma 4. (*Vanishing viscosity.*) If, for (1.1) with given data (1.2), the vanishing viscosity method converges to a piecewise continuous self-similar wave pattern, then this limit is $S(U^0)$ whenever $0 \notin (U_l, U_r)$. If, however, $0 \in (U_l, U_r)$, then the solution of the viscous problem ((2.9) with (1.2)) converges to the solution corresponding to s .

This shows that (1.1) and (2.9) are not uniformly close to each other.

4.2 Details of the experiments. Due to the similarity properties of system (1.1), all families of solutions to the Riemann problem corresponding to different given values of $U_l \neq 0$ are isomorphic to each other in the sense that a 1-1 correspondence between members of any two such families is established by an appropriately chosen scaling transformation. Thus, data (3.1) cover all cases with $U_l \neq 0$. In picking interesting values of ρ we were guided by the fact that the structure of the solution $S(U^0)$ undergoes a change at $\rho = 1$; specifically, the fast wave in $S(U^0)$ is a shock for $\rho < 1$, but a rarefaction for $\rho > 1$. Similarly, at $\delta = \pi$ the structure of the solution obtained by lifting $s(u^0)$ changes at $\rho = 0.5$; $s(u^0)$ changes from a single shock, for $\rho < 0.5$, to a composite wave, i.e. a shock with contact to an adjacent rarefaction wave, for $\rho > 0.5$. These exact solutions may be read off immediately from (4.1) and (4.3).

The results plotted in Figure 1 show that for small δ there is a reasonable agreement between the solutions obtained from the random choice method and those computed by using the Godunov scheme (modulo the spreading of contact discontinuities by the latter). On the other hand, when $\delta \rightarrow \pi$, there is a marked difference between the two.

We claim that this discrepancy is fully explained by the singular behavior of the vanishing viscosity approach observed in Lemma 4. This is obvious if we view the Godunov scheme as an approximation to a viscous regularization

of (1.1) rather than to (1.1) itself. To test this explanation, we examined the influence of artificial viscosity by solving (2.9). Hence the plots in Figure 2. Note that, according to Lemma 4, the singular behavior occurs for data with $0 \in (U_l, U_r)$ (or nearly so), corresponding to $\delta \approx \pi$; this is why we picked the value $\delta = 0.9\pi$ for this series of pictures. From the pictures it is obvious that, for fixed data, reducing the amount of artificial viscosity introduced by the scheme results in closer agreement with the ideal solution obtained from S .

Let us consider a family of data

$$U_l = (1, 0) \quad U_r = (\rho \cos \delta, \rho \sin \delta) \text{ with } \rho > 0 \text{ fixed and } \delta \approx \pi. \quad (4.4)$$

Assume that there exist reasonably behaved solutions $U_{\epsilon, \delta}$ of the regularization (2.9) for data (4.4). For a fixed value of $\epsilon > 0$ one expects $U_{\epsilon, \delta}$ to tend to $U_{\epsilon, \pi}$ for $\delta \rightarrow \pi$; if ϵ is small, $U_{\epsilon, \pi}$ will not differ much from the composite wave solution obtained from s (see (4.3), as a consequence of (ii) of Lemma 3). On the other hand, if $\delta \neq \pi$ is kept fixed, Lemma 4 suggests that $U_{\epsilon, \delta}$ converge, with $\epsilon \downarrow 0$, to the dynamically stable solution obtained from S (see (4.1)). Thus, there must be a very subtle competition between dissipation (ϵ) and dynamics (δ). This is what our third series of plots, in Figure 3, is about. From the left to the right of this figure, δ tends to π ; from the top to the bottom of the figure, the dissipation is decreased until it vanishes in the last row of plots. The pictures clearly demonstrate that the conjectured competition does actually take place.

We summarize our findings in the following

Conclusion. The Riemann problem for the hyperbolic system (1.1) is well-posed. It makes sense to use its dynamically stable solution, given by (4.1), in the random choice scheme; applying this scheme to Riemann data yields the dynamically stable solution. Using other standard schemes for the Riemann problem of (1.1) shows that this solution is well approximated by the solution of the corresponding viscous problem (2.9) for any fixed small value ϵ of the viscosity, as long as the data U_l, U_r are not too close (depending on ϵ) to satisfying the condition $0 \in (U_l, U_r)$. On one hand, as the data come to meet this condition more closely, the dynamically stable solution ceases to resemble the solution of the viscous problem. On the other hand, as ϵ is decreased, the region (in (U_l, U_r) -space) of good agreement expands, ultimately (for $\epsilon \approx 0$) covering all data for which $0 \notin (U_l, U_r)$.

Study of the Cauchy problem for (1.1) and (2.9) will be presented in [13, 14].

4.3. On the relationship to physical models. The system (1.1) considered in this paper is an example from the class of systems characterized by a rotationally equivariant flux, i.e.

$$f(U) = \Phi(|U|^2)U. \quad (4.6)$$

Systems of this kind were considered by Keyfitz and Kranzer in [24], who were interested in the Riemann problem for the equations governing planar motions of an ideal elastic string. These equations are of the form (see Antman [1])

$$\begin{aligned} U_t - V_x &= 0 \\ V_t - f(U)_x &= 0 \end{aligned} \quad (4.7)$$

with f satisfying (4.6). Any system exhibiting the structure (4.7) is hyperbolic if the corresponding system

$$U_t + f(U)_x = 0 \quad (4.8)$$

is hyperbolic and the eigenvalues of Df are all positive. Moreover there is an intimate relationship between the centered solutions of (4.7) and (4.8). This is why Keyfitz and Kranzer studied the Riemann problem for (4.8) with (4.6) first, with Φ satisfying certain assumptions. In doing so, they proved a number of interesting properties. Following these lines, Shearer [37] proved unique solvability of the original problem. Nonlinear plane waves in incompressible isotropic elastic bodies are also governed by equations of the form (4.7) with symmetry (4.6) (see Antman [2]).

The equations for nonlinear plane waves in compressible isotropic hyperelastic media have the form (4.7) (see Dafermos [8]) and so do the equations governing nonlinear plane magnetohydrodynamic waves (see Freistühler [10]). In these two cases, the function f also has a rotational equivariance, which is not, however, precisely of the form (4.6).

In all these cases, the behavior of certain transverse components of the state variable is adequately modeled (in a certain sense) by (4.8) with a flux (4.6). Obviously, the leading (nonlinear) part in a Taylor expansion of any such flux at $U = 0$ is, up to a multiplicative constant, $|U|^2 U$. This is what makes (1.1) a significant model, cf. [11, 4]. (Remark: This small amplitude

expansion argument does not apply to the elastic string problem, the setting which first gave rise to a study of the Riemann problem for a system with rotationally equivariant flux. In this problem, U represents the derivative of particle position along the string; hence U cannot vanish, since $U = 0$ would correspond to an infinite shrinking of the string. One must restrict phase space away from this unphysical value. More generally, the elastic string problem cannot be hyperbolic for $|U| < 1$, see Carasso, Rascle, and Serre [6].) With the exception of the elastic string problem, small values of $|U|$ do make sense for all systems mentioned above.

In particular, small values of $|U|$ arise in magnetohydrodynamics. With respect to this field of application, the work of Brio and Wu [5] and Wu [41] has been pioneering. They have investigated the so-called intermediate shocks. Brio and Wu contend that these shocks are physical and support this claim by numerical computations. In a sense our work may be considered as providing further evidence for this claim since the shocks appearing in the composite solution to the restricted problem (1.3) correspond to (certain) intermediate MHD shocks. On the other hand, our experiments also restrict this claim considerably. Let us explain this statement. As viscosity decreases (“viscosity” corresponds to electrical resistivity in this setting), ever smaller perturbations of the data split the intermediate shock into the dynamically stable two dimensional wave pattern of a contact discontinuity plus another wave. In the inviscid limit, intermediate shocks are completely unstable against generic perturbations.

We summarize by restating the conclusion drawn in the preceding subsection, formulating this conclusion as a rule for the construction of good numerical schemes for generic rotationally degenerate systems of physical conservation laws: As long as a solution does not come close to forming discontinuities of a certain type (satisfying an analogue of the above condition $0 \in (U_l, U_r)$), the presence or absence of a small viscosity plays no important rôle in determining the physical solution. When such discontinuities occur, one must make a clear distinction between the ideal and the viscous problem, and take care that the viscosity introduced by the computational method correctly models the desired (zero or non-zero) physical viscosity.

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FIGURE 1: Results from the Random Choice and Godunov schemes for various values of ρ and δ , at time $T = 0.75$. Shown are the plots of u (—) and v (- -). We exhibit only the range $0 \leq x \leq 3$ (neglecting the trivial piece $-0.15 \leq x \leq 0$). In Figure 1.1, $\rho = 0.25$ and δ takes the values $0.25\pi, 0.75\pi, 1.0\pi$, as indicated on the top label. In Figure 1.2, $\rho = 0.75$; Figure 1.3, $\rho = 1.0$; Figure 1.4, $\rho = 1.1$.

Figure 1.1a

Figure 1.1b

Figure 1.1c

Figure 1.1d

Figure 1.1e

Figure 1.1f

Figure 1.2a

Figure 1.2b

Figure 1.2c

Figure 1.2d

Figure 1.2e

Figure 1.2f

Figure 1.3a

Figure 1.3b

Figure 1.3c

Figure 1.3d

Figure 1.3e

Figure 1.3f

Figure 1.4a

Figure 1.4b

Figure 1.4c

Figure 1.4d

Figure 1.4e

Figure 1.4f

FIGURE 2: A comparison of the Random Choice scheme, with various values of the viscosity parameter ϵ , and the Godunov scheme. For these plots, $\rho = 0.75$ and $\delta = 0.9\pi$. In Figure 2.1, $\epsilon = 0.0075$; in Figure 2.2, $\epsilon = 0.005$. Figure 2.3 shows results using the Godunov scheme. Figure 2.4 shows results from the inviscid Random Choice scheme, i.e. $\epsilon = 0.0$.

Figure 2.1

Figure 2.2

Figure 2.3

Figure 2.4

FIGURE 3: The competition between dynamics and viscosity is illustrated in this sequence of plots, which shown the results of the Random Choice scheme with different values of viscosity and the Godunov scheme. In each plot, $\rho = 0.75$ and δ takes the values $0.75\pi, 0.95\pi, 1.0\pi$. In Figure 3.1, $\epsilon = 0.0075$; in Figure 3.2, $\epsilon = 0.005$. Figure 3.3 shows results using the Godunov scheme. Figure 3.4 shows results from the inviscid Random Choice scheme.

Figure 3.1a

Figure 3.1b

Figure 3.1c

Figure 3.2a

Figure 3.2b

Figure 3.2c

Figure 3.3a

Figure 3.3b

Figure 3.3c

Figure 3.4a

Figure 3.4b

Figure 3.4c

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