**HOMOCLINIC BIFURCATION TO A TRANSITIVE
ATTRACTOR OF LORENZ TYPE, II**

By

Clark Robinson

IMA Preprint Series # 719

November 1990

HOMOCLINIC BIFURCATION TO A TRANSITIVE ATTRACTOR OF LORENZ TYPE, II

CLARK ROBINSON*

Abstract. In this paper we prove that there is a co-dimension two bifurcation of a double homoclinic connection of a fixed point with a resonance condition among the eigenvalues to a transitive attractor that is like that of the geometric model of the Lorenz equations. The two key parameters are the variation of the eigenvalues from resonance and the amount that the homoclinic connection is broken. Because we need to work near resonance of two of the eigenvalues, one of the key steps in the proof is to calculate the Poincaré-Dulac map past a fixed point in this situation. We also indicate how this bifurcation is realized for a specific cubic differential equation introduced by Rychlik which are closely related to the Lorenz equations.

Key words. Attractor, Lorenz, Homoclinic bifurcation

AMS(MOS) subject classifications (1985 revision). 58F15, 58F13

1. STATEMENT OF MAIN THEOREM

In our previous paper [7], following the work of Rychlik [9], we proved that there is a homoclinic bifurcation to a transitive attractor. We also discussed the connection with the Lorenz equations and the geometric Lorenz model. In this paper, we reconsider this same theorem, improve the proof of the Poincaré-Dulac map past the fixed point, and emphasize the fact that it is a co-dimension two bifurcation.

We start by giving the general assumptions that the parameter must satisfy at the bifurcation value, (A0) to (A5). Then, rather than give conditions that a curve of parameter values must satisfy as in the earlier paper, we describe the part of parameter space near the bifurcation value that has an attractor. This description is given in terms of two key unfolding parameters. In Section 3, we will verify the assumptions for the cubic equations in R^3 which Rychlik considered.

The 0th assumption introduces the general assumptions and notation on the the symmetry about the z -axis, and the eigenvalues and eigenvectors for the fixed point at the origin. In this paper we only consider the case when the equations have a symmetry under reflection in the z -axis, $(x, y, z) \rightarrow (-x, -y, z)$. In our previous paper, we considered one type of nonsymmetric equations. The general situation for nonsymmetric equations that allows this type of bifurcation has yet to be determined. Assumptions 1 through 5 give the conditions at the bifurcation parameter value $\eta = \eta_0$.

(A0) We consider a C^r vector field on R^3 for $r \geq 2$, X , which depends on parameters η with a fixed point at the origin, $Q = (0, 0, 0)$, for all parameter values. We assume

*Northwestern University, Evanston, IL. Partially supported by a grant from the National Science Foundation.

that the vector field is taken to itself under the action of the reflection in the z -axis, $(x, y, z) \rightarrow (-x, -y, z)$. Further, we assume that the eigenvalues of $DX(Q)$ are all real with $\lambda_{ss}(\eta) < \lambda_s(\eta) < 0 < \lambda_u(\eta)$, and with respective eigenvectors v_{ss} , v_s , and v_u , and that $v_s = (0, 0, 1)^{tr}$ is along the axis of symmetry.

With these assumptions, there are several invariant manifolds for the fixed point at the origin. We denote the one dimensional unstable manifold tangent to v_u by $W^u(Q, \eta)$, and the two dimensional stable manifold tangent to the v_s and v_{ss} by $W^s(Q, \eta)$. Next, there is a strong stable manifold tangent to v_{ss} which we denote by $W^{ss}(Q, \eta)$. This latter manifold is made up of points which converge to Q at an asymptotic rate determined by the eigenvalue λ_{ss} . All of these manifolds are C^r if the vector field is C^r , and are even real analytic if the vector field is real analytic. Finally, there is a two dimensional manifold tangent to the two most expanding directions, v_u and v_s , which we denote by $W^{us}(Q, \eta)$. This manifold is local in the stable direction but can be extended along the unstable manifold by flowing forward in time. We call this the weak unstable manifold even though it is not expanding in all directions. This manifold is at least C^1 (and C^2 with assumption (A3) on the dominance of the contraction toward $W^{us}(Q, \eta)$ given by $e^{\lambda_{ss}}$ in comparison with the greatest contraction within $W^{us}(Q, \eta)$ given by e^{λ_s} .) With this notation we can make the second assumption about the existence of a homoclinic orbit. Because we are considering equations with symmetry, if one branch of $W^u(Q, \eta_0)$ is homoclinic, it follows that both sides are homoclinic. Thus within equations with symmetry it is only a co-dimension one condition to have a double homoclinic connection.

(A1) There is a bifurcation parameter value η_0 for which there is a *double homoclinic connection* with the unstable manifold of Q contained in the stable manifold but outside the strong stable manifold, $\Gamma \equiv W^u(Q, \eta_0) \subset W^s(Q, \eta_0) \setminus W^{ss}(Q, \eta_0)$. Also because of the symmetry, both branches of $W^u(Q, \eta_0)$ are contained in the same component of $W^s(Q, \eta_0) \setminus W^{ss}(Q, \eta_0)$.

(A2) For η_0 , the two dimensional weak unstable manifold $W^{us}(Q, \eta_0)$ is *transverse* to the two dimensional stable manifold $W^s(Q, \eta_0)$ along Γ . This transversality together with the fact that $\Gamma \subset W^s(Q, \eta_0) \setminus W^{ss}(Q, \eta_0)$, implies that $W^{us}(Q, \eta_0)$ is tangent to itself at Q (when it leaves and returns to Q .) Let $P_{\eta_0}(q) = T_q W^{us}(Q, \eta_0)$ for $q \in \Gamma$. If the bundle $\{P_{\eta_0}(q) : q \in \Gamma\}$ is orientable along Γ , we set $\nu = 1$, and if it is nonorientable we set $\nu = -1$. (The orientability is the same over both branches of Γ by the symmetry of the equations.)

This condition is generically satisfied so does not add a co-dimension to the bifurcation.

(A3) We assume that for η_0 *the strong stable eigenvalue dominates* the other two eigenvalues in the sense that $\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + \lambda_u(\eta_0) < 0$ and $\lambda_{ss}(\eta_0) - 2\lambda_s(\eta_0) < 0$.

This is an open condition so does not add a co-dimension to the bifurcation. The second inequality in (A3) is what assures that the manifold $W^{us}(Q, \eta_0)$ is C^2 . It is also

redundant with the following resonance assumption (but sometimes we want to assume (A3) but not necessarily assume (A4).)

(A4) There is a one to one *resonance* between the unstable and weak stable eigenvalues for η_0 : $\lambda_u(\eta_0) + \lambda_s(\eta_0) = 0$, so $1 - |\lambda_s(\eta_0)|/\lambda_u(\eta_0) = 0$.

This resonance conditions is a co-dimension one and gives the second co-dimension which the bifurcation parameter must satisfy. The final assumption on the bifurcation parameter is the extent to which area is changed within the $P(q)$ directions (“within the attractor directions”) during one loop around Γ .

(A5) For this assumption fix $\eta = \eta_0$. Let $\{P_{\eta_0}(q) : q \in \Gamma\}$ be the continuous bundle of planes given in (A2). Let $q(t)$ be a homoclinic orbit along one of the two branches of Γ , and let $\text{div}_2(t)$ be the rate of change of area within the plane field $P_{\eta_0}(q(t))$. Define $C_{\eta_0} > 0$ by

$$\int_{-\infty}^{\infty} \text{div}_2(q(t))dt \equiv \log(C_{\eta_0}).$$

We assume that the *change of area* $C_{\eta_0} < 2$. (Note, because the equations are symmetric, the integral is independent of which branch of Γ is used. Note also that if $\lambda_u(\eta_0) + \lambda_s(\eta_0) \neq 0$, then the integral would be $\pm\infty$ and $C_{\eta_0} = \infty$ or 0 .)

It turns out that C_{η_0} has meaning in terms of a one dimensional Poincaré map, f_{η_0} , from making one trip near one of the branches of Γ . We give a few definitions to explain this fact more fully and which will also be used in the finally assumption and statement of the theorem. Let Σ be a fixed transversal to Γ out a short distance along the local unstable manifold from Q . Points in a neighborhood V of Γ on $\Sigma \setminus W^s(Q, \eta)$ will return to Σ defining a Poincaré map

$$F_\eta : V \setminus W^s(Q, \eta) \subset \Sigma \rightarrow \Sigma.$$

In the proof, it is shown that Assumption (A3) implies that there is an invariant continuous bundle of strong stable directions over Γ , $\{E^{ss}(q) : q \in \Gamma\}$ with $E^{ss}(Q) = \langle v_{ss} \rangle$. These conditions are open so this bundle exists not only over Γ for η_0 but also over a neighborhood of Γ for nearby η . The stable manifold theory then implies that there is an invariant strong stable foliation for these nearby η . Projection along the leaves of the strong stable manifolds of orbits defines a map $\pi_\eta : \Sigma \rightarrow \Sigma^1$ and a one dimensional map $f_\eta : V^1 \setminus \{0\} \subset \Sigma^1 \rightarrow \Sigma^1$ where $V^1 = \pi_\eta(V)$ and $f_\eta(\pi_\eta w) = \pi_\eta F_\eta(w)$. We use coordinates on Σ^1 so that 0 corresponds to the point on $W^s(Q, \eta)$, and the symmetry gives $f_\eta(-u) = -f_\eta(u)$.

Let $a(\eta)$ be the signed distance, as measured in Σ^1 of the negative branch of $W^u(Q, \eta)$ from $W^s(Q, \eta)$. This is equivalent to defining

$$a(\eta) = \limsup_{u < 0, u \rightarrow 0} f_\eta(u).$$

Thus the negative branch of $W^u(Q, \eta)$ intersects the positive side of Σ if $a(\eta) > 0$, and it intersects the negative side if $a(\eta) < 0$. Because of the symmetry, and the distance of the positive branch of $W^u(Q, \eta)$ from $W^s(Q, \eta)$ is $-a(\eta)$. The interval $I_\eta = [-|a(\eta)|, |a(\eta)|]$ corresponds to the interval in Σ^1 between the intersections of the two branches of $W^u(Q, \eta)$ with the transversal after projecting out the strong stable direction. With these definitions, Lemma 1 proves that $f'_{\eta_0}(0) = \nu C_{\eta_0}$. The fact that $C_{\eta_0} < 2$ means there is hope for $f_\eta([0, |a(\eta)|]) \subset I_\eta$. See Lemma 2 for details.

Let $b(\eta) = 1 - |\lambda_s(\eta)/\lambda_u(\eta)|$. This quantity measures the extent to which the two eigenvalues are no longer in resonance.

Next, we discuss the relationship between the unfolding parameters $a(\eta)$ and $b(\eta)$ must be satisfied as the homoclinic connection is broken in order for there to be an attractor. For a small neighborhood $\mathcal{N}$ of η_0 in parameter space, let

$$\mathcal{N}' = \{\eta \in \mathcal{N} : \nu a(\eta) > 0, 0 < f_\eta(a(\eta)) \leq |a(\eta)|, b(\eta) \geq 0, |f'_\eta(\pm a(\eta))| \geq 2^{\frac{1}{2}}\}.$$

Lemma 2 proves that the boundary of $\mathcal{N}'$ is contained in $\partial\mathcal{N}$, γ_1 , and γ_2 , where the latter two are given by

$$\begin{aligned} \gamma_1 : & \quad \{\eta \in \mathcal{N} : f_\eta(a(\eta)) = \nu a(\eta), b(\eta) > 0\}, \\ \gamma_2 : & \quad \begin{cases} b(\eta) = 0, & \text{if } C_{\eta_0} < 2^{\frac{1}{2}} \\ \{\eta \in \mathcal{N} : b(\eta) > 0, |f'_\eta(\pm a(\eta))| = 2^{\frac{1}{2}}\}, & \text{if } C_{\eta_0} > 2^{\frac{1}{2}}. \end{cases} \end{aligned}$$

The form of γ_2 when $C_{\eta_0} = 2^{\frac{1}{2}}$ is some combination of the two forms above. The condition related to γ_2 implies that for $\eta \in \mathcal{N}'$ that $|f'_\eta(u)| > 2^{\frac{1}{2}}$ for all points $u \in \text{interior } I_\eta$. This latter condition is what is used to imply that the attractor is transitive.

In order to insure that $\mathcal{N}' \neq \emptyset$ and $\eta_0 \in \text{closure } \mathcal{N}'$ we need to make the following assumption on the ability to vary the parameters.

(A6) We assume that the parameter space for η is large enough so that $a(\eta)$ and $b(\eta)$ can vary independently for η near η_0 .

Finally, we can state the main result.

THEOREM 1. *Assume that vector field in R^3 , depending on a parameter η is C^2 and satisfies the above assumptions (A0)–(A6). Then, there is a neighborhood $\mathcal{N}_0 \subset \mathcal{N}$ in parameter space such that, $\mathcal{N}_0 \cap \mathcal{N}' \neq \emptyset$, $\eta_0 \in \text{closure } \mathcal{N}'$, and for $\eta \in \mathcal{N}_0 \cap \mathcal{N}'$ the flow for η has a topologically transitive attractor. The two pieces γ_1 and γ_2 of the boundary of $\mathcal{N}_0 \cap \mathcal{N}'$ satisfy the following equations:*

$$\begin{aligned} \gamma_1 : & \quad \log(2/C_{\eta_0}) = \limsup\{[(b(\eta) \log(1/|a(\eta)|))] : \eta \rightarrow \eta_0, \eta \in \gamma_1\}, \\ \gamma_2 : & \quad \begin{cases} \liminf\{[(b(\eta) \log(1/|a(\eta)|))] : \eta \rightarrow \eta_0, \eta \in \gamma_2\} = \log(2^{\frac{1}{2}}/C_{\eta_0}), & \text{if } C_{\eta_0} > 2^{\frac{1}{2}} \\ b(\eta) = 0, & \text{if } C_{\eta_0} < 2^{\frac{1}{2}}. \end{cases} \end{aligned}$$

Figure 1

Graph of the allowable range of $b(\eta)$ versus $a(\eta)$. (a) Graph for $C_{\eta_0} = 2^{\frac{1}{2}}$. (Similar ranges are valid for $2^{\frac{1}{2}} < C_{\eta_0} < 2$.) (b) Graph for $C_{\eta_0} = 1$. (Similar ranges are valid for $0 < C_{\eta_0} < 2^{\frac{1}{2}}$. This is the case we prove occurs for equations (R') and (Rob).)

If the value of ν is 1 (resp. -1), then the attractor is orientable (resp. nonorientable), and the attractor appears for $a(\eta)$ positive (resp. negative), i.e., it appears before (resp. after) the unstable manifold crosses over the stable manifold. If the vector field is C^3 then the resulting one dimensional Poincaré map f_η has an ergodic invariant measure with support equal to the whole interval I_η and which is equivalent to Lebesgue on I_η .

2. PROOF OF THEOREM 1

Throughout this section, we assume that the system satisfies assumptions (A0) – (A3).

The first step is to prove the existence of a $C^{1+\mu}$ for some $0 < \mu < 1$ strong stable foliation in a neighborhood of the homoclinic connection for a perturbation of the flow. (Here $C^{1+\mu}$ means that the first derivative is μ -Hölder.) The details of this argument are carried out in [7] and follow the ideas of the earlier papers [5] and [6]. The sketch of the proof is as follows. The existence of the continuous bundle of planes $\{P_{\eta_0}(q) : q \in \Gamma\}$ can be used to show the existence of a continuous strong stable bundle by using cone that are complementary to the $P_{\eta_0}(q)$. In fact, since a trajectory in a small neighborhood of Γ spends an arbitrarily large proportion of its time in a neighborhood of the fixed point where the eigenvalues give the desired estimates, it is possible to prove the existence of a $C^{1+\mu}$ strong stable bundle, E^{ss} , in a neighborhood of the homoclinic connection. These conditions are open, so for a small perturbation such a strong stable bundle persists in this open set of phase space.

Then by stable manifold theory, there is a $C^{1+\mu}$ foliation whose tangent lines are given by these strong stable bundles. See [5] which uses [2, Theorem 4.8]. A strong stable leaf or manifold at a point ζ_0 of radius r , $W_r^{ss}(\zeta_0, \eta)$, is characterized as being the points ζ within distance r such that the distance ' between the trajectories for ζ and ζ_0 at time t converges to zero at an exponential rate of almost $\exp(t\lambda_{ss})$.

Once we know that the strong stable manifolds form a $C^{1+\mu}$ foliation, we can form the strong stable manifolds of orbits by taking the union of the strong stable manifolds of points along an orbit:

$$W_r^{ss, orbit}(\zeta_0, \eta) = \cup \{W_r^{ss}(\zeta, \eta) | \zeta = \varphi_t(\zeta_0, \eta) \text{ for } -r \leq t \leq r\}.$$

The tangent space to $W_r^{ss, orbit}(\zeta_0, \eta)$ at ζ_0 is spanned by the strong stable bundle E^{ss} at ζ_0 and the vector field $X_\eta(\zeta_0)$. Thus, these tangent spaces are $C^{1+\mu}$ away from Q . This implies that $\pi_\eta : \Sigma \rightarrow \Sigma'$ is $C^{1+\mu}$.

As a consequences of Assumption 1, the value of $\nu = 1$ (resp. $\nu = -1$) if the Poincaré map F_η preserves (resp. reverses) the orientation of the strong stable bundle. Hence, $\nu = 1$ (resp. $\nu = -1$) if f_η preserves (resp. reverses) the orientation within Σ^1 , so f_η is increasing (resp. decreasing). In terms of the flow, if $\nu = 1$ the “sheets within the attractor” return with either no twist or at least multiples of full twists, so the same side is up. On the other hand, if $\nu = -1$ they return with a half twist plus some multiple of full twists, so the sides are reversed.

We turn now to analyzing the Poincaré map for the flow. We need to find the lowest order terms for f_η and f'_η in order to show that we have an invariant set and an expansion on it. In the two dimensional case, if the flow is linearized near a saddle fixed point, then the Poincaré map past the fixed points is given by $g(u) = Cu^E$ where $E = |\lambda_s|/\lambda_u$. In our present situation, we reduce to the two dimensional case by projecting along the strong stable foliation and use the weaker stable and unstable directions near the fixed point. This quotient space can be thought of as projection onto the invariant surface tangent to $\langle v_u, v_s \rangle$ at Q . The resulting one dimensional Poincaré map past the fixed point has $\pm Cu^E$ as the lowest order terms as the follow lemma makes this precise.

LEMMA 1. Assume (A0) – (A3) are satisfied. Let $E = E_\eta = |\lambda_s(\eta)|/\lambda_u(\eta) = 1 - b(\eta)$. Let $J \subset \Sigma^1$ be a fixed small interval about 0. For η in a small neighborhood of η_0 , the induced one dimensional Poincaré map $f_\eta : J \setminus \{0\} \subset \Sigma^1 \rightarrow \Sigma^1$ has continuous derivative on $J \setminus \{0\}$, and f_η and f'_η have the following form:

$$\begin{aligned} f_\eta(u) &= (-a(\eta) + \nu C_\eta |u|^E) \text{sign}(u) + o(|u|^E) \\ f'_\eta(u) &= \nu E_\eta C_\eta |u|^{E-1} + o(|u|^{E-1}) \end{aligned}$$

with $\nu = \pm 1$ depending on whether the strong stable bundle is orientable or not, $C_{\eta_0} \geq 0$ is given by the integral in (A5), and C_η depending continuously on η . The remainder terms

Figure 2

Graph of f_η . (a) $\nu = 1$, $a(\eta) = 0.2$, Graph of f_η . (a) $\nu = 1$, $a(\eta) = 0.2$, $E = \frac{2}{3}$, and $C = 1$. (b) $\nu = -1$, $a(\eta) = -0.2$, $E = \frac{2}{3}$, and $C = 1$.

uniformly go to zero when divided by the terms indicated as u goes to zero uniformly in η . Note that for η_0 , $f_{\eta_0}(u) = \nu C_{\eta_0} u + o(|u|)$, and $f'_\eta(u) = \nu C_{\eta_0} + o(|u|^0)$ so $f'_{\eta_0}(0)$ exists and equals νC_{η_0} . Also, if $E_\eta \leq 1$ (e.g., for $\eta \in \mathcal{N}'$), then the branches of $(f_\eta^{-1})'(u)$ have Hölder extensions at $f_\eta(0\pm)$.

Remark 1. The proof of this lemma also works in the nonsymmetric case. The delicate part of the argument about the Poincaré-Dulac map past the fixed point is the same. The part of the return map outside the neighborhood depends on the branch of the unstable manifold which is followed. Therefore, the form of the map is

$$f_\eta(u) = \begin{cases} a^+(\eta) + \nu^+ C_\eta^+ |u|^E + o(|u|^E) & \text{for } u > 0 \\ a^-(\eta) + \nu^- C_\eta^- |u|^E + o(|u|^E) & \text{for } u < 0 \end{cases}$$

where $a^\pm(\eta)$, $\nu^\pm$, and $C_\eta^\pm$ all depend on whether u is positive or negative.

Remark 2. There are several ways to prove this lemma under a variety of more or less restrictive hypothesis. In [7], an elementary but messy proof was given. The equations are normalized near the fixed point with coordinates u in the unstable direction, v in the strong stable direction, and z in the weaker stable direction. The variable $\rho = uv$, which is introduced, then satisfies the scalar equation

$$\dot{\rho} = \Lambda_\eta \rho + A_2 \rho^2 + \cdots + A_n \rho^n$$

where $\Lambda_\eta = \lambda_u(\eta) + \lambda_s(\eta)$. It is then necessary to prove the expansion of the solution $\rho(\tau)$ with initial condition ρ_0 for a time $\tau \approx -\lambda_u^{-1} \log |\rho_0|$ which are uniformly valid for

$\Lambda_\eta \geq 0$. (This value of τ is the time it takes to flow past the fixed point to a second transversal.) The method used only obtained estimates on an interval in initial conditions in state space whose length decreases as the parameter $b(\eta) = 1 - E_\eta$ goes to zero. A better proof of this general type was done earlier by Roussarie, [8, Theorem F]. He obtained a higher order expansion and showed that it was valid on an interval of uniform size as the parameter varies, although it involves $x \log(x)$ terms which go to zero slowly in the position variable x . We make further comments below about how to use his theorem to obtain our result. Finally, it is possible to prove the lemma by $C^{1+\alpha}$ linearizing within an invariant two dimensional surface near the fixed point. Hartman showed this was possible in two dimensions even with resonance. This proof is also valid on an interval of fixed size and is more in keeping with other arguments used in the proof of the main theorem. We give details of this proof below. We should mention that C. Chicone and C. Pugh both suggested using this approach.

Proof using Hartman's linearization. We want to prove that the one dimensional Poincaré map is C^1 . The two dimensional Poincaré map F_η can be split into the part G_η past Q from Σ to a second transversal S , and the part H_η from S back to Σ . Since there is a bounded amount of time for the trajectory to pass from S back to Σ , H_η is C^2 . All that remains to check is the form of G_η past Q . Hartman's linearization proof works by constructing two invariant C^1 foliations in the stable and unstable directions. The difficulty is that the unstable foliation can only be shown to be C^0 in the directions transverse to the leaves if we consider the flow in three dimensions. To avoid this difficulty, we show there is an invariant two dimensional manifold $W^{us}(Q, \eta)$ in the weak stable and unstable directions, and show that the foliations are $C^{1+\beta}$ when restricted to this two dimensional manifold. The proof of this differentiability is very similar to the one we used above to get the existence of the strong stable foliation. The difference is that everything must be restricted to $W^{us}(Q, \eta)$ in order for the estimates to be true. The estimates for one dimensional Poincaré map g_η is then determined using the linearization of the flow on $W^{us}(Q, \eta)$ using these two foliations. Then $f_\eta(u) = \pi_\eta H_\eta \circ g_\eta(u)$.

The image of any point 0- in by f_η is $a(\eta)$ because $\pi_\eta H_\eta$ takes the negative branch of the unstable manifold to $a(\eta)$.

By (A3), $\lambda_{ss}(\eta) - (2 + \alpha)\lambda_s(\eta) < 0$ for small $\alpha > 0$ and η near η_0 . By the invariant manifold theorem of [3], there is a $C^{2+\alpha}$ manifold $W^{us}(Q, \eta)$ tangent to the v_u and v_s directions at Q . Its tangent space E^{us} is $C^{1+\alpha}$. Thus the derivative of the time one map of the flow $D\varphi^1$ is $C^{1+\alpha}$ when restricted to $E^{us}|W^{us}(Q, \eta)$.

Turning to the invariant subbundles, if β is chosen with $0 < \beta \leq \alpha$ and $\lambda_s(\eta_0) + \beta\lambda_u(\eta_0) < 0$ then there is a $C^{1+\beta}$ invariant subbundle $E^s \subset E^{us}|W^{us}(Q, \eta)$ for η near η_0 . The reason this is true is that $D\varphi^{-1}$ contracts toward E^s within E^{us} by a factor of almost $\exp(\lambda_s(\eta_0) - \lambda_u(\eta_0))$. The Lipschitz constant of φ^1 is about $\exp(\lambda_u(\eta_0))$. To prove the bundle is $C^{1+\beta}$ the results of [3] say we need the product of the first of these two numbers times $1 + \beta$ times the second has to be less than one, so $0 > \lambda_s(\eta_0) - \lambda_u(\eta_0) + (1 + \beta)\lambda_u(\eta_0)$.

Thus in a neighborhood of Q we will have the correct domination of the contraction rate.

Next, by taking $\beta > 0$ smaller so that $-\lambda_u(\eta_0) - \beta\lambda_s(\eta_0) < 0$, there is a $C^{1+\beta}$ invariant subbundle E^u of $E^{us}|W^{us}(Q, \eta)$ for η near η_0 . This follows because the contraction rate toward this bundle by $D\varphi^1$ is about $\exp(-\lambda_u(\eta_0) + \lambda_s(\eta_0))$. The Lipschitz constant of φ^{-1} on $W^{us}(Q, \eta)$ is about $\exp(-\lambda_s(\eta_0))$. Again, the above estimate gives us that the product of the first of these number times $1 + \beta$ times the second is less than one. This gives the stated differentiability of the bundle.

Since each of these bundles is $C^{1+\beta}$, there are two $C^{1+\beta}$ foliations tangent to them. These give coordinates on $W^{us}(Q, \eta)$. in terms of coordinates on $W^u(Q, \eta)$ and $W^s(Q, \eta)$. Since these coordinates can $C^{1+\beta}$ linearize the flow the Poincaré-Dulac map past Q in these coordinates is the same as that for the linear flow, $g_\eta(u) = C^E$ where $E = E(\eta) = |\lambda_s(\eta)|/\lambda_u(\eta)$. The map in the original coordinates can then be written as $k_\eta \circ g_\eta \circ h_\eta$ where k_η and h_η are the $C^{1+\beta}$ change of coordinates which vary continuously with respect to η . This show that the remainders go to zero as u goes to zero, uniformly in η .

The connection of C_η with the integral in (A5) is discussed in [7], and uses the formula for the derivative of the Poincaré map in terms of the integral of the divergence given in [1, §28].

The comment about the the branches of $(f_\eta^{-1})'(\zeta)$ follows by looking at the flow for negative time past Q . The above proof shows that the derivative of the inverse on either branch has the form

$$(f_\eta^{-1})'(\zeta) = \nu E_\eta C_\eta |a(\eta) - \zeta|^{1-E} + o(|a(\eta) - \zeta|^{1-E})$$

which has a Hölder extension at $\zeta = a(\eta)$. $\square$

Proof using Roussarie's expansion. To use Roussarie's result, we again need to put the equation in normal form so we need to assume the vector field is C^∞ (or need to keep track of some higher differentiability which is determined by the eigenvalues.) As is done in [7] and [8], by a "Sternberg differentiable linearization" there is a C^∞ change of coordinates that put the differential equations into the normal form

$$\begin{aligned}\dot{v} &= \lambda_{ss}v \\ \dot{u} &= \lambda_u u \\ \dot{z} &= z[\lambda_s + \lambda_u \alpha_2 \rho + \lambda_u \alpha_3 \rho^2 + \dots]\end{aligned}$$

where $\rho = uz$. The proof of the differentiable linearization is valid uniformly in the parameter by using the form given by Takens in [11], as was mentioned to us by Roussarie. This also shows that the theorem is just as valid in conjugating to a normal form as given above as it is to conjugating to a linear system. All that is necessary is for the two systems to have the same C^∞ jet at the fixed point.

As in proof 1, the only part of the Poincaré map that needs to be checked is the one dimensional map past Q . Using the normal form, this is calculated using the u and z equations. [8, Theorem F] proves the form of this map to any finite order that is uniformly valid for a range of parameters near resonance. He considers the two variables u and ρ . and looks at the two equations

$$\begin{aligned}\dot{u} &= \lambda_u u \\ \dot{\rho} &= \lambda_u [b(\eta)\rho + \alpha_2 \rho^2 + \dots],\end{aligned}$$

where $b(\eta) = [\lambda_s(\eta) + \lambda_u(\eta)]/\lambda_u(\eta)$ is as before. (He rescales time by $\lambda_u(\eta)$ which is the reason that we have factored out this term.) Next, he defines

$$\omega(b, u) = \begin{cases} \frac{u^{-b}-1}{b} & \text{for } b = b(\eta) \neq 0 \\ -\log(u) & \text{for } b(\eta) = 0. \end{cases}$$

Then he proves that the following expansion is uniformly valid in η with C^2 remainder:

$$g_\eta(u) = Cu^{1-b} + \alpha_2 u^{2-b} \omega(b, u) + \psi_\eta(u),$$

where $\psi_\eta(u)$ is C^2 and C^2 flat at $u = 0$. Remember that $E_\eta = 1 - b(\eta)$. Since $u\omega(b, u) - u\omega(0, u)$ goes to zero uniformly on a fixed interval and $u\omega(0, u) = o(u^0)$, we get the form given in the theorem. A direct calculation shows that

$$g'_\eta(u) = CEu^{E-1} + \alpha_2(1+E)u^{E-1}(u\omega) - \alpha_2 u^{E-1}u^E + \psi'_\eta(u)$$

so $f'_\eta(u)$ has the form that is given in the lemma. The other part of the proof is as is done above in the Hartman's linearization proof. $\square$

The following lemma proves the form of the boundary of $\mathcal{N}'$.

LEMMA 2. Assume that the systems satisfies (A0) – (A3).

(a.) In a small neighborhood $\mathcal{N}_0$ of η_0 , for the system for $\eta \in \mathcal{N}_0$ to have a transitive attractor with $|f'_\eta(u)| > 2^{\frac{1}{2}}$ for all $u \in \text{interior } I_\eta$, it is necessary for $\eta \in \mathcal{N}'$. If we assume $\eta_0 \in \text{closure } \mathcal{N}'$ so there are transitive attractors arbitrarily near η_0 , then (A4) is true so $b(\eta_0) = 0$ or $E_{\eta_0} = 1$, and (A5) is true in the modified sense that $C_{\eta_0} \leq 2$.

(b.) Assume the system satisfies (A0) – (A6). Then, the two pieces of the boundary of the set $\mathcal{N}'$ are defined by

$$\begin{aligned}\gamma_1 : & \quad \{\eta \in \mathcal{N} : f_\eta(a(\eta)) = \nu a(\eta), b(\eta) > 0\}, \\ \gamma_2 : & \quad \begin{cases} b(\eta) = 0, & \text{if } C_{\eta_0} < 2^{\frac{1}{2}} \\ \{\eta \in \mathcal{N} : b(\eta), |f'_\eta(\pm a(\eta))| = 2^{\frac{1}{2}}\}, & \text{if } C_{\eta_0} > 2^{\frac{1}{2}}. \end{cases}\end{aligned}$$

Further, they satisfy the following two equations:

$$\begin{aligned} \gamma_1 : \quad & \log(2/C_{\eta_0}) = \limsup\{[(b(\eta)\log(1/|a(\eta)|))] : \eta \rightarrow \eta_0, \eta \in \gamma_1\}, \\ \gamma_2 : \quad & \begin{cases} \liminf\{[(b(\eta)\log(1/|a(\eta)|))] : \eta \rightarrow \eta_0, \eta \in \gamma_2\} = \log(2^{\frac{1}{2}}/C_{\eta_0}) & \text{if } C_{\eta_0} > 2^{\frac{1}{2}} \\ b(\eta) = 0 & \text{if } C_{\eta_0} < 2^{\frac{1}{2}}. \end{cases} \end{aligned}$$

Proof. To obtain an attracting set for the flow which contains the fixed point at the origin, f_η must preserve the interval $I_\eta = [-|a(\eta)|, |a(\eta)|] \subset V^1$. For $f_\eta(u) \in I_\eta$ for small u , the form of f_η given in Lemma 1 shows that it necessary for $\nu a(\eta) > 0$. The invariance of the interval also implies that $f_\eta(\pm a(\eta)) \in I_\eta$ so $f_\eta(a(\eta)) \leq |a(\eta)|$, i.e. that the second intersection of the unstable manifolds must lie between the first intersection of the two branches after projecting out along the strong stable foliation, (The reader can check the well known result that if $f_\eta(I_\eta) \setminus I_\eta \neq \emptyset$, then there is horseshoe for F_η which is not an attracting set.) Also for Q to be part of the attracting set, we need that 0 be an element of the image, $f_\eta(I_\eta)$, so $0 \leq f_\eta(a(\eta))$. Also the fact that f_η is transitive on I_η implies that $0 < f_\eta(a(\eta))$.

The condition that the absolute value of the derivative at all points of interior I_η is greater than $2^{\frac{1}{2}}$ certainly implies that it is greater or equal to $2^{\frac{1}{2}}$ at $\pm a(\eta)$. Finally, if $b(\eta) = 1 - E_\eta < 0$ then $|f'_\eta(u)| < 1$ for small u . Thus it is necessary for $b(\eta) \geq 0$. This completes the proof that it is necessary for $\eta \in \mathcal{N}'$.

If $\eta_0 \in \text{closure } \mathcal{N}'$, and the fact that $b(\eta) \geq 0$ for $\eta \in \mathcal{N}'$ implies that $b(\eta_0) \geq 0$. To get the opposite inequality we consider $\eta \in \mathcal{N}'$ that approach η_0 and write a for $a(\eta)$. (The reader may find it easier to just consider the case $\nu = 1$.) For these η $\nu a > 0$, so $\nu f_\eta(\nu a) \leq \nu a$, $\nu f_\eta(\nu a) + \nu a \leq 2\nu a$, $|f_\eta(\nu a) + a| \leq 2|a|$, and

$$\frac{|f_\eta(\nu a) + a|}{|aC_\eta|} \leq \frac{2}{C_\eta}.$$

Because of the form of the lowest order terms of f_η given in Lemma 1, taking limits as η goes to η_0 ,

$$\begin{aligned} \limsup \left\{ b(\eta) \log \left(\frac{1}{|a(\eta)|} \right) \right\} &= \limsup \left\{ \log \left(\frac{|f_\eta(\nu a(\eta)) + a(\eta)|}{|a(\eta)C_\eta|} \right) \right\} \\ &\leq \limsup \left\{ \log \left(\frac{2}{C_\eta} \right) \right\} \\ &= \log \left(\frac{2}{C_{\eta_0}} \right). \end{aligned}$$

Since $\log(2/C_{\eta_0})$ is a constant and $\log(1/|a(\eta)|)$ goes to infinity, we also get that $b(\eta_0) \leq 0$. Combining $b(\eta_0) = 0$. Also since $b(\eta) \geq 0$ for $\eta \in \mathcal{N}'$ and $\log(1/|a(\eta)|)$ goes to infinity, we get that $C_{\eta_0} \leq 2$. This completes the proof of part (a).

On one boundary of $\mathcal{N}'$ $f_\eta(a(\eta)) = \nu a(\eta)$ as stated for γ_1 , and putting equality in the above limit as η goes to η_0 shows that these values satisfy

$$\gamma_1 : \quad \log \left(\frac{2}{C_{\eta_0}} \right) = \limsup \left\{ (b(\eta) \log \left(\frac{1}{|a(\eta)|} \right)) : \eta \rightarrow \eta_0, \eta \in \gamma_1 \right\}$$

as stated.

From the form of $f'_\eta(u)$ given in Lemma 1, it follows that the smallest value of $|f'_\eta(u)|$ occurs for $u = \pm a(\eta)$. If $C_{\eta_0} > 2^{\frac{1}{2}}$ then the calculation below shows that the η with $|f'_\eta(a(\eta))| = 2^{\frac{1}{2}}$ occurs for $b(\eta) > 0$ so it forms the other boundary.

Along the η with $|f'_\eta(a(\eta))| = 2^{1/2}$, the form of the derivative given in Lemma 1 gives the following:

$$\begin{aligned} \liminf \left\{ b(\eta) \log \left(\frac{1}{|a(\eta)|} \right) \right\} &= \liminf \left\{ \log E_\eta + b(\eta) \log \left(\frac{1}{|a(\eta)|} \right) \right\} \\ &= \liminf \left\{ \log \left(\frac{|f'_\eta(a(\eta))|}{C_\eta} \right) \right\} \\ &= \liminf \left\{ \log \left(\frac{2^{1/2}}{C_\eta} \right) \right\} \\ &= \log \left(\frac{2^{1/2}}{C_{\eta_0}} \right). \end{aligned}$$

This gives the form of γ_2 for $C_{\eta_0} > 2^{\frac{1}{2}}$. If $C_{\eta_0} < 2^{\frac{1}{2}}$ then the calculation above shows that $|f'_\eta(a(\eta))| > 2^{\frac{1}{2}}$ so the other boundary is given by $b(\eta) = 0$. This completes the proof of Lemma 2. $\square$

Using Lemma 2, for $\eta \in \mathcal{N}'$, $|f_\eta(u)| > 2^{\frac{1}{2}}$ for $u \in \text{interior } I_\eta$. This means that f_η satisfies the condition of R. Williams to be locally eventually onto and so topologically transitive. See [12] and [6]. If the flow is C^3 the fact that $f_\eta^{-1}(u)$ has a Hölder extensions at $f_\eta(0\pm)$, means that the theorem of G. Keller applies, and we get the existence of an invariant measure for the interval map. See [6] which uses the results of [4].

This completes the proof of Theorems 1.

$\square$

3. VERIFICATION OF ASSUMPTIONS FOR SPECIFIC EQUATIONS

The Lorenz equations are given by the equations

$$\begin{aligned} \dot{x} &= -\sigma x + \sigma y \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= -\beta z + xy. \end{aligned} \tag{L}$$

They arise from modeling turbulence and their numerical simulation exhibits chaotic behavior.

Later, Rychlik showed that a slight variation of the Lorenz equation could be proved to possess a transitive attractor immediately after a bifurcation from a double homoclinic connection. The equations which he considers are motivated by noting that by a change of variables the Lorenz equations, (L) , can be put in the following form:

$$(R) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y + \delta x z \\ \dot{z} &= -\gamma z + x^2, \end{aligned}$$

with $\beta = 0$, [9]. He then added the $\beta x^2 y$ term, to be able to control the unstable manifold of the origin while keeping $\delta = 0$. He determines the form of the resulting two dimensional Poincaré map, and proves that it has a $C^{1+\mu}$ strong stable foliation immediately after a homoclinic bifurcation, but not at the bifurcation value. This proof extends the type of proof in [5], where one of the approaches show how to verify that the Poincaré map had a C^1 foliation. We also remark that Rychlik considers a different bifurcation problem than the one considered in this paper and in [7]. In particular, instead of (A2), he assumes that $W^{us}(Q, \eta_0)$ and $W^s(Q, \eta_0)$ are tangent along Γ . (This is his second codimension.) With this assumption, he is able to take $\lambda_u(\eta_0) + \lambda_s(\eta_0) > 0$.

In our previous paper we verified the assumptions for the set of equations

$$(Rob) \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 + \alpha y + \beta x^2 y - \nu y z \\ \dot{z} &= -\gamma z + \delta x^2, \end{aligned}$$

with $\nu = \pm 1$. These equations can be obtained from (R) by scaling z to shift the coefficient δ from the xz term in the $\dot{y}$ equation to the x^2 term in the $\dot{z}$ equation. (This change is made because we treat a different type of perturbation problem than Rychlik.) We also change the xz term in the $\dot{y}$ equation to a yz term. Although this makes the equations farther from the Lorenz equations, they are easier to analyze.

In this paper, we return to the equations (R) but scale the equations to shift the coefficient δ to the $\dot{z}$ -equation, the αy term is replaced by $-\alpha y$ so we can take $\alpha > 0$, and we allow $-\nu xz$ in the $\dot{y}$ equation with $\nu = \pm 1$:

$$(R') \quad \begin{aligned} \dot{x} &= y \\ \dot{y} &= x - 2x^3 - \alpha y + \beta x^2 y - \nu x z \\ \dot{z} &= -\gamma z + \delta x^2. \end{aligned}$$

We can now state the theorem.

THEOREM 2. *Equations (R') satisfy assumption (A0)–(A6) for correctly chosen values of the parameters. (In particular, for some $\alpha_0 > 2^{-\frac{1}{2}}$.)*

Remark 1. It is not clear that the choice of ν in the equations correspond to the choice in assumption (A2), i.e. it determines whether the one dimensional Poincaré-Dulac map is monotonically increasing or decreasing (or the the tangent space $W^{us}(Q, \eta_0)$ is orientable or nonorientable along Γ .) The difficulty is that the proof only proves transversality by an argument which uses the fact that an analytic function that is not identically zero must be nonzero in any interval.

Remark 2. It seems unlikely that any of these results, here [7], or [9], can be applied directly to the actual Lorenz equations (L). The difficulty is finding parameter values that verify both the homoclinic connection condition (A1) and the resonance condition (A4) for the same parameter values. The addition of the extra parameter β is what gives equations (R), (R'), or (Rob) the flexibility need to satisfy these simultaneously.

Proof of Theorem 2. The parameters are $\eta = (\alpha, \beta, \gamma, \delta, \nu)$. The fixed point Q is always the origin. The linearization of the vector field is given by

$$DX(x, y, z) = \begin{pmatrix} 0 & 1 & 0 \\ 1 - 6x^2 + 2\beta xy - \nu z & -\alpha + \beta x^2 & -\nu x \\ 2\delta x & 0 & -\gamma \end{pmatrix}.$$

At the origin, the eigenvalues are $\lambda_{ss} = -\alpha/2 - (1 + \alpha^2/4)^{1/2}$, $\lambda_u = -\alpha/2 + (1 + \alpha^2/4)^{1/2}$, and $\lambda_s = -\gamma$. By picking the parameter $\gamma_0 = \lambda_u = -\alpha_0/2 + (1 + \alpha_0^2/4)^{1/2}$ at the bifurcation, we can insure that $\lambda_s(\eta_0) + \lambda_u(\eta_0) = 0$ giving assumption (A4).

To obtain (A3), we need the combination of all three eigenvalues less than zero, $\lambda_{ss}(\eta_0) - \lambda_s(\eta_0) + \lambda_u(\eta_0) < 0$:

$$\begin{aligned} 0 &> [-\alpha_0/2 - (1 + \alpha_0^2/4)^{1/2}] + 2[-\alpha_0/2 + (1 + \alpha_0^2/4)^{1/2}] \\ &> -3\alpha_0/2 + (1 + \alpha_0^2/4)^{1/2} \end{aligned}$$

or

$$\alpha_0 > 2^{-\frac{1}{2}}.$$

Thus to obtain a C^1 foliation, it is not possible to take a small perturbation of the integrable case where $\alpha = \beta = \delta = 0$. Given the resonance condition (A4), the second inequality in (A3) follows from the first.

LEMMA 3. *For either $\nu = \pm 1$, there are parameter values $(\alpha_0^\pm, \beta_0^\pm, \gamma_0^\pm, \delta_0^\pm, \nu)$, with $\alpha_0^\pm > 1/(2^{\frac{1}{2}})$, $\beta_0^\pm > 0$, $\gamma_0^\pm = -(\alpha_0^\pm/2) + [1 + ((\alpha_0^\pm)^2/4)]^{1/2}$, and $\delta_0^\pm > 0$ but near zero, for which the equations have a homoclinic connection and an invariant continuous plane field, satisfying (A1)–(A5).*

Proof. The only difference in verifying this lemma for the equations (R') and (Rob) is in the verification of (A2). We repeat the other steps because they are easy, and we give a different approach to verifying (A2).

First, taking $\alpha = \beta = \delta = 0$, we obtain a Hamiltonian system in the (x, y) -plane with energy $H = (y^2 - x^2 + x^4)/2$. Therefore, the origin has a double homoclinic connection.

Now, we increase the value of α to $\alpha_0 > 1/(2)^{1/2}$, keeping $\beta = \delta = 0$, and choosing $\gamma_0 = -\alpha_0/2 + (1 + \alpha_0^2/4)^{1/2}$. The xy -plane is still invariant so $z = 0$ along the unstable manifold. The $-\alpha y$ term in the $\dot{y}$ equation is a friction term so the unstable manifold $W^u(Q, \eta)$ stays on the same side of $W^s(Q, \eta)$ and spirals into one of the stable fixed points. Next, increase the value of β . The term $\beta x^2 y$ in the $\dot{y}$ equation is an anti-friction term for $\beta > 0$. For large enough β , $W^u(Q, \eta)$ will cross over to the other side of $W^s(Q, \eta)$. Therefore there is a value of β' which will yield a homoclinic connection. By the symmetry of the equations, it is a double homoclinic connection.

Finally, we perturb δ to $\delta_0 > 0$, but keeping the homoclinic connection. The derivative of z is given by $\dot{z} = -\gamma z + \delta x^2$ so z is positive but small along $W^u(Q, \eta)$. Thus the $-\nu x z$ term in $\dot{y}$ is a slight inward or outward push (depending on the point) along $W^u(Q, \eta)$. By adjusting the value of β' to β_0 (or adjusting α_0) we can preserve the homoclinic connection, giving (A1). The choice of α_0 and γ_0 gives (A3) and (A4).

We are left to verify (A2) and (A5). We can prove this directly for $\alpha, \beta \approx 0$. Then we use the analyticity of the manifolds to say that they can not be identically tangent for all $\alpha_0 > 2^{-\frac{1}{2}}$. Thus there must be values that satisfy this inequality for which the manifold are transverse.

Before verifying (A2) for small α and β , we note more carefully the analyticity of $W^s(Q, \eta)$ and $W^{us}(Q, \eta)$. The fact that $W^s(Q, \eta)$ is analytic and varies analytically with parameters is completely standard. The proof for $W^{us}(Q, \eta)$ is not as standard. To prove that this manifold exists locally it is necessary to make an extension which is equal to the linear map outside a neighborhood of Q (or another such construction.) It is not clear how to do this construction analytically. However the tangent space of $W^{us}(Q, \eta)$ at points of $W^u(Q, \eta)$ is determined without any such extension and so depends analytically on η . By looking at $W^s(Q, \eta)$ and $W^{us}(Q, \eta)$ where they both cross $y = 0$ the angle between these two manifolds is an analytic function of η . If this is non-zero for small α , it can not be identically equal to zero for $\alpha > 2^{-\frac{1}{2}}$. Therefore, we can choose an $\alpha_0 > 2^{-\frac{1}{2}}$ for which these manifolds are transverse.

To verify the transversality given in (A2) for α and β near zero, we let $p(t)$ be the covectors which are perpendicular to $T_q W^{us}(Q)$ at points q on Γ . Covectors satisfy the adjoint equation to the first variation equation:

$$\dot{p} = -pDX(q(t)),$$

where we write p as a row.

We first consider $\delta = 0$, and α and β both positive but near zero and chosen so there is a homoclinic orbit. We take the parameterization of $q(t)$ with $y(0) = 0$. The equations for $\dot{p}_1$ and $\dot{p}_2$ are independent of p_3 so can be solved independently for a solution $(\bar{p}_1(t), \bar{p}_2(t))$, that is perpendicular in the (x, y) -plane to the homoclinic orbit. There is then a solution $\bar{p}_3(t)$ and $\bar{p}(t) = (\bar{p}_1(t), \bar{p}_2(t), \bar{p}_3(t))$ so that $(q(t), \bar{p}(t))$ lies on the unstable manifold of $(Q, 0)$ in the space of positions and covectors.

We need to determine properties of $\bar{p}_3(t)$ and $\bar{p}(t)$ as t goes to infinity. Since $\dot{p}_3 = \nu x(t)\bar{p}_2(t) + \gamma p_3$,

$$\bar{p}_3(t) = e^{\gamma(t-t_0)}\bar{p}_3(t_0) + \nu e^{\gamma t} \int_{t_0}^t x(s)\bar{p}_2(s)e^{-\gamma s} ds.$$

As t_0 goes to minus infinity, $e^{\gamma(t-t_0)}\bar{p}_3(t_0)$ goes to zero because $\bar{p}_3(t_0)$ goes to zero at least at a rate of $e^{-|\lambda_{ss}||t_0|}$ (because $|\lambda_{ss}|$ is the most unstable eigenvalue for covectors) and $\gamma - |\lambda_{ss}| < 0$. Thus,

$$\bar{p}_3(t) = \nu e^{\gamma t} \int_{-\infty}^t x(s)\bar{p}_2(s)e^{-\gamma s} ds.$$

We want to show that $\bar{p}_3(t)$ goes to $\nu\infty$, as t goes to infinity. It is easy to see that the integral converges as t goes to infinity. For $\alpha, \beta = 0$, $x(-s) = -x(s)$ and $p_2(-s) = -p_2(s)$, so we get

$$\int_0^\infty x(s)\bar{p}_2(s)[e^{-\gamma s} - e^{\gamma s}] ds$$

which is positive since $x(s)$ is positive and $p_2(s)$ is negative for positive s . Since this integral is positive for $\alpha, \beta = 0$, it is positive for $\alpha, \beta \approx 0$. Since it is multiplied by $\nu e^{\gamma t}$ $\bar{p}_3(t)$ goes to $\nu\infty$ as t goes to infinity. On the other hand $(\bar{p}_1(t), \bar{p}_2(t))$ goes to zero (as seen by the eigenvalues at $(Q, 0)$.) This implies that the tangent plane to $W^{us}(Q, \eta)$ limits on the (x, y) -plane and is transverse to $W^s(Q, \eta)$ for these values of η with $\delta = 0$. Transversality is an open condition so is true for nearby small $\alpha, \beta, \delta > 0$ (for which there is a homoclinic connection) which proves (A2) for these parameter values. By analyticity, as mentioned above, we get (A2) true for some $\alpha_0 > 2^{-\frac{1}{2}}$.

As argued in [7], the rates of change of area near $t = \pm\infty$ show that the integral in (A5) is $-\infty$ for $\delta = 0$. Thus for small $\delta > 0$, $\log(C_{\eta_0}) \ll 0$ so $C_{\eta_0} \ll 1$. This completes the proof of Lemma 3. $\square$

The fact that $a(\eta)$ and $b(\eta)$ can be varied independently follows because increasing β (or decreasing α) makes the manifold $W^u(Q, \eta)$ cross over $W^s(Q, \eta)$, so $a(\eta)$ varies, and varying γ makes $b(\eta)$ vary. This completes the proof of Theorem 2. $\square$

REFERENCES

- [1] A. ANDRONOV, E. A. LEONTOVICH, I.I. GORDON, & A. G. MAIER, *Theory of Bifurcation of Dynamical Systems on the Plane*, Wiley, New York, 1973.
- [2] M. HIRSCH AND C. PUGH, *Stable manifolds and hyperbolic sets*, Proc. Symposium in Pure Math., 14 (1970), pp. 133–164, Amer. Math. Soc., Providence RI..
- [3] M. HIRSCH, C. PUGH, AND M. SHUB, *Invariant Manifolds*, Lecture Notes in Math., 583 (1977), Springer-Verlag, New York/Berlin/Heidelberg.
- [4] G. KELLER, *Generalized bounded variation and applications to piecewise monotonic transformations*, Z. Wahrsch. Verw. Gebiete, 69 (1985), pp. 461–478.
- [5] C. ROBINSON, *Differentiability of the stable foliations for the model Lorenz equations*, in Dynamical Systems and Turbulence (edited by D. Rand and L-S Young) Lecture Notes in Math, Springer-Verlag, New York/Berlin/Heidelberg, 1981, pp. 302–315.
- [6] C. ROBINSON, *Transitivity and invariant measures for the geometric model of the Lorenz equations*, Ergodic Theory and Dynamical Systems, 4 (1984), pp. 605–611.
- [7] C. ROBINSON, *Homoclinic bifurcation to a transitive attractor of Lorenz type*, Nonlinearity, 2 (1989), pp. 495–518.
- [8] R. ROUSSARIE, *On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields*, Bol. Soc. Mat., 17 (1986), pp. 67–101.
- [9] M. RYCHLIK, *Lorenz attractors through Sil'nikov-type bifurcation, Part I*, Ergodic Theory and Dynamical Systems (1989).
- [10] M. SHUB, *Global Stability of Dynamical Systems*, Springer-Verlag, New York / Berlin / Heidelberg, 1987.
- [11] F. TAKENS, *Partially hyperbolic fixed points*, Topology, 10 (1971), pp. 133–147.
- [12] R. WILLIAMS, *The structure of Lorenz attractors*, Publ. Math. IHES, 50 (1979), pp. 321–347.

Recent IMA Preprints

#	Author/s	Title
637	Xinfu Chen,	Generation and propagation of the interface for reaction–diffusion equations
638	Philip Korman,	Dynamics of the Lotka–Volterra systems with diffusion
639	Harlan W. Stech,	Generic Hopf bifurcation in a class of integro-differential equations
640	Stephane Laederich,	Periodic solutions of non linear differential difference equations
641	Peter J. Olver,	Canonical Forms and Integrability of BiHamiltonian Systems
642	S.A. van Gils, M.P. Krupa and W.F. Langford,	Hopf bifurcation with nonsemisimple 1:1 Resonance
643	R.D. James and D. Kinderlehrer,	Frustration in ferromagnetic materials
644	Carlos Rocha,	Properties of the attractor of a scalar parabolic P.D.E.
645	Debra Lewis,	Lagrangian block diagonalization
646	Richard C. Churchill and David L. Rod,	On the determination of Ziglin monodromy groups
647	Xinfu Chen and Avner Friedman,	A nonlocal diffusion equation arising in terminally attached polymer chains
648	Peter Gritzmann and Victor Klee,	Inner and outer j- Radii of convex bodies in finite- dimensional normed spaces
649	P. Szmolyan,	Analysis of a singularly perturbed traveling wave problem
650	Stanley Reiter and Carl P. Simon,	Decentralized dynamic processes for finding equilibrium
651	Fernando Reitich,	Singular solutions of a transmission problem in plane linear elasticity for wedge-shaped regions
652	Russell A. Johnson,	Cantor spectrum for the quasi-periodic Schrödinger equation
653	Wenxiong Liu,	Singular solutions for a convection diffusion equation with absorption
654	Deborah Brandon and William J. Hrusa,	Global existence of smooth shearing motions of a nonlinear viscoelastic fluid
655	James F. Reineck,	The connection matrix in Morse–Smale flows II
656	Claude Baesens, John Guckenheimer, Seunghwan Kim and Robert Mackay,	Simple resonance regions of torus diffeomorphisms
657	Willard Miller, Jr.,	Lecture notes in radar/sonar: Topics in Harmonic analysis with applica- tions to radar and sonar
658	Calvin H. Wilcox,	Lecture notes in radar/sonar: Sonar and Radar Echo Structure
659	Richard E. Blahut,	Lecture notes in radar/sonar: Theory of remote surveillance algorithms
660	D.V. Anosov,	Hilbert’s 21st problem (according to Bolibruch)
661	Stephane Laederich,	Ray–Singer torsion for complex manifolds and the adiabatic limit
662	Geneviève Raugel and George R. Sell,	Navier-Stokes equations in thin 3d domains: Global regularity of solutions I
663	Emanuel Parzen,	Time series, statistics, and information
664	Andrew Majda and Kevin Lamb,	Simplified equations for low Mach number combustion with strong heat release
665	Ju. S. Il’yashenko,	Global analysis of the phase portrait for the Kuramoto–Sivashinsky equation
666	James F. Reineck,	Continuation to gradient flows
667	Mohamed Sami Elbially,	Simultaneous binary collisions in the collinear N–body problem
668	John A. Jacquez and Carl P. Simon,	Aids: The epidemiological significance of two different mean rates of partner-change
669	Carl P. Simon and John A. Jacquez,	Reproduction numbers and the stability of equilibria of SI models for heterogeneous populations
670	Matthew Stafford,	Markov partitions for expanding maps of the circle
671	Ciprian Foias and Edriss S. Titi,	Determining nodes, finite difference schemes and inertial manifolds
672	M.W. Smiley,	Global attractors and approximate inertial manifolds for abstract dissipative equations
673	M.W. Smiley,	On the existence of smooth breathers for nonlinear wave equations
674	Hitay Özbay and Janos Turi,	Robust stabilization of systems governed by singular integro-differential equations
675	Mary Silber and Edgar Knobloch,	Hopf bifurcation on a square lattice
676	Christophe Golé,	Ghost circles for twist maps
677	Christophe Golé,	Ghost tori for monotone maps
678	Christophe Golé,	Monotone maps of $T^n \times R^n$ and their periodic orbits
679	E.G. Kalnins and W. Miller, Jr.,	Hypergeometric expansions of Heun polynomials
680	Victor A. Pliss and George R. Sell,	Perturbations of attractors of differential equations
681	Avner Friedman and Peter Knabner,	A transport model with micro- and macro-structure
682	E.G. Kalnins and W. Miller, Jr.,	A note on group contractions and radar ambiguity functions
683	George R. Sell,	References on dynamical systems
684	Shui-Nee Chow, Kening Lu and George R. Sell,	Smoothness of inertial manifolds
685	Shui-Nee Chow, Xiao-Biao Lin and Kening Lu,	Smooth invariant foliations in infinite dimensional spaces

- 686 **Kening Lu**, A Hartman–Grobman theorem for scalar reaction-diffusion equations
- 687 **Christophe Golé and Glen R. Hall**, Poincaré’s proof of Poincaré’s last geometric theorem
- 688 **Mario Taboada**, Approximate inertial manifolds for parabolic evolutionary equations via Yosida approximations
- 689 **Peter Rejto and Mario Taboada**, Weighted resolvent estimates for Volterra operators on unbounded intervals
- 690 **Joel D. Avrin**, Some examples of temperature bounds and concentration decay for a model of solid fuel combustion
- 691 **Susan Friedlander and Misha M. Vishik**, Lax pair formulation for the Euler equation
- 692 **H. Scott Dumas**, Ergodization rates for linear flow on the torus
- 693 **A. Eden, A.J. Milani and B. Nicolaenko**, Finite dimensional exponential attractors for semilinear wave equations with damping
- 694 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Inertial sets for dissipative evolution equations
- 695 **A. Eden, C. Foias, B. Nicolaenko & R. Temam**, Hölder continuity for the inverse of Mañé’s projection
- 696 **Michel Chipot and Charles Collins**, Numerical approximations in variational problems with potential wells
- 697 **Huanan Yang**, Nonlinear wave analysis and convergence of MUSCL schemes
- 698 **László Gerencsér and Zsuzsanna Vágó**, A strong approximation theorem for estimator processes in continuous time
- 699 **László Gerencsér**, Multiple integrals with respect to L -mixing processes
- 700 **David Kinderlehrer and Pablo Pedregal**, Weak convergence of integrands and the Young measure representation
- 701 **Bo Deng**, Symbolic dynamics for chaotic systems
- 702 **P. Galdi, D.D. Joseph, L. Preziosi, S. Rionero**, Mathematical problems for miscible, incompressible fluids with Korteweg stresses
- 703 **Charles Collins and Mitchell Luskin**, Optimal order error estimates for the finite element approximation of the solution of a nonconvex variational problem
- 704 **Peter Gritzmann and Victor Klee**, Computational complexity of inner and outer j -radii of polytopes in finite-dimensional normed spaces
- 705 **A. Ronald Gallant and George Tauchen**, A nonparametric approach to nonlinear time series analysis: estimation and simulation
- 706 **H.S. Dumas, J.A. Ellison and A.W. Sáenz**, Axial channeling in perfect crystals, the continuum model and the method of averaging
- 707 **M.A. Kaashoek and S.M. Verduyn Lunel**, Characteristic matrices and spectral properties of evolutionary systems
- 708 **Xinfu Chen**, Generation and Propagation of interfaces in reaction diffusion systems
- 709 **Avner Friedman and Bei Hu**, Homogenization approach to light scattering from polymer-dispersed liquid crystal films
- 710 **Yoshihisa Morita and Shuichi Jimbo**, ODEs on inertial manifolds for reaction-diffusion systems in a singularly perturbed domain with several thin channels
- 711 **Wenxiong Liu**, Blow-up behavior for semilinear heat equations: multi-dimensional case
- 712 **Hi Jun Choe**, Hölder continuity for solutions of certain degenerate parabolic systems
- 713 **Hi Jun Choe**, Regularity for certain degenerate elliptic double obstacle problems
- 714 **Fernando Reitich**, On the slow motion of the interface of layered solutions to the scalar Ginzburg–Landau equation
- 715 **Xinfu Chen and Fernando Reitich**, Local existence and uniqueness of solutions of the Stefan problem with surface tension and kinetic undercooling
- 716 **C.C. Lim, J.M. Pimbley, C. Schmeiser and D.W. Schwendeman**, Rotating waves for semiconductor inverter
- 717 **W. Balser, B.L.J. Braaksma, J.-P. Ramis and Y. Sibuya**, Multisummability of formal power series solutions of linear ordinary differential equations
- 718 **Peter J. Olver and Chehrzad Shakiban**, Dissipative decomposition of partial differential equations
- 719 **Clark Robinson**, Homoclinic bifurcation to a transitive attractor of Lorenz type, II
- 720 **Michelle Schatzman**, A simple proof of convergence of the QR algorithm for normal matrices without shifts
- 721 **Ian M. Anderson, Niky Kamran and Peter J. Olver**, Internal, external and generalized symmetries
- 722 **C. Foias and J.C. Saut**, Asymptotic integration of Navier–Stokes equations with potential forces. I
- 723 **Ling Ma**, The convergence of semidiscrete methods for a system of reaction-diffusion equations
- 724 **Adelina Georgescu**, Models of asymptotic approximation
- 725 **A. Makagon and H. Salehi**, On bounded and harmonizable solutions on infinite order $arma$ systems
- 726 **San-Yih Lin and Yan-Shin Chin**, An upwind finite-volume scheme with a triangular mesh for conservation laws
- 727 **J.M. Ball, P.J. Holmes, R.D. James, R.L. Pego & P.J. Swart**, On the dynamics of fine structure
- 728 **KangPing Chen and Daniel D. Joseph**, Lubrication theory and long waves