**MULTISUMMABILITY OF FORMAL POWER SERIES
SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL
EQUATIONS**

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1. Preliminaries: Throughout this paper, let a linear differential operator

$$(1.1) \quad \mathcal{L} = \delta - \mathbf{A}(z) \quad (\delta = d/dz)$$

be arbitrarily given, where z is a complex independent variable and $\mathbf{A}$ is an n -by- n matrix whose entries are meromorphic at $z = 0$, i.e.

$$(1.2) \quad \mathbf{A} \in \mathbf{M}_n(\mathbb{C}(z)[z^{-1}])$$

We are interested in multisummability properties of formal power series $\mathbf{f} \in \mathbb{C}[[z]]^n$ such that $\mathcal{L}[\mathbf{f}] \in (\mathbb{C}(z)[z^{-1}])^n$. To state the main problem, we need some preparations. (In particular, for the definition of multisummability, see definition 4.7.)

DEFINITION 1.1: Let s be a nonnegative number. A formal power series $\mathbf{f} = \sum_{m \geq 0} \mathbf{a}_m z^m$ is said to be of Gevrey order s if there exist two nonnegative numbers C and A such that

$$(1.3) \quad |\mathbf{a}_m| \leq C (m!)^s A^m \quad \text{for } m = 0, 1, 2, \dots$$

We denote by $\mathbb{C}[[z]]_s$ the set of all power series of Gevrey order s .

DEFINITION 1.2: A function ϕ of z is said to admit an asymptotic expansion of Gevrey order s ($s \geq 0$) as $z \rightarrow 0$ on a sectorial domain

$$\mathcal{D}(R_0, a, b) = \{z; a < \arg z < b, 0 < |z| < R_0\}$$

if

(i) ϕ is holomorphic on $\mathcal{D}(R_0, a, b)$,

(ii) there exists a formal power series

$$(1.4) \quad f = \sum_{m \geq 0} a_m z^m \in \mathbb{C}[[z]]_s$$

such that

$$(1.5) \quad \left| \phi(z) - \sum_{N-1 \leq m \leq 0} a_m z^m \right| \leq K_{R, \alpha, \beta} (N!)^s (B_{R, \alpha, \beta})^N |z|^N$$

on $\mathcal{D}(R, \alpha, \beta)$ for $N = 1, 2, \dots$ and every (R, α, β) satisfying the inequalities

$$(1.6) \quad 0 < R < R_0, \quad a < \alpha < \beta < b,$$

where $K_{R, \alpha, \beta}$ and $B_{R, \alpha, \beta}$ are some nonnegative numbers depending on

(R, α, β) .

We denote by $\mathcal{A}_s(R_0, a, b)$ the set of all functions admitting asymptotic expansions of Gevrey order s as $z \rightarrow 0$ on the sectorial domain $\mathcal{D}(R_0, a, b)$. Also we set

$$(1.7) \quad J(\phi) = f.$$

Then the mapping

$$(1.8) \quad J : \mathcal{A}_s(R_0, a, b) \rightarrow \mathbb{C}[[z]]_s$$

is a homomorphism of commutative differential algebras over $\mathbb{C}$. Set

$$(1.9) \quad \mathcal{A}_{s0}(R_0, a, b) = \text{Kernel}(J : \mathcal{A}_s(R_0, a, b) \rightarrow \mathbb{C}[[z]]_s).$$

LEMMA 1.3 (G.N. Watson [9]): *A function $\phi(z)$ belongs to $\mathcal{A}_{s0}(R_0, a, b)$ if and only if*

(i) ϕ is holomorphic on $\mathcal{D}(R_0, a, b)$,

(ii) an inequality

$$(1.10) \quad |\phi(z)| \leq \mu_{R,a,b} \exp\{-\lambda_{R,a,b} |z|^{-1/s}\}$$

holds on $\mathcal{D}(R, \alpha, \beta)$ for every (R, α, β) satisfying inequalities (1.6), where $\mu_{R,a,b}$ is a nonnegative number and $\lambda_{R,a,b}$ is a positive number, and they are determined by (R, α, β) .

COROLLARY 1.4 (G.N. Watson [9]): *Homomorphism (1.8) is injective, i.e.*

$$(1.11) \quad \mathcal{A}_{s0}(R_0, a, b) = 0,$$

if and only if

$$(1.12) \quad b - a > s\pi.$$

DEFINITION 1.5: Let k be a positive number. A formal power series $f \in \mathbb{C}[[z]]$ is k -summable in a direction $\arg z = d$ if

$$(1.13) \quad f \in \text{Image}\left(\bigcup_{\epsilon} \mathcal{A}_{1/k}(R_0, d - \pi/(2k) - \epsilon, d + \pi/(2k) + \epsilon) \rightarrow \mathbb{C}[[z]]_{1/k}\right)$$

for some positive number R_0 and ϵ .

We denote by $\mathbb{C}(z)_{kd}$ the set of all k -summable power series in the direction d . It is known that

$$(1) \quad \mathbb{C}(z) \subset \mathbb{C}(z)_{kd},$$

(2) $\mathbb{C}\{z\}_{k,d}$ is a commutative differential algebra over $\mathbb{C}$.

The mapping $J : \mathcal{A}_{1/k}(R_0, d-\pi/(2k)-\epsilon, d+\pi/(2k)+\epsilon) \rightarrow \mathbb{C}\{z\}_{k,d}$ is injective. For a given $f \in \mathbb{C}\{z\}_{k,d}$, there exist two positive numbers R_0 and ϵ and a unique $\phi \in \mathcal{A}_{1/k}(R_0, d-\pi/(2k)-\epsilon, d+\pi/(2k)+\epsilon)$ such that $J(\phi) = f$. We say that ϕ is the *sum* of f in the sense of k -summability in the direction d (cf. J.-P. Ramis [6] or Y. Sibuya [8]).

REMARK 1.6: If $f \in \mathbb{C}\{z\}_{k,d}$, then there exists a positive number r such that $f \in \mathbb{C}\{z\}_{k,d'}$ for every d' satisfying the inequality $|d - d'| < r$. Furthermore the sum of f in the direction d' is identical with the sum of f in the direction d .

2. Gevrey properties of formal solutions: We can define the Newton polygon $N(\mathcal{L})$ of operator $\mathcal{L}$ given by (1.1) (cf. J.-P. Ramis [5] or Y. Sibuya [8]). The polygon $N(\mathcal{L})$ has at most a finite number of positive slopes:

$$(2.1) \quad 0 < h_q < h_{q-1} < \dots < h_2 < h_1 < +\infty.$$

There is no positive slope if and only if $z = 0$ is at worst a regular singular point of the differential equation: $\mathcal{L}[y] = 0$. If $z = 0$ is a regular singular point and $\mathcal{L}[f] \in (\mathbb{C}\{z\}[z^{-1}])^n$, then $f \in \mathbb{C}\{z\}^n$. Hence we assume that the polygon $N(\mathcal{L})$ has at least a positive slope, i.e.

$$(2.2) \quad q \geq 1.$$

We summarize in the following lemmas some fundamental facts concerning solutions of linear differential equations.

LEMMA 2.1: If $\phi \in (\mathcal{A}_s(R_0, a, b))^n$ satisfies the conditions

$$(1) \quad \mathcal{L}[\phi] = 0 \text{ on } \mathcal{D}(R_0, a, b),$$

$$(2) \quad J[\phi] = 0,$$

then

$$(2.3) \quad \phi \in (\mathcal{A}_{1/k, 0}(R_0, a, b))^n,$$

where

$$(2.4) \quad k \in \{h_1, h_2, \dots, h_q\}.$$

LEMMA 2.2 (J.-P. Ramis [5]): *If $f \in \mathbb{C}[[z]]^n$ satisfies the condition*

$$(2.5) \quad \mathcal{L}[f] \in (\mathbb{C}(z)[z^{-1}])^n,$$

then

$$(2.6) \quad f \in (\mathbb{C}[[z]])_{1/k}^n,$$

where k satisfies condition (2.4).

LEMMA 2.3 (J.-P. Ramis and Y. Sibuya [7]): *If $f \in (\mathbb{C}[[z]]_s)^n$ satisfies condition (2.5) and if $\arg z = d$ is a direction arbitrarily given, then there exist (R_0, a, b) and $\phi \in (\mathcal{A}_s(R_0, a, b))^n$ such that*

$$(i) \quad a < d < b,$$

$$(ii) \quad J(\phi) = f,$$

$$(iii) \quad \mathcal{L}[\phi] = \mathcal{L}[f].$$

3. Formal fundamental solutions and singular directions: Let

$$(3.1) \quad H(z) = F(z) G(z), \quad G(z) = z^L \exp[Q(z)]$$

be a fixed but arbitrarily selected formal fundamental solution of the homogeneous differential equation.

$$(3.2) \quad \mathcal{L}[X] = 0 ,$$

where X is an n -by- n unknown matrix, and

$$(i) \quad F(z) \in M_n(\mathbb{C}[[t]]) , \quad t^\sigma = z$$

with a positive integer σ ,

$$(ii) \quad L \in M_n(\mathbb{C}) ,$$

$$(iii) \quad Q(z) = \text{diag}[q_1(z), q_2(z), \dots, q_n(z)]$$

$$\text{with } q_k(z) \in t^{-1}\mathbb{C}[t^{-1}] ,$$

$$(iv) \quad \det F \neq 0 \text{ as an element of } \mathbb{C}[[t]] , \text{ and}$$

$$(v) \quad L Q(z) = Q(z) L .$$

For each λ such that q_λ is not identically equal to zero, consider the (rational) degree of q_λ with respect to z^{-1} . This is a nonzero rational number. The set of all of these rational numbers is identical with the set of all positive slopes $\{h_q, h_{q-1}, \dots, h_2, h_1\}$ of the Newton polygon $N(\mathcal{L})$ (cf. (2.1)). Note that $\sigma h_j \in \mathbb{Z}$ ($j = 1, 2, \dots, q$).

A direction $\arg z = d$ is called a *singular direction* of level h_λ if there exists a positive integer j such that

$$(1) \quad \deg_{1/2} q_j(z) = h_\lambda ,$$

- (2) $\operatorname{Re}(q_j(z)) \leq 0$ for $|\arg z - d| < \pi/(2h_\lambda)$ and $|z| < \rho_0(\arg z)$ for a suitable positive-valued continuous function $\rho_0(\theta)$ on the interval $|\theta - d| < \pi/(2h_\lambda)$.

A direction $\arg z = d$ is said to be *singular* if it is a singular direction of level h_λ for some λ . There exist only a finite number of singular directions (mod 2π).

4. Main result, Part I (Multisummability): This paper consists of two parts. In the first part, we shall prove the following theorem.

THEOREM 4.1: *Let*

$$(4.1) \quad 1/2 \leq h_q < h_{q-1} < \dots < h_2 < h_1 < +\infty$$

be all of positive slopes of the Newton polygon $N(\mathcal{L})$ of the operator $\mathcal{L}$. If a formal power series $f \in \mathbb{C}[[z]]^n$ satisfies the condition

$$(4.2) \quad \mathcal{L}[f] \in (\mathbb{C}(z)[z^{-1}])^n$$

and if a direction $\arg z = d$ is not singular, then there exist

$$(4.3) \quad f_\lambda \in (\mathbb{C}(z)_{h_\lambda d})^n \quad (\lambda = 1, 2, \dots, q)$$

such that

$$(4.4) \quad \begin{cases} f = \sum_{q \geq \lambda \geq 1} f_\lambda, \\ \mathcal{L}[f_\lambda] \in (\mathbb{C}(z)[z^{-1}])^n \quad (\lambda = 1, 2, \dots, q). \end{cases}$$

In the case when $0 < h_\ell < 1/2$ for some ℓ , we change the independent variable z by $z = \xi^m$ so that $m h_\ell \geq 1/2$ for all ℓ . Theorem 4.1 follows immediately from lemma 2.2 and the lemma below.

LEMMA 4.2: *If a formal power series $f \in (\mathbb{C}[[z]])_{1/h_\ell}^n$ satisfies condition (4.2) and a direction $\arg z = d$ is not singular of level h_ℓ and if $h_\ell \geq 1/2$, then there exist*

$$(4.5) \quad g \in (\mathbb{C}[[z]])_{1/h_{\ell-1}}^n \quad \text{and} \quad f_\ell \in (\mathbb{C}(z)_{h_\ell d})^n$$

such that

$$(4.6) \quad f = g + f_\ell \quad \text{and} \quad \mathcal{L}[g] \in (\mathbb{C}(z)[z^{-1}])^n.$$

REMARK 4.3: It is known that, if $\ell = 1$ and if the direction $\arg z = d$ is not singular of level h_1 , then (4.2) implies that f is h_1 -summable in the direction d (cf. J.-P. Ramis and Y. Sibuya [7]). Proof of lemma 4.2 will be given in sections 7 and 8.

Let $\mathcal{K} = \{k_1, k_2, \dots, k_N\}$ be a set of a finite number of positive numbers not less than $1/2$. We assume that $\mathcal{K}$ is fixed but arbitrarily given. Also let $\arg z = d$ be a fixed but arbitrarily given direction. Set

$$(4.7) \quad \begin{aligned} \mathbb{C}(z)_{\mathcal{K}d} &= \mathbb{C}(z)_{k_1 d} + \mathbb{C}(z)_{k_2 d} + \dots + \mathbb{C}(z)_{k_N d} \\ &= \left\{ f \in \mathbb{C}[[z]] ; f = \sum_{N \geq \ell \geq 1} f_\ell, f_\ell \in \mathbb{C}(z)_{k_\ell d} \right\}. \end{aligned}$$

Then we can prove the following theorem.

THEOREM 4.4: *For every finite set $\mathcal{K}$ of positive numbers not less than $1/2$, $\mathbb{C}(z)_{\mathcal{K}d}$ is a differential subalgebra of $\mathbb{C}[[z]]$.*

To prove this theorem it suffices to prove the lemma below.

LEMMA 4.5: For every pair (k, k') of positive numbers not less than $1/2$ and a pair (f, g) of power series such that $f \in \mathbb{C}(z)_{k, d}$ and $g \in \mathbb{C}(z)_{k', d}$, there exists another pair (F, G) of power series such that $F \in \mathbb{C}(z)_{k, d}$, $G \in \mathbb{C}(z)_{k', d}$ and

$$(4.8) \quad fg = F + G.$$

This lemma will be proved in section 9.

For $f = \sum_{N \geq \lambda \geq 1} f_\lambda$ ($f_\lambda \in \mathbb{C}(z)_{k_\lambda, d}$, $\lambda = 1, 2, \dots, N$), we define the sum of f in the direction d by $\sum_{N \geq \lambda \geq 1} \phi_\lambda$, where ϕ_λ is the sum of f_λ in the direction d . It is known that this sum is independent of the representation of f as a sum $\sum_{N \geq \lambda \geq 1} f_\lambda$ (cf. J. Ecalle [2] and J. Martinet-J.-P. Ramis [4]).

COROLLARY 4.6: If we set

$$(4.9) \quad \mathbb{C}(z)_{(d)} = \bigcup_{\mathcal{K}} \mathbb{C}(z)_{\mathcal{K}, d}$$

where $\bigcup_{\mathcal{K}}$ denotes the union over all finite sets $\mathcal{K}$ of positive numbers not less than $1/2$, then $\mathbb{C}(z)_{(d)}$ is a differential subalgebra of $\mathbb{C}[[z]]$.

DEFINITION 4.7: A formal power series f is said to be $\mathcal{K}$ -multisummable in the direction d if $f \in \mathbb{C}(z)_{\mathcal{K}, d}$. Also f is said to be multisummable in the direction d if $f \in \mathbb{C}(z)_{(d)}$.

Recently W. Balser proved that this definition of multisummability is equivalent to the definition given by J. Ecalle (cf. J. Ecalle [2] and J. Martinet-J.-P. Ramis [4]). W. Balser will discuss those results elsewhere.

5. Some known facts concerning formal fundamental solutions:

Let $H(z)$ be the same formal fundamental solution of homogeneous equation (3.2) as was given by (3.1) above. Let us assume that not all q_k are identically equal to one another.

Set

$$(5.1) \quad q_{\ell k}(z) = q_{\ell}(z) - q_k(z) \quad (1 \leq k, \ell \leq n).$$

For each such ℓ, k for which $q_{\ell k}(z)$ is not identically equal to zero, consider the (rational) degree of $q_{\ell k}(z)$ with respect to z^{-1} . This is a nonzero rational number. The set of these rational numbers is finite. We denote its entries by k_j ($j=1, 2, \dots, p$), numerated so that

$$(5.2) \quad 0 < k_p < k_{p-1} < \dots < k_2 < k_1 < +\infty.$$

Note that $\sigma k_j \in \mathbb{Z}$ ($j = 1, 2, \dots, p$). For $j = 1, 2, \dots, p$, if (ℓ, k) ($1 \leq k, \ell \leq n$) is such that $\deg_{1/2} q_{\ell k}(z) = k_j$, then we call (ℓ, k) a pair of level k_j .

For an n -by- n matrix C , the notation $C^{(j)}$ denotes the n -by- n matrix obtained from C by replacing those (ℓ, k) -entries by zero for which (ℓ, k) is of level not less than k_j . For convenience let $C^{(0)} = C$. If the diagonal entries of $Q(z)$ are ordered appropriately as we can assume without any loss of generality, then $C^{(j)}$ will become a direct sum of diagonal blocks of C in the block structure of level k_j (cf. W.Balser [1] or W.B.Jurkat [3]).

A direction $\arg z = \tau$ is called a Stokes direction if there exists a pair (ℓ, k) such that $q_{\ell k}(z)$ is not identically equal to zero and that $\operatorname{Re}(q_{\ell k}(z))$ changes its sign across the direction τ . Let $\{\tau_v; v \in \mathbb{Z}\}$ be the set of all Stokes directions. We assume without any loss of generality that

$$(5.3) \quad \tau_{v-1} < \tau_v, \quad \tau_{-1} < 0 \leq \tau_0 \quad (\text{cf. W.Balser [1]}).$$

We denote by $\mathcal{D}(a, b)$ a sectorial domain $\mathcal{D}(R, a, b)$ with "suitable" positive number R . Sectorial domains

$$(5.4) \quad \mathcal{S}_v = \mathcal{D}(\tau_{v-1}, \tau_{v+1}) \quad (v \in \mathbb{Z})$$

are called normal sectors. On every normal sector $\mathcal{S}_v$, there exists an n -by- n matrix $Y_v(z)$ such that

$$(i) \quad Y_v(z) \text{ and } Y_v(z)^{-1} \text{ are holomorphic on } \mathcal{S}_v,$$

$$(ii) \quad \mathcal{L}[Y_v] = 0 \text{ on } \mathcal{S}_v,$$

$$(iii) \quad Y_v(t^\sigma) G(t^\sigma)^{-1} \in \mathbf{M}_n(\mathcal{A}_s(R, (\tau_{v-1})/\sigma, (\tau_{v+1})/\sigma))$$

for some $s \in \{1/(\sigma k_1), 1/(\sigma k_2), \dots, 1/(\sigma k_p)\}$ and a positive number $R > 0$,

$$(iv) \quad J[Y_v G^{-1}] = F(t^\sigma).$$

The set $\{Y_v(z), v \in \mathbb{Z}\}$ is called an admissible system of fundamental solutions of equation

$$(5.5) \quad \mathcal{L}[X] = 0.$$

There exists a system $\{W_v, v \in \mathbb{Z}\}$ of n -by- n matrices such that

$$(5.6) \quad W_v \in \text{GL}(n, \mathbb{C}) \quad (v \in \mathbb{Z}),$$

$$(5.7) \quad Y_{v-1}(z) = Y_v(z) W_v \quad (v \in \mathbb{Z}).$$

The system $\{W_v, v \in \mathbb{Z}\}$ is called the system of Stokes multipliers corresponding to the admissible system $\{Y_v, v \in \mathbb{Z}\}$.

A direction $\arg z = d$ is said to be *admissible* if $d + (\pi/(2k_j))$ and $d - (\pi/(2k_j))$ are not Stokes directions for every $j = 1, 2, \dots, p$.

6. Main result, Part II (Structure of solutions of homogeneous equations): In the second part of this paper, we shall study the multisummability properties of fundamental solutions of equation (5.5).

Let $\{W_v, v \in \mathbb{Z}\}$ be the system of Stokes multipliers corresponding to an admissible system $\{Y_v, v \in \mathbb{Z}\}$ of fundamental solutions. It is obvious that, for every v , we have

$$(6.1) \quad \exp[Q(z)] W_v \exp[-Q(z)] - I \in M_n(\mathcal{A}_{s0}(\mathcal{I}_{v-1} \cap \mathcal{I}_v)),$$

for some $s > 0$ such that $s \in \{1/k_1, 1/k_2, \dots, 1/k_p\}$, where $\mathcal{A}_{s0}(\mathcal{I}_{v-1} \cap \mathcal{I}_v) = \mathcal{A}_{s0}(R, \tau_{v-1}, \tau_v)$ for some $R > 0$, and I is the n -by- n identity matrix.

A system $\{X_v, v \in \mathbb{Z}\}$ of fundamental solutions is admissible if and only if, for every v , we have

$$(6.2) \quad \exp[Q(z)] C_v \exp[-Q(z)] - I \in M_n(\mathcal{A}_{s0}(\mathcal{I}_v)),$$

for some $s > 0$ such that $s \in \{1/k_1, 1/k_2, \dots, 1/k_p\}$, where

$$(6.3) \quad C_v = Y_v(z)^{-1} X_v(z) \in GL(n, \mathbb{C}) \quad (v \in \mathbb{Z}),$$

and $\mathcal{A}_{s0}(\mathcal{I}_v) = \mathcal{A}_{s0}(R, \tau_{v-1}, \tau_{v+1})$ for some $R > 0$.

Let $\{V_v, v \in \mathbb{Z}\}$ be the system of Stokes multipliers corresponding to an admissible system $\{X_v, v \in \mathbb{Z}\}$ of fundamental solutions. Then

$$(6.4) \quad W_v C_{v-1} = C_v V_v \quad (v \in \mathbb{Z}),$$

where C_v is defined by (6.3). Also it can be verified easily that

$$(6.5) \quad W_v^{(j)} C_{v-1}^{(j)} = C_v^{(j)} V_v^{(j)} \quad (v \in \mathbb{Z}; j = 0, 1, 2, \dots, p)$$

(cf. W.Balser [1] or W.Jurkat [3]).

The basic tool in the study of multisummability properties of fundamental solutions is the following theorem.

THEOREM 6.1: *For any given direction $\arg z = d$, there exists an admissible system $\{X_v(z), v \in \mathbb{Z}\}$ of fundamental solutions of equation (5.5) such that the corresponding Stokes multipliers $\{V_v, v \in \mathbb{Z}\}$ satisfies the conditions*

$$(6.6) \quad V_v^{(j-1)} = \mathbb{I}$$

for every pair (j, v) such that

$$(6.7) \quad d - \pi/(2k_j) < \tau_{v-1} < \tau_v < d + \pi/(2k_j),$$

where $j = 1, 2, \dots, p$.

The system $\{X_v(z), v \in \mathbb{Z}\}$ in theorem 6.1 is not uniquely associated with the direction d . In the following theorem, we clarify the degree of freedom of selecting such a system.

Theorem 6.1 will be proved in section 10.

REMARK 6.2: If v satisfies the condition:

$$d - \pi/(2k_1) < \tau_{v-1} < \tau_v < d + \pi/(2k_1),$$

then (6.6) implies that $V_v = \mathbb{I}$ and hence that $X_v(z)$ is independent of v

as long as $\mathbf{v}$ satisfies condition

$$(6.8) \quad d - \pi/(2k_1) < \tau_{\mathbf{v}} < d + \pi/(2k_1) .$$

THEOREM 6.3: *For an admissible direction $\arg z = d$, let $\{X_{\mathbf{v}}(z), \mathbf{v} \in \mathbb{Z}\}$ be an admissible system satisfying condition (6.6) for every pair $(j, \mathbf{v})$ such that (6.7) holds. Let $\{Y_{\mathbf{v}}(z), \mathbf{v} \in \mathbb{Z}\}$ be another admissible system of fundamental solutions of (5.5). Set*

$$(6.9) \quad C_{\mathbf{v}} = Y_{\mathbf{v}}(z)^{-1} X_{\mathbf{v}}(z) \in GL(n, \mathbb{C}) \quad (\mathbf{v} \in \mathbb{Z}) .$$

Then, $\{Y_{\mathbf{v}}(z), \mathbf{v} \in \mathbb{Z}\}$ satisfies condition (6.6) for every pair $(j, \mathbf{v})$ such that (6.7) holds, if and only if

$$(6.10) \quad C_{\mathbf{v}}^{(j-1)} = \mathbf{I}$$

for every pair $(j, \mathbf{v})$ such that

$$(6.11) \quad d - \pi/(2k_j) < \tau_{\mathbf{v}} < d + \pi/(2k_j),$$

where $j = 1, 2, \dots, p$.

REMARK 6.4: Now it is clear that, as long as condition (6.8) is satisfied, the fundamental solution $X_{\mathbf{v}}(z)$ is uniquely determined by the direction d . Let us denote by $X(z; d)$ the unique solution $X_{\mathbf{v}}(z)$ with $\mathbf{v}$ satisfying condition (6.8). Set

$$(6.12) \quad \Psi(z; d) = X(z; d) G(z)^{-1} .$$

Theorem 6.3 will be proved in section 11. The basic result concerning the multisummability of formal fundamental solutions of (5.5) is the theorem below.

THEOREM 6.5: Let $H(z) = F(z) G(z)$ ($G(z) = z^L \exp[Q(z)]$) be the same as (3.1), and let a direction $\arg z = d$ be admissible in the sense given at the end of section 5. Then $F(t^\sigma)$ is $\{\sigma k_1, \sigma k_2, \dots, \sigma k_p\}$ -multisummable in the direction $\arg t = d/\sigma$ (cf. definition 4.6). Furthermore, the sum of $F(t^\sigma)$ in the direction d/σ is given by $\Psi(t^\sigma; d)$.

Theorem 6.5 will be proved in section 12. Next, we shall investigate how the matrix $\Psi(z; d)$ depends on the direction d . We summarize our observation in the theorem below.

THEOREM 6.6: For two admissible directions d_1 and d_2 such that $d_1 < d_2$, the followings hold:

(a) If every θ in the interval $d_1 < \theta < d_2$ is admissible then

$$(6.13) \quad \Psi(z; d_1) = \Psi(z; d_2).$$

(b) If exactly one θ in the interval $d_1 < \theta < d_2$ is not admissible, then

$$(6.14) \quad \exp[Q(z)] V^{(j-1)} \exp[-Q(z)] = \mathbf{I}$$

$$= \mathbf{M}_n(\omega_{1/k_p}^0(R, d_2 - (\pi/2k_j), d_1 + (\pi/2k_j))),$$

where $j = 1, 2, \dots, p$ and

$$(6.15) \quad \Psi(z; d_1)G(z) = \Psi(z; d_2)G(z)V, \quad V \in GL(n, \mathbb{C}),$$

and R is any positive number.

REMARK 6.7: If we set

$$(6.16) \quad \mathcal{U}(d_1, d_2) = \{ V \in GL(n, \mathbb{C}); V \text{ satisfies (6.14)} \},$$

then $\mathcal{U}(d_1, d_2)$ is a subgroup of the multiplicative group $GL(n, \mathbb{C})$.

By virtue of the results due to W. Balser [1], it can be verified that the system $\{X(z; d); d \text{ is an admissible direction}\}$ is uniquely characterized by their asymptotic behaviour in small sectors bisected by directions d respectively and by the specific form of the corresponding Stokes multipliers which is given in theorem 6.6.

As to individual $X(z; d)$, we shall prove the following characterization.

THEOREM 6.8: *For a given admissible direction $\arg z = d$ and a given fundamental solution $Y(z)$ of (5.5) such that*

$$(6.17) \quad \Psi(t^\sigma) = Y(t^\sigma)G(t^\sigma)^{-1} \in M_n(\mathcal{A}_s(R, (d/\sigma) - \epsilon, (d/\sigma) + \epsilon))$$

for some $s \in \{1/(\sigma k_1), 1/(\sigma k_2), \dots, 1/(\sigma k_p)\}$ and some positive numbers R and ϵ , and that

$$(6.18) \quad J(\Psi) = F,$$

the following conditions are equivalent:

$$(1) \quad \Psi(z) = \Psi(z; d),$$

$$(2) \quad \Psi(t^\sigma) \text{ is the sum of } F(t^\sigma) \text{ as a } (\sigma k_1, \sigma k_2, \dots, \sigma k_p)\text{-multisummable power series in the direction } \arg t = d/\sigma,$$

$$(3) \quad \text{for every } \nu \text{ such that}$$

$$(6.19) \quad d - \pi/(2k_p) < \tau_\nu < d + \pi/(2k_p),$$

there exist n -by- n matrices $\Psi_\nu(z)$ satisfying the following conditions:

$$(i) \quad \Psi_\nu(t^\sigma) \in M_n(\mathcal{A}_{1/(\sigma k_p)}(R, (\tau_{\nu-1}/\sigma), (\tau_{\nu+1}/\sigma))) \text{ for some } R > 0,$$

$$(ii) \quad \Psi_\nu(z) = \Psi(z) \text{ whenever } d - \pi/(2k_1) < \tau_\nu < d + \pi/(2k_1),$$

$$(iii) \quad \Psi_{v-1}(z) - \Psi_v(z) \in M_n(\mathcal{A}_{\frac{1}{k}, 0}^{1/k}(\mathcal{S}_{v-1} \cap \mathcal{S}_v)) \quad (j = 2, 3, \dots, p)$$

$$\text{if } d - \pi/(2k_j) < \tau_{v-1} < \tau_v < d + \pi/(2k_j).$$

Theorems 6.6 and 6.8 will be proved in section 13.

7. Proof of lemma 4.2: In this section we shall prove lemma 4.2. It is clear that theorem 4.1 follows from lemma 4.2. We shall prove lemma 4.2 by utilizing the following lemma:

LEMMA 7.1: *If a formal power series $f \in (\mathbb{C}[[z]])_{1/h_\ell}^n$ satisfies condition*

$$(7.1) \quad \mathcal{L}[f] \in (\mathbb{C}(z)[z^{-1}])^n$$

and if a direction $\arg z = d$ is not singular of level h_ℓ , then there exist

(i) *a finite number of open sectorial domains:*

$$(7.2) \quad \mathcal{V}_v = \mathcal{D}(\alpha_v, \beta_v) \quad (v = 1, 2, \dots, N)$$

such that

$$(7.3) \quad \begin{cases} \alpha_1 < d - \pi/(2h_\ell), & d + \pi/(2h_\ell) < \beta_N, \\ \alpha_v < \beta_{v-1} < \alpha_{v+1} < \beta_v & (v = 1, 2, \dots, N-1), \end{cases}$$

(ii) *a finite number of functions ϕ_v ($v = 1, 2, \dots, N$) such that*

$$(7.4) \quad \begin{cases} \phi_v \in (\mathcal{A}_{1/h_\ell}^{1/h_\ell}(\mathcal{V}_v))^n, \\ \phi_v - \phi_{v-1} \in (\mathcal{A}_{\frac{1}{h_\ell}, 0}^{1/h_\ell}(\mathcal{V}_{v-1} \cap \mathcal{V}_v))^n, \end{cases}$$

and that

$$(7.5) \quad J(\phi_v) = f \quad \text{and} \quad \mathcal{L}[\phi_v] = \mathcal{L}[f]$$

for $v = 1, 2, \dots, N$.

In this lemma we do not assume that $h_x \geq 1/2$. Proof of lemma 7.1 will be given in section 8.

Let us derive lemma 4.2 from lemma 7.1. For each v , let us fix a point z_v on the boundary of $\mathcal{V}_{v-1} \cap \mathcal{V}_v$ so that

$$(7.6) \quad \alpha_v < \arg z_v < \beta_{v-1} \quad (v = 2, \dots, N-1).$$

Denote by P_v the line segment joining 0 to z_v .

The integral

$$(7.7) \quad - (1/(2\pi i)) \sum_{N \geq \mu \geq 2} \int_{P_\mu} [(\phi_\mu(\zeta) - \phi_{\mu-1}(\zeta))/(\zeta - z)] d\zeta$$

defines N functions ψ_v on $\mathcal{V}_v$ respectively such that

$$(7.8) \quad \begin{cases} \psi_v \in (\mathcal{A}_{1/h}(\mathcal{V}_v))^n, \\ \psi_v - \psi_{v-1} = \phi_v - \phi_{v-1} \quad \text{on } \mathcal{V}_{v-1} \cap \mathcal{V}_v, \\ \mathcal{L}[\psi_v] \in (\mathbb{C}[z][z^{-1}])^n. \end{cases}$$

Since

$$\phi_v - \psi_v = \phi_{v-1} - \psi_{v-1} \quad \text{on } \mathcal{V}_{v-1} \cap \mathcal{V}_v,$$

we define a function ϕ such that

$$(7.9) \quad \phi \in (\mathcal{A}_{1/h_2}(\bigcup_v \mathcal{V}_v))^n$$

by setting

$$(7.10) \quad \phi = \phi_v - \psi_v \quad \text{on} \quad \mathcal{V}_v.$$

Set

$$(7.11) \quad f_x = J(\phi).$$

Then condition (7.3) implies that $f_x \in (\mathbb{C}[z]_{h_2/d})^n$. Since $\phi_v = \psi_v + \phi$, lemma 4.2 follows immediately. q.e.d.

8. Proof of lemma 7.1: In lemma 7.1 the formal power series $f(z)$ is given in advance. Therefore, without any loss of generality, we can assume that σ in section 3 (cf. (i) of that section) is equal to 1 so that we do not use any fractional powers of z . If we assume that $\sigma = 1$, then $h_1, h_2, \dots, h_q$ are integers.

It is known that there exists an n -by- n matrix $P(z)$ such that

$$(a) \quad P(z) \in GL(n, \mathbb{C}[[z]]_{1/h_1} [z^{-1}]),$$

$$(b) \quad B(z) = P(z)^{-1} A(z) P(z) - P(z)^{-1} \delta P(z) = \text{diag}(B_1(z), B_2(z))$$

where

$$(c) \quad B_1(z) \in M_{n_1}(\mathbb{C}[z][z^{-1}]) \text{ and } B_2(z) \in M_{n_2}(\mathbb{C}[z][z^{-1}]), \quad (n_1 + n_2 = n),$$

(d) the positive slopes of the Newton polygon of $\mathcal{X}_1 = \delta - B_1(z)$ are $h_1, h_2, \dots, h_{\ell-1}$,

(e) the positive slopes of the Newton polygon of $\mathcal{X}_2 = \delta - B_2(z)$ are $h_{\ell}, h_{\ell+1}, \dots, h_q$.

(cf. J.-P. Ramis [6] and Y. Sibuya [8]).

Set

$$(8.1) \quad \mathcal{X} = \delta - B(z)$$

and

$$(8.2) \quad f(z) = P(z)g(z).$$

Then

$$g(z) \in (\mathbb{C}[[z]]_{1/h_{\ell}}[z^{-1}])^n$$

and

$$(8.3) \quad \mathcal{L}[f] = P(z)\mathcal{X}[g].$$

Hence we have also

$$(8.4) \quad \mathcal{X}[g] \in (\mathbb{C}[[z]]_{1/h_{\ell-1}}[z^{-1}])^n.$$

Since the mapping:

$$\mathcal{X}: (\mathbb{C}[[z]]_{1/h_{\ell-1}}[z^{-1}])^n \rightarrow (\mathbb{C}[[z]]_{1/h_{\ell-1}}[z^{-1}])^n / (\mathbb{C}(z)[z^{-1}])^n$$

is surjective (cf. J.-P. Ramis [5] and Y. Sibuya [8]), there exists

$\mathbf{u} \in (\mathbb{C}[[z]]_{1/h_{\ell-1}}[z^{-1}])^n$ such that

$$(8.5) \quad \mathcal{X}[\mathbf{g} + \mathbf{u}] \in (\mathbb{C}(z)[z^{-1}])^n.$$

Note that

$$(8.6) \quad \mathcal{X}[\mathbf{u}] = -P(z)^{-1}\mathcal{X}[\mathbf{f}] + \mathcal{X}[\mathbf{g} + \mathbf{u}].$$

Set $\mathbf{v} = \mathbf{g} + \mathbf{u}$. Then $\mathbf{v} \in (\mathbb{C}[[z]]_{1/h_{\ell}}[z^{-1}])^n$. Write $\mathbf{v}$ in the form:

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{bmatrix},$$

where

$$\mathbf{v}_1 \in (\mathbb{C}[[z]]_{1/h_{\ell}}[z^{-1}])^{n_1}, \quad \mathbf{v}_2 \in (\mathbb{C}[[z]]_{1/h_{\ell}}[z^{-1}])^{n_2}.$$

Furthermore, we have

$$(8.7) \quad \mathcal{X}_1[\mathbf{v}_1] \in (\mathbb{C}(z)[z^{-1}])^{n_1}, \quad \mathcal{X}_2[\mathbf{v}_2] \in (\mathbb{C}(z)[z^{-1}])^{n_2}.$$

Now utilizing the assumption that d is not singular of level h_{ℓ} and the properties of the Newton polygons of $\mathcal{X}_1$ and $\mathcal{X}_2$, we conclude that

$$(8.8) \quad \mathbf{v}_1 \in (\mathbb{C}[[z]]_{1/h_{\ell-1}}[z^{-1}])^{n_1}, \quad \mathbf{v}_2 \in (\mathbb{C}(z)_{h_{\ell}d}[z^{-1}])^{n_2}$$

(cf. lemma 2.2 and remark 4.3). Also note that

$$(8.9) \quad \mathbf{f} = P(z)\mathbf{v} - P(z)\mathbf{u}.$$

Now by utilizing the existence theorem of the Gevrey asymptotic solutions (cf. J.-P. Ramis and Y. Sibuya [7]), we can find a finite number of open sectorial domains:

$$\mathcal{V}_v = \mathcal{D}(\alpha_v, \beta_v) \quad (v = 1, 2, \dots, N)$$

satisfying condition (7.3), N n -by- n matrices $\Phi_1, \Phi_2, \dots, \Phi_N$, and $2N+1$ functions $\lambda_1, \lambda_2, \dots, \lambda_N, \mu_{1,1}, \mu_{1,2}, \dots, \mu_{1,N}$ and μ_2 such that

$$(i) \quad \Phi_v \in M_n(\mathcal{A}_{\ell-1}^{1/h}(\mathcal{V}_v)), \quad \lambda_v \in (\mathcal{A}_{\ell-1}^{1/h}(\mathcal{V}_v))^n, \\ \mu_{1,v} \in (\mathcal{A}_{\ell-1}^{1/h}(\mathcal{V}_v))^n, \quad \mu_2 \in (\mathcal{A}_{\ell}^{1/h}(\bigcup_v \mathcal{V}_v))^n,$$

$$(ii) \quad J(\Phi_v) = P, \quad J(\lambda_v) = \mathbf{u}, \quad J(\mu_{1,v}) = \mathbf{v}_1, \quad J(\mu_2) = \mathbf{v}_2,$$

$$(iii) \quad \delta(\Phi_v) = A\Phi_v - \Phi_v B \text{ on } \mathcal{V}_v,$$

$$\mathcal{X}[\lambda_v] = -\Phi_v^{-1} \mathcal{Z}[f] + \mathcal{X}[\mathbf{v}] \text{ on } \mathcal{V}_v$$

$$\mathcal{X}_1[\mu_{1,v}] = \mathcal{X}_1[\mathbf{v}_1] \text{ on } \mathcal{V}_v,$$

$$\mathcal{X}_2[\mu_2] = \mathcal{X}_2[\mathbf{v}_2] \text{ on } \bigcup_v \mathcal{V}_v.$$

Note that μ_2 is the sum of $\mathbf{v}_2$ in the direction d .

Set

$$\Psi_v = \begin{bmatrix} \mu_{1,v} \\ \mu_2 \end{bmatrix} \quad \text{and} \quad \phi_v = \Phi_v \Psi_v - \Phi_v \lambda_v \in (\mathcal{A}_{\ell-1}^{1/h}(\mathcal{V}_v))^n.$$

Then $J(\phi_v) = P_v - P_u = f$. Furthermore

$$(a) \quad \phi_v - \phi_{v-1} = (\Phi_v - \Phi_{v-1}) \Psi_v + \Phi_{v-1} (\Psi_v - \Psi_{v-1}) \\ - (\Phi_v \lambda_v - \Phi_{v-1} \lambda_{v-1})$$

$$= (\mathcal{A}_{1/h}^{(0)}(\mathcal{F}_{v-1} \cap \mathcal{F}_v))^n,$$

and

$$(b) \quad \mathcal{L}[\phi_v] = \Phi_v \mathcal{L}[\Psi_v - \lambda_v] = \Phi_v \mathcal{L}[v] + \mathcal{L}[f] - \Phi_v \mathcal{L}[v] = \mathcal{L}[f].$$

q.e.d.

9. Proof of lemma 4.5: In this section, we shall prove lemma 4.5. To do this, we can assume without any loss of generality that

$$(9.1) \quad 1/2 \leq k' < k.$$

(Note that, if $k = k'$, then fg is k -summable.) Since $f \in \mathbb{C}\{z\}_{k,d}$ and $g \in \mathbb{C}\{z\}_{k',d}$, there exist two positive numbers R and ε and two functions ϕ_0 and γ such that

$$(9.2) \quad \begin{cases} \phi_0 \in \mathcal{A}_{1/k}(R, d - \varepsilon - \pi/(2k), d + \varepsilon + \pi/(2k)), \\ \gamma \in \mathcal{A}_{1/k'}(R, d - \varepsilon - \pi/(2k'), d + \varepsilon + \pi/(2k')), \end{cases}$$

and that

$$(9.3) \quad J(\phi_0) = f, \quad J(\gamma) = g.$$

Set

$$(9.4) \quad \begin{cases} \mathcal{V}_0 = \mathcal{D}(R, d - \varepsilon - \pi/(2k), d + \varepsilon + \pi/(2k)), \\ \mathcal{V} = \mathcal{D}(R, d - \varepsilon - \pi/(2k'), d + \varepsilon + \pi/(2k')). \end{cases}$$

Then $\mathcal{V}_0 \subset \mathcal{V}$. Now choose a covering

$$(9.5) \quad \{ \mathcal{V}_v = \mathcal{D}(R, \alpha_v, \beta_v); \quad v = -N, -N+1, \dots, 0, 1, 2, \dots, N \}$$

of $\mathcal{V}$ such that

$$(9.6) \quad \begin{cases} 0 < \beta_v - \alpha_v < \pi/k & \text{for } v \neq 0, \\ \alpha_{-N} = d - \varepsilon - \pi/(2k'), \quad \beta_N = d + \varepsilon + \pi/(2k'), \\ \alpha_v < \beta_{v-1} < \alpha_{v+1} < \beta_v & (-N \leq v < N-1). \end{cases}$$

Then there exist $2N$ functions ϕ_v ($-N \leq v \leq N$, $v \neq 0$) such that

$$(9.7) \quad \phi_v \in \mathcal{A}_{1/k}(\mathcal{V}_v), \quad J(\phi_v) = J(\phi_0).$$

Now if we choose paths $\mathfrak{P}_v$ of integration as in section 7, the integral

$$(9.8) \quad - (1/(2\pi i)) \sum_{N \geq \mu \geq -N+1} \int_{\mathfrak{P}_\mu} [(\phi_\mu(\zeta) - \phi_{\mu-1}(\zeta)) \gamma(\zeta)/(\zeta-z)] d\zeta$$

defines $2N+1$ functions ψ_v ($-N \leq v \leq N$) on $\mathcal{V}_v$ respectively such that

$$(9.9) \quad \begin{cases} \psi_v \in \mathcal{A}_{1/k}(\mathcal{V}_v), \\ \psi_v - \psi_{v-1} = (\phi_v - \phi_{v-1}) \gamma & \text{on } \mathcal{V}_{v-1} \cap \mathcal{V}_v. \end{cases}$$

If we set

$$(9.10) \quad \psi(z) = \phi_v(z) \gamma(z) - \psi_v(z) \quad \text{on } \mathcal{V}_v,$$

then

$$(9.11) \quad \psi \in \mathcal{A}_{1/k'}(\mathcal{V}) ,$$

and hence

$$(9.12) \quad J(\psi) \in \mathbb{C}[z]_{k', d} .$$

On the other hand, since $\psi_0 \in \mathcal{A}_{1/k}(\mathcal{V}_0)$, we have

$$(9.13) \quad J(\psi_0) \in \mathbb{C}[z]_{k, d}$$

This implies that

$$(9.14) \quad f g = J(\phi_0(z) \gamma(z)) = J(\psi) + J(\psi_0) .$$

q.e.d

10. Proof of theorem 6.1: We shall use the same notations as we defined in sections 5 and 6. To prove theorem 6.1 by induction, we note that, for any n -by- n matrix C , we have $C^{(p)} = \mathbf{I}$. Hence condition (6.6) is always satisfied for " $j = p + 1$ ". Let us assume that there exists an admissible system $\{Y_v(z), v \in \mathbb{Z}\}$ whose corresponding Stokes multipliers $\{W_v, v \in \mathbb{Z}\}$ satisfies the condition (6.6), i.e.

$$W_v^{(j-1)} = \mathbf{I}$$

for every pair (j, ν) such that

$$(10.1) \quad d - \pi/(2k_j) < \tau_{\nu-1} < \tau_\nu < d + \pi/(2k_j), \quad j \geq j_0.$$

We shall modify the system $\{Y_\nu(z), \nu \in \mathbb{Z}\}$ so that the corresponding Stokes multipliers satisfy condition (6.6) for $j \geq j_0$. To do this, let us set

$$(10.2) \quad X_\nu(z) = Y_\nu(z) C_\nu \quad (\nu \in \mathbb{Z}),$$

where $C_\nu \in GL(n, \mathbb{C})$ ($\nu \in \mathbb{Z}$). In order that $\{X_\nu(z), \nu \in \mathbb{Z}\}$ be admissible, it is necessary and sufficient that the matrices C_ν satisfy condition (6.2), i.e.

$$\exp[Q(z)] C_\nu \exp[-Q(z)] - \mathbf{I} \in M_n(\mathcal{S}_0(\mathcal{S}_\nu))$$

for some $s \in \{1/k_1, 1/k_2, \dots, 1/k_p\}$. Let $\{V_\nu, \nu \in \mathbb{Z}\}$ be the Stokes multipliers corresponding to the admissible system $\{X_\nu(z), \nu \in \mathbb{Z}\}$. Then we have (6.5), i.e.

$$W_\nu(j) C_{\nu-1}(j) = C_\nu(j) V_\nu(j) \quad (\nu \in \mathbb{Z}, j = 0, 1, 2, \dots, p).$$

Therefore, if we choose $\{C_\nu, \nu \in \mathbb{Z}\}$ so that $C_\nu(j_0) = \mathbf{I}$ ($\nu \in \mathbb{Z}$), then $\{V_\nu, \nu \in \mathbb{Z}\}$ also satisfy condition (6.6) for (10.1). Note that $C^{(j_0)} = \mathbf{I}$ implies $C^{(j)} = \mathbf{I}$ ($j \geq j_0$) for any n -by- n matrix C .

Let ν_0 be the smallest ν satisfying the conditions:

$$(10.3) \quad \begin{cases} d - \pi/(2k_{j_0}) < \tau_{\nu-1} < \tau_\nu < d + \pi/(2k_{j_0}), \\ W_\nu(j_0-1) \neq \mathbf{I} \end{cases}$$

According to the lemma on p. 80 and the proposition on p. 82 of W. Jurkat [3], there exist unique n -by- n matrices C and $V \in GL(n, \mathbb{C})$ satisfying the following conditions:

$$(10.4) \quad \begin{cases} W_{v_0}^{(j_0-1)} = C V, & C^{(j_0)} = \mathbf{I}, & V^{(j_0)} = \mathbf{I}, \\ C = C^{(j_0-1)}, & V = V^{(j_0-1)}, \\ \exp[Q(z)] C \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{s0}(\mathcal{S}_{v_0})), \\ \exp[Q(z)] V \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{s0}(\tau_{v_0} - \pi/k_{j_0}, \tau_{v_0})). \end{cases}$$

Utilizing the fact that

$$(a) \quad \exp[Q(z)] W_{v_0}^{(j_0-1)} \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{s0}(\mathcal{S}_{v_0-1} \cap \mathcal{S}_{v_0})),$$

$$(b) \quad W_{v_0}^{(j_0)} = \mathbf{I} \quad \text{and} \quad W_{v_0}^{(j_0-1)} \neq \mathbf{I},$$

we can show in the case when $V \neq \mathbf{I}$ that, if

$$(10.5) \quad v \leq v_0 - 1 \quad \text{and} \quad d - \pi/(2k_{j_0}) < \tau_v,$$

then $\mathcal{S}_v$ is contained in the sectorial domain $\mathcal{D}(\tau_{v_0} - \pi/k_{j_0}, \tau_{v_0})$.

Therefore we have always

$$\exp[Q(z)] V \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{s0}(\mathcal{S}_v)).$$

Set

$$(10.6) \quad C_v = \begin{cases} C & \text{if } v = v_0, \\ V^{-1} & \text{if } v \text{ satisfies (10.5),} \\ \mathbf{I} & \text{otherwise.} \end{cases}$$

Then it is easily verify that $C_v^{(j_0)} = \mathbf{I}$ for every v , and that

$$V_v^{(j_0-1)} = \mathbf{I}$$

for every $v \geq v_0$ such that

$$d - \pi/(2k_{j_0}) < \tau_{v-1} < \tau_v < d + \pi/(2k_{j_0}).$$

In this way, we can complete the proof of theorem 6.1.

REMARK 10.1: In the statement of theorem 6.1, we restricted (j, v) by the strict inequality (6.7). Hence we did not need the assumption that d is an admissible direction.

11. Proof of theorem 6.3: Let $\{W_v, v \in \mathbb{Z}\}$ be the Stokes multipliers corresponding to the system $\{Y_v(z), v \in \mathbb{Z}\}$. Then we have

$$W_v^{(j)} C_{v-1}^{(j)} = C_v^{(j)} V_v^{(j)} \quad (v \in \mathbb{Z}, j = 0, 1, 2, \dots, p).$$

Suppose that $C_v^{(j-1)} = \mathbf{I}$ for every (j, v) satisfying condition

$$(6.11) \quad d - \pi/(2k_j) < \tau_v < d + \pi/(2k_j).$$

Then, $W_v^{(j-1)} = V_v^{(j-1)} = \mathbf{I}$ for every (j, v) satisfying condition

$$(6.7) \quad d - \pi/(2k_j) < \tau_{v-1} < \tau_v < d + \pi/(2k_j).$$

Conversely, suppose that $W_v^{(j-1)} = \mathbf{I}$ for every (j, v) satisfying condition (6.7). Then $C_{v-1}^{(j-1)} = C_v^{(j-1)}$ for every (j, v) satisfying condition (6.7). Hence, for a fixed $j = 0, 1, 2, \dots, p$, the matrices $C_v^{(j-1)}$ are independent of v as long as (6.11) is satisfied. Set

$$(11.1) \quad \mathcal{C}_j = C_v^{(j-1)}$$

for all $\mathbf{v}$ satisfying (6.11). Since

$$\exp[Q(z)] C_{\mathbf{v}} \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{SO}(\mathcal{S}_{\mathbf{v}})),$$

we have

$$\exp[Q(z)] \mathcal{C}_j \exp[-Q(z)] - \mathbf{I} \in \mathbf{M}_n(\mathcal{A}_{SO}(R, d - \pi/(2k_j), d + \pi/(2k_j)))$$

for any $R > 0$. Now note that $\mathcal{C}_j = \mathcal{C}_j^{(j-1)}$. Furthermore, since d is admissible, we must have $\mathcal{C}_j = \mathcal{C}_j^{(j)}$. This in turn implies that

$$(11.2) \quad C_{\mathbf{v}}^{(j-1)} = C_{\mathbf{v}}^{(j)}$$

for all $\mathbf{v}$ satisfying (6.11).

Note that, if $\mathbf{v}$ satisfies (6.11), then $\mathbf{v}$ also satisfies (6.11) with j replaced by $j+1$. Hence (11.2) implies that $C_{\mathbf{v}}^{(j-1)} = C_{\mathbf{v}}^{(p)} = \mathbf{I}$. q.e.d.

12. Proof of theorem 6.5: First of all, set

$$(12.1) \quad \mathcal{E} = \{ \mathbf{v} ; d - \pi/(2k_p) < \tau_{\mathbf{v}} < d + \pi/(2k_p) \}.$$

Then

$$(12.2) \quad \{ z ; d - \pi/(2k_p) \leq \arg z \leq d + \pi/(2k_p), 0 < |z| < R \} \subset \bigcup_{\mathbf{v} \in \mathcal{E}} \mathcal{S}_{\mathbf{v}},$$

since d is admissible. As in section 7, let us fix a point $z_{\mathbf{v}}$ on the boundary of $\mathcal{S}_{\mathbf{v}-1} \cap \mathcal{S}_{\mathbf{v}}$ so that

$$(12.3) \quad \tau_{\mathbf{v}-1} < \arg z_{\mathbf{v}} < \tau_{\mathbf{v}} \quad (\mathbf{v} \in \mathcal{E}).$$

Denote by $\mathcal{P}_{\mathbf{v}}$ the line segment joining 0 to $z_{\mathbf{v}}$.

As in section 7, we shall consider the integral

$$(12.4) \quad (1/(2\pi i)) \sum_{v, v-1 \in \mathcal{J}} \int_{\mathcal{P}_v^\sigma} [(\Psi_{v-1}(\rho^\sigma) - \Psi_v(\rho^\sigma))/(\rho-t)] d\rho,$$

where $\mathcal{P}_v^\sigma = \{ t ; t^\sigma \in \mathcal{P}_v \}$ and

$$(12.5) \quad \Psi_v(t^\sigma) = X_v(t^\sigma) G(t^\sigma)^{-1} \in \mathbf{M}_n(\mathcal{A}_{1/\sigma k_p}(\mathcal{S}_v^\sigma))$$

where $\mathcal{S}_v^\sigma = \{ t ; t^\sigma \in \mathcal{S}_v \}$. Note that, $J(\Psi_v(t^\sigma)) = F(t^\sigma)$ and

$$(12.6) \quad \Psi_{v-1}(\zeta) - \Psi_v(\zeta) = \Psi_v(\zeta) [G(\zeta) V_v G(\zeta)^{-1} - \mathbf{I}].$$

For each v such that v and $v-1 \in \mathcal{J}$, let j_v denote the smallest j such that

$$(12.7) \quad d - \pi/(2k_j) < t_{v-1} < t_v < d + \pi/(2k_j).$$

Since the Stokes multipliers $\{V_v, v \in \mathbf{Z}\}$ satisfy condition (6.6) for every pair (j, v) such that (12.7) holds, we can show that

$$(12.8) \quad \Psi_{v-1}(\zeta) - \Psi_v(\zeta) \in \mathbf{M}_n(\mathcal{A}_{1/k_{j_v-1}}^0(\mathcal{S}_{v-1} \cap \mathcal{S}_v)).$$

Hence we have

$$(12.9) \quad \int_{\mathcal{P}_v^\sigma} [(\Psi_{v-1}(\rho^\sigma) - \Psi_v(\rho^\sigma))/(\rho-t)] d\rho \\ \in \mathbf{M}_n(\mathcal{A}_{1/\sigma k_{j_v-1}}(R, d/\sigma - \varepsilon - \pi/(2\sigma k_{j_v-1}), d/\sigma + \varepsilon + \pi/(2\sigma k_{j_v-1})))$$

for some R and $\varepsilon > 0$. Note that $\mathcal{S}_{v-1} \cap \mathcal{S}_v$ is not contained in the sectorial domain $\mathcal{D}(d - \pi/(2k_{j_v-1}), d + \pi/(2k_{j_v-1}))$. From (12.9) we conclude that

$$(12.10) \quad J\left(\int_{\mathcal{P}_v^\sigma} [(\Psi_{v-1}(p^\sigma) - \Psi_v(p^\sigma))/(p-t)] dp\right) = M_n(\mathbb{C}(z)_{\sigma k_p, \frac{1}{\sigma}-1} d/\sigma).$$

On each sectorial domain $\mathcal{S}_v$ ($v \in \mathcal{E}$), integral (12.4) defines a function $\Gamma_v(z)$ such that

$$(i) \quad \Gamma_v(t^\sigma) = M_n(\mathcal{A}_{1/(\sigma k_p)}(\mathcal{S}_v^\sigma)),$$

$$(ii) \quad \Gamma_v(z) - \Gamma_{v-1}(z) = \Psi_v(z) - \Psi_{v-1}(z) \quad \text{on } \mathcal{S}_{v-1} \cap \mathcal{S}_v.$$

Therefore by setting

$$(12.11) \quad \Psi(z) = \Psi_v(z) - \Gamma_v(z) \quad \text{on } \mathcal{S}_v,$$

we can define a function such that

$$(12.12) \quad \Psi(t^\sigma) = M_n(\mathcal{A}_{1/\sigma k_p}(R, d/\sigma - \varepsilon - \pi/(2\sigma k_p), d/\sigma + \varepsilon + \pi/(2\sigma k_p)))$$

for some R and $\varepsilon > 0$. This implies that

$$(12.13) \quad J(\Psi(t^\sigma)) = M_n(\mathbb{C}(z)_{\sigma k_p} d/\sigma).$$

Finally, as we showed in section 6 (cf. remarks 6.2 and 6.4),

$$(12.14) \quad \Psi_v(z) = \Psi(z; d)$$

if $d - \pi/(2k_1) < \tau_v < d + \pi/(2k_1)$. Since $\Psi_v(z) = \Gamma_v(z) + \Psi(z)$,

theorem 6.5 follows immediately. $q.e.d.$

13. Proofs of theorems 6.6 and 6.8: Part (a) of theorem 6.6 follows from the uniqueness of the sum of a multisummable power series. To prove

Part (b) of theorem 6.6, let $\{X_v(z), v \in \mathbb{Z}\}$ and $\{Y_v(z), v \in \mathbb{Z}\}$ be the admissible systems which satisfy the conditions stated in theorem 6.1 and are determined uniquely by the directions d_1 and d_2 respectively. Also let $\{V_v, v \in \mathbb{Z}\}$ and $\{W_v, v \in \mathbb{Z}\}$ be the Stokes multipliers corresponding to $\{X_v(z), v \in \mathbb{Z}\}$ and $\{Y_v(z), v \in \mathbb{Z}\}$ respectively. Then if

$$(13.1) \quad d_2 - \pi/(2k_j) < \tau_{v-1} < \tau_v < d_1 + \pi/(2k_j),$$

we have

$$(13.2) \quad V_v(j-1) = I, \quad W_v(j-1) = I.$$

Set

$$(13.3) \quad C_v = Y_v(z)^{-1} X_v(z) \in GL(n, \mathbb{C}) \quad (v \in \mathbb{Z}).$$

Then

$$(13.4) \quad W_v(j-1) C_{v-1}(j-1) = C_v(j-1) V_v(j-1).$$

Hence if (13.1) is satisfied, we have

$$(13.5) \quad C_{v-1}(j-1) = C_v(j-1).$$

This in turn implies that, if we have

$$(13.6) \quad d_2 - \pi/(2k_j) < \tau_v < d_1 + \pi/(2k_j)$$

and

$$(13.7) \quad d_2 - \pi/(2k_1) < \tau_\mu < d_1 + \pi/(2k_1),$$

then

$$(13.8) \quad C_{\mu}^{(j-1)} = C_{\nu}^{(j-1)}$$

Note that the sectorial domains $\mathcal{S}_{\nu}$ with ν satisfying (13.6) cover the sectorial domain $\mathcal{D}(d_2 - \pi/(2k_j), d_1 + \pi/(2k_j))$. Hence

$$(13.9) \quad C_{\mu} \subseteq \mathcal{U}(d_1, d_2) \quad (\text{cf. remark 6.7}).$$

Finally, note that, if μ satisfies condition (13.7), we have

$$(13.10) \quad X_{\nu}(z)G(z)^{-1} = \mathbf{Y}(z; d_1), \quad Y_{\nu}(z)G(z)^{-1} = \mathbf{Y}(z; d_2),$$

and hence $C_{\mu} = V$, where V is given by (6.15). q.e.d.

Let us now look at theorem 6.8. The equivalence between (i) and (ii) is evident. On the other hand, we can prove the equivalence between (i) and (iii) by utilizing a method similar to the proof of theorem 6.5.

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