**HOMOGENIZATION APPROACH TO LIGHT
SCATTERING FROM POLYMER-DISPERSED
LIQUID CRYSTAL FILMS**

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HOMOGENIZATION APPROACH TO LIGHT SCATTERING FROM POLYMER-DISPERSED LIQUID CRYSTAL FILMS*

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Abstract. A polymer-dispersed liquid crystal film consists of a large number of liquid crystal droplets dispersed in polymer matrices. We consider the scattering of a plane wave by such a film. Using the approach of homogenization we derive an asymptotic series for the electric-magnetic field u of the form $u_0 + au_1 + a^2u_2 + \dots$ as $a \rightarrow 0$, where a is the edge of the cubes, each containing a single droplet. The term $u_0 = u_0(x)$ accounts for reflection and transmission through the homogenized material, whereas the terms $u_j = u_j(x, x/a)$ ($j \geq 1$) represent the scattering effects.

Key words. Scattering, Maxwell's equations, homogenization, liquid crystal droplets

AMS(MOS) subject classifications. 35P25, 73K20, 78A45, 78A57.

§1. Introduction. Polymer-dispersed liquid crystal (PDLC) films consist of liquid crystal (LC) microdroplets dispersed in polymer matrices. They can be switched electrically from a cloudy, light-scattering state (“off state”) to a transparent state (“on state”); see [5] [6] [11]. To optimize the light scattering performance of PDLC film, one has to analyze the dependence of the scattered light on several parameters: the refractive indices of the polymer material and of the liquid crystal droplets in their off and on states, the volume fraction and size distribution of the liquid crystal droplets in the film, the thickness of the film, and the temperature.

The radius of the droplets is $0.1\mu m - 10\mu m$, and the volume fraction they occupy in the film is typically $0.1 - 0.5$. Often there will be at least 20 droplets across the thickness of the film. Hence the number of droplets per $1cm^2$ of film area is typically at least $2 \cdot 10^9$. This enormous number of scatterers makes the task of solving the complete scattering problem nearly impossible. Indeed, as pointed out in [10], an exact analysis of light scattering by PDLC films is not possible because no theory of multiple scattering is available.

Single scattering by a nematic droplet has been studied in [15] [16] [17]. The analysis is based on the Rayleigh–Gans approximation [3] which assumes, in particular, that

$$(1.1) \quad \frac{\rho}{\lambda} \ll 1$$

where ρ is the radius of the droplet and λ is the wavelength of monochromatic incident plane-wave. The results of this analysis have been used in [8] [9] [10] to study the scattering by PDLC film, assuming that the droplets form a collection of optically isotropic scatterers

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with negligible multiple scattering. Comparison of the analysis with experiments indicates (cf. [8])) that multiple scattering is important in PDLC films even in their on state.

In this paper we approach the multiple scattering problems from the point of view of homogenization, as suggested in [7; Chap. 7]. Accordingly, we view the film as a periodic array consisting of cubes of edge a ; each cube hosting precisely one droplet of radius ρ . The dielectric matrix is

$$(1.2) \quad \varepsilon_p = \begin{pmatrix} n_p^2 & 0 & 0 \\ 0 & n_p^2 & 0 \\ 0 & 0 & n_p^2 \end{pmatrix} \quad \text{in the polymer}$$

and

$$(1.3) \quad \varepsilon_d^1 = \begin{pmatrix} n_0^2 & 0 & 0 \\ 0 & n_0^2 & 0 \\ 0 & 0 & n_e^2 \end{pmatrix}, \quad \varepsilon_d^2 = \begin{pmatrix} n_e^2 & 0 & 0 \\ 0 & n_0^2 & 0 \\ 0 & 0 & n_0^2 \end{pmatrix}$$

in the droplet at “on state” and “off state” respectively. Typically $n_p = 1.5$, $n_0 = 1.52$ and $n_e = 1.5 - 1.7$. As $a \rightarrow 0$ (but ρ/a remains fixed) the material in the film can be approximated by “homogenized” material with anisotropic dielectric matrix

$$(1.4) \quad \varepsilon = \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix};$$

the q_{ij} are constants and ε is a symmetric positive definite matrix.

Writing $u = (e, h)$ where e is the electric field and h is the magnetic field, we shall derive an asymptotic series

$$(1.5) \quad u(x) = u_0(x) + au_1\left(x, \frac{x}{a}\right) + a^2u_2\left(x, \frac{x}{a}\right) + \cdots$$

for small a , provided (1.1) is satisfied.

The homogenization problem for the Maxwell equation has been studied (in terms of the component h) in [2] provided x belongs to a bounded domain Ω and h satisfies some “nice” boundary conditions such as

$$(1.6) \quad n \times h = 0 \quad \text{on} \quad \partial\Omega \quad (n = \text{normal to } \partial\Omega).$$

Arguments of coerciveness and compactness can then be used to deduce that a unique solution h to the Maxwell equation exists, and $h \rightarrow h_0$ as $a \rightarrow 0$, where h_0 is the solution to the homogenized problem.

The situation in the present paper is quite different since

(i) Ω is the whole space $\mathbb{R}^3$, and homogenization takes place only in a part of $\mathbb{R}^3$, namely, in the infinite slab $\{0 < x_1 < d\}$ occupied by the film;

(ii) The boundary conditions are not “very nice;” they consist of transmission conditions at $\{x_1 = 0\}$ and $\{x_1 = d\}$ and of a radiation condition at infinity.

The radiation condition states that the solution u has the form of incident wave plus a sum of waves outgoing from the film as $x_1 \rightarrow -\infty$, and u is a sum of waves outgoing from the film as $x_1 \rightarrow +\infty$.

There are no existence or uniqueness theorems for the present scattering problem. For a related problem of scattering by a domain with periodic surface, existence and uniqueness have recently been proved by Chen and Friedman [4]. However, the scattering domain in [4] is isotropic and homogeneous. For anisotropic material the Green function is very complicated [12] and it is doubtful whether the methods of [4] (which rely on explicit representation of solutions by means of Green’s function) can be extended to such problems and in particular to the present one of scattering by PDLC film.

Our approach in this paper is formal. We assume that a solution u of the multiple scattering problem exists, and we develop a method for computing the terms in the expansion (1.5). The method involves solving scattering problems by a film with dielectric matrix (1.4) and with radiation condition at ∞ . It also exploits some integral representation as in [4].

In §2 we describe the homogenization approach. The first term u_0 is obtained in §3. In §4 we derive the differential equations and transmission conditions (across $x_1 = 0$ and $x_1 = d$) for $u_1, u_2, \dots$. In §§5,6 we show how to compute u_1 and in §7 we indicate the form of the other terms $u_2, u_3, \dots$. In $\{x_1 < 0\}$ each u_j ($j \geq 1$) has the form $u_j = (e_j, h_j)$ where e_j is the electric field, h_j is the magnetic field and for small a ,

$$\begin{aligned}
 h_j \left(x, \frac{x}{a} \right) &= H_{00}^{(j)} e^{i(-\alpha_1 x_1 + \alpha_2 x_2)} \\
 (1.7) \quad &+ \sum_{(m,n) \neq (0,0)} H_{mn}^{(j)} e^{(\gamma_{mn} - \gamma^2)^{1/2} x_1 + i[(\alpha_2 + \frac{2\pi m}{a})x_2 + \frac{2\pi n}{a} x_3]}
 \end{aligned}$$

where the $H_{mn}^{(j)}$ depend on a only,

$$|H_{mn}^{(j)}| \leq \frac{C_k a^{-2}}{(m^2 + n^2)^k} \quad \forall k \geq 1$$

and $\gamma = (\alpha_1^2 + \alpha_2^2)^{1/2}$,

$$\gamma_{mn} = \left\{ \left(\alpha_2 + \frac{2\pi m}{a} \right)^2 + \left(\frac{2\pi n}{a} \right)^2 \right\}^{1/2}.$$

Here $(\alpha_1, \alpha_2, 0)$ is the wave vector of the incident light; it is assumed to be such that $\gamma_{mn} \neq \gamma$ for all m, n . We note that the scattering terms ($m + n \neq 0$) are decaying exponentially in the distance from the slab.

A representation similar to (1.7) (with x_1 replaced by $d - x_1$) holds in $\{x_1 > d\}$.

§2. The homogenization approach. We consider time harmonic solutions

$$\vec{E}(x, t) = e^{-i\omega t} e(x), \quad \vec{H}(x, t) = e^{-i\omega t} h(x)$$

to the Maxwell equations. The magnetic permeability μ is assumed to be constant throughout the whole space $\mathbb{R}^3$; in fact we take $\mu = 1$. The dielectric coefficient ε is the constant 1 outside the film

$$\Omega = \{0 < x_1 < d\}.$$

Inside the film, ε is a matrix given by

$$\varepsilon = \begin{pmatrix} n_p^2 & 0 & 0 \\ 0 & n_p^2 & 0 \\ 0 & 0 & n_p^2 \end{pmatrix} \quad \text{in the polymer,}$$

$$\varepsilon = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \quad \text{in the droplets}$$

where n_j are as in (1.3).

We assume that the droplets are uniformly distributed in the film, such that in each cube

$$ja < x_1 < (j+1)a \quad \left(j = 0, 1, \dots, N, \quad N = \frac{d}{a} \right)$$

$$k_2 a < x_2 < (k_2 + 1)a \quad (k_2 = 0, \pm 1, \pm 2, \dots)$$

$$k_3 a < x_3 < (k_3 + 1)a \quad (k_3 = 0, \pm 1, \pm 2, \dots)$$

there is a single droplet of radius ρ ($\rho < a/2$); one droplet is obtained from another by translation

$$x_1 \rightarrow x_1 + ja, \quad x_2 \rightarrow x_2 + k_2 a, \quad x_3 \rightarrow x_3 + k_3 a$$

where j, k_2, k_3 are integers. Thus, in the film,

$$\varepsilon_{ij} = \varepsilon_{ij} \left(\frac{x}{a} \right)$$

where $\varepsilon_{ij}(y)$ is periodic of period 1 in y_1, y_2, y_3 .

Introducing the unit cube

$$Y = \{y = (y_1, y_2, y_3), \quad 0 \leq y_j \leq 1\}$$

and the region D in Y corresponding to the location of the drop, we have:

$$(\varepsilon_{ij}(y)) = \begin{pmatrix} n_p^2 & 0 & 0 \\ 0 & n_p^2 & 0 \\ 0 & 0 & n_p^2 \end{pmatrix} \quad \text{in } Y \setminus D, \quad (\varepsilon_{ij}(y)) = \begin{pmatrix} n_1^2 & 0 & 0 \\ 0 & n_2^2 & 0 \\ 0 & 0 & n_3^2 \end{pmatrix} \quad \text{in } D.$$

We assume that $\overline{D}$ is contained in the interior of Y .

Next introduce the regions

$$R_1 = \{x_1 < 0\}, \quad R_2 = \{0 < x_1 < d\}, \quad R_3 = \{d < x_1 < \infty\};$$

the incident plane wave will come from R_1 .

The Maxwell equations are

$$(2.1) \quad \nabla \times e - i \frac{\omega}{c} h = 0,$$

$$(2.2) \quad \nabla \times h + i \frac{\varepsilon \omega}{c} e = 0,$$

$$(2.3) \quad \nabla \cdot h = 0, \quad \nabla \cdot (\varepsilon e) = 0.$$

In each region R_j , (2.3) follow from (2.1) and (2.2). The weak form of (2.1)–(2.3) leads to the following transmission conditions:

$$(2.4) \quad \begin{aligned} &h \text{ is continuous across } \{x_1 = 0\}, \{x_1 = d\}, \\ &e_2, e_3 \text{ and } (\varepsilon e)_1 \text{ are continuous across } \{x_1 = 0\}, \{x_1 = d\}, \end{aligned}$$

where the subindex refers to the component of the vector. The system (2.1)–(2.3) is thus equivalent to (2.1), (2.2) and (2.4).

REMARK 2.1. As in [4; Lemma 1.1], of the six transmission conditions (2.4) at $x_1 = 0$ or $x_1 = d$, two follow from the remaining four and the Maxwell equations. These four conditions are:

$$(2.5) \quad n \times h \text{ and } n \times e \text{ are continuous across } \{x_1 = 0\}, \{x_1 = d\} .$$

Thus in the sequel we need only to consider the system (2.1), (2.2), (2.5).

From (2.1), (2.2) we get

$$(2.6) \quad \operatorname{rot} (\varepsilon^{-1} \operatorname{rot} h) - \frac{\omega^2}{c^2} h = 0$$

where rot means $\nabla \times$.

Consider the equation

$$(2.7) \quad \operatorname{rot} (\varepsilon^{-1} \operatorname{rot} h) + \lambda h = 0 \quad (\lambda > 0)$$

in a bounded domain G in R_2 ($\varepsilon^{-1} = \varepsilon^{-1}(x/a)$) with suitable boundary conditions, such as

$$n \times h = 0 \quad \text{on} \quad \partial G ,$$

or

$$n \times \operatorname{rot} h = 0 \quad \text{on} \quad \partial G .$$

As proved in [2; pp. 139–147], this problem is coercive and has a unique solution h_a and $h_a \rightarrow h_0$ (weakly in suitable norms) as $a \rightarrow 0$ where h_0 is the solution to the homogenized problem

$$\operatorname{rot} (Q^{-1} \operatorname{rot} h_0) + \lambda h_0 = 0 \quad \text{in} \quad G$$

with the same boundary conditions as h_a . The matrix Q is defined as follows:

Set

$$\mathcal{L}w = \sum_{i,j=1}^3 \frac{\partial}{\partial y_i} \left(\varepsilon_{ij}(y) \frac{\partial w}{\partial y_j} \right) , \quad y \in Y$$

and denote by θ^k ($k = 1, 2, 3$) the solution to the elliptic problem

$$(2.8) \quad \begin{aligned} \mathcal{L}\theta^k &= \mathcal{L}y_k \quad \text{in} \quad Y , \\ \theta^k &\text{ is } Y\text{-periodic} \end{aligned}$$

(this means that $\theta^k(y_1, y_2, y_3)$ satisfies the differential equation for all $y \in \mathbb{R}^3$ and is periodic in each y_j of period 1). Set

$$(2.9) \quad b_{ij} = \varepsilon_{ij} - \sum_{k=1}^3 \varepsilon_{ik} \frac{\partial \theta^j}{\partial y_k} .$$

Then

$$(2.10) \quad Q = (q_{ij}) , \quad q_{ij} = \int_Y b_{ij}(y) dy .$$

The matrix Q is symmetric and positive definite.

The above result holds also for (2.6) (instead of (2.7)) provided ω^2/c^2 is not an eigenvalue of the corresponding problem; in the sequel this is always assumed to be the case.

The above results tells us what the dielectric matrix should be in R_2 , namely, Q . In the empty space the dielectric coefficient is of course 1. From (1.5) we then deduce that $u_0 = (e_0, h_0)$ where e_0, h_0 satisfy (2.1), (2.2), (2.5) with

$$(2.11) \quad \varepsilon = Q \text{ in } R_2, \quad \varepsilon = 1 \text{ in } R_1 \cup R_3 .$$

§3. Solving for u_0 . In this section we solve the problems (2.1), (2.2), (2.5) with ε given by (2.11).

Consider an incident (incoming) plane wave in R_1 :

$$e_i = a_0 e^{i\alpha \cdot x} , \quad h_i = b_0 e^{i\alpha \cdot x}$$

with wave vector $\alpha = (\alpha_1, \alpha_2, 0)$, $\alpha_1 > 0$; for simplicity we may take $\alpha_3 = 0$, since the geometry is symmetric with respect to the x_2 and x_3 axes. Set

$$M_0 = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix} , \quad a_0 = \begin{pmatrix} a_1^0 \\ a_2^0 \\ a_3^0 \end{pmatrix}$$

Then (2.1), (2.2) in R_1 reduce to

$$(3.1) \quad \alpha_1^2 + \alpha_2^2 = \gamma^2 ,$$

$$(3.2) \quad M_0 b_0 = -\gamma a_0 , \quad M_0 a_0 = \gamma b_0$$

where $\gamma = \omega/c$. Notice that $-\gamma^2$ is an eigenvalue of M_0^2 . From (3.2) it follows that $M_0^2 a_0 = -\gamma^2 a_0$, or

$$(3.3) \quad \alpha_1 a_1^0 + \alpha_2 a_2^0 = 0$$

The incident wave gives rise to reflected and transmitted waves. The reflected wave is given by

$$e_r = a e^{i(-\alpha_1 x_1 + \alpha_2 x_2)} \quad , \quad h_r = b e^{i(-\alpha_1 x_1 + \alpha_2 x_2)}$$

where

$$(3.4) \quad M_1 b = -\gamma a \quad , \quad M_1 a = \gamma b$$

and

$$M_1 = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & \alpha_1 \\ -\alpha_2 & -\alpha_1 & 0 \end{pmatrix} .$$

The general solution of (3.4) is given by

$$(3.5) \quad a = -u' \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ 0 \end{pmatrix} - v' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where u', v' are arbitrary real numbers.

The transmitted waves in R_2 have the form

$$e = \sum_{j=1}^4 u_j a_j^* e^{i(\beta_j x_1 + \alpha_2 x_2)} \quad ,$$

$$h = \sum_{j=1}^4 u_j b_j^* e^{i(\beta_j x_1 + \alpha_2 x_2)}$$

where $a = a_j^*$, $b = b_j^*$ and $\beta = \beta_j$ satisfy:

$$(3.6) \quad Q^{-1} M_2(\beta) b = -\gamma a \quad , \quad M_2(\beta) a = \gamma b$$

with

$$M_2(\beta) = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & -\beta \\ -\alpha_2 & \beta & 0 \end{pmatrix}; \text{ clearly } M_2(\beta)^2 = \begin{pmatrix} -\alpha_2^2 & \beta\alpha_2 & 0 \\ \beta\alpha_2 & -\beta^2 & 0 \\ 0 & 0 & -\alpha_2^2 - \beta^2 \end{pmatrix}$$

and eliminating b_j^* we find that β_j, a_j^* solve the eigenvalue problem

$$(3.7) \quad (M_2^2(\beta_j) + \gamma^2 Q)a_j^* = 0 .$$

Since the determinant in (3.7) is a polynomial of degree 4 in β_j , we expect that in the generic case there are four simple eigenvalues β_j .

The transmission conditions (2.5) at $\{x_1 = 0\}$ become:

$$(3.8) \quad n \times \left[\sum_{j=1}^4 M_2(\beta_j) u_j a_j^* \right] = n \times (Ma + M_0 a_0) ,$$

$$(3.9) \quad n \times \left[\sum_{j=1}^4 u_j a_j^* \right] = n \times (a + a_0) .$$

Rewriting (3.8), (3.9) in the form

$$(3.10) \quad n \times \left[\sum_{j=1}^4 (M_2(\beta_j) a_j^*) u_j + u' M_1 \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ 0 \end{pmatrix} + v' M_1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = n \times M_0 a_0 ,$$

$$(3.11) \quad n \times \left[\sum_{j=1}^4 (G^{-1} a_j^*) u_j + u' \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ 0 \end{pmatrix} + v' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = n \times a_0$$

we get a linear system of four equations in six unknowns.

Similarly we can get reduce the transmission conditions at $\{x_1 = d\}$ to a linear system of four equations, for the unknown u_j and two additional variables u'', v'' . To begin with, we introduce the outgoing wave in R_3 , which must have the form

$$e = a'' e^{i(\alpha_1 x_1 + \alpha_2 x_2)}$$

$$h = b'' e^{i(\alpha_1 x_1 + \alpha_2 x_2)}$$

where

$$(3.12) \quad M_0 b'' = -\gamma a'' , \quad M_0 a'' = \gamma b'' .$$

The general solution for a'' is:

$$(3.13) \quad a'' = -u'' \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ 0 \end{pmatrix} - v'' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

where u'', v'' are arbitrary real numbers. The transmission conditions at $\{x_1 = d\}$ (cf. (3.8), (3.9)) are

$$(3.14) \quad n \times \sum_{j=1}^4 M_2(\beta_j) u_j a_j^* e^{\beta_j d} = n \times M_0 a'' e^{i\alpha_1 d} ,$$

$$(3.15) \quad n \times \sum_{j=1}^4 a_j^* e^{i\beta_j d} = n \times a'' e^{i\alpha_1 d} ,$$

or, upon using (3.13),

$$(3.16) \quad n \times \left[\sum_{j=1}^4 (M_2(\beta_j) a_j^* e^{i\beta_j d}) u_j + u'' M_0 \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ 0 \end{pmatrix} e^{i\alpha_1 d} + v'' M_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\alpha_1 d} \right] = 0 ,$$

$$(3.17) \quad n \times \left[\sum_{j=1}^4 (a_j^* e^{i\beta_j d}) u_j + u'' \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ 0 \end{pmatrix} e^{i\alpha_1 d} + v'' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\alpha_1 d} \right] = 0 .$$

So far we have reduced the problems of solving $u_0 = (e_0, h_0)$ to the system of 8 linear equations (3.10), (3.11), (3.16), (3.17) in 8 unknowns u_j, u', v', u'', v'' .

THEOREM 3.1. *The linear system (3.10), (3.11), (3.16), (3.17) can be reduced to a linear system of four equations in the four unknowns u_j ; furthermore, the coefficient matrix is analytic in $\beta_j, \alpha_k, \gamma, d$ and its determinant does not vanish identically.*

This implies that the determinant can vanish only for discrete set of the parameters α_k, γ, d ; for all other values of the parameters there exists a unique solution u_0 .

Proof. Consider first the system (3.10), (3.11) and write the augmented matrix in the form

$$\left(\begin{array}{ccccc|c} \dots & -\beta_j a_3^{*j} & \dots & 0 & \alpha_1 & -\alpha_1 a_3^0 \\ \dots & -\alpha_2 a_1^{*j} + \beta_j a_2^{*j} & \dots & -\gamma^2 & 0 & -\alpha_2 a_1^0 + \alpha_1 a_2^0 \\ \dots & a_2^{*j} & \dots & \alpha_1 & 0 & a_2^0 \\ \dots & a_3^{*j} & \dots & 0 & 1 & a_3^0 \end{array} \right)$$

where a_k^{*j} is the k -th component of a_j^* . We now perform several elementary operations on the last matrix:

add (row 4) $\times (-\alpha_1)$ to row 1,

add (row 2) $\times (\alpha_1/\gamma^2)$ to row 3,

rotate the last row to be on top of row 1

We then get

$$\left(\begin{array}{ccccc|c} \dots & a_3^{*j} & \dots & 0 & 1 & a_3^0 \\ \dots & (-\beta_j - a_1) a_3^{*j} & \dots & 0 & 0 & -2\alpha_1 a_3^0 \\ \dots & -\alpha_2 a_1^{*j} + \beta_j a_2^{*j} & \dots & -\gamma^2 & 0 & -\alpha_2 a_1^0 + \alpha_1 a_2^0 \\ \dots & a_2^{*j} + \frac{\alpha_1}{\gamma^2} (-\alpha_2 a_1^{*j} + \beta_j a_2^{*j}) & \dots & 0 & 0 & a_2^0 + \frac{\alpha_1^2 a_2^0 - \alpha_1 \alpha_2 a_1^0}{\gamma^2} \end{array} \right)$$

The relations (3.3) and (3.1). show that the right-hand side in the fourth row is equal to $2a_2^0$. We conclude that the augmented matrix can be simplified to

$$(3.18) \quad \left(\begin{array}{ccccc|c} \dots & a_3^{*j} & \dots & 0 & 1 & \alpha_3^0 \\ \dots & (-\beta_j - a_1) a_3^{*j} & \dots & 0 & 0 & -2\alpha_1 a_3^0 \\ \dots & -\alpha_2 a_1^{*j} + \beta_j a_2^{*j} & \dots & -\gamma^2 & 0 & -\alpha_2 a_1^0 + \alpha_1 a_2^0 \\ \dots & \gamma^2 a_2^{*j} - \alpha_1 \alpha_2 a_1^{*j} + \alpha_1 \beta_j a_2^{*j} & \dots & 0 & 0 & 2\gamma^2 a_2^0 \end{array} \right)$$

We now turn to the homogeneous systems (3.16), (3.17), and proceed as before, with α_1 replaced by $-\alpha_1$, in performing the elementary operations on the coefficients matrix. Thus we obtain

$$(3.19) \quad \begin{pmatrix} \dots & \alpha_3^{*j} e^{i\beta_j d} & \dots & 0 & e^{i\alpha_1 d} & 0 \\ \dots & (-\beta_j + \alpha_1) a_3^{*j} e^{i\beta_j d} & \dots & 0 & 0 & 0 \\ \dots & (-\alpha_2 a_1^{*j} + \beta_j a_2^{*j}) e^{i\beta_j d} & \dots & -\gamma^2 e^{i\alpha_1 d} & 0 & 0 \\ \dots & (\gamma^2 a_2^{*j} + \alpha_1 \alpha_2 a_1^{*j} - \alpha_1 \beta_j a_2^{*j}) e^{i\beta_j d} & \dots & 0 & 0 & 0 \end{pmatrix}$$

From the first and third equations in (3.18) we can express v' and u' in terms of the u_j . Similarly from the first and third equations in (3.19) we can express v'' , and u'' uniquely in terms of the u_j . Thus the combined linear simplifies to

$$(3.20) \quad \begin{aligned} \sum_{j=1}^4 (\beta_j + \alpha_1) a_3^{*j} u_j &= 2\alpha_1 a_3^0 , \\ \sum_{j=1}^4 (\gamma^2 a_2^{*j} + \alpha_1 \beta_j a_2^{*j} - \alpha_1 \alpha_2 a_1^{*j}) u_j &= 2\gamma^2 a_2^0 , \\ \sum_{j=1}^4 (-\beta_j + \alpha_1) a_3^{*j} e^{i\beta_j d} u_j &= 0 , \\ \sum_{j=1}^4 (\gamma^2 a_2^{*j} - \alpha_1 \beta_j a_2^{*j} + \alpha_1 \alpha_2 a_1^{*j}) e^{i\beta_j d} u_j &= 0 . \end{aligned}$$

and it is easy to check that the determinant of the coefficients does not vanish identically. This completes the proof of Theorem 3.1.

4. Higher order terms. For any function $w(y)$ ($y \in Y$), set

$$\mathcal{M}w = \int_I w(y) dy .$$

In this section we shall derive differential equations in the region R_2 for the terms $u_1, u_2, \dots$ in (1.5). For simplicity we shall do this only for the magnetic components $h_1, h_2, \dots$. In

the next section we shall show how to actually solve $h_1, h_2, \dots$ in $\mathbb{R}^3$. Once the h_j have been determined, the electric fields e_j can be determined by means of (2.2)

Recall that

$$(4.1) \quad \text{rot}(\varepsilon^{-1} \left(\frac{x}{a} \right) \text{rot } h) - \gamma^2 h = 0 .$$

Following [2] we write

$$(4.2) \quad h = h_0(x) + ah_1(x, y) + a^2 h_2(x, y) + \dots \quad \text{in } R_2 ,$$

where $y = x/a$, and introduce the operators

$$A_1 = \text{rot}_y(\varepsilon^{-1}(y) \text{rot}_y) ,$$

$$A_2 = \text{rot}_y \varepsilon^{-1}(y) \text{rot}_x + \text{rot}_x \varepsilon^{-1}(y) \text{rot}_y ,$$

$$A_3 = \text{rot}_x \varepsilon^{-1}(y) \text{rot}_x - \gamma^2 .$$

Substituting formally (4.2) into (4.1) and comparing powers of a , we get

$$(4.3) \quad A_1 h_0 = 0 ,$$

$$(4.4) \quad A_1 h_1 + A_2 h_0 = 0 ,$$

$$(4.5) \quad A_1 h_2 + A_2 h_1 + A_3 h_0 = 0 ,$$

and generally

$$(4.6) \quad A_1 h_{n+2} + A_2 h_{n+1} + A_3 h_n = 0 \quad (n \geq 1) .$$

We already know that (formally) h_0 satisfies

$$(4.7) \quad \text{rot}_x(Q^{-1} \text{rot } h_0) - \gamma^2 h_0 = 0 .$$

To derive an equation for h_1 we proceed as in [2]. Taking div_x on both sides of (4.5) and observing (by (4.7)) that

$$(4.8) \quad \text{div}_x h_0(x) = 0 ,$$

we obtain

$$(4.9) \quad \text{div}_x \text{rot}_y(\varepsilon^{-1} \text{rot}_y h_2 + \varepsilon^{-1} \text{rot}_x h_1) = 0$$

where $\varepsilon = \varepsilon(y)$.

Taking div_y on both sides of (4.5), we get

$$(4.10) \quad \operatorname{div}_y(\operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_y h_2) + \operatorname{div}_y(\operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_x h_1) - \gamma^2 \operatorname{div}_y h_1 = 0 .$$

Since $\operatorname{div}_y \operatorname{rot}_x = -\operatorname{div}_x \operatorname{rot}_y$, equations (4.9) and (4.10) imply that $\operatorname{div}_y h_1 = 0$. Together with (4.4) we see that h_1 is the solution to the problem:

$$(4.11) \quad \begin{aligned} \operatorname{rot}_y(\varepsilon^{-1} \operatorname{rot}_y h_1 + \varepsilon^{-1} \operatorname{rot}_x h_0) &= 0 , \\ \operatorname{div}_y h_1 &= 0 , \\ h_1 &\text{ is } Y\text{-periodic} . \end{aligned}$$

Denote by e_j the unit vector δ_{ij} . The problem

$$(4.12) \quad \begin{aligned} \operatorname{rot}_y(\varepsilon^{-1} \operatorname{rot}_y \chi^j(y)) &= \operatorname{rot}_y(\varepsilon^{-1} e_j) , \\ \operatorname{div}_y \chi^j(y) &= 0 , \\ \chi^j &\text{ is } Y\text{-periodic} , \\ \mathcal{M}(\chi^j) &= 0 \end{aligned}$$

has a unique solution (by [2]) since $\operatorname{div}_y(\operatorname{rot}_y(\varepsilon^{-1} e_j)) = 0$. Indeed, the solution is the minimizer of the functional

$$J(\varphi) = \frac{1}{2} (\varepsilon^{-1} \operatorname{rot}_y \varphi, \operatorname{rot}_y \varphi) - (\varepsilon^{-1} e_j, \operatorname{rot}_y \varphi)$$

in the space

$$\{\operatorname{div} \varphi = 0, \quad \varphi \text{ is } Y\text{-periodic} , \quad \mathcal{M}(\varphi) = 0\} .$$

Comparing (4.11) with (4.12) it is clear that

$$(4.13) \quad h_1(x, y) = - \sum_{j=1}^3 \chi^j(y) (\operatorname{rot}_x h_0)_j + \tilde{h}_1(x)$$

where $(\operatorname{rot}_x h_0)_j$ is the j -th component of $\operatorname{rot}_x h_0$, and $\tilde{h}_1(x)$ is still to be determined.

Substituting (4.13) into (4.5) we find after some rearrangements that

$$\begin{aligned}
& \operatorname{rot}_y [\varepsilon^{-1} \operatorname{rot}_y h_2 + \varepsilon^{-1} \operatorname{rot}_x \tilde{h}_1(x)] \\
&= \sum_j \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_x (\chi^j(y) (\operatorname{rot}_x h_0)_j)) \\
(4.14) \quad &+ \sum_j \operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_y (\chi^j(y) (\operatorname{rot}_x h_0)_j) - \operatorname{rot}_x (\varepsilon^{-1} \operatorname{rot}_x h_0) + \gamma^2 h_0 \\
&\equiv \operatorname{rot}_y b_1(x, y) + b_2(x, y) \equiv b(x, y) .
\end{aligned}$$

Note that since $\varepsilon(y)$ is discontinuous across ∂Y , the same is true of b_1 .

Clearly b is Y -periodic and using $\operatorname{div}_y \operatorname{rot}_x = -\operatorname{div}_x \operatorname{rot}_y$, we find that

$$\begin{aligned}
\operatorname{div}_y b(x, y) &= \operatorname{div}_y b_2(x, y) \\
&= - \sum_j \operatorname{div}_x \operatorname{rot}_y \varepsilon^{-1} \operatorname{rot}_y (\chi^j(y) (\operatorname{rot}_x h_0)_j) + \operatorname{div}_x \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_x h_0) \\
&= - \sum_j \operatorname{div}_x \operatorname{rot}_y (\varepsilon^{-1} e_j (\operatorname{rot}_x h_0)_j) + \operatorname{div}_x \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_x h_0)
\end{aligned}$$

by (4.12), and the right-hand side vanishes, i.e., $\operatorname{div}_y b(x, y) = 0$. Since $b(x, y)$ is also Y -periodic, for any x there is a unique solution φ_2 of

$$\begin{aligned}
& \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_y \varphi_2(x, y)) = b(x, y) , \\
& \operatorname{div}_y \varphi_2 = 0 , \\
(4.15) \quad & \varphi_2 \text{ is } Y\text{-periodic} , \\
& \mathcal{M}(\varphi_2) = 0 .
\end{aligned}$$

We can now rewrite (4.14) as

$$(4.16) \quad \operatorname{rot}_y [\varepsilon^{-1} \operatorname{rot}_y (h_2 - \varphi_2) + \varepsilon^{-1} \operatorname{rot}_x \tilde{h}_1(x)] = 0 .$$

Let

$$(4.17) \quad w = \varepsilon^{-1} [\operatorname{rot}_y (h_2 - \varphi_2) + \operatorname{rot}_x \tilde{h}_1(x)] .$$

Then

$$(4.18) \quad \operatorname{rot}_y w = 0 ,$$

$$(4.19) \quad \operatorname{div}_y \varepsilon w = 0 .$$

Proceeding as in [2; p. 142] we can derive for w the expression

$$(4.20) \quad w = (I - \operatorname{grad}_y \theta) \mathcal{M}(w)$$

where $(\operatorname{grad}_y \theta) \xi = \operatorname{grad}_y (\theta \xi)$ and

$$(4.21) \quad \begin{aligned} -\operatorname{div}_y (\varepsilon \operatorname{grad}_y \theta^j) &= -\operatorname{div}_y (\varepsilon e_j) , \\ \theta^j &\text{ is } Y\text{-periodic} , \\ \mathcal{M}(\theta^j) &= 0 . \end{aligned}$$

From (4.20) we get

$$(4.22) \quad \mathcal{M}(\varepsilon w) = \operatorname{rot}_x \tilde{h}_1(x) .$$

Substituting w from (4.20) into (4.22) we obtain

$$\mathcal{M}(\varepsilon(I - \operatorname{grad}_y \theta)) \mathcal{M}(w) = \operatorname{rot}_x \tilde{h}_1(x) ,$$

i.e.,

$$\begin{aligned} \mathcal{M}(w) &= \mathcal{M}(\varepsilon(I - \operatorname{grad}_y \theta))^{-1} \operatorname{rot}_x \tilde{h}_1(x) \\ &= Q^{-1} \operatorname{rot}_x \tilde{h}_1(x) . \end{aligned}$$

Thus $\tilde{h}_1$ satisfies the differential equation

$$(4.23) \quad \operatorname{rot}_x (Q^{-1} \operatorname{rot}_x \tilde{h}_1(x)) = \operatorname{rot}_x \mathcal{M}(w) .$$

Applying $\mathcal{M}$ to (4.6) with $n = 1$ we find that

$$\mathcal{M}(\operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_y h_2 + \operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_x h_1 - \gamma^2 h_1) = 0 .$$

Substituting h_1 from (4.13) into (4.23) we can write (since $\mathcal{M}(\chi^j(y)) = 0$)

$$\begin{aligned} & \operatorname{rot}_x \mathcal{M}[\varepsilon^{-1} \operatorname{rot}_y (h_2 - \varphi_2)] + \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_x \tilde{h}_1(x)) - \gamma^2 \tilde{h}_1(x) \\ &= \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_y \varphi_2) + \sum_j \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_x \chi^j(y)) (\operatorname{rot}_x h_0)_j , \end{aligned}$$

i.e. (by (4.17))

$$\begin{aligned} & \operatorname{rot}_x \mathcal{M}(w) - \gamma^2 \tilde{h}_1(x) \\ (4.24) \quad &= \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_y \varphi_2) + \sum_j \operatorname{rot}_x \mathcal{M}[\varepsilon^{-1} \operatorname{rot}_x \tilde{\chi}^j(y) (\operatorname{rot}_x h_0)_j] \equiv c(x) . \end{aligned}$$

Combining this with (4.23), we finally get the desired equation for $\tilde{h}_1$:

$$(4.25) \quad \operatorname{rot}_x (Q^{-1} \operatorname{rot}_x \tilde{h}_1(x)) - \gamma^2 \tilde{h}_1(x) = c(x) \quad \text{in } R_2 .$$

Since $\operatorname{div}_x c(x) = 0$, we deduce from (4.25) that

$$(4.26) \quad \operatorname{div}_x \tilde{h}_1(x) = 0 \quad \text{in } R_2 .$$

We next proceed to derive a differential equation for h_2 . First from (4.16) we deduce that

$$(4.27) \quad h_2(x, y) = \varphi_2(x, y) - \sum_{j=1}^3 \chi^j(y) (\operatorname{rot}_x \tilde{h}_1)_j + \tilde{h}_2(x) .$$

Substituting this into (4.6) with $n = 1$, we obtain

$$\begin{aligned} & \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_y h_3 + \varepsilon^{-1} \operatorname{rot}_x \tilde{h}_2(x)) \\ &= \sum_j \operatorname{rot}_y \varepsilon^{-1} \operatorname{rot}_x (\chi^j(y) \operatorname{rot}_x \tilde{h}_1)_j - \operatorname{rot}_y (\varepsilon^{-1} \operatorname{rot}_x \varphi_2(x, y)) \\ & \quad + \sum_j \operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_y (\chi^j(y) (\operatorname{rot}_x \tilde{h}_1)_j) - \operatorname{rot}_x \varepsilon^{-1} \operatorname{rot}_y \varphi_2(x, y) \\ & \quad - \operatorname{rot}_x (\varepsilon^{-1} \operatorname{rot}_x h_1(x, y)) + \gamma^2 h_1(x, y) \equiv b^*(x, y) . \end{aligned}$$

Proceeding similarly to the calculation of $\operatorname{div}_y b(x, y)$ above and using (4.15) and $\operatorname{div}_y h_1 = 0$, we find that

$$\operatorname{div}_y b^*(x, y) = 0 .$$

Since $b^*(x, y)$ is also Y -periodic, there exists a unique solution $\varphi_3(x, y)$ of

$$(4.28) \quad \begin{aligned} \operatorname{rot}(\varepsilon^{-1} \operatorname{rot}_y \varphi_3(x, y)) &= b^*(x, y) , \\ \operatorname{div}_y \varphi_3 &= 0 , \end{aligned}$$

$$\varphi_3 \text{ is } Y\text{-periodic} , \mathcal{M}(\varphi_3) = 0$$

for each $x \in R_2$.

We can now write

$$(4.29) \quad \operatorname{rot}_y(\varepsilon^{-1} \operatorname{rot}_y(h_3 - \varphi_3) + \varepsilon^{-1} \operatorname{rot}_x \tilde{h}_2(x)) = 0 .$$

The argument which led from (4.16) to (4.23) can be applied to (4.29) to deduce that

$$(4.30) \quad \operatorname{rot}_x(Q^{-1} \operatorname{rot}_x \tilde{h}_2(x)) = \operatorname{rot}_x \mathcal{M}(w)$$

where

$$w = \varepsilon^{-1} \operatorname{rot}_y(h_3 - \varphi_3) + \varepsilon^{-1} \operatorname{rot}_x \tilde{h}_2(x) .$$

Taking $\mathcal{M}$ on both sides of (4.6) with $n = 2$ then yields

$$\begin{aligned} \operatorname{rot}_x \mathcal{M}(w) - \gamma^2 \tilde{h}_2(x) &= \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_y \varphi_3) \\ &+ \sum_j \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_x \chi^j(y)) (\operatorname{rot}_x \tilde{h}_1(x)_j) \\ &- \operatorname{rot}_x \mathcal{M}(\varepsilon^{-1} \operatorname{rot}_x \varphi_2(x, y)) \equiv c^*(x) . \end{aligned}$$

Using (4.30) we find the desired equation for $\tilde{h}_2$:

$$(4.31) \quad \operatorname{rot}_x Q^{-1} \operatorname{rot}_x \tilde{h}_2(x) - \gamma^2 \tilde{h}_2(x) = c^*(x) \quad \text{in } R^2 .$$

Since $\operatorname{div}_x c^*(x) = 0$, we also have

$$(4.32) \quad \operatorname{div}_x \tilde{h}_2(x) = 0 \quad \text{in } R_2 .$$

We can now proceed to derive equations for $h_3, h_4, \dots$ in exactly the same way as for h_2 .

So far we considered the expansion of h in R_2 only. Now we turn to the expansion of h in R_1 and R_3 :

$$(4.33) \quad h = \bar{h}_0(x) + a\bar{h}_1(x, y) + a^2\bar{h}_2(x, y) + \dots \quad \text{in } R_1, y = \frac{x}{a},$$

and

$$(4.34) \quad h = \hat{h}_0(x) + a\hat{h}_1(x, y) + a^2\hat{h}_2(x, y) + \dots \quad \text{in } R_3, y = \frac{x}{a}.$$

The transmission conditions (2.5) at $x_1 = 0$ are:

$$(4.35) \quad n \times (\bar{h}_0 + a\bar{h}_1 + a^2\bar{h}_2 + \dots) = n \times (h_0 + ah_1 + a^2h_2 + \dots),$$

$$(4.36) \quad \begin{aligned} & n \times \{\nabla_x \times [\bar{h}_0 + a\bar{h}_1 + a^2\bar{h}_2 + \dots]\} + n \times \{\nabla_y \times [\bar{h}_1 + a\bar{h}_2 + \dots]\} \\ &= n \times \{Q^{-1}\nabla_x \times [h_0 + ah_1 + a^2h_2 + \dots]\} + n \times \{Q^{-1}\nabla_y \times [h_1 + ah_2 + \dots]\}. \end{aligned}$$

Note that in (4.36) we take Q rather than $\varepsilon(y)$. This is natural in view of the fact established in [2; p. 89] (for elliptic equations) that for a Neumann boundary condition one should replace $\varepsilon^{-1}(y)\frac{\partial}{\partial n}$ by $Q^{-1}\frac{\partial}{\partial n}$ when computing the boundary condition of the limit function h_0 ; this should naturally be the case also for the higher order terms $h_1, h_2, \dots$.

Conditions similar to (4.35), (4.36) hold at $x_1 = d$. The transmission conditions (2.5) for $h_0, \bar{h}_0$ and $\hat{h}_0$ are precisely the zeroth order equations in (4.35), (4.36) and in the corresponding equations at $x_1 = d$ provided we think of $\frac{1}{a}\nabla_y$ as being of degree zero in a . The first order conditions at $x_1 = 0$ are

$$(4.37) \quad n \times \bar{h}_1 = n \times h_1,$$

$$(4.38) \quad n \times (\nabla_x \times \bar{h}_1) + n \times \left(\frac{1}{a} \nabla_y \times \bar{h}_1 \right) = n \times Q^{-1}\nabla_x \times h_1 + n \times \left(\frac{1}{a} Q^{-1}\nabla_y \times h_1 \right),$$

Since we have not yet determined $\tilde{h}_1$, we do not yet know the right-hand sides of (4.37), (4.38). For this reason we proceed formally to consider these relations valid at $x_1 = 0$ for all $y \in Y$ and we take average on y ; this seems to be a reasonable approximation. Recalling that $\tilde{h}_1 = \mathcal{M}h_1(x, \cdot)$ and setting $\widetilde{\bar{h}_1} = \mathcal{M}\bar{h}_1(x, \cdot)$, we then get on $x_1 = 0$:

$$(4.39) \quad n \times \widetilde{\bar{h}_1} = n \times h_1,$$

$$(4.40) \quad n \times \nabla_x \times \widetilde{\bar{h}_1} = n \times Q^{-1}\nabla_x \times \tilde{h}_1.$$

In the next section we shall compute $\tilde{h}_1(x)$ and in Section 6 we shall show how to compute $\bar{h}_1(x, x/a)$.

§5. **Computation of $\tilde{h}_1$.** We shall exploit the special form of $c(x)$, which follows from (4.24), (4.14) and the form of h_0 :

$$(5.1) \quad c(x) = \text{rot}_x \sum_{j=1}^4 \vec{d}_j e^{i(\beta_j x_1 + \alpha_2 x_2)}$$

where $\vec{d}_j$ are constant vectors. We shall assume from now on that

$$(5.2) \quad \beta_j \neq 0 \quad (1 \leq j \leq 4) .$$

Fix $\eta > 0$ such that

$$(5.3) \quad \frac{n\pi}{1+2\eta} \neq \beta_j \quad (1 \leq j \leq 4, \quad n = 0, \pm 1, \pm 2, \dots)$$

(here we need the assumption (5.2)). Fix also a C^∞ function $\zeta(x_1)$ such that

$$\zeta(x_1) = \begin{cases} 1 & \text{if } 0 \leq x_1 \leq d \\ 0 & \text{if } x_1 \leq -\eta \text{ or } x_1 \geq d + \eta . \end{cases}$$

We can expand $\zeta(x_1)e^{i\beta_j x_1}$ into a Fourier series in $[-\eta, d + \eta]$:

$$(5.4) \quad \begin{aligned} \zeta(x_1)e^{i\beta_j x_1} &= \sum_{n=-\infty}^{\infty} a_n^j e^{i\frac{n\pi}{d+2\eta} (x_1+\eta)} \\ &= \sum_{n=-\infty}^{\infty} \tilde{a}_n^j e^{i\frac{n\pi}{d+2\eta} x_1} ; \end{aligned}$$

clearly

$$(5.5) \quad |\tilde{a}_n^j| \leq \frac{C_k}{n^k} \quad \forall k \geq 1 .$$

For $0 \leq x_1 \leq d$ we have:

$$(5.6) \quad c(x) = \sum_{j=1}^4 \sum_{n=-\infty}^{\infty} \text{rot} \left[\tilde{a}_n^j \vec{d}_j e^{i\left(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2\right)} \right] .$$

Defining

$$M(\lambda) = \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & -\lambda \\ -\alpha_2 & \lambda & 0 \end{pmatrix}$$

it follows from (5.3) that

$$\det \left(M^2 \left(\frac{n\pi}{d+2\eta} \right) + \mu^2 Q \right) \neq 0 \quad (n = 0 \pm 1, +2, \dots) .$$

Since $\det(AB + I) = \det(BA + I)$, we deduce that

$$(5.7) \quad \det \left(M \left(\frac{n\pi}{d+2\eta} \right) Q^{-1} M \left(\frac{n\pi}{d+2\eta} \right) + \gamma^2 I \right) \neq 0 \quad \forall n$$

and, further, as easily seen,

$$(5.8) \quad \det \left(M \left(\frac{n\pi}{d+2\eta} \right) Q^{-1} M \left(\frac{n\pi}{d+2\eta} \right) + \gamma^2 I \right) \geq \sigma n^4$$

for all large $|n|$ ($\sigma > 0$) .

We try to find a special solution of (4.25) in the form

$$(5.9) \quad \tilde{h}_1^*(x) = \sum_{j=1}^4 \sum_{n=-\infty}^{\infty} b_{jn} e^{i \left(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2 \right)} .$$

Substituting into (4.25) and using (5.6) we obtain, after comparing coefficients,

$$\left[i M \left(\frac{n\pi}{d+2\eta} \right) Q^{-1} i M \left(\frac{n\pi}{d+2\eta} \right) - \gamma^2 \right] b_{jn} = i M \left(\frac{n\pi}{d+2\eta} \right) \tilde{a}_n^j \vec{d}_j .$$

Recalling (5.7) we deduce that the b_{jn} are uniquely determined,

$$(5.10) \quad b_{jn} = -i \left[M \left(\frac{n\pi}{d+2\eta} \right) Q^{-1} M \left(\frac{n\pi}{d+2\eta} \right) + \gamma^2 \right]^{-1} M \left(\frac{n\pi}{d+2\eta} \right) \tilde{a}_n^j \vec{d}_j ;$$

furthermore, by (5.5), (5.8),

$$(5.11) \quad |b_{jn}| \leq \frac{C_k}{n^k} \quad \forall k \geq 1 .$$

Since

$$w = b_{jn} e^{i\left(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2\right)}$$

solves the equation

$$\text{rot}_x(Q \text{rot}_x w) - \gamma^2 w = \text{rot}_x \tilde{a}_n^j \vec{d}_j e^{i\left(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2\right)},$$

we immediately conclude that $\text{div}_x w = 0$, or

$$(5.12) \quad b_{jn}^{(1)} \frac{n\pi}{d+2\eta} + b_{jn}^{(2)} \alpha_2 = 0$$

where $b_{jn}^{(k)}$ is the k -th component of b_{jn} .

Set

$$b_n = \sum_{j=1}^4 b_{jn}.$$

Then

$$(5.13) \quad \tilde{h}_1^*(x) = \sum_{n=-\infty}^{\infty} b_n e^{i\left(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2\right)}$$

and

$$(5.14) \quad b_n^{(1)} \frac{n\pi}{d+2\eta} + b_n^{(2)} \alpha_2 = 0 \quad (b_n = (b_n^{(1)}, b_n^{(2)}, b_n^{(3)})) .$$

In region R_1 , we must seek $\widetilde{h}_1$ in the form

$$(5.15) \quad \widetilde{h}_1 = \sum_{n=-\infty}^{\infty} \bar{c}_n e^{i(-\alpha_1 x_1 + \alpha_2 x_2)}$$

where

$$\bar{c}_n = u'_n \begin{pmatrix} \alpha_2 \\ \alpha_1 \\ 0 \end{pmatrix} + v'_n \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and in region R_3 we must seek $\widehat{h}_1$ in the form

$$(5.16) \quad \widehat{h}_1 = \sum_{n=-\infty}^{\infty} \hat{c}_n e^{i(\alpha_1 x_1 + \alpha_2 x_2)}$$

where

$$\hat{c}_n = u_n'' \begin{pmatrix} \alpha_2 \\ -\alpha_1 \\ 0 \end{pmatrix} + v_n'' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

In region R_2 the general solution $\tilde{h}_1$ of (4.25) is a sum of the special solution $\tilde{h}_1^*$ and the general solution of the homogeneous equation:

$$(5.17) \quad \tilde{h}_1(x) = \sum_{n=-\infty}^{\infty} \left\{ b_n e^{i(\frac{n\pi}{d+2\eta} x_1 + \alpha_2 x_2)} + \sum_{j=1}^4 u_j^n b_j^* e^{i(\beta_j x_1 + \alpha_2 x_2)} \right\}$$

where the β_j and b_j^* are the same as in §3, i.e.,

$$b_j^* = \frac{1}{\gamma} M(\beta_j) a_j^* , \quad (M^2(\beta_j) + \gamma^2 Q) a_j^* = 0 .$$

We now fix n and write the transmission conditions (4.39), (4.40) for $\tilde{h}_1$ and $\tilde{\bar{h}}_1$ at $x_1 = 0$:

$$\begin{aligned} n \times u_n' \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} + n \times v_n' \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} &= n \times \left(b_n + \sum_{j=1}^4 u_j^n b_j^* \right) , \\ \dots \end{aligned}$$

Similar conditions can be written at $x_1 = d$. The combined system of 8 equations in 8 unknown can be analyzed as in §3, except that we have the additional term b_n . Since however b_n satisfies the condition (5.12), the system can be reduced, as in §3, to four linear equations in four unknowns, with coefficients matrix whose determinant does not vanish in general.

We have so far computed the function $\tilde{h}_1(x)$ as well as the corresponding waves $\tilde{\bar{h}}_1(x), \hat{\bar{h}}_1(x)$ in $\{x_1 < 0\}$ and $\{x_1 > d\}$. Thus in particular, $h_1(x, y)$ is completely determined in R_2 .

§6. Computation of $\bar{h}_1$. Set

$$H_1(x) = h_1 \left(x, \frac{x}{a} \right) , \quad \bar{H}_1(x) = \bar{h}_1 \left(x, \frac{x}{a} \right) .$$

Then the transmission conditions (4.37), (4.38) at $x_1 = 0$ (with $y = x/a$) can be written in the form

$$(6.1) \quad n \times \overline{H}_1 = n \times H_1 ,$$

$$(6.2) \quad n \times (\nabla_x \times \overline{H}_1) = n \times (Q^{-1} \nabla_x \times H_1) .$$

Further, for any a , we wish $\overline{H}_1$ to be a solution of the Maxwell equations in R_1 and to satisfy the correct radiation condition at $x_1 = -\infty$, i.e., $\overline{H}_1$ should be a sum of waves outgoing from R_2 .

Before constructing $\overline{H}_1$ and the corresponding electric field $\overline{E}_1$, notice that the right-hand sides of (6.1), (6.2) have the form

$$(6.3) \quad A\sigma\left(\frac{x_2}{a}, \frac{x_3}{a}\right) e^{i\alpha_2 x_2}$$

where $A = A_1 + A_2/a$, A_1, A_2 are constants independent of a , and $\sigma(y_1, y_2)$ is periodic in y_1 and y_2 of period 1; for the right-hand side of (6.1), $A_2 = 0$. We also note that since the functions $\chi^j(y)$ are C^∞ outside the set where the $\varepsilon_{ij}(y)$ are discontinuous, the functions $\sigma(y_2, y_3)$ are C^∞ .

The Maxwell equations in R_1 with (6.1) (6.2) and the radiation condition at $x_1 = -\infty$ is overdetermined. This is because we took average over (4.37) (4.38) in Y when we independently determined $h_1\left(x, \frac{x}{a}\right)$ in R_2 . The procedure seems reasonable. Now if the solutions $\overline{H}(x), \overline{E}(x)$ exists, then they must satisfy (at least to a good approximation):

$$(6.4) \quad \overline{H}(x) = \int_{\{y_1=0\}} \left\{ \frac{i\omega}{c} \overline{J}(y') \Phi(x - y') + \overline{I}(y') \times \nabla_{y'} \Phi(x - y') \right. \\ \left. + \frac{c}{i\omega} \nabla_x [\overline{J}(y') \cdot \nabla_{y'} \Phi(x - y')] \right\} dy_2 dy_3 ,$$

$$(6.5) \quad \overline{E}(x) = \int_{\{y_1=0\}} \left\{ \frac{i\omega}{c} \overline{I}(y') \Phi(x - y') + \overline{J}(y') \times \nabla_{y'} \Phi(x - y') \right. \\ \left. + \frac{c}{i\omega} \nabla_x [\overline{I}(y') \cdot \nabla_{y'} \Phi(x - y')] \right\} dy_2 dy_3 ,$$

where $y' = (0, y_2, y_3)$, $n = (1, 0, 0)$, $\overline{I} = -n \times \overline{H}_1$, $\overline{J} = n \times \overline{E}_1$ and

$$(6.6) \quad \Phi(z) = \frac{1}{4\pi|z|} e^{i\gamma|z|} \quad (z \in \mathbf{R}^3) .$$

We shall now assume that

$$(6.7) \quad \gamma_{mn} \neq \gamma \quad \text{for} \quad m, n = 0, \pm 1, \pm 2, \dots$$

where

$$(6.8) \quad \gamma_{mn} = \left\{ \left(\alpha_2 + \frac{2\pi m}{a} \right)^2 + \left(\frac{2\pi n}{a} \right)^2 \right\}^{1/2}.$$

and begin to evaluate the right-hand side of (6.4).

THEOREM 6.1. *The function $\bar{H}$ given by (6.4) satisfies:*

$$(6.9) \quad \begin{aligned} \bar{H}(x) = & \sum_{0 \leq |\gamma_{mn}| < \gamma} H_{mn} e^{i[-(\gamma^2 - \gamma_{mn}^2)^{1/2} x_1 + (\alpha_2 + \frac{2\pi m}{a}) x_2 + \frac{2\pi n}{a} x_3]} \\ & + \sum_{|\gamma_{mn}| > \gamma} H_{mn} e^{(\gamma_{mn}^2 - \gamma^2)^{1/2} x_1 + i[(\alpha_2 + \frac{2\pi m}{a}) x_2 + \frac{2\pi n}{a} x_3]} \end{aligned}$$

where H_{mn} are functions of a satisfying

$$(6.10) \quad |H_{mn}| \leq \frac{C_k a^{-2}}{(m^2 + n^2 + 1)^k} \quad \forall k \geq 1.$$

A similar assertion holds in R_3 , with x_1 replaced by $d - x_1$.

Note that the second sum in (6.9) is decaying exponentially as $x_1 \rightarrow -\infty$; the first sum consists of a finite number of terms, and in fact just one term ($m = n = 0$) if a is small enough.

Proof. Consider an integral of the form

$$(6.11) \quad L = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\gamma|x-y'|}}{|x-y'|} \sigma\left(\frac{y_2}{a}, \frac{y_3}{a}\right) e^{i\alpha_2 y_2} dy_2 dy_3.$$

where $\sigma(z_1, z_2)$ is C^∞ and periodic in z_1, z_2 . Observe from (6.5) that $\bar{H}$ is a linear combination of integrals of the form $L = L(x)$ and their first and second derivatives. Consequently, if we can establish that L has the expansion (6.9) with $|H_{mn}| \leq C_k/(m^2 + n^2)^k$, then the assertion of the theorem follows.

We expand σ into its Fourier series

$$(6.12) \quad \sigma(z_2, z_3) = \sum A_{mn} e^{2\pi i(m z_2 + n z_3)}$$

where

$$(6.13) \quad |A_{mn}| \leq \frac{C_k}{(m^2 + n^2)^k} \quad \forall k \geq 1$$

(since $\sigma \in C^\infty$). Then

$$(6.14) \quad L = \frac{1}{4\pi} \sum A_{mn} e^{i[(\alpha_2 + \frac{2\pi m}{a})x_2 + \frac{2\pi n}{a}x_3]} \Gamma_{mn}$$

where

$$(6.15) \quad \Gamma_{mn} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{i\gamma(x_1^2 + \xi_2^2 + \xi_3^2)^{1/2}}}{(x_1^2 + \xi_2^2 + \xi_3^2)^{1/2}} e^{-i[(\alpha_2 + \frac{2\pi m}{a})\xi_2 + \frac{2\pi n}{a}\xi_3]} d\xi_2 d\xi_3 .$$

From [1; (2.10), (2.12)] we see that

$$\Gamma_{mn} = H(\alpha)$$

provided in $F(\alpha, z_2, z_3)$ (in [1]) we replace q by γ and

$$e^{i[(\alpha - q_1)y - q_2 z_2]} \quad \text{by} \quad e^{i[-(\alpha_2 + \frac{2\pi m}{a})y + \frac{2\pi n}{a}z_2]} \quad (\text{in (2.10)}) ,$$

i.e., $\alpha = q_1 - (\alpha_2 + m/a)$, $q_2 = m/a$. The derivation of (2.16), (2.17) for $H(\alpha)$ in [1] can then be carried out with the only difference being that now

$$(6.16) \quad A = \tilde{\alpha} = [(\alpha - q_1)^2 + q_2^2]^{1/2} = \left\{ \left(\alpha_2 + \frac{2\pi m}{a} \right)^2 \left(\frac{2\pi n}{a} \right)^2 \right\}^{1/2} = \gamma_{mn} .$$

Consequently

$$(6.17) \quad \begin{aligned} \Gamma_{mn} &= \frac{\pi i}{(\gamma^2 - A^2)^{1/2}} e^{i(\gamma^2 - A^2)|x_1|} & \text{if} & \quad |A| < \gamma \\ &= \frac{\pi}{(A^2 - \gamma^2)^{1/2}} e^{-(A^2 - \gamma^2)^{1/2}|x_1|} & \text{if} & \quad \gamma < |A| . \end{aligned}$$

Substituting this into (6.14) and recalling (6.13), and that $|x_1| = -x_1$ in R_1 , the assertion of the theorem follows.

§7. Higher order terms. Similarly we can compute $\hat{h}_1$ in R_3 . Next we can proceed to determine h_2 . Here again we first determine $\tilde{h}_2(x)$ by solving the corresponding Maxwell equations in all of $\mathbb{R}^3$ (this is similar to what we may have done for $\tilde{h}_1$). This determines, in particular, the values of h_2 in all of R_2 and we can then determine $\bar{h}_2(x, x/a)$ in R_1 by an integral representation.

The same procedure applies also to h_3, h_4 , etc. We summarize:

THEOREM 7.1. For any positive integer N , the sum of the first $N + 1$ terms in the asymptotic series of h in R_1 is given by

$$\bar{h}_i(x) + \sum_{j=1}^N a^j \bar{h}_j \left(x, \frac{x}{a} \right) ,$$

and the $\bar{h}_j \left(x, \frac{x}{a} \right)$ have the form (if a is small enough)

$$(7.1) \quad h_j \left(x, \frac{x}{a} \right) = H_{00}^{(j)} e^{i(-\alpha_1 x_1 + \alpha_2 x_2)} + \sum_{(m,n) \neq (0,0)} H_{mn}^{(j)} e^{(\gamma_{mn} - \gamma^2)^{1/2} x_1 + i[(\alpha_2 + \frac{2\pi m}{a})x_2 + \frac{2\pi n}{a}x_3]}$$

where $H_{mn}^{(j)}$ are functions of a satisfying

$$(7.2) \quad |H_{mn}^{(j)}| \leq \frac{C_k a^{-2}}{(m^2 + n^2)^k} \quad \text{for any } k \geq 1 .$$

Notice if a is small enough then there is exactly one term in (6.9) which is oscillating as $x_1 \rightarrow -\infty$, namely the term with $m = n = 0$. All other terms decay exponentially as $x_1 \rightarrow -\infty$. This explains the form (7.1).

A result similar to Theorem 7.1 holds in R_3 .

REMARK 7.1. The homogenization formulas have been established for a fixed incoming wave with fixed wave length $\lambda = 2\pi/\gamma$ and small a , or ρ . Thus $\rho/\lambda \ll 1$ (as in the Rayleigh-Gans approximation). It is not clear to what extent the formulas obtained here are valid if ρ/λ is not “very small.” A particular case of interest is when $\rho/\lambda = 1/7$ (as in [10]). In this case, if $a = 8\rho$ (which means that the volume fraction of droplets in the film is $\frac{1}{20}$) then

$$a = \frac{8\lambda}{7} = \frac{8}{7} \frac{2\pi}{\alpha_1} \quad \text{if } \alpha_2 = 0 .$$

It follows that $\frac{2\pi}{a} < \alpha_1$ and therefore the series in (6.9) (and the corresponding series for h_j in (7.1)) contain non-decaying terms with $m = 0$, $n = 1$ and $m = 1$, $n = 0$.

Conclusion. We have made the Ansatz (1.5) in each of three regions R_j and obtained formulas which can be used to compute the solution. In particular, the structure of the magnetic field in R_1 is given by (7.1) (Theorem 7.1). The backscattering terms appear in the sum $\sum_{(m,n) \neq (0,0)}$. A similar formula holds for the forward scattering in R_3 . All the scattering terms are decaying exponentially as functions of the distance from the film.

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